THE DERIVE - NEWSLETTER #115

ISSN 1990-7079

THE BULLETIN OF THE DERIVE USER GROUP

+ CAS-TI

Contents:

September 2019

Some links which might be of interest:

Multimedia Math Resources (Cecilia Arias)
A collection of digital tools, practice sites, databases with lesson plans, problems, and projects, and iPad math apps for grades Pre-K to 12.

https://www.pinterest.at/ariascec/multimedia-math-resources//

Art of Problem Solving
(Best of Sonnhard Graubner)

https://artofproblemsolving.com/downloads/printable_post_collections/37242

Visit: **http://vixra.org//**
which is an archive of 31553 e-prints in Science and Mathematics serving the *whole* scientific community, e.g.

A Set of Formulas for primes (Plouffe): http://vixra.org/pdf/1812.0494v4.pdf

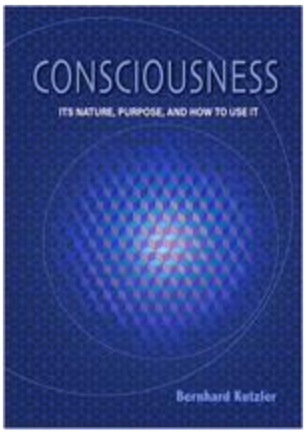
A famous paper on Lambert W-function

https://cs.uwaterloo.ca/research/tr/1993/03/W.pdf

And this is a forum for Symbolic Mathematic

https://groups.google.com/forum/#!forum/sci.math.symbolic

It is a pleasure to announce:



Bernhard Kutzler (DUG Member#1) researched CONSCIOUSNESS and collected his results in a book:

https://www.bernhardkutzler.com/book.html

Dear DUG Members,

You can find two pictures in my letter. They shall announce two great talks given at ACA2019 (Applications of Computer Algebra) in Montréal, this July. Our member Michel Beaudin (one of the organizers) had the idea to ask Simon Plouffe (right) for his permission to publish his plenary lecture as a contribution in our newsletter. And fortunately, he agreed and sent his ppt-presentation. We can feel very proud to have such a great and famous mathematician among our contributors.

Asking Helmut Heugl (left) for his permission to include his lecture was an easy job – just a short phone call. So, we have two very contrary articles: high-level mathematics and high-level didactics united in one bulletin. Helmut's "Right or Wrong?" shall accomplish his paper and might raise a discussion among the readers.

I split both lectures in order to get some variety in our newsletter. In this sense I added a contribution, which I found in my stack of collected DUG-papers: Colin Kennedy produced nice graphs of ammonites and other animals varying the parameters of only one function. You may remember Piotr Trebisz' articles on snail shells in earlier newsletters.

Please notice the links provided in our Forum and the announcement of Alfred Roulier (page 3). Karsten Schmidt faced problems working with DERIVE after performing a Windows 10 update. Günter Schödl, our expert for such cases could help. This might be helpful for other DERIVE users, too. See page 4.

It is a special pleasure for me to present a book written by our DUG-member#1 and organizer of many DERIVE & TI-events (TIME conferences a.o.), Bernhard Kutzler. He changed his life dramatically from a mathematician and business man to a researcher of our "Consciousness". And this is the title of his book (in German "Bewusstsein"). It is available in English and German as well. On YouTube you can follow an extended discussion between Bernhard and Hannes Kutzler.

Much luck Bernhard for your new life, thanks for your immense support for so many years

Best regards and wishes to all of you

Josef

Pictures are provided by Gilles Picard from ETSMTL, many tanks Gilles.

The *DERIVE-NEWSLETTER* is the Bulletin of the *DERIVE* & CAS-*TI User Group*. It is published at least four times a year with a content of 40 pages minimum. The goals of the *DNL* are to enable the exchange of experiences made with *DERIVE*, *TI*-CAS and other CAS as well to create a group to discuss the possibilities of new methodical and didactical manners in teaching mathematics.

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Contributions:
Please send all contributions to the Editor. Non-English speakers are encouraged to write their contributions in English to reinforce the international touch of the *DNL*. It must be said, though, that non-English articles will be warmly welcomed nonetheless. Your contributions will be edited but not assessed. By submitting articles, the author gives his consent for reprinting it in the *DNL*. The more contributions you will send, the more lively and richer in contents the *DERIVE* & CAS-*TI Newsletter* will be.

Next issue: December 2019

Preview: Contributions waiting to be published

Some simulations of Random Experiments, J. Böhm, AUT, Lorenz Kopp, GER
Wonderful World of Pedal Curves, J. Böhm, AUT
Simulating a Graphing Calculator in *DERIVE*, J. Böhm, AUT
Cubics, Quartics – Interesting features, T. Koller & J. Böhm, AUT
Logos of Companies as an Inspiration for Math Teaching
Exciting Surfaces in the FAZ
BooleanPlots.mth, P. Schofield, UK
Old traditional examples for a CAS – What´s new? J. Böhm, AUT
Mandelbrot and Newton with *DERIVE*, Roman Hašek, CZ
Tutorials for the NSpireCAS, G. Herweyers, BEL
Dirac Algebra, Clifford Algebra, Vector-Matrix-Extension, D. R. Lunsford, USA
A New Approach to Taylor Series, D. Oertel, GER
Statistics of Shuffling Cards, Charge in a Magnetic Field, H. Ludwig, GER
Selected Lectures from TIME 2016
More Applications of TI-Innovator™ Hub and TI-Innovator™ Rover
Surfaces and their Duals, Cayley Symmetroid, J. Böhm, AUT
Affine Mappings –Treated Systematically, H. Nieder, GER
The Penney-Ante Game, MacDonald Phillips, USA
Hyper Operations for DERIVE, J Angres and others, GER
Investigations of Lottery Game Outcomes, W Pröpper, GER
A Traversing Bridge (Eine Rollbrücke), W. Alvermann, GER
DERIVE Bugs?, D. Welz, GER
Tweening & Morphing with TI-NspireCX-II-T, J. Böhm. AUT

Impressum:
Medieninhaber: *DERIVE* User Group, A-3042 Würmla, D´Lust 1, AUSTRIA
Richtung: Fachzeitschrift
Herausgeber: Mag. Josef Böhm

Erratum:

Dear Josef,

In my modest contribution about Functional Derivative the editor substituted "*consecutive forces*" for "*conservative forces*".

Best regards,

Francisco Marcelo Fernández

The Editor is sorry for the mistake and apologizes,
Josef

Mail from Alfred Roulier, Switzerland

Liebe T3-Freunde

In den nun 14 Jahren meiner Mitarbeit bei T^3-Schweiz ist allerhand Material entstanden. Vieles davon wurde in die TI-Materialdatenbank aufgenommen. Dennoch schlug Hans Kammer vor, dass wir beide einen Verleger für diese Arbeiten suchen. Unsere Anfrage beim HEP-Verlag wurde aber abgelehnt, weil ihr Kundenspektrum zu wenig kompatibel mit dem Niveau unserer Arbeiten ist. Ich habe nun mithilfe eines Informatikstudenten eine Webseite zusammengestellt.

Die URL lautet www.alfred-roulier.ch

Sie gibt m.E. auf anschauliche Weise gut wieder, was wir da in all den Jahren zusammengebastelt haben. Wenn ihr in euren Kreisen die URL an mögliche Interessenten weiterleitet, bin ich natürlich dankbar. Grossen Dank geht aber an euch alle, die ihr mir in den TI-Workshops als Gesprächspartner massgeblich geholfen habt.

Da ich in diesem Jahr 80-jährig werde, betrachte ich die Webseite auch ein wenig als Abschluss bei T^3. Sollte allerdings noch diese oder jene Idee Gestalt annehmen, werde ich mich über Hans Kammer melden.
Mit vielem Dank und herzlichen Grüssen
Alfred Roulier

Short Translation:

At the occasion of his 80th birthday Alfred Roulier offers a website containing his materials produced within his 14 years of activity for T^3 Switzerland. You will find two sections: Science (Astronomy, Physics and Mathematics) and Jazz. Download many pdf- and tsn-files from the Science-section and 47 (!) mp3-files from the Jazz-section.
Enjoy!

Many thanks to Alfred Roulier and best wishes for many fruitful years (Science & Music) on behalf of the DUG-community,
Josef

Mail from Karsten Schmidt, Germany

Karsten attended ACA2019 in Montréal and had problems installing DERIVE 6 on his laptop after a Windows10 update. He tried installing the patch recommended by Günter Schödl in earlier DNLs, but he failed. I sent Karsten's complaints to Günter and he – reliable as ever – answered immediately (from Sweden!). (I translate his answer, Josef)

Günter Schödl mailed:

Hello Josef!

1. Remove all installation remains of DERIVE, i.e. delete the TI-folder in the program files-folder. This is important, when trying a new installation (after a former try)!
2. Download from Von https://support.microsoft.com/de-de/help/917607/feature-not-included-help-not-supported-error-opening-help-windows the suitable winhlp32 (for 8.1).
3. Install DERIVE as usual.
4. Install winhlp32 using install.cmd and start it with right click as administrator.

The error message appears when install.cmd has not been run as administrator, because then write permission is missed.

Just now I installed DERIVE (latest WIN-version, June update), everything works perfect.

Best regards from Sweden,
Günter

Karsten's reaction

Dear Josef,
when it became clear for me that this would need a greater action (until now it was only necessary to run install.cmd as admin, which was not sufficient this time), I waited until being back from Montréal.

Yesterday I deinstalled DERIVE, possibly the first time in my life. It does not work via system control (as usual), but then I found unwise.exe in the DERIVE folder and tried after saving all mth-files (and to go for sure, the compatibility files, too). Then I deleted the TI-folder and rebooted.

I had downloaded Windows8.1-KB917607-x64(1).msu into the Download-folder which also contains install.cmd. Then I installed DERIVE (this not the first time in my life), launched install.cmd as admin and copied back my mth-files. The compatibility files were still there (in folder C:\Users\KS\AppData\Local\VirtualStore\Program Files (x86)\TI Education\Derive 6\Math).

Now DERIVE runs properly as ever, and the Help-function works, too.

Many thanks to Günter Schödl for his quick and effective assistance! And also, many thanks for your support.

Inspecting the error message from my first attempts once more, I think that the problem was caused by the missing winhlp32.exe.mui-file. It might have been deleted when performing the greater Windows-update.

Best regards to you and Noor – Karsten.

This contribution is based on a plenary lecture given at ACA2019 in Montréal in July. We can be very proud and happy that Simon Plouffe gave permitted to publish his talk in our bulletin. If you want to get some information, then "google" for him and you will find numerous papers of this great and famous mathematician.
Simon provided the slides (72) of his talk and I tried to make an article to save space. I am very grateful that Simon was very patient to answer my questions and gave additional explanations. I added some comments and links to papers which are connected with his findings. Enjoy and be amazed, Josef

π, the primes and the Lambert W-function

Simon Plouffe [simon.plouffe@gmail.com]
Université de Nantes(IUT), Nantes, France

Abstract

The talk is divided into two parts. The first part will show new formulas for primes like

$$691 = 2^4 \sum_{n=1}^{\infty} \frac{n^{11}}{e^{\pi n} - 1} - 2^{16} \sum_{n=1}^{\infty} \frac{n^{11}}{e^{4\pi n} - 1}$$

At the same time, the prime is well approximated with the formula:

$$691 \approx \frac{2^4 11!}{\pi^{12}}$$

In fact, the prime is given exactly by

$$691 = \frac{2^4 11!}{\pi^{12}} \left(1 + \frac{1}{3^{12}} + \frac{1}{5^{12}} + \frac{1}{7^{12}} + \cdots\right)$$

Using the bootstrap method, one can do the same for many primes. This leads to a conjecture about the representation of all the primes using π and a simple function of *n*. And speaking of primes, I will show a set of formulas that can generate an infinity of primes using a recurrence equation function. If $\{x\}$ is the rounded value of *x* and $S_0 = 43.804\ldots$, then $S_{n+1} = \left\{S_n^{5/4}\right\}$will generate an infinity of primes, beginning with

113, 367, 1607, 10177, 102217, 1827697, 67201679, 6084503671, …

Here, the exponent 5/4 can be made as close as we want to 1.

The second part will talk about Lambert's function and present a model for the number of prime numbers and the *n*th prime number, at the same time we will present two new formulas for the *n*th zero of the Zeta function and the number of zeros.

Inspired

One day I came across this formula of Ramanujan (notebooks):

$$\zeta(3) = \frac{7\pi^3}{180} - 2\sum_{n=1}^{\infty} \frac{1}{n^3(e^{2\pi n}-1)}$$

I tried to understand this formula. For doing it I made a series of experiments with one of my favorite programs: *lindep*[1] or *PSQL*[2], that is an Integer Relation algorithm. *lindep* is part of *PariGP* and now *PSLQ* is no longer a cryptic FORTRAN animal but is a part of Maple.

I made an interface within Maple to write a FORTRAN source from one inquiry, compile it on the host computer, run it and come back with the answer. I made one for Mathematica and Maple, too.

