THE DERIVE - NEWSLETTER #107

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+ CAS-TI

Contents:

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Mail from Francisco M. Fernandez, Argentina

Dear Josef,

I am attaching a Derive file that explains a problem that I found when calculating an integral. Is there any reason why Derive behaves in this way?

Best regards,

Marcelo

Derive cannot calculate this integral

$$\int_0^{\infty} - \frac{\sqrt{2}\cdot\sqrt{\pi}\cdot e^{c^2/(2\cdot b) - 2\cdot a\cdot\rho^2}\cdot\rho^{2\cdot s-1}\cdot(16\cdot a^2\cdot b^2\cdot\rho^4 - 16\cdot a\cdot b^2\cdot\rho^2\cdot(s+1) - 4\cdot b^3\cdot(\lambda^2+\rho^2) - 4\cdot b^2\cdot \sim}{8\cdot b^{5/2}}$$

$$\frac{(m^2 - s^2 + \rho^4) - b\cdot\rho^2 - c^2\cdot\rho^2)}{} \, d\rho$$

but if we expand with respect to λ it can do it

… (bulky expression !!)

also expanding with respect to b

$$\frac{2^{(1-2\cdot s)/2}\cdot\sqrt{\pi}\cdot a\cdot\sqrt{b}\cdot e^{c^2/(2\cdot b)}\cdot a^{-s-1}\cdot(2\cdot a\cdot\lambda^2 + s)\cdot(s-1)!}{8\cdot a} +$$

$$\frac{2^{(1-2\cdot s)/2}\cdot\sqrt{\pi}\cdot a^2\cdot e^{c^2/(2\cdot b)}\cdot a^{-s-2}\cdot(4\cdot a^2\cdot(m^2+s) + s\cdot(s+1))\cdot(s-1)!}{16\cdot a^2\cdot\sqrt{b}} +$$

$$\frac{2^{(1-2\cdot s)/2}\cdot\sqrt{\pi}\cdot a\cdot e^{c^2/(2\cdot b)}\cdot a^{-s-1}\cdot s!}{32\cdot a\cdot b^{3/2}} + \frac{2^{(1-2\cdot s)/2}\cdot\sqrt{\pi}\cdot a\cdot c^2\cdot e^{c^2/(2\cdot b)}\cdot a^{-s-1}\cdot s!}{32\cdot a\cdot b^{5/2}}$$

DNL: I could not reproduce the first calculation (Memory exhausted), the second (wrt *b*) one worked, Dear DUG Members,

Yes, I know this DNL is long overdue. There are two articles which kept me very busy. When I received Marcelo's mail asking for *Resultant and Discriminant* I had no idea what he might have in mind. I knew the "resultant" as sum of two vectors and the "discriminant" in connection with a quadratic (and weak memories on a discriminant of a cubic). I believe that some of you didn't hear about like me.

So, I started a web- and book research. And I found out that the resultant of polynomials is playing a very important role in CAS (e.g. for finding common factors of polynomials). The mathematics behind was really beyond my knowledge as school teacher.

But it was an exciting journey into the world of Resultant & Discriminant. (I missed exploring the Subresultants, which might also be important in theory and applications.) I hope that you will enjoy my explorations. I was happy that I found the opportunity to apply the collaboration between DERIVE and David Parker's great DPGraph (http://www.dpgraph.com/) and Peter Scofield's tools.

I am not quite sure that my resultant-function is programmed in the most elegant way and I didn't provide a TI-Nspire realization. It should not be too difficult to do it. Make a try and if possible keep me informed.

The second part of my "Attractors"-contribution brought some work because I wanted to make the Strange Attractors appearing on the TI-Nspire screens, too. It is possible as you will see – less points but quite impressive.

W. Alvermann sent a fine paper on Reuleaux-Triangles and J. Staacke has an interesting question for TI-Nspire Users (see the Forum and next DNL).

I close my letter with pictures from our wonderful autumn – it is not the Indian Summer but end of summer in Upper Austria.

Best regards and wishes

Josef

The *DERIVE-NEWSLETTER* is the Bulletin of the *DERIVE* & CAS-*TI User Group*. It is published at least four times a year with a content of 40 pages minimum. The goals of the *DNL* are to enable the exchange of experiences made with *DERIVE*, *TI*-CAS and other CAS as well to create a group to discuss the possibilities of new methodical and didactical manners in teaching mathematics.

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Contributions:
Please send all contributions to the Editor. Non-English speakers are encouraged to write their contributions in English to reinforce the international touch of the *DNL*. It must be said, though, that non-English articles will be warmly welcomed nonetheless. Your contributions will be edited but not assessed. By submitting articles, the author gives his consent for reprinting it in the *DNL*. The more contributions you will send, the more lively and richer in contents the *DERIVE* & CAS-*TI Newsletter* will be.

Next issue: December 2017

Preview: Contributions waiting to be published

Some simulations of Random Experiments, J. Böhm, AUT, Lorenz Kopp, GER
Wonderful World of Pedal Curves, J. Böhm, AUT
Tools for 3D-Problems, P. Lüke-Rosendahl, GER
Simulating a Graphing Calculator in *DERIVE*, J. Böhm, AUT
Graphics World, Currency Change, P. Charland, CAN
Cubics, Quartics – Interesting features, T. Koller & J. Böhm, AUT
Logos of Companies as an Inspiration for Math Teaching
Exciting Surfaces in the FAZ / Pierre Charland´s Graphics Gallery
BooleanPlots.mth, P. Schofield, UK
Old traditional examples for a CAS – What´s new? J. Böhm, AUT
Mandelbrot and Newton with *DERIVE*, Roman Hašek, CZK
Tutorials for the NSpireCAS, G. Herweyers, BEL
Some Projects with Students, R. Schröder, GER
Dirac Algebra, Clifford Algebra, D. R. Lunsford, USA
A New Approach to Taylor Series, D. Oertel, GER
Rational Hooks, J. Lechner, AUT
Statistics of Shuffling Cards, H. Ludwig, GER
Charge in a Magnetic Field, H. Ludwig, GER
Factoring Trinomials, D. McDougall, CAN
Selected Lectures from TIME 2016
Reuleaux Triangle, W. Alvermann, GER

and others

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Sehr geehrter Herr Böhm,

… Mich interessiert, ob es eine Möglichkeit gibt wie beim V200, im Programm Menüs (mehrseitige Auswahl, die als Programmkopf zum Beispiel stehen bleiben und nicht durch scrollen verschwinden) zu verwenden. Ich habe ihnen einmal das Programm *anageo* angeheftet. Dort ist ein Menü eingearbeitet. Kann man das auch mit LUA realisieren?

Danke und beste Grüße

J. Staacke

Dear Mr Böhm,

… I wonder if there is a possibility to use menus like with the V200 (choice with more pages – top-down-menus – which remain on the top of the screen). I attach the V200-program *anageo* as an example. Is it possible to realize this or similar with LUA?

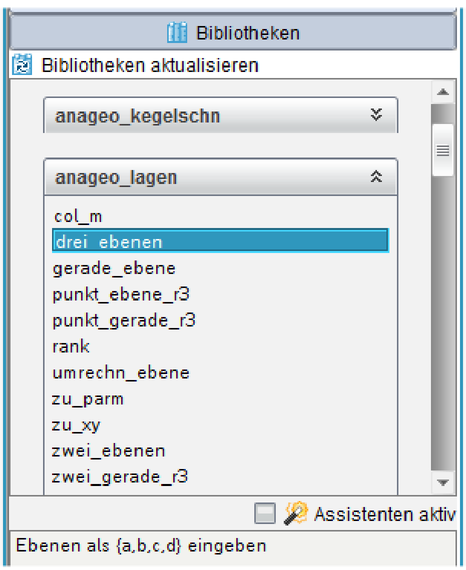
Many thanks and best regards

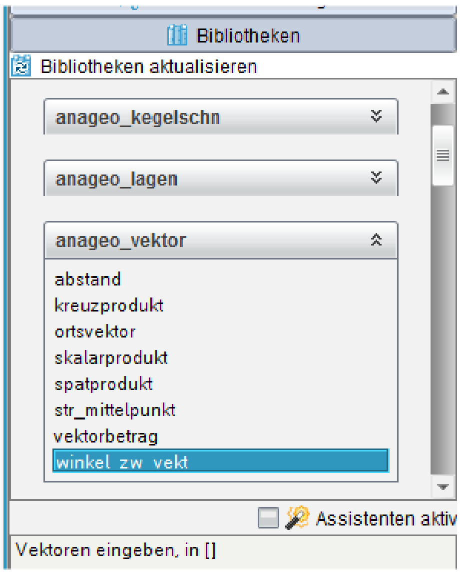
J. Staacke

Dear Mr Staacke,

… I asked some colleagues and we all agree that unfortunately it is impossible to create similar menus on the Nspire screen. (This is one of some other features of the V200 which I am still missing on the Nspire …). It might be possible with LUA – I received a menu driven program package for TI-Innovator from Steve Arnold, but this is not so easy to realize. I recommend to split the package into libraries and store them in the MyLib-folder. Then you can access the programs and functions from everywhere. I started producing sample libraries (now in German, later in English, too).

I extended your library with *drei_ebenen*() to investigate the intersection of three planes in space (was not so easy, and will be presented in one of the next DNLs).





Peter Lüke-Rosendahl, GER

“It is not a geometric sequence …“

0
48
1680
57120
1940448
65918160
2239277040
76069501248

Dear Josef,

…

This about number 48:

Find numbers *x* with:

1) $x+1$ is a square number
2) $\frac{x}{2}+1$ is a square number, too.

48 is such a number. There are others, see attachment.

The quotient of two subsequent numbers is of special interest. It starts with 35, then 34, 33.9714.., 33.970588… 33.9705653, 33.97056273, …………

Seems to converge!

Best regards

Peter

The conjecture of the possible convergence of the quotient was noticed by my son when I sent the "48"-problem for my 7 years old grandson, who likes calculating and puzzling. I thought that he could find the next number using a table of square numbers.

DNL: … I found out some strange patterns. The limit looks exciting. Is there a formula for this sequence? Where did you find this interesting problem?

Peter: … This formula is what I am looking for. I found the problem in *H.F. Ullman*, „Einsteins Rätsel Universum“, (www.ullmannmedien.com) page 18.

DNL: … I was lucky and found your sequence in the “Bible for Sequences” (Sloane’s). The respective link is https://oeis.org/A008845. Don’t follow the link if you want to work on your own .

Peter: Great, I think that I will have a look!

DNL: … Now it is an easy job to “*DERIVE*“ the limit! See the attached file together with a short *DERIVE* program. I wonder how one could find this formula?

Regards to you, your son and your grandson and thanks for the very nice problem

$$a(n) := \frac{-6 + (3 - 2\cdot\sqrt{2})\cdot(17 + 12\cdot\sqrt{2})^{-n} + (3 + 2\cdot\sqrt{2})\cdot(17 + 12\cdot\sqrt{2})^{n}}{4}$$

```
    peter(n, k, n1, n2, list) :=
      Prog
        k := 1
        list := []
        Loop
          If k > n exit
#1:       n1 := k^2 - 1
          n2 := n1/2 + 1
          If INTEGER?(√n2)
             list := APPEND(list, [[n1, k, √n2]])
          k :+ 2
        list
```

#3: peter(500000) =

$$\begin{bmatrix} 0 & 1 & 1 \\ 48 & 7 & 5 \\ 1680 & 41 & 29 \\ 57120 & 239 & 169 \\ 1940448 & 1393 & 985 \\ 65918160 & 8119 & 5741 \\ 2239277040 & 47321 & 33461 \\ 76069501248 & 275807 & 195025 \end{bmatrix}$$

1145 sec

#4: table :=

$$\begin{bmatrix} 0 & 1 & 1 \\ 48 & 7 & 5 \\ 1680 & 41 & 29 \\ 57120 & 239 & 169 \\ 1940448 & 1393 & 985 \\ 65918160 & 8119 & 5741 \\ 2239277040 & 47321 & 33461 \\ 76069501248 & 275807 & 195025 \end{bmatrix}$$

#5: [c1 := table↓↓1, c2 := table↓↓2, c3 := table↓↓3]

#6: c1 := [0, 48, 1680, 57120, 1940448, 65918160, 2239277040, 76069501248]

#7: c2 = [1, 7, 41, 239, 1393, 8119, 47321, 275807]

#8: c3 = [1, 5, 29, 169, 985, 5741, 33461, 195025]

#9: $$a(n) := \frac{-6 + (3 - 2\cdot\sqrt{2})\cdot(17 + 12\cdot\sqrt{2})^{-n} + (3 + 2\cdot\sqrt{2})\cdot(17 + 12\cdot\sqrt{2})^{n}}{4}$$

#10: TABLE(a(n), n, 1, 7) =

$$\begin{bmatrix} 1 & 48 \\ 2 & 1680 \\ 3 & 57120 \\ 4 & 1940448 \\ 5 & 65918160 \\ 6 & 2239277040 \\ 7 & 76069501248 \end{bmatrix}$$

#11: $$\lim_{n\to\infty} \frac{a(n + 1)}{a(n)} = 12\cdot\sqrt{2} + 17$$

#12: $$\lim_{n\to\infty} \frac{a(n + 1)}{a(n)} = 33.97056274$$

It was mid-June when I received a mail from Argentina asking for a *DERIVE* program to calculate the "discriminant" and "resultant". I must admit that I had no idea about the resultant and the only discriminant I knew was the one of the quadratic equation $b^2 - 4ac$ and I believed to remember that there is a discriminant for the cubic equation, too. When I informed using websites, downloaded pdf-files and other resources I found out that Maxima; MAPLE and *MATHEMATICA* have implemented functions for resultant and discriminant as well.

This is the extended story of our very intense email- and files exchange, enjoy.

From Marcelo June 15, 2017

Dear Josef,

I could not find any Derive program for the calculation of the discriminant of a polynomial or the resultant of two polynomials.

Do you know if someone has provided something like that?

Best regards,

Francisco Marcelo Fernández

From Josef June 16, 2017

Dear Marcelo,

as far as I do know there is no program for your problems. I will put your request into the next DNL-issue. Can you send one or two examples - then I will try to do it on my own.

Best regards to Argentina,

Josef

From Josef June 16, 2017

Dear Marcelo,

here I am once more. I found some examples and I will try to produce a program to calculate the resultant of two polynomials (determinant of the respective Sylvester Matrix).

Regards

Josef

From Marcelo June 16, 2017

Dear Josef,

Thank you very much for your interest in my problems. I am lazy and started using Maple that has built in functions for resultant and discriminant, but I think that it would be useful to have such programs in Derive.

Best regards,

Marcelo

From Josef June 18, 2017

Dear Marcelo,

I believe that I could program functions resultant and discriminant for polynomials. Please check the attached *DERIVE* file.

Best regards

Josef

For all of you who are in the same situation as I was in June, I'd like to give some information which I found:

Wikipedia:

In mathematics, the **resultant** of two polynomials is a polynomial expression of their coefficients, which is equal to zero if and only if the polynomials have a common root (possibly in a field extension), or, equivalently, a common factor (over their field of coefficients). In some older texts, the resultant is also called **eliminant**.

It is calculated as the determinant of the **Sylvester matrix**.

So, first of all, we need to know, what is the Sylvester matrix of two polynomials? And then it will be interesting, what we can do with *resultant* and *discriminant*?

Let $p(x)$ and $q(x)$ two polynomials.

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 = \prod_{i=1}^{n}(x-\alpha_i) \text{ and}$$

$$q(x) = b_m x^m + b_{m-1}x^{m-1} + \ldots + b_1 x + b_0 = \prod_{j=1}^{m}(x-\beta_j) .$$

The **Sylvester matrix** is a $(m+n)\times(m+n)$ matrix defined by the coefficients of the polynomials as follows. Its determinant is the resultant $R(p,q)$.

$$\begin{pmatrix} a_n & a_{n-1} & \cdots & a_1 & a_0 & & & \\ & a_n & a_{n-1} & & & & & \\ \vdots & & \ddots & \ddots & & & & \\ & & & & & & a_1 & a_0 \\ b_m & b_{m-1} & \cdots & & b_0 & & & \\ & b_m & b_{m-1} & \ddots & \ddots & b_0 & & \\ \vdots & & & & & & \ddots & \\ & & & & & & b_1 & b_0 \end{pmatrix}$$

The spaces are filled with zeros. An example shall illustrate this:

$$\left[f := 2\cdot x^4 - 1,\ g := 3\cdot x^3 + 4\cdot x - 1 \right]$$

$$\text{DET}\begin{bmatrix} 2 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & -1 \\ 3 & 0 & 4 & -1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 4 & -1 & 0 & 0 \\ 0 & 0 & 3 & 0 & 4 & -1 & 0 \\ 0 & 0 & 0 & 3 & 0 & 4 & -1 \end{bmatrix} = -713$$

Before presenting my `resultant`-function I'd like to check another interesting property of the resultant of two polynomials. Let α_i the m roots of $p(x)$ and β_j the n roots of $q(x)$ then the resultant is given by

$$a_n^{\,m} b_m^{\,n} \prod_{i=1}^{n} \prod_{j=1}^{m} (\alpha_i - \beta_j)\,.$$

The roots of g from above are not so nice, but we will try:

$$(\rho := \mathrm{SOLUTIONS}(f = 0,\ x)) = \rho := \left[\frac{2^{3/4}}{2},\ -\frac{2^{3/4}}{2},\ \frac{2^{3/4}\cdot i}{2},\ -\frac{2^{3/4}\cdot i}{2}\right]$$

$$(\sigma := \mathrm{SOLUTIONS}(g = 0,\ x)) = \sigma := \left[\frac{(4\cdot\sqrt{337} + 36)^{1/3}}{6} - \frac{(4\cdot\sqrt{337} - 36)^{1/3}}{6},\ \frac{(4\cdot\sqrt{337} - 36)^{1/3}}{12} - \frac{(4\cdot\sqrt{337} + 36)^{1/3}}{12} + i\cdot\left(\frac{\sqrt{3}\cdot(4\cdot\sqrt{337} - 36)^{1/3}}{12} + \frac{\sqrt{3}\cdot(4\cdot\sqrt{337} + 36)^{1/3}}{12}\right),\ \frac{(4\cdot\sqrt{337} - 36)^{1/3}}{12} - \frac{(4\cdot\sqrt{337} + 36)^{1/3}}{12} - i\cdot\left(\frac{\sqrt{3}\cdot(4\cdot\sqrt{337} - 36)^{1/3}}{12} + \frac{\sqrt{3}\cdot(4\cdot\sqrt{337} + 36)^{1/3}}{12}\right)\right]$$

It really works!!

$$2^3\cdot 3^4\cdot \prod_{i=1}^{4} \prod_{j=1}^{3} (\rho_i - \sigma_j) = -713$$

From Marcelo June 18, 2017

Dear Josef,

I did some simple tests and your programs seem to work properly; thank you very much.

I have just learned about these kinds of functions of polynomials and I think that they may be useful for other people as well.

Best regards,

Marcelo

From Marcelo June 21, 2017

Dear Josef,

I am attaching a Derive program that shows a difficulty that I had when trying to write a program using resultant. I think that the attached Derive 6 program is clear enough. It does not work in Derive 5 either. Curiously, I did not have problems with a similar construction using a somewhat different matrix. I think that the problem is more related with Derive than with your very nice resultant program. I hope this finding may be useful.

Best regards,

Marcelo

The final – and correct program – will be presented together with applications at the end of this "Chronicle", Josef

From Josef June 21, 2017

Really interesting.
I tried some "tricks", but it does not work.
I will include "`resultant`" together with your problem and other applications into DNL#107.
Maybe that we will get any reaction. Unfortunately, Albert Rich is no longer involved in *DERIVE* affairs - nevertheless, I will try to reach him.

Best regards and many thanks for the interesting contribution.

Can you write a short paper on the problem - together with the final solution + explication?
This would be great.
Josef

From Marcelo June 21, 2017

Dear Josef,
I do not really know how to solve that problem and for that reason I cannot provide "the final solution + explication". Some Derive programs do not work within other programs. I think that it happened to me some time ago but I do not remember the problem now. The most curious think is that in the dfw-file that I am attaching "resultant" works perfectly within "eigs". The problem is similar but I am using a nonorthogonal basis set and the characteristic polynomials are somewhat different.
Best regards,
Marcelo

From Josef June 21, 2017

Dear Marcelo,
many thanks for providing this successful operation with `resultant`. I am still working on your last example. All what I could find out is, that within your program even `poly_degree(p1)` does not work. The program hangs up.
I didn't have in mind explication of the *DERIVE* bug and its "final solution". I had in mind the mathematical problem which you are tackling with `resultant()`.
Best regards and once more thanks for the exciting communication.
Josef

From Marcelo June 22, 2017

Dear Josef,
When I learned about the resultant of two polynomials and the discriminant of a polynomial I began applying it to some well-known problems in quantum mechanics. The two examples that I sent you are about the secular equation arising from the Rayleigh-Ritz method. The first example has some error in the definition of the matrix but it is not important for the discussion of the bug in my Derive program.
Your resultant and the Kronecker product appearing in one DNL are two quite useful programs. I wrote a pedagogical paper some time ago about Kronecker (Eur. J. Phys. 37 (2016) 065403) and I am using both together in another amusing application.
Best regards,
Marcelo

From Marcelo June 22, 2017

Dear Josef,

Maybe I misunderstood your mail. Did you suggest that I write something on the application of the resultant to the variational method in quantum mechanics? At first I thought that you were asking me to write something about the strange bug in my program and its solution. That is why I answer that I could not do that. If you think that the application of resultant() to the QM problem may be interesting, just tell me which format you handle in DNL. In the past you only accepted Word files.

Best regards,

Marcelo

From Josef June 24, 2017

Hi Marcelo,

that's great. Yes, I had the application of the resultant in mind.

You can send a WORD-file or a pdf-file, does not matter.

We could combine your application and the *DERIVE*-problem. I'll try to contact Albert Rich.

Best regards

Josef

From Marcelo June 27, 2017

Dear Josef,

I am attaching a pdf file with the description of the problem and a Derive 6 file with some calculations. Please, tell me if this is what you had in mind. Note that I solved the problem by splitting the program eigs() into two programs to avoid memory exhaustion.

Best regards,

Marcelo

From Josef June 28, 2017

Dear Marcelo,

many thanks for your paper.

This is really "Upper Class Mathematics" – a "little bit" beyond my mathematics knowledge (school teacher).

Nevertheless, it will give a good impression what can be done using the resultant of two polynomials.

Best regards and many thanks for your efforts,

Josef

From Josef to Albert Rich July 12, 2017

Dear Albert,

how are you? Hope that you enjoy Hawaii? Just recently we could watch a TV-report about these lovely and marvelous islands. We are very happy that we – thanks *DERIVE* – had the opportunity to visit this wonderful part of the world many years ago.

Dear Albert, I am sorry to bother you again with a *DERIVE* question, which seems to be interesting.

I received a request from Marcelo Fernandez (Argentina) about a *DERIVE* built-in function to find the resultant of two polynomials and the discriminant of one polynomial.

I must admit that I had no idea what this could be?

What to do? I informed about resultant and discriminant and tried to produce the respective functions (programs).
And interestingly enough, both worked – compared with MATHEMATICA- and other CAS-results. This will be a nice article for DNL#107, but …
… I sent my file to Marcelo. He came back with an application and he found out that resultant-function does not work within another function in one case, but it does in another case.
We could not find the reason for that.
It would be great if you could explain this bug? (in my program, a deficiency of DERIVE, a memory problem, …?)
Mahalo Nui (hope this is correct?)
Josef

From Albert Rich to Josef July 14, 2017

Hello Josef,

It's great to hear from you, and be reassured that the Derive community remains alive and well.

Having not worked with Derive in years, it took way longer than it should have, but I finally figured out how to resolve the evaluation problem Marcelo encountered. Comments in the attached dfw file labelled ADR: describe the simple changes to the functions "resultant" and "matel" required to fix the problem.

Hope this helps.

Aloha from Hawaii Island (aka the "Big Island" of Hawaii),

Albert

From Josef July 15, 2017
Dear Marcelo,
As I promised I sent our resultant-collaboration to Albert Rich and asked for advice with respect to the strange behaviour.
He answered two days later and presented the – easy – solution. I didn't include a variable as local into the list of arguments.
The question remains why the function did work in one case and did not in the other one.
I attach the file.
Best regards
Josef

From Marcelo July 15, 2017
Dear Josef,
I think that Derive may be quite baffling. Thank you very much
Marcelo

On the next page you can find my `resultant(polynomial_1, polynomial_2, variable)`. Some applications will follow. Then I will come back to the 2nd part of the headline – the discriminant. The chronicle will be closed with Marcelo's article mentioned above together with parts of the respective *DERIVE*-file.

The first three expressions shall present the Sylvester matrices and their determinants of two polynomials depending on the chosen variable *x* or *y*.

$$\left[q1 := x^2 + a\cdot y + 1,\ q2 := x\cdot y + b + y^2\right]$$

$$\mathrm{DET}\begin{bmatrix} 1 & 0 & a\cdot y + 1 \\ y & b + y^2 & 0 \\ 0 & y & b + y^2 \end{bmatrix} = y^4 + a\cdot y^3 + y^2\cdot(2\cdot b + 1) + b^2$$

$$\mathrm{DET}\begin{bmatrix} a & x^2 + 1 & 0 \\ 0 & a & x^2 + 1 \\ 1 & x & b \end{bmatrix} = x^4 - a\cdot x^3 + 2\cdot x^2 - a\cdot x + a^2\cdot b + 1$$

Three auxiliary functions. `lc` (leading coefficient) will be used in Marcelo's application.

```
#10:  [pd(p, x) := POLY_DEGREE(p, x), pc(p, x, n) := POLY_COEFF(p, x, n)]

#11:  lc(p, x) := pc(p, x, pd(p, x))
```

$$\#12:\quad \left[pp1 := x^4 - 1,\ pp2 := x^3 + 4\right]$$

```
#13:  VECTOR(pc(pp1, x, k), k, pd(pp1, x), 0, -1) = [1, 0, 0, 0, -1]
```

$$\#14:\quad [lc(pp1, x), lc(q1, y), lc(f, x), lc(f, y)] = \left[1, a, 2, 2\cdot x^4 - 1\right]$$

This is my program:

```
      resultant(p1, p2, x, syl, n, m, cp1, cp2, i, j) :=
        Prog
          n := pd(p1, x)
          m := pd(p2, x)
          cp1 := VECTOR(pc(p1, x, k), k, n, 0, -1)
          cp2 := VECTOR(pc(p2, x, k), k, m, 0, -1)
          syl := VECTOR(VECTOR(0, i, n + m), j, n + m)
          i := 1
          Loop
            If i > m exit
            j := 1
            Loop
              If j > n + 1 exit
              syl↓i↓(j + i - 1) := cp1↓j
#17:          j :+ 1
            i :+ 1
          i := m + 1
          Loop
            If i > m + n exit
            j := 1
            Loop
              If j > m + 1 exit
              syl↓i↓(j + i - 1 - m) := cp2↓j
              j :+ 1
            i :+ 1
          i := 1
          "RETURN syl"
          DET(syl)
```

The first tests are successful (variable x by default):

$$\text{resultant}(f,\ g) = -713$$

$$\text{resultant}(q1,\ q2) = y^4 + a\cdot y^3 + y^2\cdot(2\cdot b + 1) + b^2$$

$$\text{resultant}(q1,\ q2,\ y) = x^4 - a\cdot x^3 + 2\cdot x^2 - a\cdot x + a^2\cdot b + 1$$

On page 8 I showed an interesting connection between the roots α_i and β_j of the polynomials and the resultant based on the product of the differences of the roots. But there is also a connection based on the sums of the roots;

All roots of the polynomial resulting as resultant of $q(t)$ and $p(x-t)$ are of the form $\alpha_i + \beta_j$.

$$\left[pp1 := x^4 - 1,\ pp2 := x^3 + 4\right]$$

$$(\rho := \text{SOLUTIONS}(pp1 = 0,\ x)) = \rho := [1,\ -1,\ i,\ -i]$$

$$(\sigma := \text{SOLUTIONS}(pp2 = 0,\ x)) = \sigma := \left[-2^{2/3},\ \frac{2^{2/3}}{2} + \frac{\sqrt{3}\cdot 2^{2/3}\cdot i}{2},\ \frac{2^{2/3}}{2} - \frac{\sqrt{3}\cdot 2^{2/3}\cdot i}{2}\right]$$

$$\prod_{i=1}^{4}\prod_{j=1}^{3}(\rho_i - \sigma_j) = 255$$

$$h(x) := \text{resultant}\left(\lim_{x\to t} pp1,\ \lim_{x\to x - t} pp2,\ t\right)$$

$$h(x) := x^{12} + 16\cdot x^9 - 3\cdot x^8 + 96\cdot x^6 + 192\cdot x^5 + 3\cdot x^4 + 256\cdot x^3 - 480\cdot x^2 + 48\cdot x + 255$$

```
sums := APPEND(VECTOR(VECTOR(α + β, α, ρ), β, σ))

VECTOR(h(x), x, sums)

[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]
```

My `resultant()` worked properly but when Marcelo used it as a subroutine within another function it hang up sometimes. Albert Rich found the reason. I did not include the loop variable j like loop variable i into the parameter list to make it local. Now it worked as expected.