This leads to more findings:

$$\pi = 72\sum_{n=1}^{\infty} \frac{1}{n(e^{\pi n}-1)} - 96\sum_{n=1}^{\infty} \frac{1}{n(e^{2\pi n}-1)} + 24\sum_{n=1}^{\infty} \frac{1}{n(e^{4\pi n}-1)}$$

$$\frac{1}{\pi} = 8\sum_{n=1}^{\infty} \frac{n}{e^{\pi n}-1} - 40\sum_{n=1}^{\infty} \frac{n}{e^{2\pi n}-1} + 32\sum_{n=1}^{\infty} \frac{n}{e^{4\pi n}-1}$$

$$\zeta(3) = 28\sum_{n=1}^{\infty} \frac{1}{n^3(e^{\pi n}-1)} - 37\sum_{n=1}^{\infty} \frac{1}{n^3(e^{2\pi n}-1)} + 7\sum_{n=1}^{\infty} \frac{1}{n^3(e^{4\pi n}-1)}$$

$$\zeta(5) = 24\sum_{n=1}^{\infty} \frac{1}{n^5(e^{\pi n}-1)} - \frac{259}{10}\sum_{n=1}^{\infty} \frac{1}{n^5(e^{2\pi n}-1)} - \frac{1}{10}\sum_{n=1}^{\infty} \frac{1}{n^5(e^{4\pi n}-1)}$$

You see the pattern here? $e^{\pi n}, e^{2\pi n}, e^{4\pi n}, \ldots$

And some exotics like:

$e^{\pi} - \pi = 19.99909997919\ldots$ in 1987 with my hp 15c and is still a mystery.

$$\sum_{n=1}^{\infty} \frac{n^3}{e^{2\pi n/7}-1} = 10.00000000000000190161767888663\ldots$$

$$\sum_{n=1}^{\infty} \frac{n^3}{e^{2\pi n/13}-1} \cong 119.000000000000000000000000000000959374585\ldots$$

[1] http://www.cecm.sfu.ca/organics/papers/bailey/paper/html/pari.htm
https://pari.math.u-bordeaux.fr/

PARI-GP is a small but very fast math program similar to Maple, it has less formal power than Maple BUT it is among the fastest computing calculators for mathematics to high precision. It is used in SAGE-MATH. PARI is also available as a C library to allow faster computations.

[2] http://mathworld.wolfram.com/PSLQAlgorithm.html, http://arminstraub.com/downloads/math/pslq.pdf
https://www.davidhbailey.com/dhbpapers/pslq-comp-alg.pdf

PSLQ is (to make it simple) a program that can do the inverse of finding a root of a polynomial like the Newton method.
Newton method: can find the floating-point value to any polynomial. PSLQ is a program that given a real number can find what polynomial has that number as a root. It is considered as a very efficient and important algorithm or program; it is that program that helped me to find a certain formula for Pi in 1995.

Getting back to Ramanujan again, an identity with 1.

$$24\sum_{n=1}^{\infty}\frac{n^{13}}{e^{2\pi n}-1}=1$$

In fact, there are more like that.

More formulas:

$$24\sum_{n=1}^{\infty}\frac{n^{673}}{e^{2\pi n}-1}=\text{a 1077 digit prime}$$

and

$$240\sum_{n=1}^{\infty}\frac{n^{22807}}{e^{2\pi n}-1}=\text{a 71399 digit prime}$$

Following the lead, we get

$$691=16\sum_{n=1}^{\infty}\frac{n^{11}}{e^{\pi n}-1}-2^{16}\sum_{n=1}^{\infty}\frac{n^{11}}{e^{4\pi n}-1}$$

Can we get other primes like that? … all the primes?

But why 691?

In fact, it comes from this identity with Eisenstein series (Jean-Pierre Serre, cours d' arithmetique, p 157).

$$Eis_6=1+\frac{65520}{691}\sum_{n=1}^{\infty}\sigma_{11}(n)q^n$$

Hum, $2^{16}-16=65520$, and 691 is the numerator of B_{12}, a Bernoulli number.

Depending of the person, you would prefer the first version or the second one. Here Eis_6 is the Eisenstein series (not Euler numbers) [3].

Anyhow, we have this approximation of 691 …

$$691\approx\frac{2^4 11!}{\pi^{12}}$$

Well, yes there are others

$$61\approx\frac{2^8 6!}{\pi^7}$$

and 61 is the 3rd Euler number.

[3] See http://mathworld.wolfram.com/EisensteinSeries.html and https://en.wikipedia.org/wiki/Eisenstein_series for Eisenstein series. $q=e^{2\pi i\tau}$, $\sigma_k(n)$ is the divisor function: sum of the kth powers of all divisors of n, e.g. $\sigma_{11}(6)=1^{11}+2^{11}+3^{11}+6^{11}=362976252$ (https://oeis.org/A013959).

The Euler numbers appear in the Taylor series expansions of the secant and hyperbolic secant functions. The latter is the function in the definition. They also occur in combinatorics, specifically when counting the number of alternating permutations of a set with an even number of elements (https://en.wikipedia.org/wiki/Euler_number).

$$E_{2n}=i\sum_{k=1}^{2n+1}\sum_{j=0}^{k}\binom{k}{j}\frac{(-1)^j(k-2j^{2n+1}}{2^k i^k k},i=\sqrt{-1}$$

E_{510} is a 1062 digits prime.

$$E_{510} \approx \frac{2^{512} 510!}{\pi^{511}}$$

These numbers come from the expansion of the Dirichlet Beta series

$$\beta(s) = \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)^s}$$

Let me try with DERIVE (Josef):

DIVISOR_SIGMA(11, 6) = 362976252

$$\text{BERNOULLI}(12) = -\frac{691}{2730}$$

$$\frac{2^4 \cdot 11!}{\pi^{12}} = 690.99869687720122094$$

$$\frac{2^8 \cdot 6!}{\pi^7} = 61.027187130028935209$$

```
VECTOR(EULER(2·n), n, 1, 5) = [-1, 5, -61, 1385, -50521]

EULER(6) = -61

EULER(510) =

  -16118434469171764007192875818393917359078321286821804351839758727224771919451090700064638457~
  330657765096697678674503618056898863840388535702542289156659906903370715763347144810541327633~
  316972073956849715434803063861822096556135621504128397806162915958803077371187060949327918575~
  859950046441655552773725416781583269694114763645916338194565450081168672664810841709283627741~
  537961415109052003646804513084982438685634333688494534692288789737535173469984996603825926229~
  786095475907263097821640631532809147240029912093286554544300461894882849175462285749197253451~
  896592946415583016941909144003418944696288609158067399706947018503837944945780784461303472431~
  537719323938746676396725664946691022414241138390697728603712791809670412887293866914614139655~
  754516870815197093078859736004648469790919688210902471150552627188401572739328172065638086629~
  520483274133145766818498588279616640622872820929728811471997737407465198955427262756865856329~
  915374366233660051651133163257154088446874553873607233165522255258895094389258049985996717013~
  2448633637661899695780587443307090858021

DIM(EULER(510)) = 1062

PRIME?(- EULER(510)) = true
```

Where are the Euler numbers coming from?

$$\frac{1}{\cos(x)} = 1 + \frac{x^2}{2!} + \frac{5x^4}{4!} + \frac{61x^4}{6!} + \cdots$$

$$\frac{1}{\cosh(x)} = 1 - \frac{x^2}{2!} + \frac{5x^4}{4!} - \frac{61x^4}{6!} + \cdots$$

More generally, can we find all trigonometric expressions that would lead to approximations of primes with π^k?

Could it be possible to get all primes with these intriguing expressions with π^k?

For example, with another expression (trig) we have:

$$7 \approx \frac{2^9}{\pi^3\sqrt{4+2\sqrt{2}}}$$

If we look at A006873 [4] (Number of alternating 4-signed permutations) 1, 1, 7, 47, 497, 95767, 1704527, 34741217, …

If the sequence contains a prime then if we have the asymptotic expansion of $a(n)$ it leads to one more approximation of that prime using π^k.

In this case, it is:

$$a(n) \approx \frac{n!\,8^n}{n\,\pi^n}\cdot\frac{1}{\sqrt{4+2\sqrt{2}}}$$

The exponential generating function [5] of A006873 is

$$\frac{\sin(x)+\cos(3x)}{\cos(4x)}$$

But, how the expression of $a(n)$ is found?

1) expand $\frac{\sin(x)+\cos(3x)}{\cos(4x)}$ into a series
2) Collect coefficients of the exponential generating function. (with $n!$)
3) Compute the ratio of $\frac{a(n+1)}{a(n)}$
4) Compute the first differences
5) Identify the constant 2.5464790894703253723 02…
6) The constant is $\frac{8}{\pi}$
7) Retro engineer the expression to $a(n)$
8) $\sqrt{4+2\sqrt{2}}$ is found in $\frac{\Gamma(n)8^n}{\pi^n}=\sqrt{4+2\sqrt{2}}$

Building one by one each prime from this idea:

$$1 \approx \frac{3\cdot 13!}{2\pi^{14}} \qquad 3 \approx \frac{64\sqrt{2}}{\pi^3}$$

$$5 \approx \frac{2^6\cdot 4!}{\pi^5} \qquad 11 \approx \frac{768}{\pi^4} \qquad 11 \approx \frac{1944\cdot\sqrt{3}}{\pi^5}$$

$$17 \approx \frac{4\cdot 8!}{\pi^8} \qquad 31 \approx \frac{4\cdot 10!}{5\pi^{10}}$$

Not much a pattern found here!

- We have some primes with Euler numbers via the Beta Dirichlet series, some Bernoulli numbers …
- Can't we just generate primes with this information? Not exactly!
- Can't we just generate some primes with <any> formula?
- What are the known formulae anyway? Which one is the most efficient?

[4] https://oeis.org/A006873
[5] http://www.math.uwaterloo.ca/~dgwagner/co430II.pdf

Who	Year	Comment	Efficiency	How many primes
Eratosthenes	-276 to -194	Sieve	Practical	Computable infinity
Mersenne	1536	Primes of the form 2^p-1	Practical, exact	51
Fermat	1640	Fermat's little theorem	Weak Probable primes	Computable infinity
Euler	1772	Second degree polynomial	Practical	40
Mills and Wright	1947 and 1951	Double exponential	Practical	Less than 5 known exactly
Wilson	Circa 1780	Uses p!	Theoretical	Very few primes
Jones, Sato, Wada, Wiens	1976	25th degree polynomial with 26 variables	Theoretical	Very few primes
John H. Conway	1987	FRACTRAN	Theoretical	Very few primes
Dress, Landreau	2010	6th degree polynomial	Practical	58
Benoit Perichon and al.	2010	26 primes in arithmetic progression	Practical	26
Tomas Oliveira e Silva et al.	2019	Sieve optimized, fastest known prime generating program	Practical	Computable infinity
Fridman et al.	2019	Prime generating constant	Practical, limited to precision	Computable infinity
A064648	2019	Engel expansion of 0.705230171...	Practical, limited to precision	Computable infinity
Simon Plouffe	2019	Efficient Mills-Wright-like formula	Practical, limited to precision	Computable infinity

The 6th degree polynomial took months to find [6]:

$$P(x) = \frac{1}{72}x^6 - \frac{5}{24}x^5 - \frac{1493}{72}x^4 + \frac{1027}{8}x^3 + \frac{100471}{18}x^2 - \frac{11971}{6}x - 57347$$

It delivers 58 prime numbers for *x* from -42 to 15.

I'd like to produce 8 of them using DERIVE. Josef:

$$p(x) := \frac{x^6}{72} - \frac{5}{24}\cdot x^5 - \frac{1493}{72}\cdot x^4 + \frac{1027}{8}\cdot x^3 + \frac{100471}{18}\cdot x^2 - \frac{11971}{6}\cdot x - 57347$$

```
prs := VECTOR(|p(x)|, x, -2, 5)

prs := [32381, 49919, 57347, 53653, 38321, 11351, 26731, 74873]

VECTOR(PRIME?(p_), p_, prs) = [true, true, true, true, true, true, true, true]
```

The formula of Mills is another good example [7].

If $A = 1.3063778838630806904686144926 \ldots$ then $\lfloor A^{3^n} \rfloor$ is always prime [8].

$a(n)$ = 2, 11, 1361, 2521008887, 16022236204009818131831320183, 4113101149215104800030529537915953170486139623539759933135949994882770404074832568499, …

$a(21)$ is a number 1.214 billion digits long. (The triple size at each iteration.)

Let me try with DERIVE (mc is Mills' constant):

```
mc := 1.306377883863080690468614492602605712916784585156713644368053759966434

VECTOR(FLOOR(mc^(3^n)), n, 5)

[2, 11, 1361, 2.521008887·10^9, 1.6022236204009818131831320183·10^28]

PRIME?(16022236204009818131831320183) = true
```

Here is the algorithm of Mills seen in the eye of reverse engineering:

Begin with $p = 2$

a) New prime = Next Prime(p^3)
b) Got to a)

You get the sequence 2, 11, 1361, 2521008887, 1602223620400981131831320183, …

DERIVE again:

```
ITERATES(NEXT_PRIME(p_^3), p_, 2, 5)

[2, 11, 1361, 2521008887, 16022236204009818131831320183,
  4113101149215104800030529537915953170486139623539759933135949994882770404074832568499]
```

Interesting websites for more information:

[6] https://fr.wikipedia.org/wiki/Formules_pour_les_nombres_premiers
[7] https://en.wikipedia.org/wiki/Formula_for_primes#Mills%27_formula
[8] https://oeis.org/A0510211
https://primes.utm.edu/glossary/page.php?sort=MillsConstant

The formula of Wright is even worse (E. M. Wright formula from 1951)

If $g_0 = \propto = 1.9287800$ ... and $g_{n+1} = 2^{g_n}$ then $\lfloor g_n \rfloor = \left\lfloor 2^{\cdots 2^{2^{\propto}}} \right\rfloor$ is always prime.

$a(n)$ = 3, 13, 16381. The fourth term is 4932 digits long.