Ok, this was quite nice until here. The question is, what can we do with the resultant? I found in my internet recherche that most CAS have it implemented – *DERIVE* does not – and it is widely used. (Maybe that calculation and use of the resultant is implemented internally in *DERIVE*, too?)

In *Modern Computer Algebra* (J. von zur Gathen, J. Gerhard) dedicated more than 50 pages to the resultant and modular gcd algorithms. (“The resultant and gcd computation”). I must admit that this was as a whole too heavy mathematics for me.

But I was successful searching – and finding – some applications of the resultant in Wikipedia and some other publications (see the references) which are not so complicated.

1 Elimination

Let's take the following nonlinear system as first example:

$$25x^2 - x^2y^2 = 9$$
$$4x^2 + y = 0$$

DERIVE has no problem to solve this system (same with TI-NspireCAS):

$$\left[eq1 := x^2 \cdot y^2 - 25 \cdot x^2 + 9,\ eq2 := 4 \cdot x^2 + y\right]$$

SOLUTIONS(eq1 ∧ eq2, [x, y])

$$\begin{bmatrix} 1 & -4 \\ -1 & -4 \\ \frac{\sqrt{\sqrt{13} - 2}}{2} & 2 - \sqrt{13} \\ -\frac{\sqrt{\sqrt{13} - 2}}{2} & 2 - \sqrt{13} \\ \frac{i \cdot \sqrt{\sqrt{13} + 2}}{2} & \sqrt{13} + 2 \\ -\frac{i \cdot \sqrt{\sqrt{13} + 2}}{2} & \sqrt{13} + 2 \end{bmatrix}$$

I could imagine that the resultant is built-in in the CAS. It works as follows:

$$\text{resultant}(eq1, eq2) = (y^3 - 25 \cdot y - 36)^2$$

$$\text{resultant}(eq1, eq2, y) = 16 \cdot x^6 - 25 \cdot x^2 + 9$$

The resultant of these two bivariate polynomials is a single univariate polynomial, so the variable x has been eliminated. This resulting polynomial shares properties with its parents, f(x y) and g(x; y), but is easier to analyze than its parents.

Since two polynomials share a root if and only if their resultant is zero, we take the resultant polynomial and set it equal to zero. By finding roots of the resultant relative to x, we are finding values of y that will make the resultant relative to x zero, therefore making the two polynomials share a non-constant factor in the variable x. (*Henry Woody, Polynomial Resultants*).

We produce canditates for possible solutions by solving the resultants for *x* and *y*:

$$x\_cand := \text{SOLUTIONS}(16 \cdot x^6 - 25 \cdot x^2 + 9, x)$$

$$x\_cand := \left[1, -1, \frac{\sqrt{\sqrt{13} - 2}}{2}, -\frac{\sqrt{\sqrt{13} - 2}}{2}, \frac{i \cdot \sqrt{\sqrt{13} + 2}}{2}, -\frac{i \cdot \sqrt{\sqrt{13} + 2}}{2}\right]$$

$$\text{y\_cand} := \text{SOLUTIONS}((y^3 - 25\cdot y - 36)^2,\ y)$$

$$\text{y\_cand} := [-4,\ \sqrt{13} + 2,\ 2 - \sqrt{13}]$$

Next expression presents all possible combinations of the candidates:

```
VECTOR(VECTOR([u, v], u, x_cand), v, y_cand)
```

$$\left[\begin{bmatrix} 1 & -4 \\ -1 & -4 \\ \dfrac{\sqrt{(\sqrt{13} - 2)}}{2} & -4 \\ -\dfrac{\sqrt{(\sqrt{13} - 2)}}{2} & -4 \\ \dfrac{i\cdot\sqrt{(\sqrt{13} + 2)}}{2} & -4 \\ -\dfrac{i\cdot\sqrt{(\sqrt{13} + 2)}}{2} & -4 \end{bmatrix}, \begin{bmatrix} 1 & \sqrt{13} + 2 \\ -1 & \sqrt{13} + 2 \\ \dfrac{\sqrt{(\sqrt{13} - 2)}}{2} & \sqrt{13} + 2 \\ -\dfrac{\sqrt{(\sqrt{13} - 2)}}{2} & \sqrt{13} + 2 \\ \dfrac{i\cdot\sqrt{(\sqrt{13} + 2)}}{2} & \sqrt{13} + 2 \\ -\dfrac{i\cdot\sqrt{(\sqrt{13} + 2)}}{2} & \sqrt{13} + 2 \end{bmatrix}, \begin{bmatrix} 1 & 2 - \sqrt{13} \\ -1 & 2 - \sqrt{13} \\ \dfrac{\sqrt{(\sqrt{13} - 2)}}{2} & 2 - \sqrt{13} \\ -\dfrac{\sqrt{(\sqrt{13} - 2)}}{2} & 2 - \sqrt{13} \\ \dfrac{i\cdot\sqrt{(\sqrt{13} + 2)}}{2} & 2 - \sqrt{13} \\ -\dfrac{i\cdot\sqrt{(\sqrt{13} + 2)}}{2} & 2 - \sqrt{13} \end{bmatrix}\right]$$

What remains is, selecting those pairs which satisfy both equations, i.e. eq1 = 0 and eq2 = 0.

$$(\text{SELECT}(w_3 = [0,\ 0],\ w,\ \text{APPEND}(\text{VECTOR}(\text{VECTOR}([u,\ v,\ [\text{SUBST}(eq1,\ [x,\ y],\ [u,\ v]),\ \text{SUBST}(eq2,\ [x,\ y],\ [u,\ v])]],\ u,\ \text{x\_cand}),\ v,\ \text{y\_cand})))\downarrow\downarrow[1,\ 2]$$

This is the set of all solutions.

$$\begin{bmatrix} 1 & -4 \\ -1 & -4 \\ \dfrac{i\cdot\sqrt{(\sqrt{13} + 2)}}{2} & \sqrt{13} + 2 \\ -\dfrac{i\cdot\sqrt{(\sqrt{13} + 2)}}{2} & \sqrt{13} + 2 \\ \dfrac{\sqrt{(\sqrt{13} - 2)}}{2} & 2 - \sqrt{13} \\ -\dfrac{\sqrt{(\sqrt{13} - 2)}}{2} & 2 - \sqrt{13} \end{bmatrix}$$

The next example is a “little bit” more demanding.

We would be glad to have at least one real solution of the system $ff(x,y) = 0$ and $gg(x,y) = 0$

$$ff(x,\ y) := 3\cdot x^2\cdot y^2 - 4\cdot y^3 - x^2 + 4\cdot y + x\cdot y^2 + 2$$

$$gg(x,\ y) := 2\cdot x\cdot y + 3\cdot y^3 - 2\cdot x^3 - 2\cdot x - 5\cdot x^2\cdot y^2 - 5$$

```
resultant(ff(x, y), gg(x, y), x)
```

$$363\cdot y^{12} - 77\cdot y^{11} - 1282\cdot y^{10} + 321\cdot y^{9} + 1620\cdot y^{8} - 757\cdot y^{7} - 728\cdot y^{6} + 1015\cdot y^{5} - 156\cdot y^{4} - 779\cdot y^{3} + 178\cdot y^{2} + 315\cdot y + 47$$

$$\text{NSOLVE}(363\cdot y^{12} - 77\cdot y^{11} - 1282\cdot y^{10} + 321\cdot y^{9} + 1620\cdot y^{8} - 757\cdot y^{7} - 728\cdot y^{6} + 1015\cdot y^{5} - 156\cdot y^{4} - 779\cdot y^{3} + 178\cdot y^{2} + 315\cdot y + 47, y, \text{Real})$$

```
y = 1.209364839 ∨ y = 1.195689338

NSOLVE(ff(x, 1.209364839) = 0, x)

x = -0.2273588786 ∨ x = 0.2273588786

[gg(-0.2273588786, 1.209364839), gg(0.2273588786, 1.209364839)]

[-0.1433932235, 0]
```

One solution is $x \approx 0.22736$, $y \approx 1.20936$.

Try:

$$\text{SOLVE}\left(\left[3\cdot x^{2}\cdot y^{2} - 4\cdot y^{3} - x^{2} + 4\cdot y + x^{2}\cdot y + 2,\ 2\cdot x\cdot y + 3\cdot y^{3} - 2\cdot x^{3} - 2\cdot x - 5\cdot x^{2}\cdot y^{2} - 5\right], [x, y]\right)$$

We can extend this approach for polynomial systems of more variables.

$$\text{eqq1} := x^{2} + x\cdot y + z^{2} + z\cdot x - 10$$

```
eqq2 := x·y·z + x + y + z - 20

eqq3 := 2·x + 3·y - z

eqq4 := resultant(eqq1, eqq2, x)

eqq5 := resultant(eqq2, eqq3, x)

eqq6 := resultant(eqq4, eqq5, y)

SOLVE([eqq1, eqq2, eqq3], [x, y, z])
```

Compare the result of eqq6 with the direct *DERIVE* calculation using SOLVE.

Comment in Wikipedia with regard to this generalization for extended systems:

For getting a correct algorithm two complements have to be added to the method. Firstly, at each step, a linear change of variable may be needed in order that the degrees of the polynomials in the last variable are the same as their total degree. Secondly, if, at any step, the resultant is zero, this means that the polynomials have a common factor and that the solutions split in two components. One, were the common factor is zero, and the other which is obtained by factoring out this common factor before continuing.

A headline in *Pablo A. Parrilo*'s paper is:

2 Implicitation of rational curves

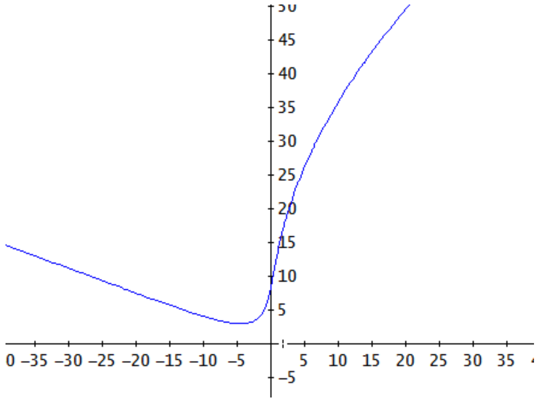
The only content of this paragraph is: "To be completed". So let me try to complete!

A curve is given by its parameter form as follows:

$$\left[\frac{t^3}{2} - t^2 + 2\cdot t - 1,\ 2\cdot t^2 + 4\cdot t + 5\right]$$

We would like to have it in implicit form.

My – personal – trick was until now to solve the system:



$$\text{SOLVE}\left(\left[x = \frac{t^3}{2} - t^2 + 2\cdot t - 1,\ y = 2\cdot t^2 + 4\cdot t + 5\right], [x, t]\right)$$

$$\left[x = -\frac{\sqrt{2}\cdot((y + 19)\cdot\sqrt{(y - 3)} + 5\cdot\sqrt{2}\cdot y + 3\cdot\sqrt{2})}{8} \wedge t = -\frac{\sqrt{2}\cdot\sqrt{(y - 3)}}{2} - 1,\ x = \frac{\sqrt{2}\cdot((y + 19)\cdot\sqrt{(y - 3)} - 5\cdot\sqrt{2}\cdot y - 3\cdot\sqrt{2})}{8} \wedge t = \frac{\sqrt{2}\cdot\sqrt{(y - 3)}}{2} - 1\right]$$

This gives the graph in two parts. I can proceed with some manipulations of the two expressions (remove the squares, etc) in order to achieve finally the requested form.

$$4\cdot(4\cdot x + 5\cdot y + 3)^2 = 2\cdot(y - 3)\cdot(y + 19)^2$$

Applying resultant is much easier:

$$\text{resultant}\left(x - \left(\frac{t^3}{2} - t^2 + 2\cdot t - 1\right),\ y - (2\cdot t^2 + 4\cdot t + 5),\ t\right) = 0$$

$$-\frac{32\cdot x^2 + 16\cdot x\cdot(5\cdot y + 3) - y^3 + 15\cdot y^2 - 187\cdot y + 1101}{4} = 0$$

It is an easy job to show that both equations are equivalent.

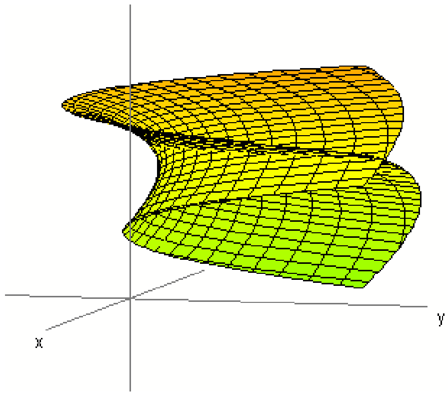
The question is: Can we extend this for three dimensions, too?

Yes,we can!

Take the follwing surface:

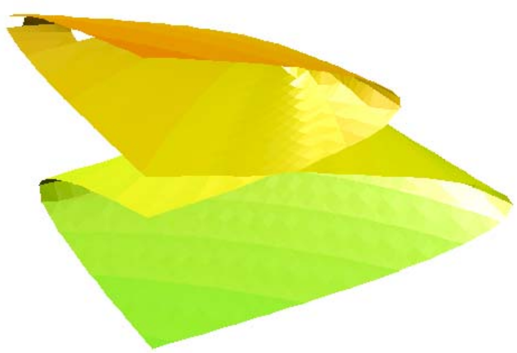
$$\left[5 + 4\cdot s^2 - t,\ 3 - s + 2\cdot t^2,\ 10 + 2\cdot s - t\right]$$

I know that we cannot do implicit 3D-plots. But let me try and then I will find a way to plot the resulting implicit form.




```
surfs1 := resultant(5 + 4·s^2 – t – x, 3 – s + 2·t^2 – y, t)

surfs2 := resultant(5 + 4·s^2 – t – x, 10 + 2·s – t – z, t)

resultant(surfs1, surfs2, s)

64·(16·x^2 – x·(16·y + 32·z^2 – 624·z + 3149) + 4·y^2 – 2·y·(8·z^2 – 176·z +
  935) + 16·z^4 – 640·z^3 + 9820·z^2 – 68419·z + 182389)

TimesOperator := Asterisk

64*(16*x^2 – x*(16*y + 32*z^2 – 624*z + 3149) + 4*y^2 – 2*y*(8*z^2 – 176*z +
  935) + 16*z^4 – 640*z^3 + 9820*z^2 – 68419*z + 182389)
```

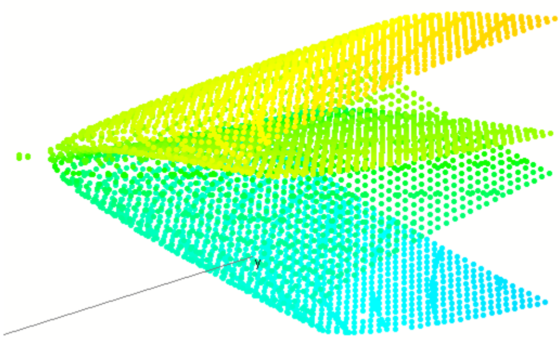
I eliminate parameters *s* and *t*. As it is impossible to solve for $z = z(x,y)$ we cannot plot the surface in its implicit form. But there is DPGraph (see DNL#5 and DNL#46), which supports *DERIVE*. I change the multiplication operator to an asterisk and click on  in the menu bar. Then I find my self in DPGraph. Some adjustments are necessary to obtain the right plot region and to fix the plot properties … and this is it:

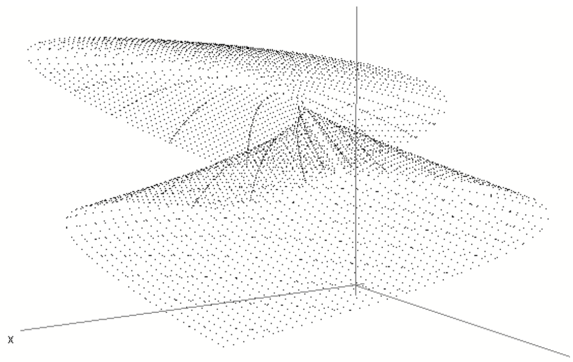


It is true that we cannot produce a 3D implicit plot with *DERIVE* but at this occasion I remember a tool provided by Peter Schofied in DNL#64, page33, in his article *Implicit Plots in 3D* (published 2006).

I loaded file IMPLICIT_Peter.mth and Peter made it possible!!

```
ImplicitPts(64·(16·x^2 - x·(16·y + 32·z^2 - 624·z + 3149) + 4·y^2 - 2·y·(8·z^2
  - 176·z + 935) + 16·z^4 - 640·z^3 + 9820·z^2 - 68419·z + 182389), [1, 1,
  1], [20, 20, 20], 0.5)

ImplicitDots(64·(16·x^2 - x·(16·y + 32·z^2 - 624·z + 3149) + 4·y^2 -
  2·y·(8·z^2 - 176·z + 935) + 16·z^4 - 640·z^3 + 9820·z^2 - 68419·z + 182389),
  [1, 1, 1], [20, 20, 20], 0.5)
```





3 Random Matrices

This is the next headline in Parillos's paper and again: "To be completed". I searched in my books and on the web as well. I could find only one paper which might fit to this objective.

https://math.berkeley.edu/~rezakhan/randommatrix.pdf

4 GCD of polynomials

This should be an important application. In the papers given below you will find resultant and subresultant mentioned very often together with many proofs. I believe that this is too special for our Newsletter and would lead us too far away from everydays mathematics.

http://www.informatik.uni-leipzig.de/~graebe/vorlesungen/polynome/ch3.pdf

Next link is quite interesting because it leads to a Computer Algebra lecture given at the famous RISC-Institute which was founded by Bruno Buchberger (Father of the Gröbner Bases).

http://www.risc.jku.at/education/courses/ws2010/ca/

http://www.risc.jku.at/education/courses/ws2011/ca/3-gcd.pdf

http://www.risc.jku.at/education/courses/ws2010/ca/4-resultants.pdf

I will turn now to the "discriminant" of a polynomial.

Again Wikipedia:

The **discriminant** of a polynomial is a polynomial function of its coefficients, which allows deducing some properties of the roots without computing them.

The discriminant Δ of a polynomial $p(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ can be defined in two ways:

- by its roots α_i: $\Delta = a_n^{2n-2}\prod_{i<j}(\alpha_i - \alpha_j)^2 = (-1)^{\frac{n(n-1)}{2}} a_n^{2n-2}\prod_{i\neq j}(\alpha_i - \alpha_j)$

- by the resultant: $\Delta = \frac{(-1)^{\frac{n(n-1)}{2}}}{a_n} res(p(x), p'(x)).$

The DERIVE realisation:

$$\text{discriminant}(p, x) := \frac{(-1)^{\text{pd}(p,\ x)\cdot(\text{pd}(p,\ x)\ -\ 1)/2}\cdot 1}{\text{pc}(p,\ x,\ \text{pd}(p,\ x))}\cdot\text{resultant}\left(p,\ \frac{d}{dx}\ p,\ x\right)$$

$$\text{discriminant}(a\cdot x^2 + b\cdot x + c) = b^2 - 4\cdot a\cdot c$$

$$\text{discriminant}(a\cdot x^3 + b\cdot x^2 + c\cdot x + d) = -27\cdot a^2\cdot d^2 + 18\cdot a\cdot b\cdot c\cdot d - 4\cdot a\cdot c^3 - 4\cdot b^3\cdot d + b^2\cdot c^2$$

$$\text{discriminant}(x^3 + p\cdot x + q) = -4\cdot p^3 - 27\cdot q^2$$

We know the importance of the discriminant for a quadratic, similar conclusions can be drawn for cubic and quartic.

If Δ is negative, then the cubic has one real and two complex solutions, if it is positive then there are three real solutions and if it is zero, then we have at least two equal zeros. See examples:

$$\text{discriminant}(2\cdot x^3 + 2\cdot x^2 + 5\cdot x - 10) = -14980$$

$$\text{SOLUTIONS}(2\cdot x^3 + 2\cdot x^2 + 5\cdot x - 10,\ x) = [1.064,\ -1.032 + 1.905\cdot i,\ -1.032 - 1.905\cdot i]$$

$$\text{discriminant}(2\cdot x^3 - 10\cdot x^2 + 5\cdot x + 20) = 2300$$

$$\text{SOLUTIONS}(2\cdot x^3 - 10\cdot x^2 + 5\cdot x + 20,\ x) = [-1.092,\ 3.398,\ 2.693]$$

$$\text{discriminant}(28\cdot x^3 - 64\cdot x^2 + 3\cdot x + 45) = 0$$

$$\text{SOLUTIONS}(28\cdot x^3 - 64\cdot x^2 + 3\cdot x + 45,\ x) = \left[\frac{3}{2},\ -\frac{5}{7}\right]$$

I will skip the quartic, it can be found in textbooks and in the web.

Finally, we will have a look on the “roots-definition”:

$$\text{discriminant}(2\cdot x^4 - 1) = -2048$$

$$(\xi 2 := \text{SOLUTIONS}(2\cdot x^4 - 1,\ x)) = \xi 2 := \left[\frac{2^{3/4}}{2},\ -\frac{2^{3/4}}{2},\ \frac{2^{3/4}\cdot i}{2},\ -\frac{2^{3/4}\cdot i}{2}\right]$$

$$2^{2\cdot 4-2}\cdot(\xi 2_1 - \xi 2_2)^2\cdot(\xi 2_1 - \xi 2_3)^2\cdot(\xi 2_1 - \xi 2_4)^2\cdot(\xi 2_2 - \xi 2_3)^2\cdot(\xi 2_2 - \xi 2_4)^2\cdot(\xi 2_3 - \xi 2_4)^2$$

-2048

By the way, a negative discriminant of a quartic indicates two real and two complex roots, a positive one indicates four real or four complex roots.

It was just for (programming-) fun to have a function available which selects the pairs of roots (α_i,α_j) with $i < j$ in order to have an easier way calculating the product of the squares.

$$\text{pa}(n) := \text{SORT}(\text{SELECT}(u_1 < u_2,\ u,\ \text{APPEND}(\text{VECTOR}(\text{VECTOR}([i,\ j],\ i,\ 1,\ n - 1),\ j,\ 2,\ n))))$$

$$\text{pa}(4) = \begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 1 & 4 \\ 2 & 3 \\ 2 & 4 \\ 3 & 4 \end{bmatrix}$$

$$2^{4\cdot 2-2}\cdot\prod\left((\xi 2_{u,1} - \xi 2_{u,2})^2,\ u,\ \text{pa}(4)\right) = -2048$$

Additional references (Web resources)

Svante Janson, Resultant and Discriminant of Polynomials
http://studylib.net/doc/8187084/resultant-and-discriminant-of-polynomials-1.-resultant

Andries E. Brouwer, Resultant and Discriminant
https://www.win.tue.nl/~aeb/2WF02/resultant.pdf

Henry Woody, Polynomial Resultants
http://buzzard.ups.edu/courses/2016spring/projects/woody-resultants-ups-434-2016.pdf

Pablo A. Parillo (Chapter 6 of a whole course)
https://ocw.mit.edu/courses/electrical-engineering-and-computer-science/6-972-algebraic-techniques-and-semidefinite-optimization-spring-2006/lecture-notes/lecture_06.pdf

Attracted by (STRANGE) Attractors (2)

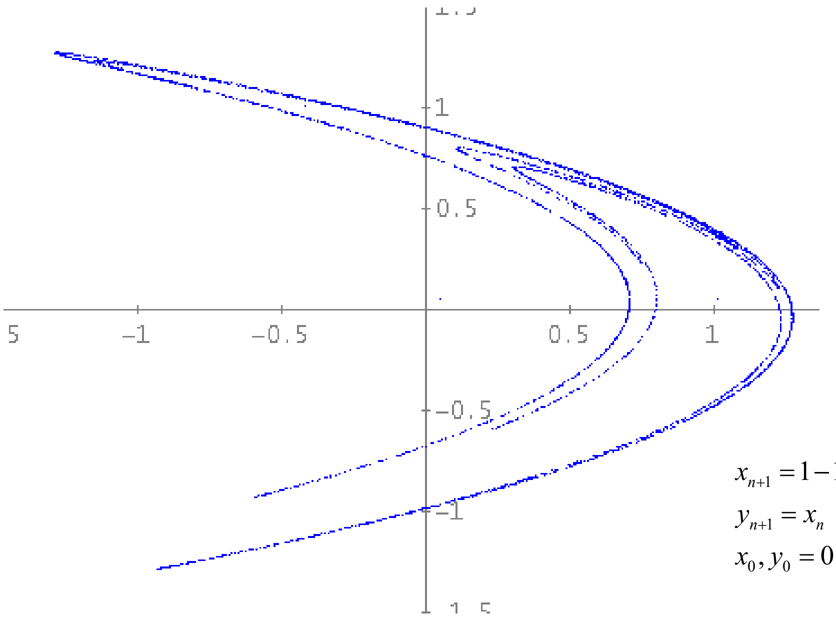
An illustrated guided tour from well-known to unknown attractors

Your Tour Guide: Josef Böhm

5 A famous two-dimensional quadratic mapping

Let’s have a look to Julien C. Sprott. The attractor I have in mind is hidden behind his code “EWM?MPMMWMMMM”.

```
decode(EWM?MPMMWMMMM) = [1, 0, -1.4, 0, 0.3, 0, 0, 1, 0, 0, 0, 0]

qmap_2(EWM?MPMMWMMMM, 4000)
```

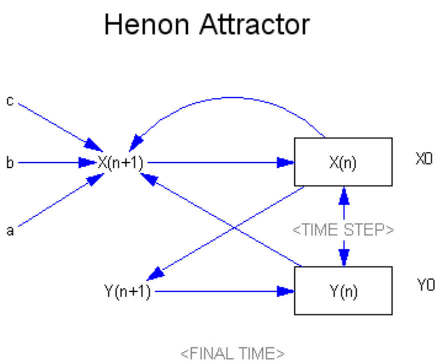


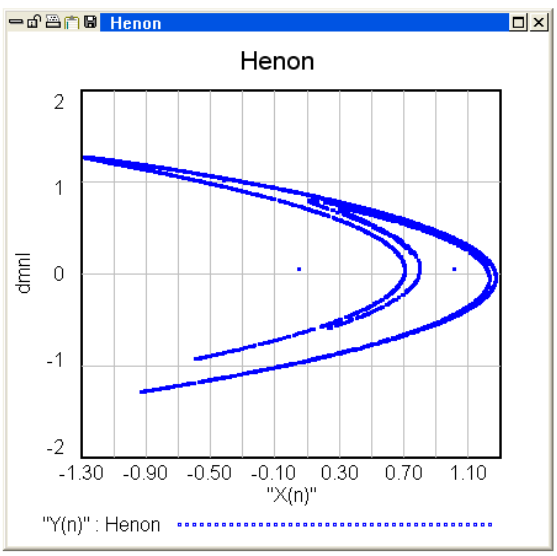
$$x_{n+1} = 1 - 1.4 x_n^2 + 0.3 y_n$$
$$y_{n+1} = x_n$$
$$x_0, y_0 = 0.05$$

You will surely recognize the famous Hénon Attractor.