No one was able to compute the fifth term of this sequence.

In both cases, it is a good idea but not practical at all.

But if we use **Sylvester's sequence** rather A000058 [9] in the OEIS catalogue is

1, 2, 7, 43, 1807, 3263443, 106500566950807, …

Has the property that

$$1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{7} + \frac{1}{43} + \frac{1}{1807} + \cdots$$

Called the Pierce expansion or Egyptian fractions expansion of 0.99999999999…

[9] https://oeis.org/A000058

The sequence is given by the recurrence

$$S_{n+1} = S_n^2 - S_n + 1$$

So, starting at 2 we get 3, then, 7, 43, …

But what if we start differently by having

$$S_0 = 1.618141809324092\ldots$$

will produce 2, 3, 7, 43, 1811, 3277913, … all primes. Nice, but it grows too fast. The length doubles at each step.

The number $S_0 = 1.618141809324092\ldots$ was found using Simulated Annealing + Monte Carlo. Simulated Annealing is what we call "le recruit simulé" [10].

Simulated Annealing + Monte Carlo

1) First, we choose a starting value and exponent (preferably a rational fraction for technical reasons).
2) Use Monte-Carlo method with Simulated Annealing, in plain English:
   We keep only the values that show primes and ignore the rest. Once we have a series of 4-5 primes then we are ready for the next step.
3) We use a formula for forward calculation and backward.

One example:

Hypothesis: there exists an infinite sequence of primes generated by $\{c \cdot n^n\}$, c real and { } is the nearest integer.

Yes, if $c = 0.2655883729431433908\ldots$ then the sequence ($n \geq 3$) is

7, 67, 829, 12391, 218723, 4455833, 1028944377, …

But fails after 19 terms at $n = 22$. The sequence is finite.

We go back to Mills' model:

What if we use a smaller exponent and test if it works?

When $a_0 = 43.80468771580293481\ldots$ then if $a_{n+1} = a_n^{\frac{5}{4}}$, and use { } to isolate primes, this is now sequence A323176 [11]:

113, 367, 1607, 10177, 102217, 1827697, 67201679, 6084503671,…

Now, if you want an even smaller exponent choosing carefully a_0, would it work, too?

Let's try: $a_{n+1} = a_n^{\frac{11}{10}}$.

Then if $a_0 = 1000000000000000000000000000000049.3122107477645$, we have the following prime sequence:

[10] Wikipedia: **Simulated annealing** (SA) is a probabilistic technique for approximating the global optimum of a given function. Specifically, it is a metaheuristic to approximate global optimization in a large search space for an optimization problem.

[11] http://oeis.org/A323176

1000000000000000000000000000000049
158489319246111348520210137339236753
524807460249772597364312157022725894401
3908408957924020300919472370957356345933709
70990461585528724931289825118059422005340095813
…

```
PRIME?(3908408957924020300919472370957356345933709) = true

PRIME?(70990461585528724931289825118059422005340095813) = true
```

Continuing according to this idea …

With $a_{n+1} = a_n^{\frac{101}{100}}$ then if $a_0 = 100^{500} + 961.49937633785074906096890050…$

I could compute 100 terms of this sequence: $a(100)$ is a 1340 digits prime (only).

http://plouffe.fr/Record%20100%20primes%20sequence.txt

I use a formula for forward calculation and backward calculation:

Forward calculation: next smallest prime to $\{a(n)^e\}$.

Backward calculation (to check): Previous prime = solve for x in $x^e - S(n+1)$.

This is the simulated annealing – *High speed guessing with a filter f*:

Guess first (real) value

- (0) Apply $f(S_0)$.
- (1) Is $f(S_0)$ prime?
- (2) If yes, then keep prime in list, if not go to (0) with a new starting value S_0.

The machine: core i7 at 4.4 Ghz with a 220 TB jbod
– 283 Billion floating point operations per second.

Finally, what could be the sequence with the smallest initial value, like 2?

Let’s try:

$a(n) = \{2^{d^n}\}$ where $d = 1.300768041481769105525256…$ (sequence A306317) [12]

2, 3, 5, 7, 13, 29, 79, 293, 1619, 14947, 269237, 11570443, 1540936027, …

Let’ try using DERIVE, too, taking the first 63 digits of *d*

```
d := 1.30076870414817691055252567828266106688423996320151467218595488
```

$$\mathrm{VECTOR}\left(\mathrm{ROUND}\left(2^{d^n}\right),\ n,\ 1,\ 12\right)$$

```
[2, 3, 5, 7, 13, 29, 79, 293, 1619, 14947, 269237, 11570443]
```

[12] http://oeis.org/A306317

Can we go backward, too? Like from any specific prime number using this algorithm?

Let's say from $10^{100} + 267$ to 2?

Yes, if the exponent α is inverted:

When $\alpha = 0.385622566415290\ldots$

then we have the sequence $10^{100} + 267$, 742123524365563, 542489, 163, 7, 2.

Here $a(0) = 2.1322219996628413452$ and the exponent $1/\alpha = 2.5932092490404286167308\ldots$

```
d_ := 0.38562256641529

VECTOR(ROUND((10^100 + 267)^(d_^n)), n, 2, 6)

[742123822526586, 542489, 163, 7, 2]

VECTOR(ROUND(2.1322219996628413452^((1/d_)^n)), n, 0, 3)

[2, 7, 163, 542489]
```

In 1902, a certain M. Cipolla published a formula for the n^{th} prime number:

$$p_n = n\,(\ln(n) + \ln(\ln(n)) - 1)$$

On the other hand, the formula for the number of primes less or equal to n is

$$\pi(n) = \frac{n}{\ln(n)} \qquad (n \longrightarrow \infty)$$

One formula being the inverse of each other.

Actually, no.

Very recently [4,5], a number of people began to realize that these inverses are not as they appear (References will be given at the end of the article in DNL#116.)

$$\text{If } \pi(n) \approx \frac{n}{\ln(n)} = y$$

then the inverse is $-yW\left(-\frac{1}{y}\right)$ (or for n) to simplify the notation.

This means that $p(n) \approx -nW_{-1}\left(-\frac{1}{n}\right)$[13].

Knowing that the value of $W\left(-\frac{1}{n}\right)$ has to be with W_{-1} and not $W\left(0, -\frac{1}{n}\right)$.

Now there is a big question about p_n, $\pi(n)$ and the precision.

As we know, the P.N.T. (Prime Number Theorem) is a major item. But in term of precision it is very rough. It is true, yes, but when $n \longrightarrow \infty$, the same with p_n.

We go back to the classic equations.

[13] Here $W_{-1}(x)$ is the second real branch of the Lambert W function, which is a strictly negative function defined on the domain $x \in [-1/e, 0]$.

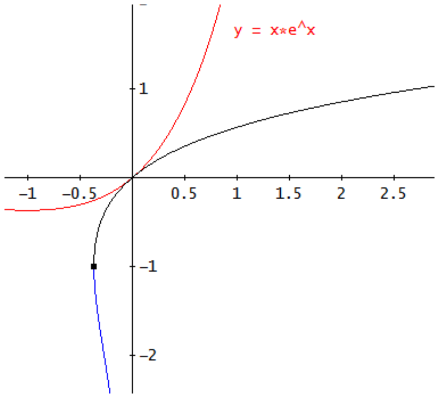
Comment (Josef):
Lambert W function is the inverse function of

$$f(x) = x \cdot e^x.$$

So, it is defined implicitly by the equation

$$x = W(x) \cdot e^{W(x)}.$$

The plot shows $f(x)$ (in red) and its inverse: one real branch W_0 (black) and the other one W_{-1} (blue)



$$\pi(n) \approx \frac{n}{\ln(n)}$$

is the classic equation; we change it for (see Dusart thesis [2010] for details):

$$\pi(n) \approx \frac{n}{\ln(n) - 1}$$

If we compute the inverse of $\frac{n}{\ln(n)-1} = y$, then we receive (with y renamed):

$$p(n) \approx -n \cdot W_{-1}\left(-\frac{e}{n}\right).$$

With the n^{th} prime we have the formula $p_n \approx n\ln(n)$ and further $\pi(n) \approx \frac{n}{W_0(n)}$.

I will interrupt here and continue presenting Simon Plouffe's lecture in the next DNL. All references will be given at the end of the lecture. I recommend to get more information about Lambert W function. Download "*On the Lambert W function*" from https://cs.uwaterloo.ca/research/tr/1993/03/W.pdf.

Additional Comments (Josef):

MATHEMATICA gives:

Solve[n/Log[n]==y,n]

$$\left\{\left\{n \to -y\,\text{ProductLog}\left[-\frac{1}{y}\right]\right\}\right\}$$

Solve[n/(Log[n]-1)==y,n]

$$\left\{\left\{n \to -y\,\text{ProductLog}\left[-\frac{e}{y}\right]\right\}\right\}$$

(`ProductLog` is the *MATHEMATICA*-expression for Lambert W.)

Lambert W is not implemented in *DERIVE*. In an earlier issue of DNL (DNL#74 from 2009) Jim FitzSimons presents a procedure to calculate function values of Lambert W. You can find it in MTH74.zip on the DUG-website.

Juri Rolf showed in MNU 65/2 (2012) how to solve the equation $x^y = y^x$ applying the Lambert W function. Among other ideas he referred to the paper mentioned above: it is possible to prove the following identities using Lambert W function:

$$x \log x = a \iff x = e^{W(a)}$$
$$x\, b^x = a \iff x = \frac{W(a \log b)}{\log b}$$
$$x^{x^a} = b \iff x = e^{\frac{W(a \log b)}{a}}$$
$$a^x = x + b \iff x = -b - \frac{W(-a^{-b} \log a)}{\log a}$$

First of all, I wanted to check these identities using *MATHEMATICA*:

```
Solve[x*Log[x] == a, x]
```

$$\left\{\left\{x \to \frac{a}{\text{ProductLog}[a]}\right\}\right\}$$

```
Solve[x*b^x == a, x]
```

$$\left\{\left\{x \to \frac{\text{ProductLog}[a\,\text{Log}[b]]}{\text{Log}[b]}\right\}\right\}$$

$$\left\{\left\{x \to \left(\frac{a\,\text{Log}[b]}{\text{ProductLog}[a\,\text{Log}[b]]}\right)^{\frac{1}{a}}\right\}\right\}$$

```
Solve[a^x == x + b, x]
```

$$\left\{\left\{x \to -b - \frac{\text{ProductLog}[-a^{-b}\,\text{Log}[a]]}{\text{Log}[a]}\right\}\right\}$$

I can use Rolf's hint and try the first two examples:

Hint: ♣ $e^{♣}$ = ♥ ⇔ ♣ = W(♥).

$x \cdot \log x = a$ $\log x \cdot e^{\log x} = a$ | apply Hint! $\log x = W(a)$ $x = e^{W(a)}$ Using $a = W(a) \cdot e^{W(a)}$ gives the result provided by *Mathematica*.	$x \cdot b^x = a$ $x \cdot (e^{\log b})^x = a \| \cdot \log b$ $x \cdot \log b \cdot (e^{x \cdot \log b}) = a \cdot \log b$ | apply Hint! $x \cdot \log b = e^{W(a \cdot \log b)}$ $x = \frac{e^{W(a \cdot \log b)}}{\log b}$

Applications of Computer Algebra – ACA 2019
Montreal, July 16 – 20, 2019

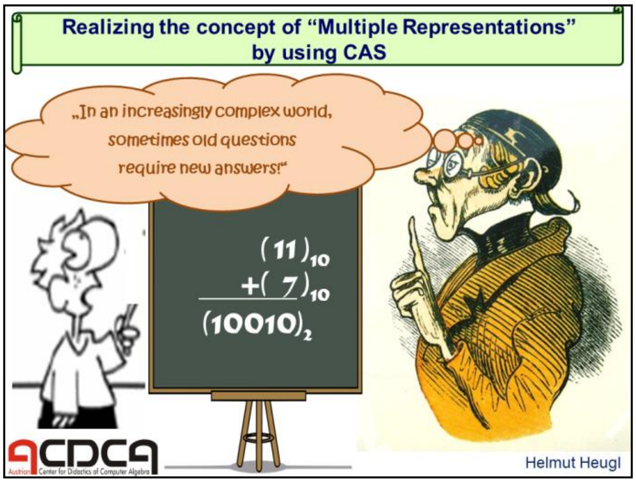
Realizing the Concept of "Multiple Representations" by using CAS

Helmut Heugl[1]

Mathematical concepts are presented in multiple modes of representation (or "prototypes") such as text, graphs and diagrams, tables, algebraic expressions and computer simulations. A prime goal of teaching is to help learners develop an understanding of the mathematical concepts by considering and using these different representational modes and levels. Several prototypes of a concept provide complementary information [1].

Therefore, it is not enough to become acquainted with and to understand the information of a certain representation mode. A central cognitive activity on the way to mathematical concepts is to build links between representation modes of a concept. In traditional mathematics education prototypes mostly are available in a serial way. The main importance of technology tools is that the learner can use several prototypes parallely.

By using examples of Algebra and Analysis I will show the role of CAS when building links between several representation modes of a concept or when solving problems [2].



"***Old questions require new answers***"!

2004 I have given a lecture here in Montreal and I I have spoken about of the results of our CAS-research projects.

One result we have called [Heugl, 1996]
"The Window-Shuttle-Method"

Meanwhile we can find many investigations and new answers of this important didactical principle, but the new answers also have a new heading:

"**The principle of multiple representations**"

The potential for computer-based aids to learning mathematics remains high, although the current contribution of technology to didactic and content innovations is at least in our country frustratingly low. Discussions about the role of technology are too often based on what computers can do rather than on research-based investigations of how students learn with technology. In this lecture I will discuss the question. "How do students learn with modern digital media which can be characterized as „Multi-Representations-Programs "(MRP) and what is the role of CAS among the offered modes of representations.