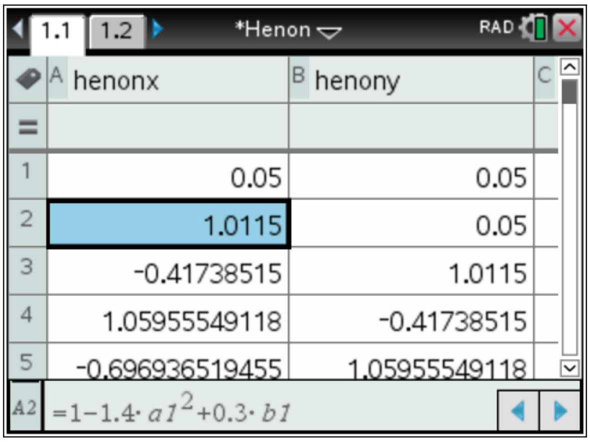
Please allow a side step. I use a program for modelling dynamic systems -VENSIM PLE- (http://vensim.com/vensim-software/) for plotting the Hénon attractor:

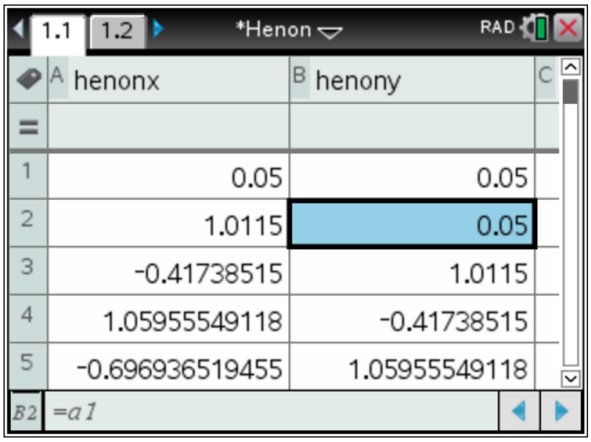
$$x_{n+1} = 1 + a\,x_n^2 + b\,y_n$$
$$y_{n+1} = x_n$$
x_0, y_0 given



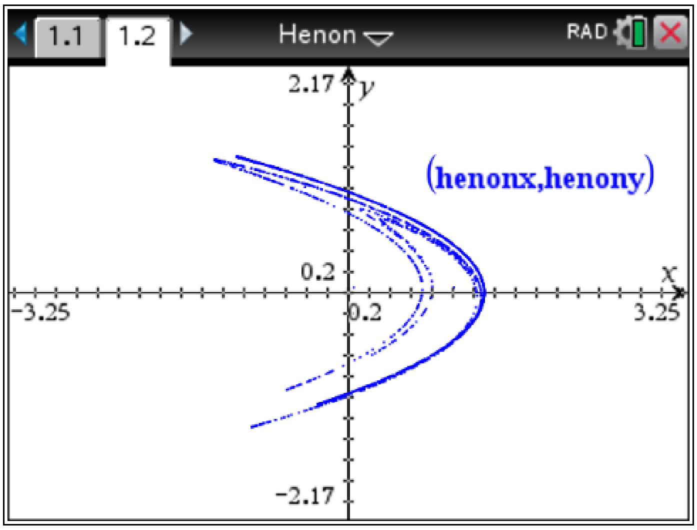


The easiest way to create the Hénon attractor on a TI-Nspire screen is defining the two recursive equations in the spreadsheet application as shown below.





We can copy down 2500 rows maximum (with the software) and then plot the scatter diagram. We recognize that the graph looks a little bit different (maybe some inaccuracy?).

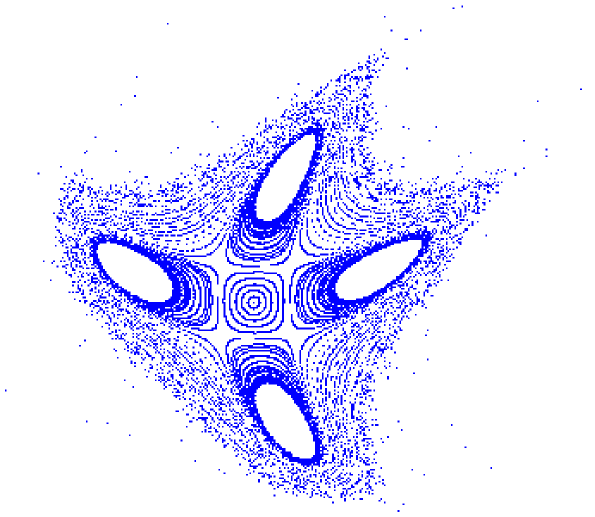


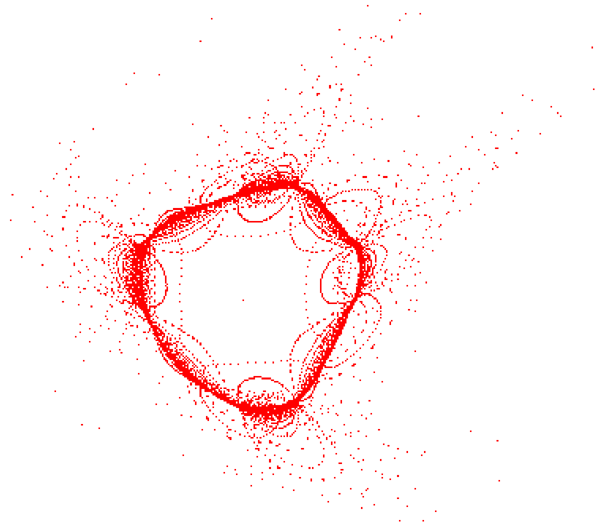
Before showing other quadratic mappings, I'd like to present other Hénon systems.

```
henon(start, α, n, list := [], i, xn, yn, xo, yo) :=
  Prog
    i := 1
    [xn := start↓1, yn := start↓2]
    Loop
      If i > n
         RETURN list
      list := APPEND(list, [[xn, yn]])
      [xo := xn, yo := yn]
      xn := xo·COS(α) - (yo - xo^2)·SIN(α)
      yn := xo·SIN(α) + (yo - xo^2)·COS(α)
      If ABS(xn) > 10000
         xn := 0
      If ABS(yn) > 10000
         yn := 0
      i :+ 1
```

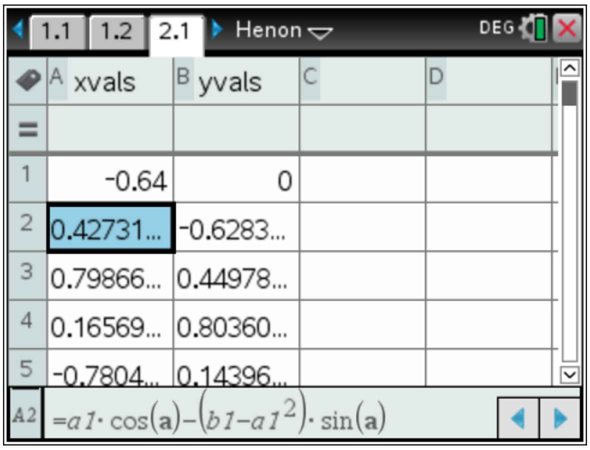
These are not attractors but variations of the initial point result in exciting graphs of chaotic distributed points in the plane forming unexpected patters.

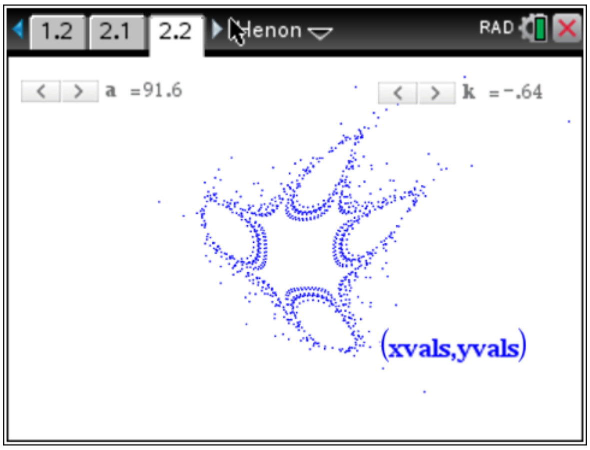
```
VECTOR(henon([k, 0], 90.57·1°, 400), k, -0.7, 0.7, 0.025)

VECTOR(henon([0.5, 0], k·1°, 400), k, 75, 95, 0.5)
```

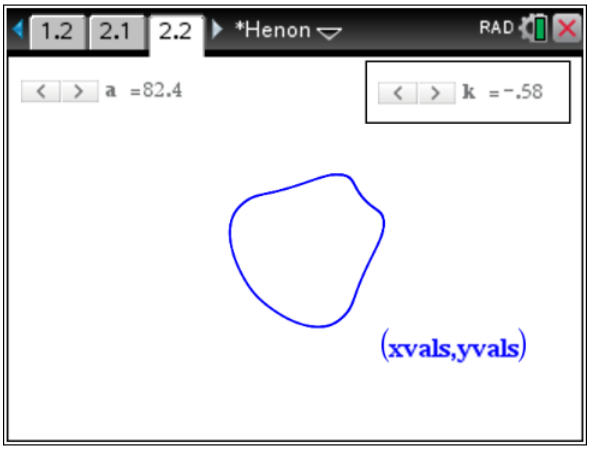


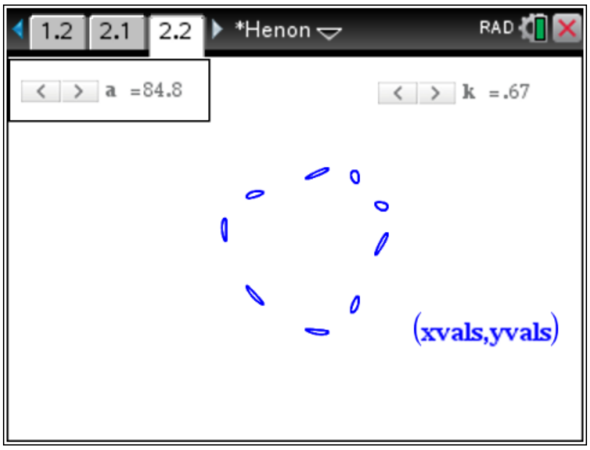


We can vary the initial point (left) or the angle (right). Unfortunately, we cannot produce the families of scatter diagrams with the TI-Nspire-environment. But what we can do is varying the parameters by a slider and observe how the "members of the family" are changing.