1. Mathematical prerequisites

Before describing the principle of "Multiple Representations" and showing examples concerning the role of CAS some mathematical prerequisites are necessary.

We cannot discuss the question "Why CAS?" without asking "Why mathematics?" Diverse answers can be found in the mathematical literature and also in every mathematics curriculum of every country all over the world. I prefer the answer of my important mentor, a famous Austrian mathematician, Bruno Buchberger:

> *"mathematical thinking technology is the essence of science and the essence of a technology-based society"*

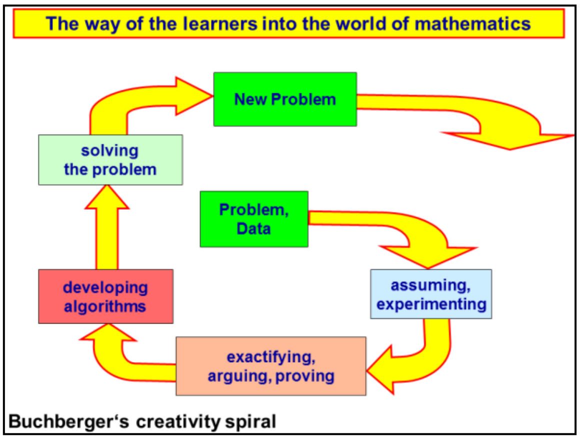
Interpreting a mathematics curriculum we should not require: "*our students have to learn integrals!",* we should question: *„what thinking technology students are gaining when learning integrals?"* This definition answers not only the question "Why learning mathematics?", it influences also the question: "How learning mathematics?" and therefore it plays an important role in the topic of my lecture.

The way of the learners into the world of mathematics:

For describing the learner's way into the world of mathematics Bruno Buchberger uses a spiral as a model [Buchberger, 2002].

The spiral begins with observations, data material or **problems**, the solution of which can generate the development of algorithms or in the creation of new concepts.

By analyzing, **experimenting** or generally through heuristic strategies, assumptions are found for solving the given problem.



Buchberger's creativity spiral

By proving and substantiating, in other words, by **exactifying,** one enters into the next stage of the spiral: Theorems and sentences which can now be ensured and proved.

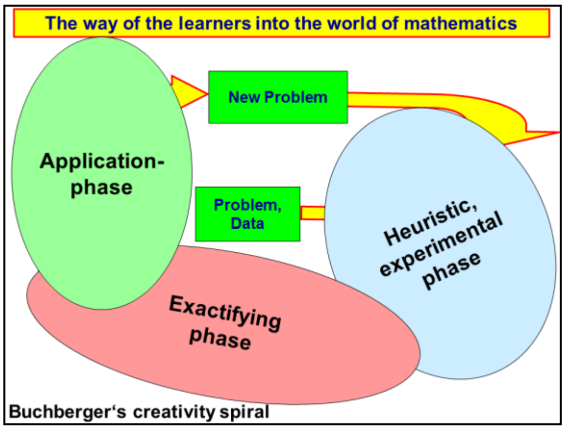
Thus, supported by acquired knowledge, one proceeds **to develop** those **algorithms** or programs which are necessary for problem solving. Testing of and consolidating the developed algorithms by practicing is an integral part of this stage.

The actual part of the spiral ends with the next step. The insight and strategies are now used **to solve the initial problem** or related problems.

When **new problems** evolve and new additional knowledge is necessary or new algorithms need to be developed, then the stages of the spiral are repeated once again.

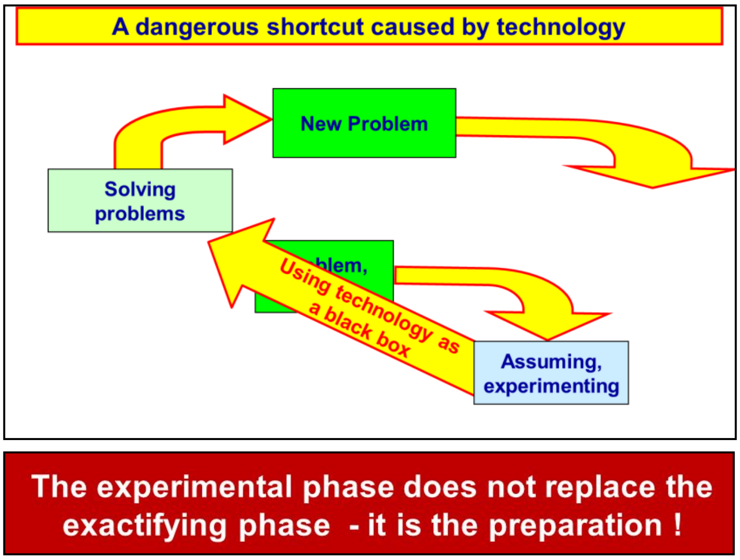
The several steps in course of one pass of the spiral can be seen as activities in **three phases**:

- The heuristic, experimental phase
- The exactifying phase
- The application phase



Buchberger's creativity spiral

Observing mathematic classes where technology is used, we find **a dangerous shortcut** of the creativity spiral:



The pupil could form assumptions in the heuristic phase, skip over the abstract phase, over the corroboration of algorithms and the practicing of calculating skills and then using the CAS as a Black Box immediately turn to the applications.

This way of doing mathematics does not establish the thinking technology that is demanded in our curriculum.

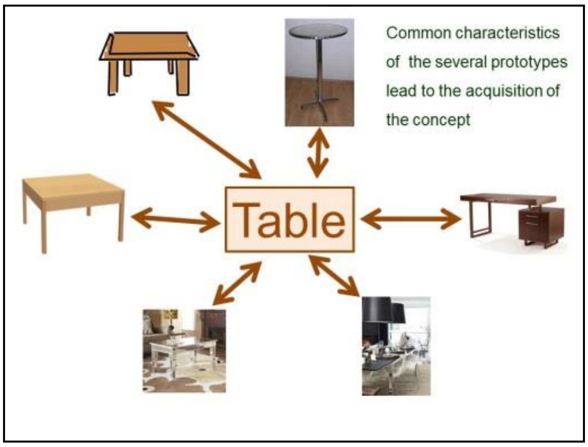
2. The „principle of multiple representations“

There is an important gap between mathematical concepts and concepts in other scientific fields. We cannot see them, study them through a microscope or take a picture of them. The only way of gaining access to them is using signs, words or symbols, expressions or drawings. With other words we need representations to develop them and to work with them.

2.1 Learning by exploring prototypes (representations)

General concepts become cognitively available through concrete representations or in our words through prototypes of the concept [Dörfler, 1991}

Think about the concept “table”. How does a child acquire this concept? Not by an exact definition given by mother or father. The child experiences several prototypes of tables and the common characteristics of the several prototypes lead to the acquisition of the concept.



Mathematical representations can be categorized in two main classes – symbols and icons

- **Descriptive representations** include information given by words or sentences in the natural language or symbols and objects in the mathematical language or data e.g. offered in tables
- **Depictive representations** provide visual information like graphs, diagrams, geometric objects, flow charts, etc.

2.2 Learning by interacting between „prototypes“(representations)

The availability of single prototypes is not enough for a successful learning process. The central cognitive activity is the permanent variation of the representations and the interaction between the several prototypes. The analysis of the use of representations in context of mathematics education shows that a close interaction between depictive and descriptive representations is needed order to make the best use of both kinds of representations for successful thinking and problem solving.

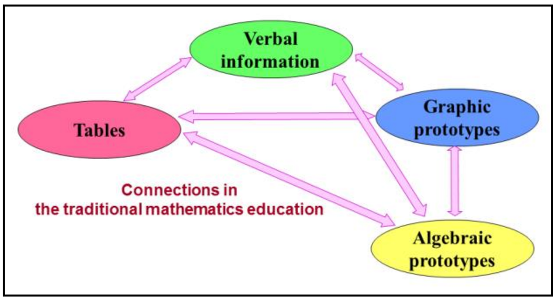
> ***We consider the combined use of different representations - especially descriptive and depictive ones – as a key concept for teaching mathematics and for thinking and problem solving in mathematics*** [Schnotz, 2010]

This quotation can be seen as a **definition of the “Principle of multiple representations”**

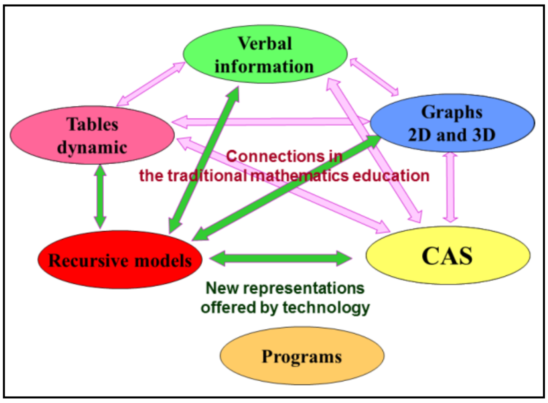
On the way of the learners into the world of mathematics the concept of functions plays a central role.

Following prototypes (representations) of functions are used:

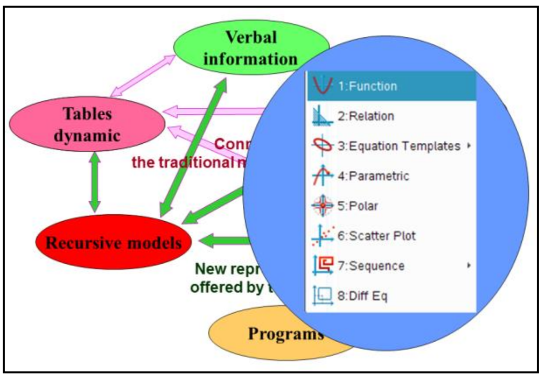
In traditional mathematics education following prototypes (representations) of functions are used, some descriptive and others depictive.



The computer as a medium of prototypes*:*
The computer changes the various prototypes and offers those which would not be available without the computer.

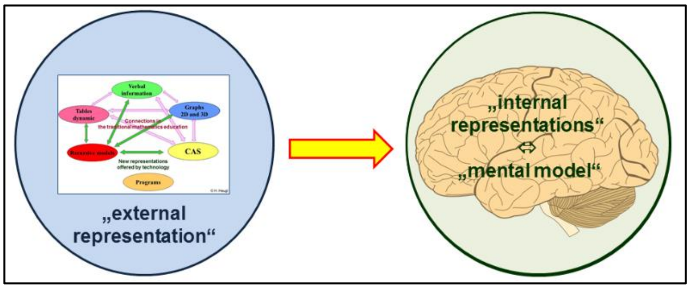


If we use a learning media like TI Nspire CAS, GeoGebra or Casio Classpad and look at the graphic prototype we can see a lot of graphic representations



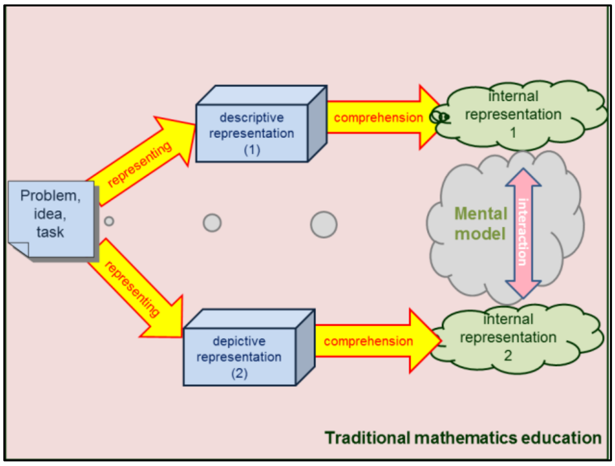
2.3 From external to internal representations

The sense and purpose of learning by using multiple representations is the construction of internal (mental) representations on the basis of intentionally created external representations. The distinction between descriptive and depictive representations applies also to the internal (mental) representations constructed during comprehension [Kintsch, 1998].



I will try to visualize the difference between the development of a mental model on the one hand without using technology in traditional mathematics education and on the other hand when using technology:

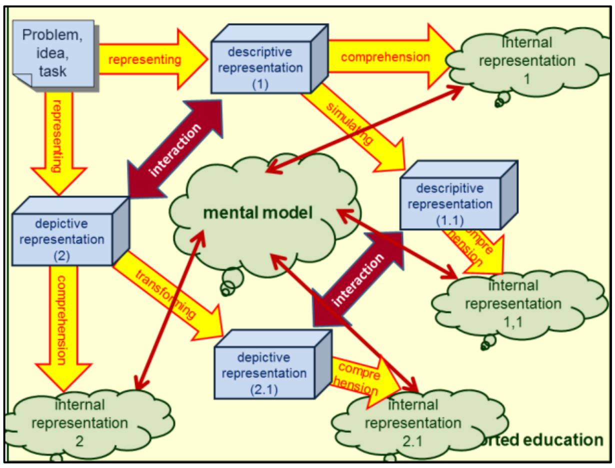
Development of a mental model in traditional mathematics education



More frequently only one representation (either descriptive or depictive) is developed. Rarely the interaction between several representations can be observed.

One of the reasons is that representations can only be developed serially and therefore interactions between representations are more difficult.