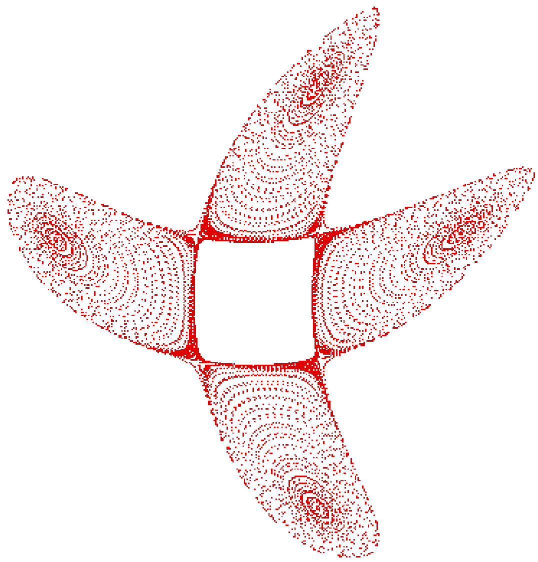


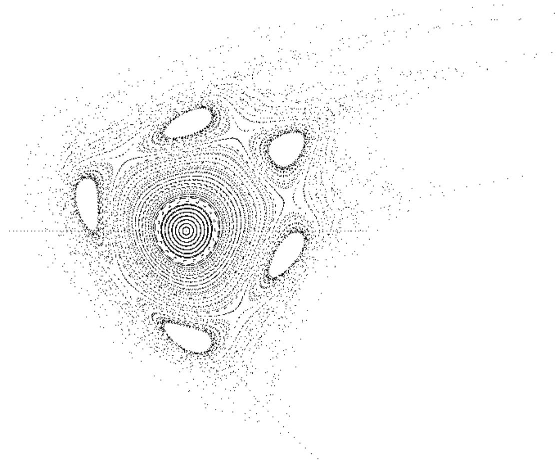


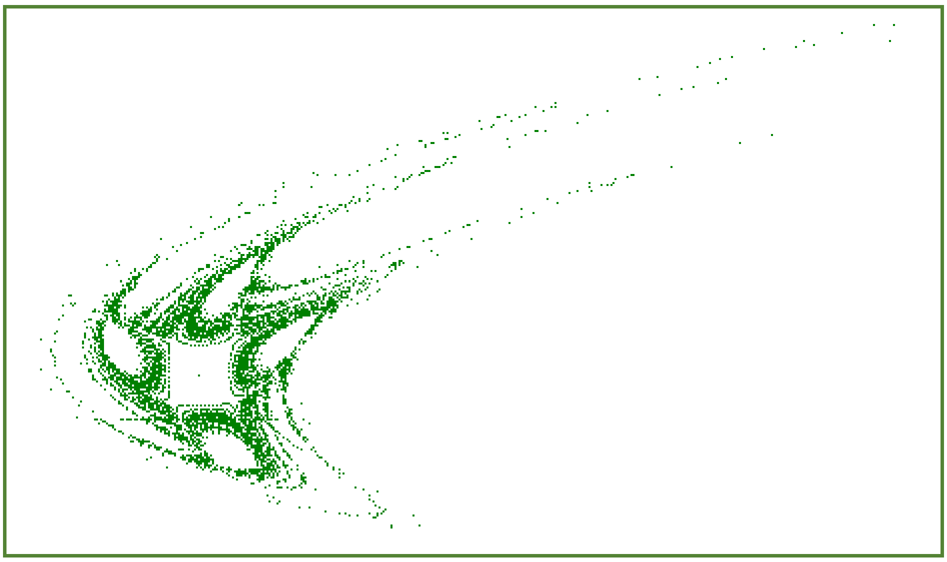


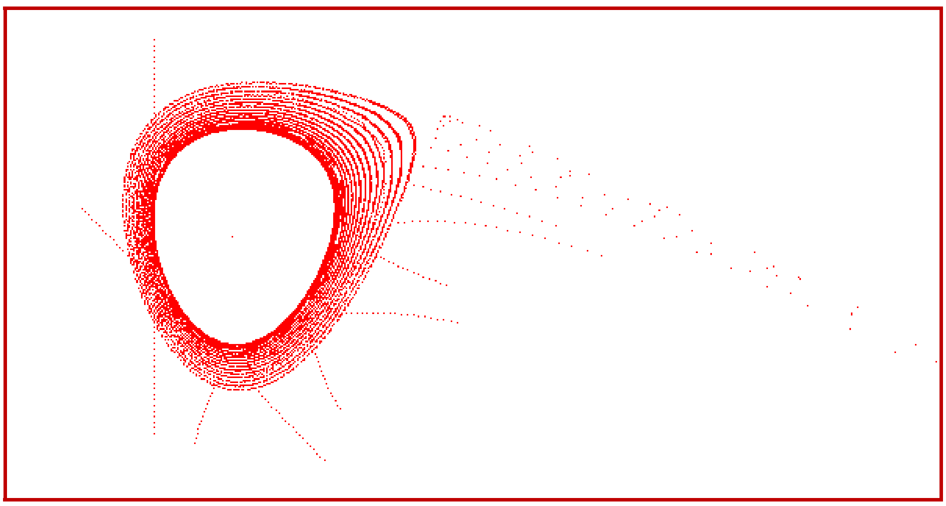


More nice *DERIVE*-generated Hénon families:



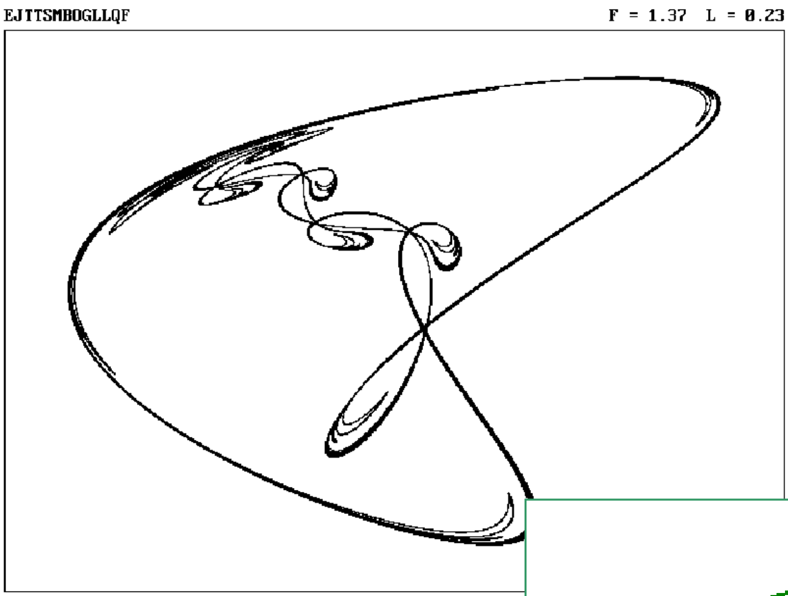


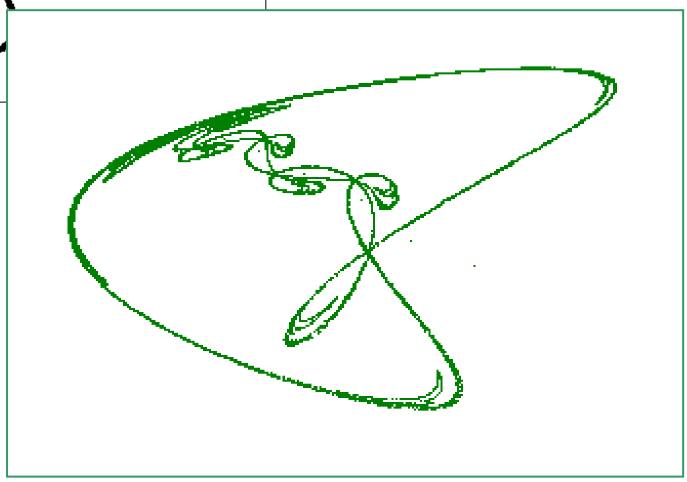




6 Generalized polynomial maps

The next program (*DERIVE* and TI-Nspire) is to plot polynomial maps. As a reference for my program `map` I used one of the many graphs in Sprott's book .





`map(EJTTSMBOGLLQF, 20000)`

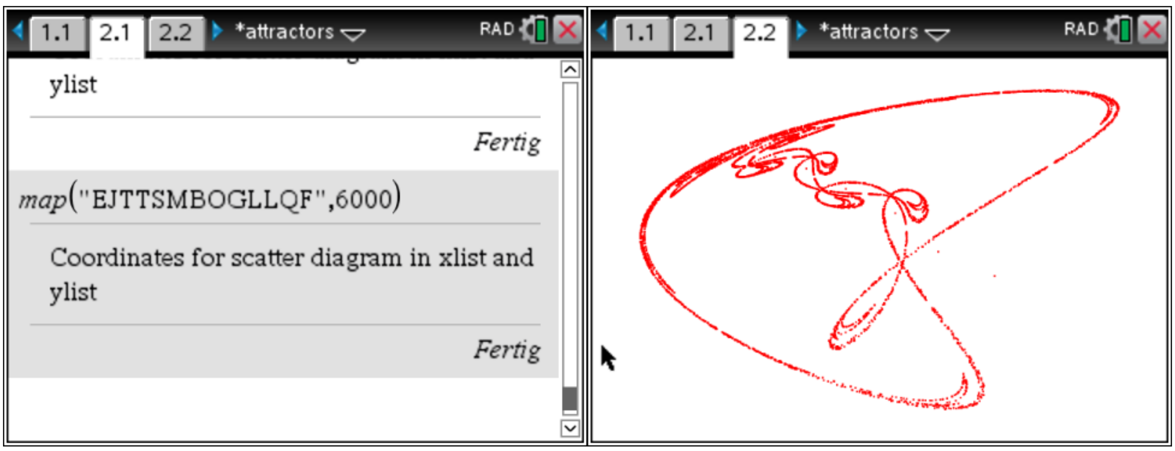
My graph produced with my program `map` looks quite the same (20 000 points!).

Sprott's program returns the Lyapunov Exponent and the fractal dimension (which I didn't include in my investigations).

His L =0.23 is confirmed by my `ljap_e("EJTTSMBOGLLQF",1000)`.

`ljap_e(EJTTSMBOGLLQF, 1000) = 0.24121`

TI_Nspire's *map*() cannon plot 20000 points, but as you can see 6000 points can do it also.



It is possible to produce the plots using the spreadsheet app, too. But here we can only produce 2500 points maximum – and it might be that we would miss some accuracy. I prefer the program.

Both map-programs understand Sprott's codes and lists of coefficients as well.

Sprott lists all his maps in an appendix. The two-dimensional cubic (case F) map needs 20 coefficients.

Case F: D = 2, O = 3, M = 20

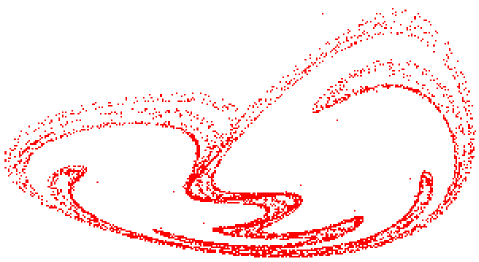
$$X = a_1 + a_2X + a_3X^2 + a_4X^3 + a_5X^2Y + a_6XY + a_7XY^2 + a_8Y + a_9Y^2 + a_{10}Y^3$$

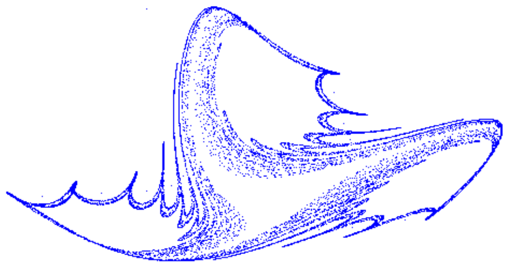
$$Y = a_{11} + a_{12}X + a_{13}X^2 + a_{14}X^3 + a_{15}X^2Y + a_{16}XY + a_{17}XY^2 + a_{18}Y + a_{19}Y^2 + a_{20}Y^3$$

The list of coefficients reads in the order [a_1, a_2, …, a_{20}].

Some examples from Sprott's collection

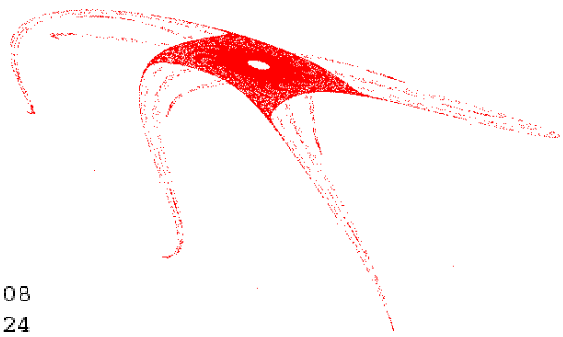
Cubic maps





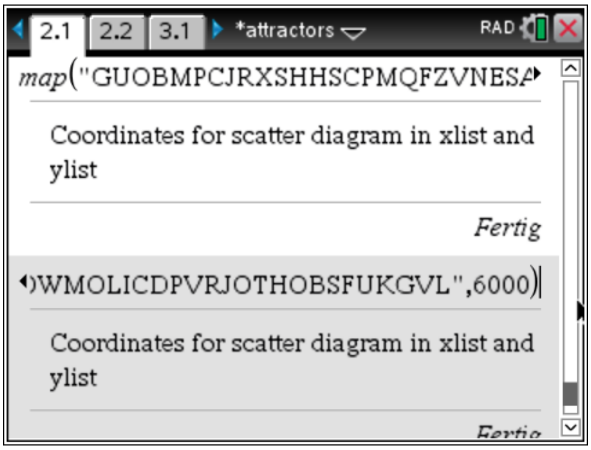
Quartic and Quintic maps

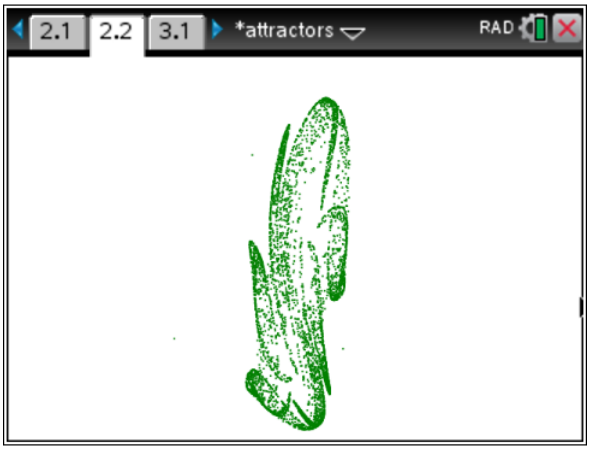
30 000 points with DERIVE

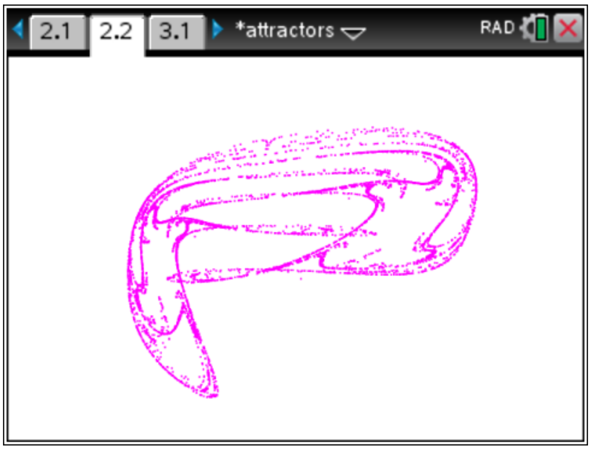


```
GUMMMMMMMMMMMMWEODFMMMMMMMMMMMM 1.02 0.08
GUOBMPCJRXSHHSCPMQFZVNESALEKOHY 1.49 0.24
GXEMONYFKDJMDTPNSLGHQLHOOTOQBUN 1.20 0.18
HCJBKUPMMMMMMMMMMWMMMMICMMMMMMMMMMMMMMMMMMM 1.39 0.07
HHANQRENHONYATQYPTNXKNMNQEGDWKYPNSMMMODAOBC 1.13 0.01
HIFZLMPJUBERQBKLRRDOWMOLICDPVRJOTHOBSFUKGVL 1.39 0.24
```

See on the next page the 6000 points plot of the second and the last code from above (GUOBM … and HIFZ …)







You can see that we can obtain satisfying results with TI-Nspire, too.

I don't want to print the code of the `map`-program – it needs too much space. Instead of this I'll present the next program which is much more demanding and interesting.

7 Exciting searches for "attractive" polynomial maps

It is great to try the many promising codes and then admire the fantastic figures appearing on the screen. Then it is clear that one has the strong desire to find his own attractors. Just taking any numbers as coefficients is not very promising. So let's apply our `sa(order,number of tries)` – program.

On the next page you can find a demonstration how it works:

I present the TI-Nspire procedure followed by one *DERIVE* search.

With the Nspire it is recommended to enter a randseed number (at least for the first run) in order to start new investigations. With only ENTER the next random number will be generated based on the built-in randseed.

The default initial point for searching and plotting as well is [0.05,0.05]. If you want to choose another point (be careful!) then you can add this point as third argument in the *DERIVE* program, but not in TI-Nspire.

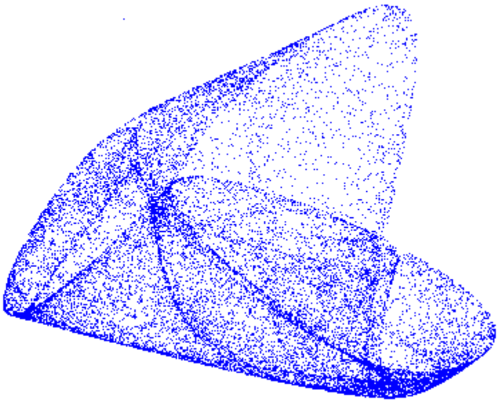
The last important difference lies in the fact that you don't need any work around for plotting the scatter diagram in *DERIVE* and not to forget, we can generate and plot much more points!

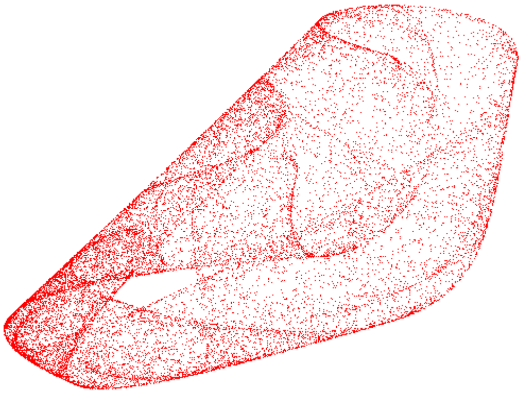
```
                                                                    Fertig
sa(3,100000)
Enter  any pos. integer or ENTER: 100
no attractor found
                                                                    Fertig
sa(3,200000)
Enter  any pos. integer or ENTER: 120
⁻0.145952542364  ";"  ⁻0.22  0.98  ⁻0.29  ⁻4.63  0.36  ⁻0.39  ⁻0.41  ⁻0.8   3.98  ⁻4.94  ⁻0.04
⁻1.42068835283   ";"   0.32 ⁻0.26   0.69  ⁻4.35  0.93  ⁻1.3   ⁻0.63  ⁻0.42 ⁻0.08   3.66   0.28
⁻1.29392738871   ";"  ⁻0.03  0.04  ⁻3.15  ⁻4.83  4.    1.46   4.34  ⁻1.15 ⁻1.77  ⁻0.55   0.14
0.116311525048   ";"   0.52 ⁻1.62   3.33  ⁻1.27  2.24  1.78  ⁻3.28  ⁻1.05  1.53  ⁻1.24   0.
                                                                    Fertig
map(mat▸list([0.52  ⁻1.62  3.33  ⁻1.27  2.24  1.78  ⁻3.28  ⁻1.05  1.53  ⁻1.24  0.  ⁻1.33  2.42  1.87
Enter initial point {x0,y0} or ENTER:
Coordinates for scatter diagram in xlist and ylist
                                                                    Fertig
```

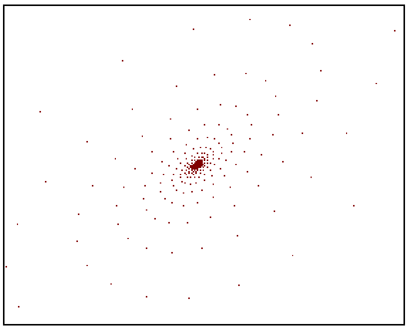
This is the Nspire Calculator screen (software) and the respective polynomial map of order 3. The positive LE indicates a "strange attractor".

Let's do the same with *DERIVE*: I was lucky and found two "suspicious" Lyapunov exponents and plotted both quadratic maps (20000 iteration points).

```
sa(2, 100000)

[ -0.49671    [-0.06, 2.43, -4.8, 0.86, -0.16, 0.82, 0.19, 0.9, -3.91, -0.36, -0.26, -3.79] ]
[ -0.026602   [-0.37, 1.2, 4.09, -4.9, -1.79, -1.98, -0.45, -1.15, 0.05, -0.42, 1.39, 1.84]  ]
[ -0.46675    [0.65, -1.77, 2.31, -0.34, 0.39, 3.48, 0.3, -1.02, 1.4, 0.78, -1.75, 3.48]     ]
[  0.42153    [0.4, -2.59, 1.37, -3.28, -2.65, -4.31, -0.24, -1.52, 3.13, 0.07, 1.61, 1.67]  ]
[  0.24404    [-0.02, 0.72, 0.2, -1.42, 0.71, 3.33, -0.25, 2.4, 0.56, 0.21, -0.62, -0.77]    ]

map([0.4, -2.59, 1.37, -3.28, -2.65, -4.31, -0.24, -1.52, 3.13, 0.07, 1.61, 1.67], 20000)

map([-0.02, 0.72, 0.2, -1.42, 0.71, 3.33, -0.25, 2.4, 0.56, 0.21, -0.62, -0.77], 20000, [0.03, 0.03])
```







Nice graphs, aren't they?

I plot also 10000 points of one list having a negative LE $\lambda = -0.0266$. As you can see the graph is not very appealing – it is a point attractor.

We can search for attractors generated by two-dimensional polynomial systems up to order `o` = 4. `cf` is the list of the randomly generated coefficients from (-5, +5) whose number `cfs` depends on the order `o` of the system. The distance to the separated points which is necessary for calculating the Lyapunov exponent is fixed with 10^{-6}, and the initial point is fixed at [0.05, 0.05] – but it can be changed.

The result is a list of the polynomial coefficients together with the respective Lyapunov exponents. All other combinations of numbers are tries leading to unbounded iterations.

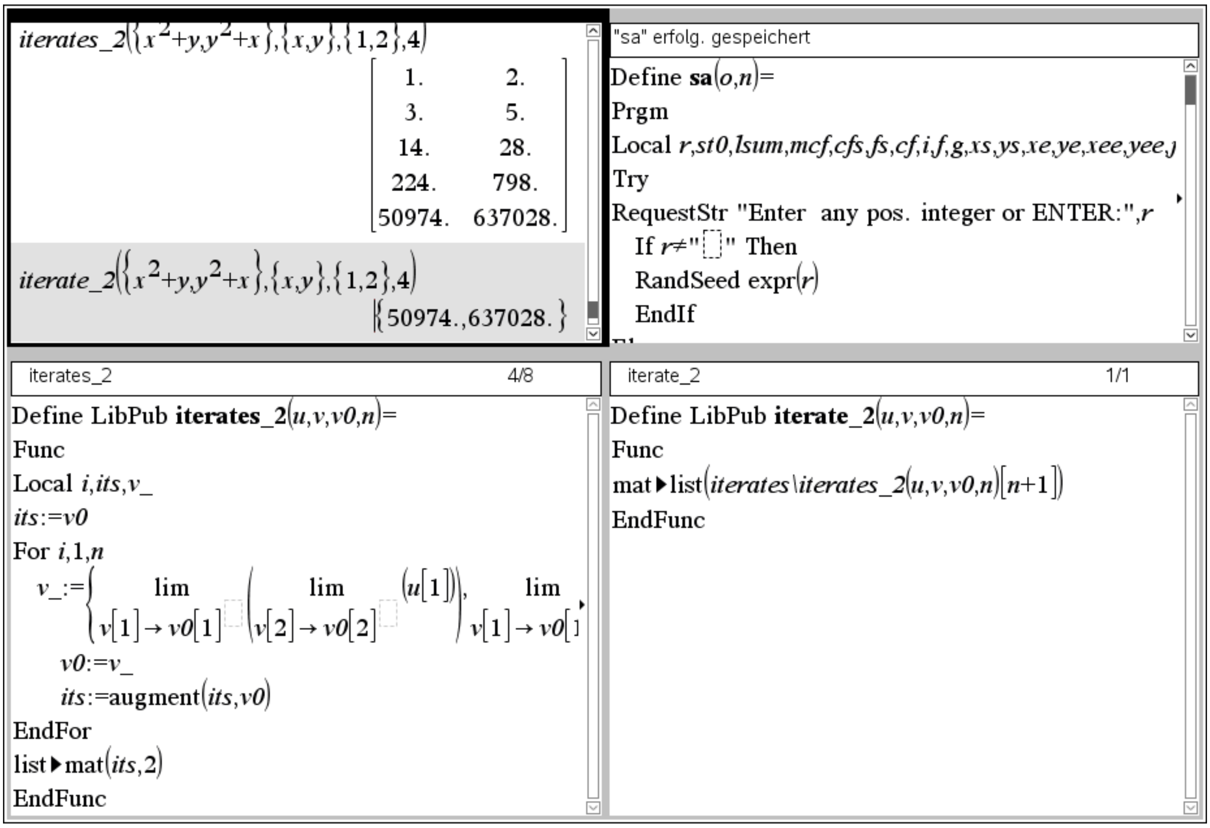
This is the moment when discovering begins and the experiment becomes exciting. Make some thousand tries and wait for the results. Feel happy if there are positive LEs appearing, because they promise STRANGE ATTRACTORS. Then plot some thousand iteration points, lean back and enjoy the moment when it is very likely that you are the very first person seeing this object appearing.

For our "search attractor"-program we will use ITERATE as we did in the last DNL. As you might know, unfortunately ITERATE(S) is not implemented in the TI-Nspire tool box. So, I produced iterates_2() and iterate_2() for TI-NspireCAS. Please compare DERIVE's ITERATE(S) and the TI-Nspire's one.

$$\text{ITERATES}\left(\left[x^2 + y,\ y^2 + x\right],\ [x,\ y],\ [1,\ 2],\ 4\right)$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 5 \\ 14 & 28 \\ 224 & 798 \\ 50974 & 637028 \end{bmatrix}$$

$$\text{ITERATE}\left(\left[x^2 + y,\ y^2 + x\right],\ [x,\ y],\ [1,\ 2],\ 4\right) = [50974,\ 637028]$$



I put the "iterates" into a library in order to have it available at any time.

I reprint sa(o,n) for TI-Nspire. It is "inspired" by my *DERIVE*-program which was "inspired" by a huge BASIC program provided by Julien C. Sprott. The *DERIVE* version and both versions of the map-program will not be presented here. You are invited to inspect (and use!) them (can be found in MTH107.zip).

```
Define sa(o,n)=
Prgm
Local r,st0,lsum,mcf,cfs,fs,cf,i,f,g,xs,ys,xe,ye,xee,yee,j,dl2,rs,nrow,vals,xn,yn,λ
Try
RequestStr "Enter  any pos. integer or ENTER:",r
  If r≠"" Then
  RandSeed expr(r)
  EndIf
Else
Goto next
EndTry
Lbl next
cfs:=(o+1)·(o+2)
```

$$mcf:=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & x & x^2 & x\cdot y & y & y^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & x & x^2 & x^3 & x^2\cdot y & x\cdot y & x\cdot y^2 & y & y^2 & y^3 & 0 & 0 & 0 & 0 & 0 \\ 1 & x & x^2 & x^3 & x^4 & x^3\cdot y & x^2\cdot y & x^2\cdot y^2 & x\cdot y & x\cdot y^2 & x\cdot y^3 & y & y^2 & y^3 & y^4 \end{bmatrix}$$

```
fs:=mat▶list(mcf[o])
vals:={}
```

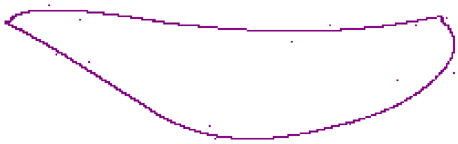
```
For i,1,n
cf:=floor(100·(10·rand(cfs)-5))/100
f:=sum(left(cf,cfs/2)·fs): g:=sum(right(cf,cfs/2)·fs)
lsum:=0
st0:=iterate_2({f,g},{x,y},{0.05,0.05},5)
xs:=st0[1]:ys:=st0[2]
xe:=xs+1.E-6:ye:=ys
For j,1,2000
  If |xs|>10 or |ys|>10 Then
  Goto nexttry
  EndIf
  xn:= lim_(y→ys)( lim_(x→xs)(f)):yn:= lim_(y→ys)( lim_(x→xs)(g))
  xee:= lim_(y→ye)( lim_(x→xe)(f)):yee:= lim_(y→ye)( lim_(x→xe)(g))
  xe:=xee:ye:=yee
  dl2:=(xn-xe)^2+(yn-ye)^2
  rs:=1/(10^6·√(dl2))
  xs:=xn:ys:=yn
  xe:=xs+rs·(xe-xs):ye:=ys+rs·(ye-ys)
  lsum:=lsum+ln(10^12·dl2)
EndFor
λ:=(1/(2·ln(2))·lsum)/j
nrow:=augment(augment({λ},{";"}),cf)
vals:=augment(vals,nrow)
Lbl nexttry
EndFor
If dim(vals)=0 Then
 Disp "no attractor found"
Else
Disp list▶mat(vals,cfs+2)
EndIf
EndPrgm
```

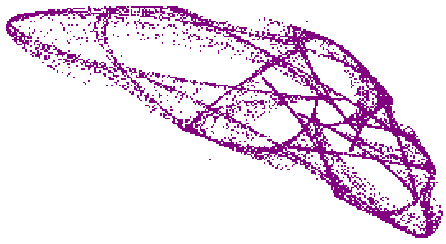
It is easy to change the program in such way that only maps with positive Lyapunov exponents will be listed. I will do this in the next programs.