Development of a mental model when using modern digital media
(⇔ “multi representation tools”)



(1) Depictive and descriptive representations can be developed parallelly and therefore interactions are much easier.

(2) Representations can be changed by transforming with technology or by simulating ⇨ new internal representations are developed

(3) The resulting mental model is based on several interacting internal representations.

Advantages of the use of technology:

- ☞ The development of representations is much easier
- ☞ Technology offers new representations e.g. recursive models, 3D-graphic, a.s.o.
- ☞ Transforming representations and especially simulation is at first possible by using technology
- ☞ Interaction between representations is easier and representations are available parallelly
- ☞ Technology supported interaction between representations produces better mental models

Modern digital media are „Multi-Representations-Programs"(MRP) ⇨

> **Modern digital media are a prerequisite for the realization of the principle of multiple representations.**

3. The role of CAS for realizing „the principle of multiple representations"

3.1 Interaction between verbal representations and symbolic representations of CAS

Symbolic representations are also elements of a language the mathematical language:

> ***Mathematics is a language*** *and like other languages it has its own grammar, syntax, vocabulary, word order, synonyms, conventions, etc.*
>
> *This language is both a means of communication and an instrument of thought.*
>
> *[W. Esty, 1997]*

The development of this language over the centuries:

The ancient mathematical language was close or identical to the native language	In the course of history more and more new language elements, grammar and syntax were developed
"If a straight line be cut at random, the square on the whole is equal to the squares on the segments and twice the rectangle contained by the segments." (Euclid, Elements, II.4, 300 B.C.)	The equivalent rule using algebraic language elements: $(a+b)^2 = a^2 + b^2 + 2.a.b$
"The area of any circle is equal to a right-angled triangle in which one of the sides about the right angle is equal to the radius, and the other to the circumference of the circle." (Archimedes, On the Sphere and the Cylinder, 220B.C.)	$A = r.2\pi\ r/2 = r^2\pi$

These examples and our investigations in the classroom corroborate J. Piaget's thesis that the genesis of knowledge in the sciences and in the individual follows the same mechanisms. Also students during their way into mathematics acquire more and more new language elements and necessary rules for using the language of mathematics for problem solving.

If we maintain that the main role of mathematics is problem solving, consisting of the activities modelling – operating – interpreting, then a main goal of mathematics learning is the translation process from a problem formulated in the native language to a mathematical model written in the language of mathematics.

Technology and especially CAS support the translation process from the native language into the language of mathematics.

Example 1: Derivation of Cramer's rule

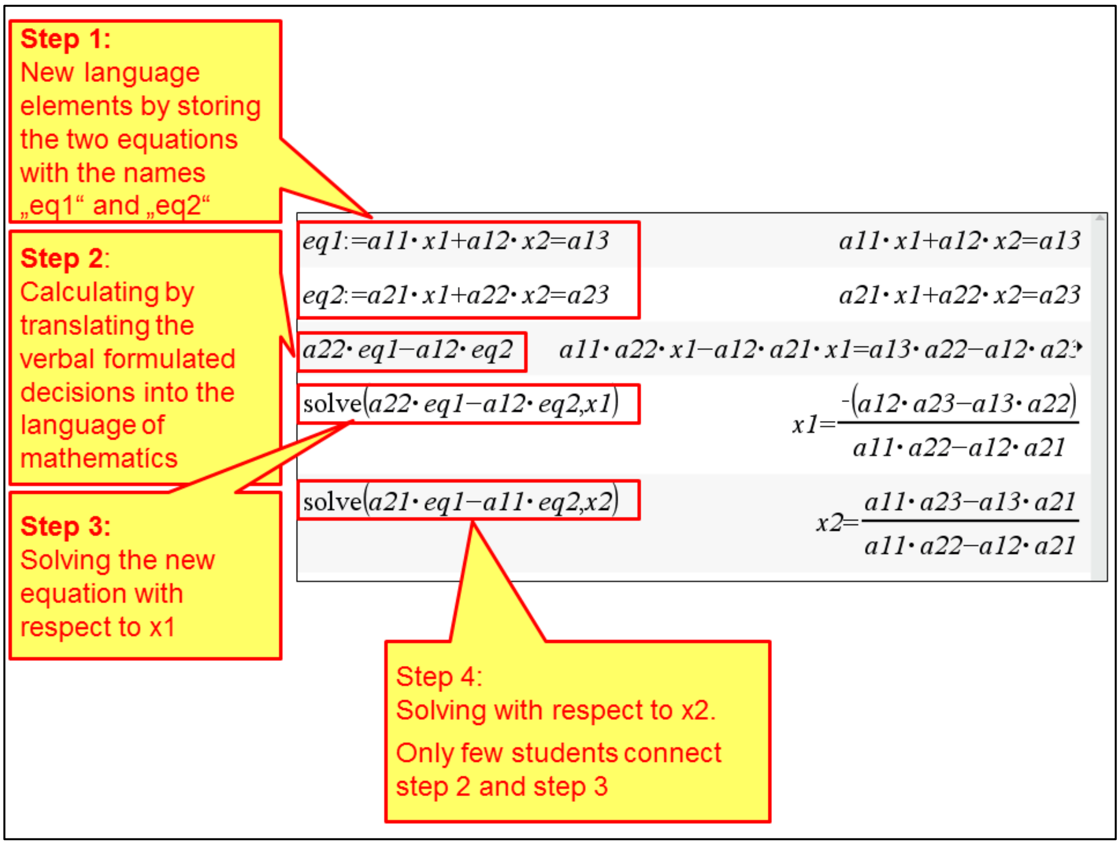
Given is a system of equations with two equations and two variables:

$$(1)\quad a11\cdot x1 + a12\cdot x2 = a13$$

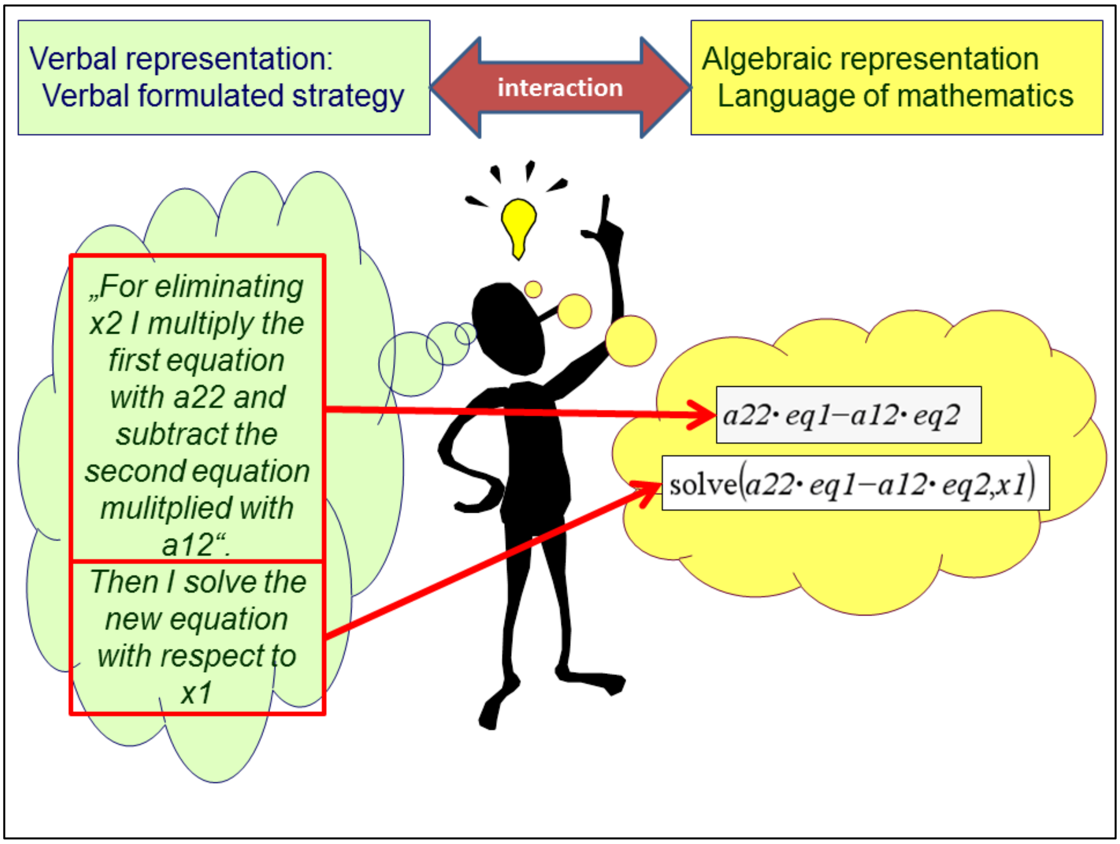
$$(2)\quad a21\cdot x1 + a22\cdot x2 = a23$$

Task:

Find a solution by using CAS.



The advantages of the use of CAS



The verbal formulated decisions of the verbal representation are directly translated into the language of mathematics.

Example 2: Flu epidemic

For investigating the development of a flu wave in a town with 5000 inhabitants the number of sufferers E with respect to the time t (in days) can be described with a cubic polynomial function.

Following information are known:

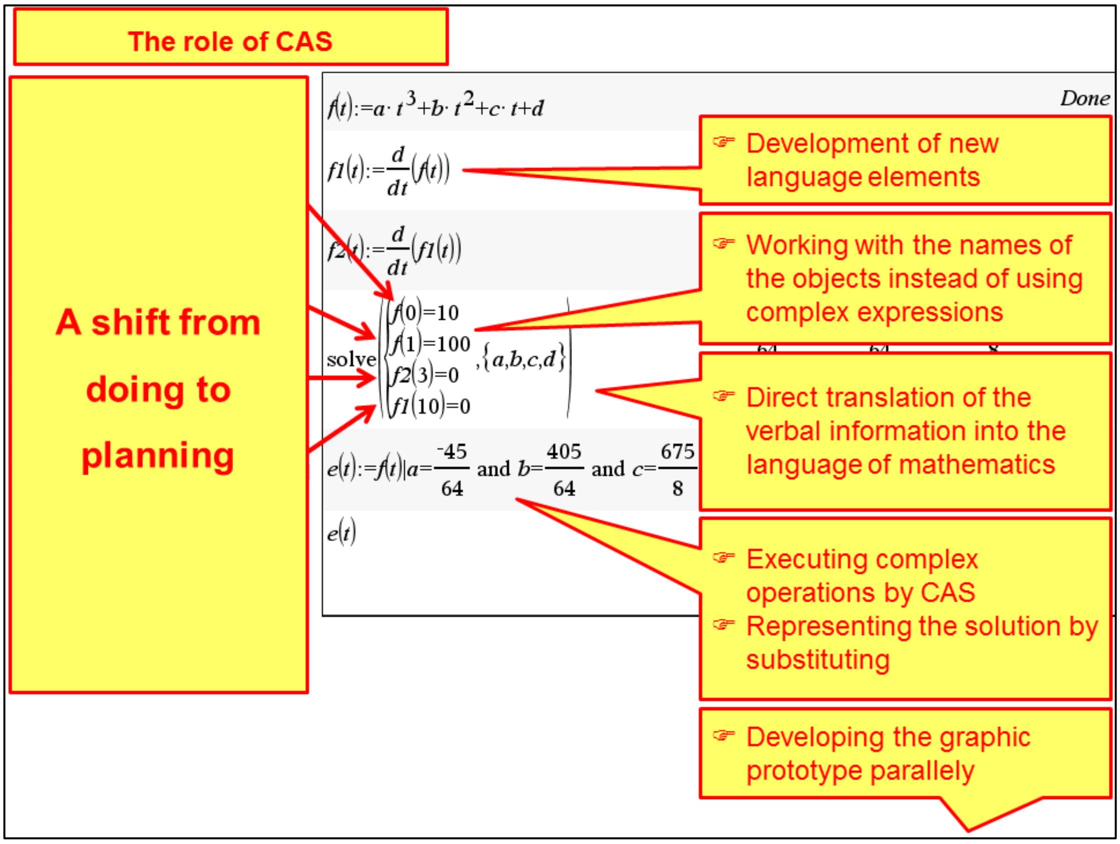
- At the beginning of the study 10 persons are down with influenza
- After one day the number has increased to 100 persons
- At the third day the growth rate is strongest
- At the 10^{th} day the flu wave (the number of sufferers) has is maximum

Task:

a) Determine the equation of the function E and draw the graph

b) Find the day where the progressive increase of the number of sufferers turns into a degressive increase

The role of CAS:



Conclusion 3.1: The direct translation of verbal formulated decisions and arguments into symbolic representations by using CAS causes a new quality of interactions between two modes of representation. A consequence is a more stable mental model which is better suited for problem solving.

3.2 Interaction between graphic representations and CAS

A short look at results of brain research: *"2,5 million nerve fibers are connecting several parts of the body with the brain. 2 million of them are belonging to the optic nerves"* [Hanisch, Sattelberger, 2008]

A consequence is that human beings are perfectly equipped for visualization. The graphic mode of representation or more general depictive representations play an important role for the concept of multiple representations.

But the creation of a depictive representation is not sufficient for successful cognitive problem solving.

Operating on depictive representations implies a close interaction between description and depiction because procedures on a representation are usually guided by a descriptive representation. Accordingly, learners should be taught to closely interconnect descriptive and depictive representations when solving problems.

Mathematical procedures and operations are executed in the symbolic mathematical language and therefore CAS are the appropriate descriptive modes of representation.