Let's have another run searching for cubic maps:

```
sa(3, 100000)

⎡          -5
⎢ 1.6391·10     [-0.22, -0.81, -1.6, -3.42, -2.05, -2.91, -1.39, 1.71, 0.82
⎢
⎢   0.39000     [0.09, 1.33, 1.36, 1.63, -4.77, -4.65, -4.76, -2.13, 0.92, ·
⎢
⎢  -0.35505    [-0.61, 0.27, 4.72, 2.51, -3.15, -0.85, 0.62, 0.47, -3.55, 4
⎢
⎣   0.14003       [0.11, -0.23, -1.81, 2.58, 2.6, 2.15, -0.16, 1.53, -3.97,
```

I plot one of the maps with a positive LE (right) and then the one map with its LE very close to zero. A LE < 0 indicates a point attractor and LE = 0 announces a curve attractor. The left graph is a confirmation of this.

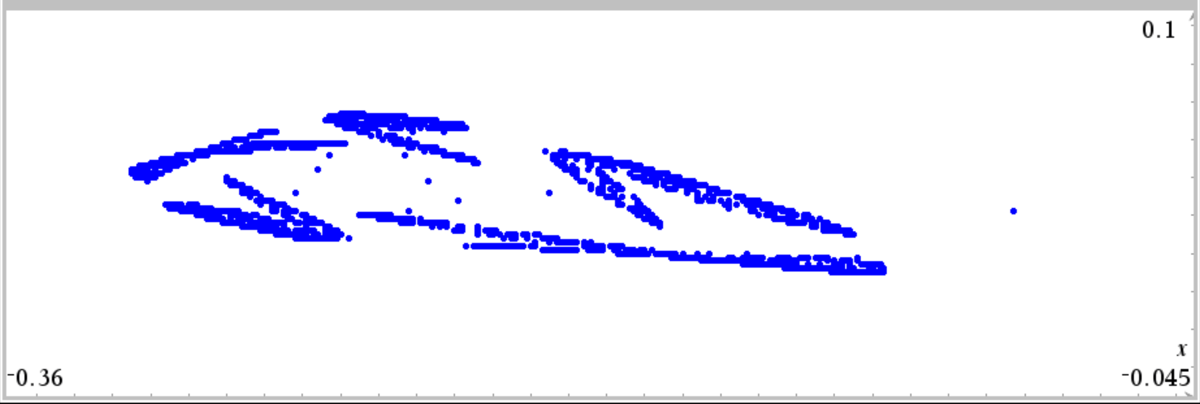




Browsing in Sprott's book one gets an impression about his tremendous and time consuming work collecting all his wonderful attractors … and this is just the beginning of our real "Attractors' Tour".

Last search for now: We'll find a quartic map with TI-Nspire and plot 4000 points of it.

```
sa(4,100000)

Enter any pos. integer or ENTER: 1111
⎡-0.160860139704 ";" -0.02 -0.13 0.29 -2.44 1.32 2.13 -3.22 -1.52 -3.03 2.11 2.37 (
⎢-0.657261817366 ";" -0.1 0.81 0.78 1.9 -0.68 3.02 1.37 3.99 -3.57 -2.74 0.85 (▸
⎢0.151827896219 ";" 0.06 2.23 4.26 -4.46 4.1 -0.24 2.69 -1.83 -2.92 -0.4 3.17 -
⎣-0.443529675608 ";" -0.02 -1.09 -1.57 -1.08 1.89 -1.7 -3.21 1.15 -2.02 -3.56 4.66 (
                                                                          Fertig
map(mat▸list([0.06 2.23 4.26 -4.46 4.1 -0.24 2.69 -1.83 -2.92 -0.4 3.17 -3.11 1.43 -2.4▸

Enter initial point {x0,y0} or ENTER:
```



7 Two special polynomial maps (and attractors, of course)

Some special structures of the polynomial functions result in special forms of attractors. Sprott called two of them "*Symplectic Attractors*" and "*Plaid Attractors*".

Symplectic Attractors

$$x_{n+1} = a + b\,x_n + c\,x_n^2 + d\,x_n^3 + e\,x_n^4 + f\,x_n^5 \pm y_n$$
$$y_{n+1} = g \mp x_n$$

Plaid Attractors

$$x_{n+1} = a + b\,y_n + c\,y_n^2 + d\,y_n^3 + e\,y_n^4$$
$$y_{n+1} = f + g\,x_n + h\,x_n^2 + i\,x_n^3 + k\,x_n^4$$

We will search for both types of attractors and then admire the results. (I changed the program to only return the coefficients giving a positive LE.)

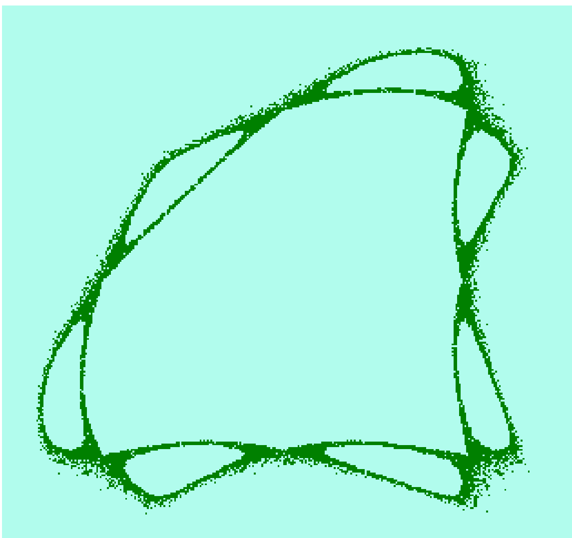
```
sa_symp(3, 5000)
```

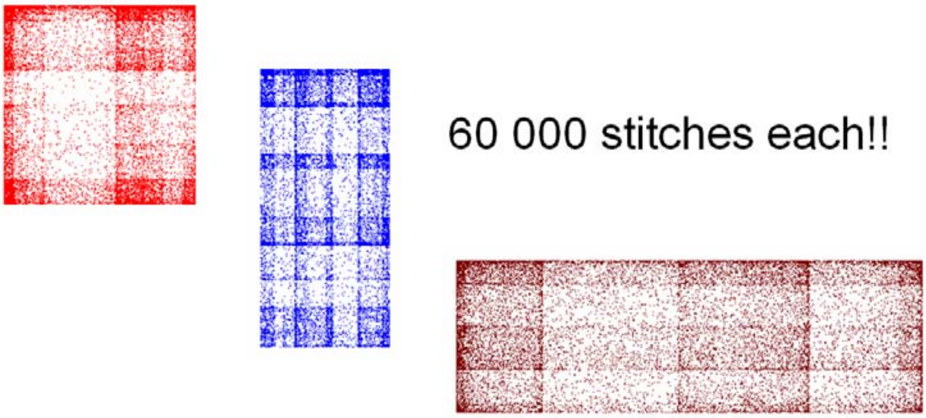
0.0107637	[-0.36, -0.8, 0.73, 0.42, 1, 0.67, -1]
0.00957944	[0.17, 1.55, -2.1, -2.88, -1, -0.04, 1]
0.0132139	[0.63, 0.71, -0.1, -0.95, -1, 0.53, 1]
0.0117269	[-0.14, 0.07, 1.09, 0.24, 1, -1.15, -1]
0.0491747	[-3.64, -0.44, 0.2, 0.01, 1, 2.69, -1]

```
sa_plaid(4, 10000)
```

0.287029	[-0.42, 1.87, -0.88, -2.33, 3.78, -0.09, -0.39, 0.57, -2.65, -3.18]
0.28711	[0.5, 0.39, -2.01, -3.63, -2.2, -0.52, 1.63, -0.94, 4.06, -4.12]
0.250974	[0.05, -1.16, 3.49, -1.12, -2.39, 0.24, -3.9, 4.85, -2.81, -1.27]
0.294624	[0.16, 0.55, -1.72, -3.12, 1.47, -0.25, 2.62, 2.01, -2.15, -0.71]
0.0995453	[-0.49, -0.38, 1.45, -2.04, 1.19, 0.05, -3.2, -1.26, 2.98, 3.03]

```
map_symp([-3.64, -0.44, 0.2, 0.01, 1, 2.69, -1], 20000)
```

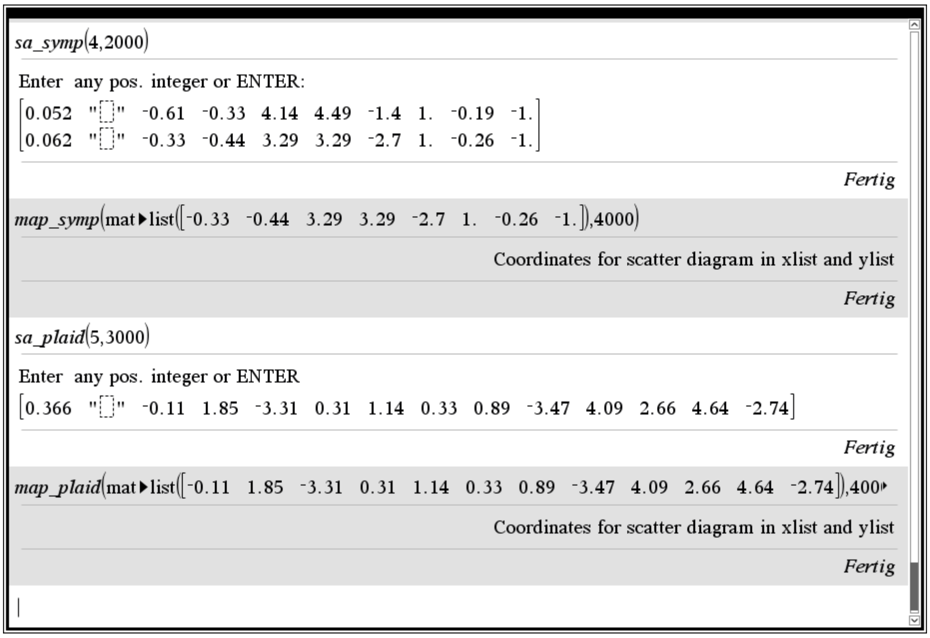




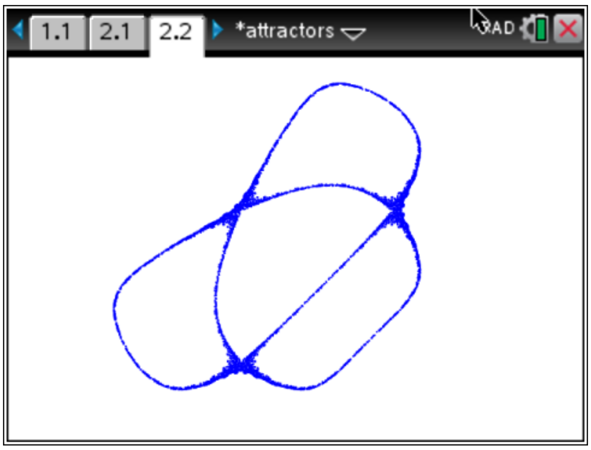
You will understand why Sprott called them “Plaid-attractors”. I must admit that I don’t have any idea for “symplectic”. I found this attribute in Mathematics Encyclopedias but I couldn’t find any connection to the iteration formulae – silly me!?

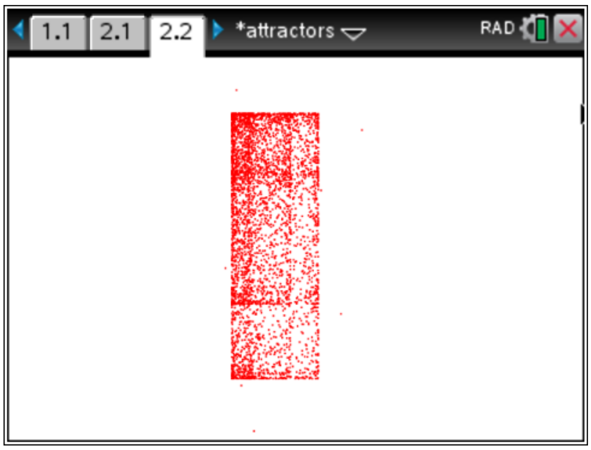
Julien C. Sprott included all his attractors (generating and plotting as well) into one huge menu-guided BASIS program. I liked to split it up. So, I separated the “symplectics” and the “plaids” from the other generalized polynomial maps.

What about them on the TI-Nspire screen?



```
sa_symp(4,2000)
Enter any pos. integer or ENTER:
[0.052 "[]" -0.61 -0.33 4.14 4.49 -1.4 1. -0.19 -1.]
[0.062 "[]" -0.33 -0.44 3.29 3.29 -2.7 1. -0.26 -1.]
                                                        Fertig
map_symp(mat▸list([-0.33 -0.44 3.29 3.29 -2.7 1. -0.26 -1.]),4000)
                  Coordinates for scatter diagram in xlist and ylist
                                                        Fertig
sa_plaid(5,3000)
Enter any pos. integer or ENTER
[0.366 "[]" -0.11 1.85 -3.31 0.31 1.14 0.33 0.89 -3.47 4.09 2.66 4.64 -2.74]
                                                        Fertig
map_plaid(mat▸list([-0.11 1.85 -3.31 0.31 1.14 0.33 0.89 -3.47 4.09 2.66 4.64 -2.74]),400▸
                  Coordinates for scatter diagram in xlist and ylist
                                                        Fertig
```





In the next DNL we will explore the attractors in 3D space.