In earlier DNLs presented Piotr Trebisz "Mathematic Models for Snail Shells" (DNL#81 and following). Here we have another model: One function produces various figures. Josef

Ammonites, Gastropods and Bivalves

Colin Kennedy, UK

Recent contributions to the Derive Newsletter by Walter Wegscheider (DNL #60) and Susan Jackman (DNL #61) have shown how to produce coil-like 'creatures' using Derive. I was interested because I have recently read 'Climbing Mount Improbable' by Richard Dawkins in which he gives examples of computer-generated shells resembling ammonites, gastropods and bivalves. He does not give much detail as to how these were generated, but refers to two papers by D. M. Raup in the Journal of Palaeontology (1966, 40, 1178-1190 and 1967, 41, 43-65). I have not seen these papers but Hawkins does give a description of the factors which control shell shape.

All shells are based on a logarithmic spiral which is controlled by three parameters called (by Hawkins) flare (f), verm (v) and spire (s). Flare (f) determines how rapidly the spiral expands while verm (v) determines whether the coils will be non-touching and worm-like, just touching, or overlapping. For instance, by keeping f = 2 then when v = 0.5 the coils will be just touching but if v = 0.3 the coils will overlap. Spire (s) determines how much the coils are pulled out in a direction (here the z axis) perpendicular to the 'plane' of the ammonite thus allowing gastropod type shells to be produced. A very high value of flare (f) can give a shape similar to a bivalve.

The vector formula I have developed for Derive will produce 'shells' similar to those in Hawkins' book. I have retained the parameters f, v and s which have the same functions as those described by Hawkins, although their precise use in formulae used by Hawkins is likely to be different. I have also introduced a fourth parameter, q (for squash!), in order to squash (or lengthen) the otherwise circular cross sections of the 'ammonite' coils along the z axis (i.e., perpendicular to the plane of the ammonite) as some ammonite shells show this characteristic.

The Derive vector is-

$$\text{amm}(f,\ v,\ sp,\ q) :=$$

$$\left[\frac{1}{2}\cdot(1 + v + (1 - v)\cdot \text{COS}(t))\cdot f^{s/(2\cdot\pi)},\ s,\ \left(\frac{1}{2}\cdot(1 - v)\cdot \text{SIN}(t)\cdot f^{s/(2\cdot\pi)} + \frac{sp\cdot s}{2\cdot\pi}\right)\cdot q\right]$$

All images require cylindrical coordinates to be set. For the 'ammonites' and 'bivalves' s is plotted from -4π to 4π and for 'gastropods' from -6π to 6π. t is plotted from 0 to 2π. To avoid too chunky an image s needs about 100 panels and t about 40 panels.

Various ammonites are shown below together with the parameter values used.

For presenting the figures with TI-Nspire we have to convert the cylindrical coordinates into cartesian ones. This is no problem. The graphs are not as smooth as the DERIVE generated ones, because we have only a 50 x 50 grid (panels in DERIVE) for the parameter curves. We have to adapt the plot range for the various surfaces. Josef

$$\mathbf{r}(f,v):=\frac{1}{2}\cdot(1+v+(1-v)\cdot\cos(t))\cdot f^{\frac{u}{2\cdot\pi}} \;\blacktriangleright\; \textit{Done}$$

$$\mathbf{amm}(f,v,sp,q):=\left\{\mathbf{r}(f,v)\cdot\cos(u),\mathbf{r}(f,v)\cdot\sin(u),\left(\frac{1}{2}\cdot(1-v)\cdot\sin(t)\cdot f^{\frac{u}{2\cdot\pi}}+\frac{sp\cdot u}{2\cdot\pi}\right)\cdot q\right\} \;\blacktriangleright\; \textit{Done}$$

$\mathbf{am1}:=\mathbf{amm}(2,0.5,0,1)$

$$\blacktriangleright\; \left\{0.25\cdot(\cos(t)+3.)\cdot\cos(u)\cdot 2^{\frac{u}{2\cdot\pi}},0.25\cdot(\cos(t)+3.)\cdot\sin(u)\cdot 2^{\frac{u}{2\cdot\pi}},0.25\cdot\sin(t)\cdot 2^{\frac{u}{2\cdot\pi}}\right\}$$

$\mathbf{xp1}(t,u):=\mathbf{am1}[1]:\mathbf{yp1}(t,u):=\mathbf{am1}[2]:\mathbf{zp1}(t,u):=\mathbf{am1}[3] \;\blacktriangleright\; \textit{Done}$

$\mathbf{am2}:=\mathbf{amm}(2,0.25,0,1)$

$$\blacktriangleright\; \left\{0.375\cdot(\cos(t)+1.66667)\cdot\cos(u)\cdot 2^{\frac{u}{2\cdot\pi}},0.375\cdot(\cos(t)+1.66667)\cdot\sin(u)\cdot 2^{\frac{u}{2\cdot\pi}},0.375\cdot\sin(t)\cdot 2\right.$$

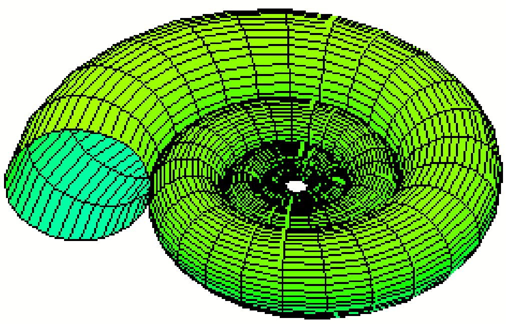
$\mathbf{xp2}(t,u):=\mathbf{am2}[1]:\mathbf{yp2}(t,u):=\mathbf{am2}[2]:\mathbf{zp2}(t,u):=\mathbf{am2}[3] \;\blacktriangleright\; \textit{Done}$

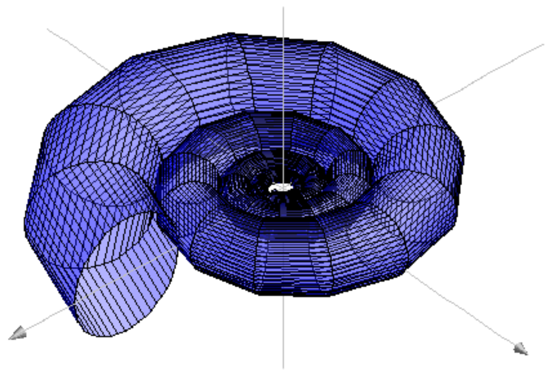
$\mathbf{am3}:=\mathbf{amm}(2,0.5,0,0.3)$

$$\blacktriangleright\; \left\{0.25\cdot(\cos(t)+3.)\cdot\cos(u)\cdot 2^{\frac{u}{2\cdot\pi}},0.25\cdot(\cos(t)+3.)\cdot\sin(u)\cdot 2^{\frac{u}{2\cdot\pi}},0.075\cdot\sin(t)\cdot 2^{\frac{u}{2\cdot\pi}}\right\}$$

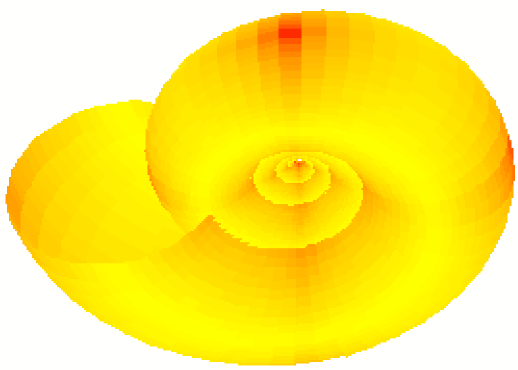
$\mathbf{xp3}(t,u):=\mathbf{am3}[1]:\mathbf{yp3}(t,u):=\mathbf{am3}[2]:\mathbf{zp3}(t,u):=\mathbf{am3}[3] \;\blacktriangleright\; \textit{Done}$

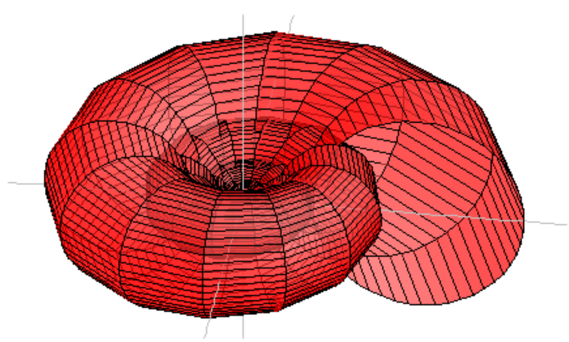
It is necessary to store the list of coordinates as a constant. xp1(t,u):=amm(2,0.5,0.1)[1] does not work; xp1(t,u):=am1[1] works.





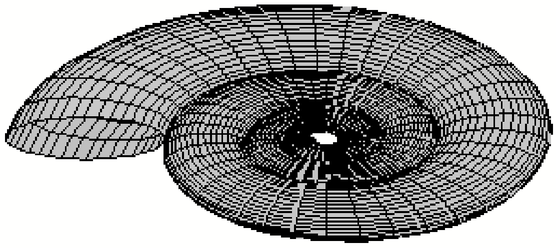
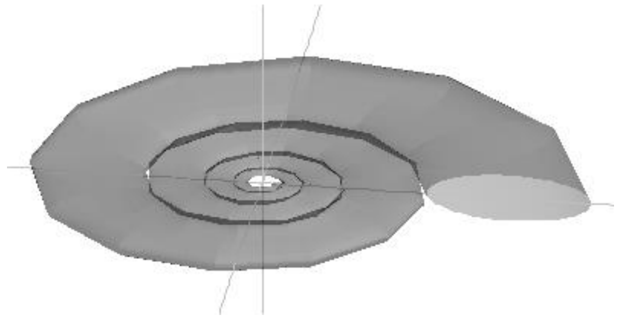
amm(2, 0.5, 0, 1)



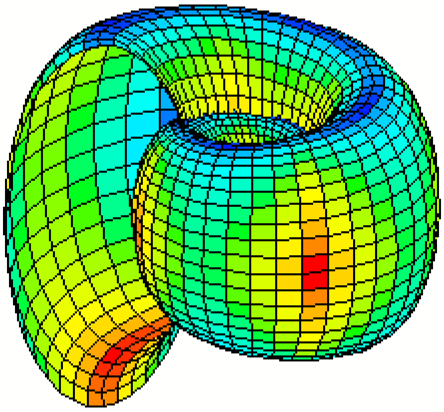
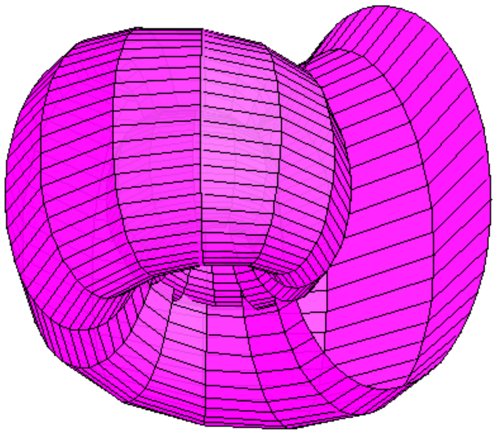


amm(2, 0.25, 0, 1)

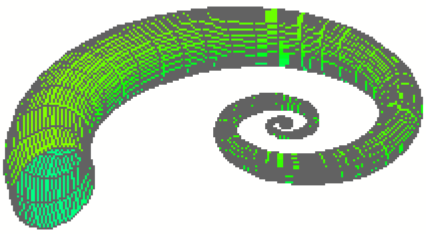
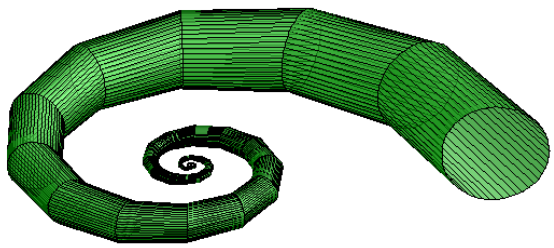
Some more graphs with DERIVE and TI-Nspire:

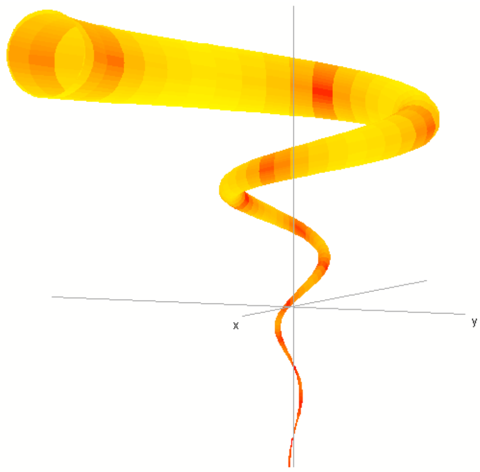
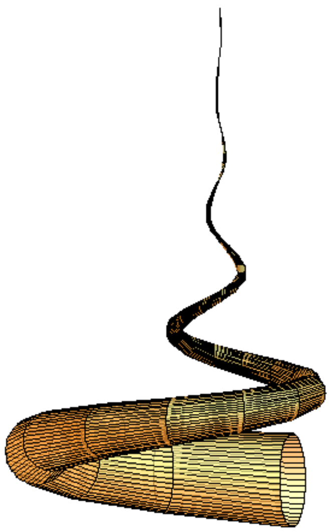
amm(2, 0.5, 0, 0.3)

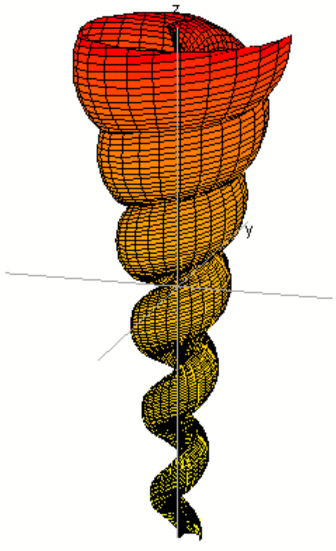
amm(2, 0.25, 0, 2)

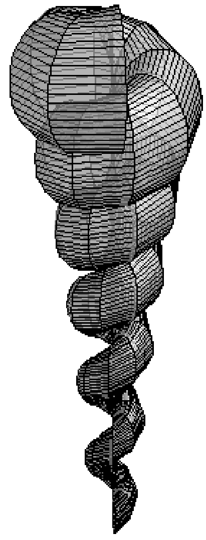



amm(4, 0.7, 0, 1)

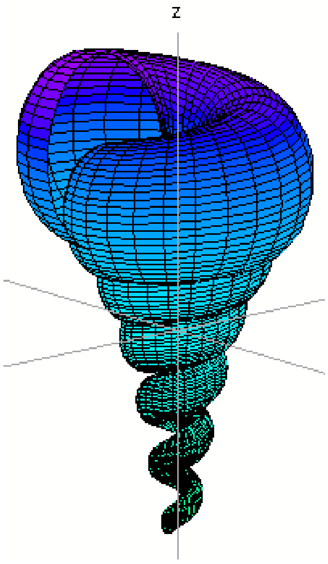



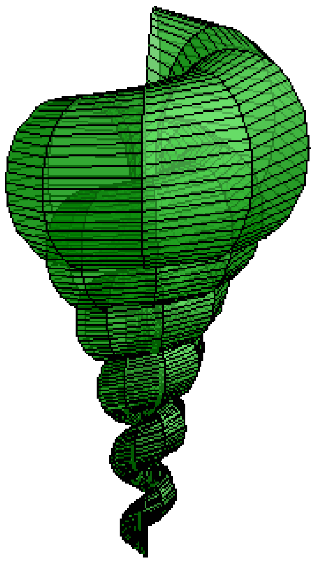
amm(4, 0.7, 6, 1)





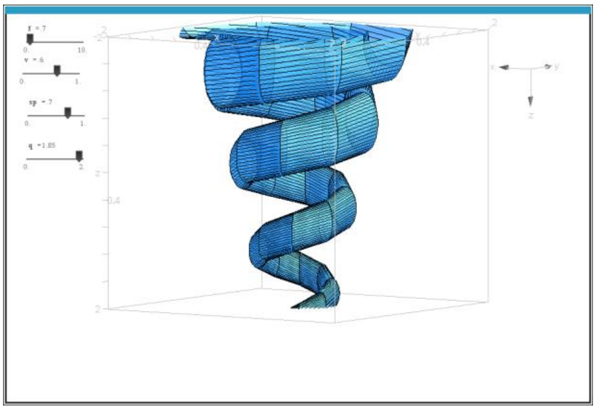
amm(1.3, 0, 1.2, 1.4)

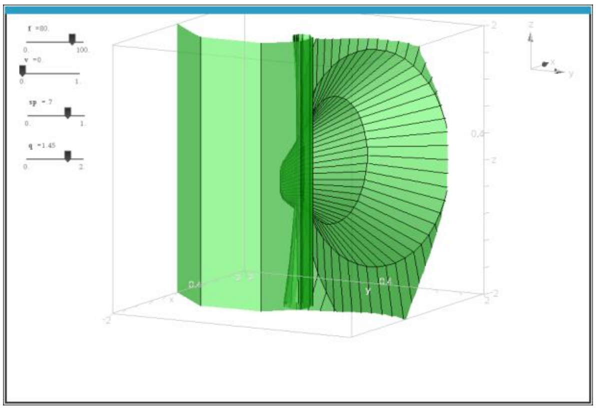


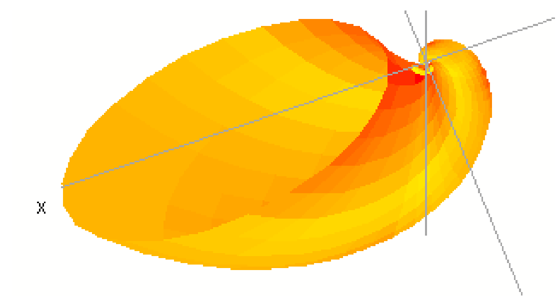


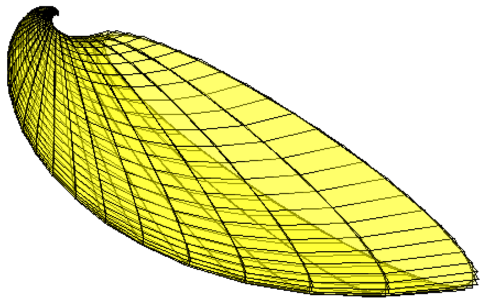
amm(1.4, 0.1, 0.8, 1.35)

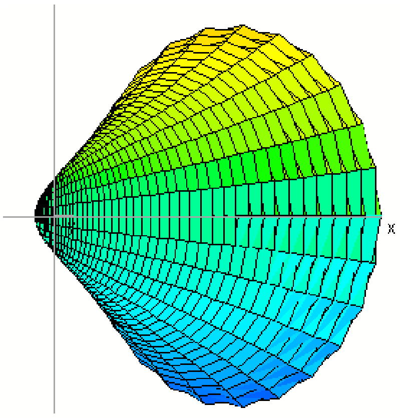
The best way to explore this zoo is to insert four slider bars in the plot window, one for each of the four parameters **f**lare, **v**erm, **s**pire and s**q**uash. But remember, please be kind to small animals!

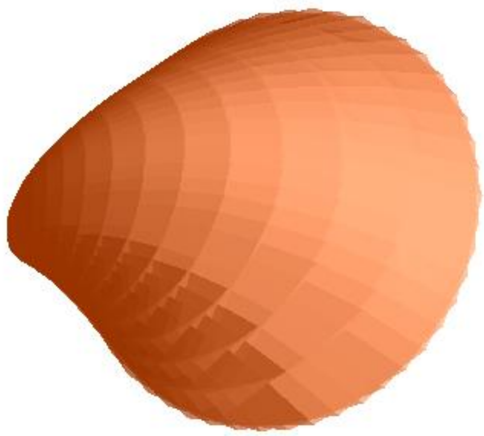


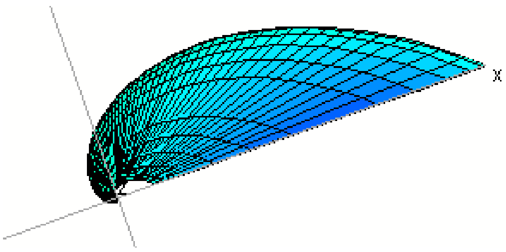






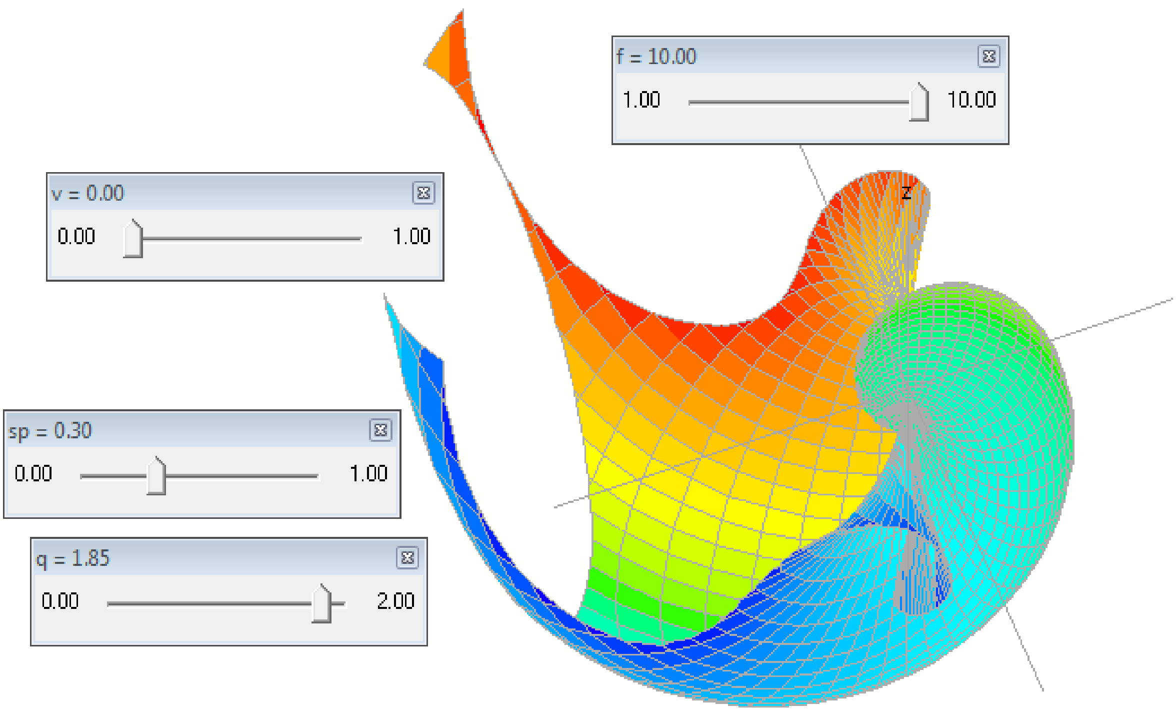






amm(1000, 0.1, 2, 1.3)

The DERIVE sliders:



Right or Wrong? Unexpected results when using CAS

Helmut Heugl, Austria & Michel Beaudin, CAN & Josef Böhm, Austria

1. A delicate problem of the final exam in Thüringen in Germany caused by the use of CAS

This problem is especially interesting for my didactical strategies when using technology. For me one of the most important changes when using CAS is that the learners develop own language elements of their mathematical language and for calculating they use the names of expressions instead of these complex expressions. The following shows that just this innovation also can cause mistakes.

In the central final exam in Thüringen CAS is obligatory and students are using TI Nspire CAS.

Final exam 2018 part B (CAS)

Analysis example 2 part a):

Given is an array of curves by the function $f_k : x \to \dfrac{80x}{k^2+x^2}$ with $D=\mathbb{R}$ and $k \in \mathbb{R}^+$. The graph of f_k is named as G_k.

Task: Determine with respect to k the position and sort of the extreme points of G_k

.Following solutions of students have been found:

Solution 1:

$f(x,k):=\dfrac{80\cdot x}{x^2+k^2}$	*Fertig*
$\dfrac{d}{dx}(f(x,k))\to fa(x,k)$	*Fertig*
$\dfrac{d}{dx}(fa(x,k))\to fb(x,k)$	*Fertig*
⚠ solve$(fa(x,k)=0,x)$	$x=^-k$ or $x=k$
$fb(k,k)$	$\dfrac{80}{k^3}$
⚠ $fb(^-k,k)$	$\dfrac{40}{k^3}$

Solution 2:

$f(x):=\dfrac{80\cdot x}{x^2+k^2}$	*Fertig*
$\dfrac{d}{dx}(f(x))\to fa(x)$	*Fertig*
$\dfrac{d}{dx}(fa(x))\to fb(x)$	*Fertig*
⚠ solve$(fa(x)=0,x)$	$x=^-k$ or $x=k$
$fb(k)$	$\dfrac{80}{k^3}$
⚠ $fb(^-k)$	$\dfrac{40}{k^3}$
⚠ $fa(x)$	$\dfrac{^-80\cdot(x^2-k^2)}{(x^2+k^2)^2}$
⚠ $fb(x)$	$\dfrac{160\cdot x\cdot(x^2-3\cdot k^2)}{(x^2+k^2)^3}$
$fb(k)$	$\dfrac{80}{k^3}$
⚠ $fb(^-k)$	$\dfrac{40}{k^3}$

Solution 3:

$f(x,k):=\frac{80\cdot x}{k^2+x^2}$	*Fertig*
$\frac{d}{dx}(f(x,k))$	$\frac{-80\cdot(x^2-k^2)}{(x^2+k^2)^2}$
$\frac{-80\cdot(x^2-k^2)}{(x^2+k^2)^2}\to fa(x,k)$	*Fertig*
$\frac{d}{dx}(fa(x,k))$	$\frac{160\cdot x\cdot(x^2-3\cdot k^2)}{(x^2+k^2)^3}$

$\frac{160\cdot x\cdot(x^2-3\cdot k^2)}{(x^2+k^2)^3}\to fb(x,k)$	*Fertig*
$\text{solve}(fa(x,k)=0,x)$	$x=-k$ or $x=k$
$fb(k,k)$	$\frac{-40}{k^3}$
$fb(-k,k)$	$\frac{40}{k^3}$

What is right? What is wrong?

The only correct solution is solution 3 although all students have used the same mathematical strategy – first derivative, zeros of the derivative, second derivative, a.s.o.

My Analysis:

1. The way of traditionally educated students:

$f(x)$	$\frac{80\cdot x}{x^2+k^2}$
$\frac{d}{dx}(f(x))$	$\frac{-80\cdot(x^2-k^2)}{(x^2+k^2)^2}$
$\frac{d}{dx}\left(\frac{-80\cdot(x^2-k^2)}{(x^2+k^2)^2}\right)$	$\frac{160\cdot x\cdot(x^2-3\cdot k^2)}{(x^2+k^2)^3}$
$\frac{160\cdot x\cdot(x^2-3\cdot k^2)}{(x^2+k^2)^3}\|x=k$	$\frac{-40}{k^3}$

They would calculate the derivative of f with respect to x and using this expression they calculate the second derivative with respect to x.

Then they substitute x by k and get the value of the second derivative

2. The way of most of the CAS users:

$f(x)$	$\frac{80\cdot x}{x^2+k^2}$
$fa(x):=\frac{d}{dx}(f(x))$	*Done*
$fb(x):=\frac{d}{dx}(fa(x))$	*Done*
$fb(k)$	$\frac{80}{k^3}$

They develop new language elements by defining functions and find the derivatives by using the names of the functions instead of the complex expressions.

Then they look at the function value of the second derivative for x = k.

$fb(x):=\frac{d}{dx}\left(\boxed{\frac{d}{dx}(fa(x))}\right)$	*Done*
$\boxed{fb(k):=\frac{d}{dk}\left(\frac{d}{dk}(fa(k))\right)}$	*Done*
$\frac{d}{dk}(fa(k))$	$\frac{-40}{k^2}$
$fb(k):=\frac{d}{dk}\left(\frac{-40}{k^2}\right)$	*Done*
$fb(k)$	$\frac{80}{k^3}$

This means:

fb(x) is stored in that way and the calculation of the first derivative is not executed.

The mistake occurs when looking for the function value of the second derivative fb(k) because now the CAS tool calculates the derivative with respect to k.

The same error cannot occur when using GeoGebra

Datei Bearbeiten Ansicht Einstellungen Werkzeuge Fenster Hilfe

1	f(x):=(80x)/(k^2+x^2) $\rightarrow\ f(x) := 80 \cdot \frac{x}{k^2+x^2}$
2	fa(x):=Ableitung(f(x)) $\rightarrow\ fa(x) := -160 \cdot \frac{x^2}{(k^2+x^2)^2} + \frac{80}{k^2+x^2}$
3	fb(x):=Ableitung(Ableitung(f(x))) $\rightarrow\ fb(x) := 640 \cdot \frac{x^3}{(k^2+x^2)^3} - 480 \cdot \frac{x}{(k^2+x^2)^2}$
4	fb(k) $\rightarrow\ -\frac{40}{k^3}$
5	fc(x):=Ableitung(fa(x)) $\rightarrow\ fc(x) := 640 \cdot \frac{x^3}{(k^2+x^2)^3} - 480 \cdot \frac{x}{(k^2+x^2)^2}$
6	fc(k) $\rightarrow\ -\frac{40}{k^3}$

When using the command "Ableitung" (= "Derivative") the calculation is always executed immediately.

The way of GeoGebra corresponds to the way of calculating by hand.

Conclusion:

An important fact of my didactical theory is to consider mathematics as a language which is developed by the learners during their way into the world of mathematics. Developing a language means developing new language elements and new rules and using these new language elements and rules for problem solving.

This philosophy can be better realized when using tools like TI Nspire CAS which also have additional didactical advantage, the use of the notes application which allows the connection of text and mathematical expressions.

Therefore for me this example is especially worrying because I have to recognize that the strategy to use names of expressions instead of complex expressions lead to wrong results.

But unexpected results can also be a motivation not only to believe in the results of the technology but to critically questioning the way of solution.

3. An example which I have given to my students to test their testing strategies

The goal of this example, especially of the phase 2 of the given problem, is that the students should not uncritically accept solutions of the technological tool. They should learn that arguing and reasoning accompanies the whole problem-solving process.

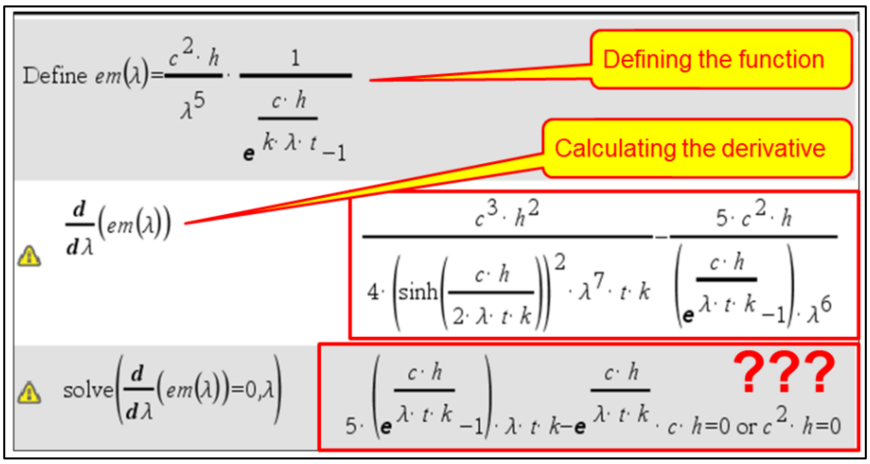
Example: Planck's radiation formula - surprises by using the strategy of substitution [Dorninger, 1988]

The emission “em” of a black corpus is a function of the wavelength λ.

$$em(\lambda) = \frac{c^2 \cdot h}{\lambda^5} \cdot \frac{1}{e^{\frac{c \cdot h}{k \cdot \lambda \cdot t}} - 1} \qquad \text{c,h,k,t are constants } (\neq 0)$$

The task: Determine the maximum value of the function.

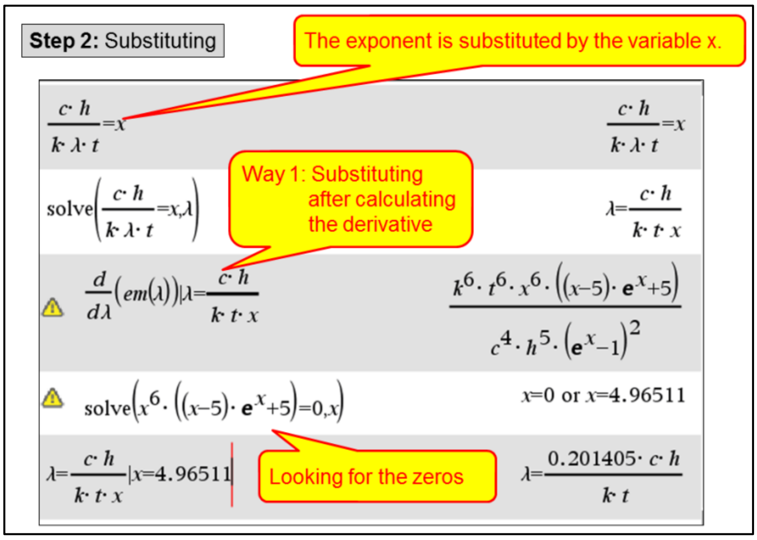
Phase 1: Solving the problem by using the strategy of substituting



Step 1: Looking for the zeros of the derivative.

Solving the equation shows the limits of the CAS-tool, the first solution is not usable.

The mathematical **competence of substituting** is necessary.



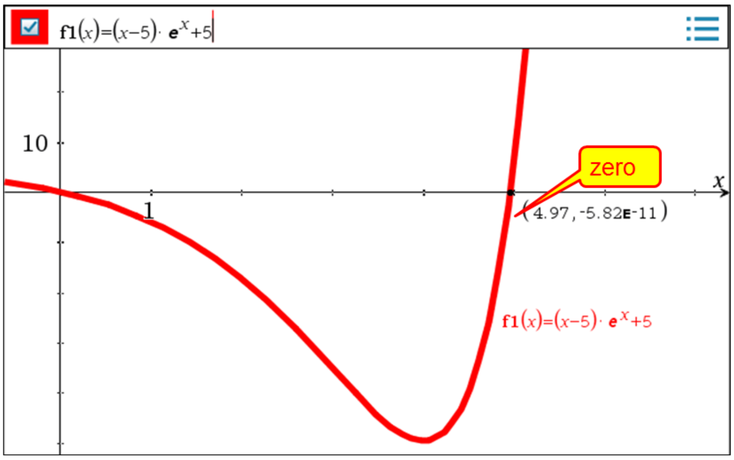
Step 2: Substituting

The exponent is substituted by the variable x.

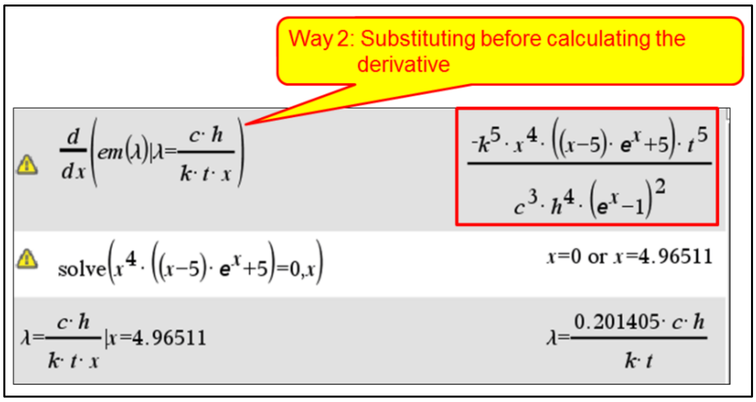
Two several ways are thinkable:

Way 1: Substituting after calculating the derivative

Students think: When looking for the zeros of the derivative it will do to look for the zeros of the numerator. A reasonable value of x is 4.96511. When substituting for x we get the solution for λ (with respect to the constants c, h, k, t).



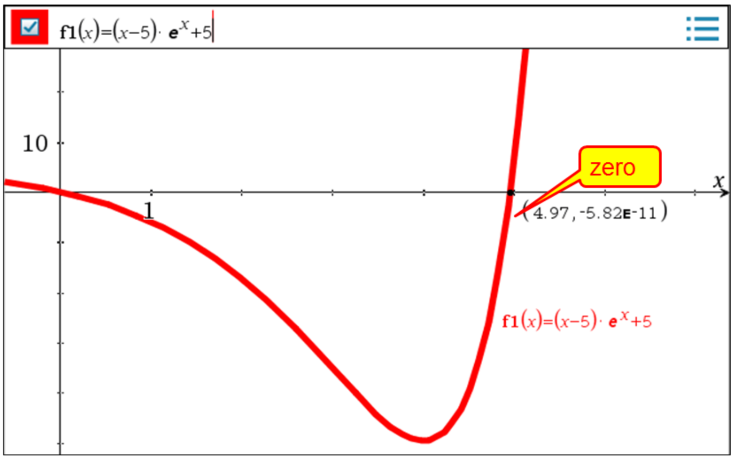
Step 3: Another option is to solve the equation $(x - 5).e^x + 5 = 0$ graphically by looking for the zeros of the function f1.



Way 2: Substituting before calculating the derivative

The numerical solution is the same.

But the derivative with respect to x is not the same as in way 1!



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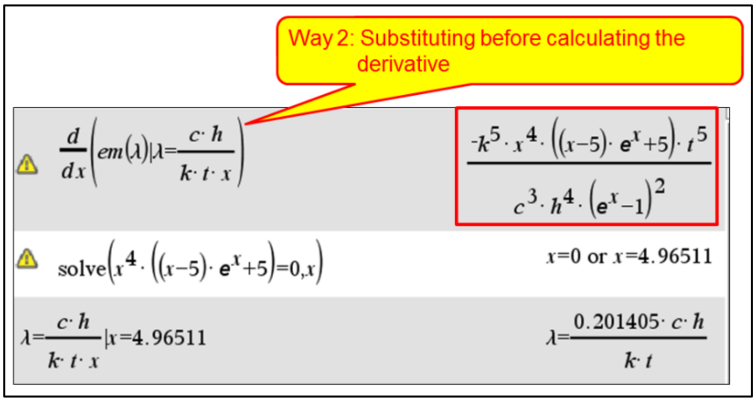


Way 2: Substituting before calculating the derivative

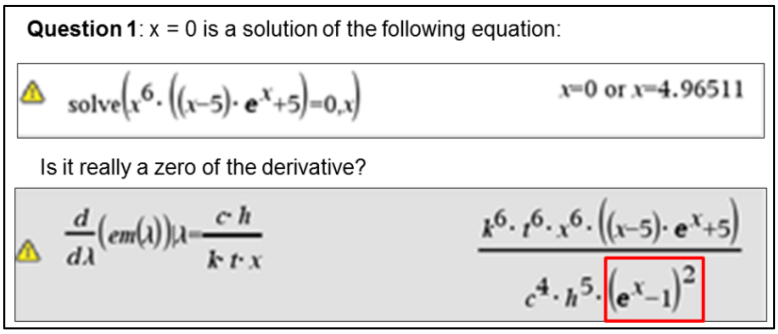
The numerical solution is the same.

But the derivative with respect to x is not the same as in way 1!

Phase 2: Reflecting about the mathematical results, justifying single steps of the way to the solution

Possible questions are:

Question 1: Is x=0 also a mathematical solution?



Is it enough to look at the numerator when looking for the zeros of the derivative?

Is x = 0 also a solution?

At first, we have to look for the domain of the function by looking for the zeros of the denominator and we can see that for x = 0 the term $(e^x – 1)$ is 0.

At first, we have to look for the domain of the function by looking for the zeros of the denominator and we can see that for x = 0 the term $(e^x – 1)$ is 0.

Question 2: Why are the derivatives with respect to x of way 1 and way 2 not equal?

$$\frac{d}{d\lambda}(em(\lambda))|\lambda=\frac{c\cdot h}{k\cdot t\cdot x} \qquad \boxed{\frac{k^{6}\cdot t^{6}\cdot x^{6}\cdot\left((x-5)\cdot \mathbf{e}^{x}+5\right)}{c^{4}\cdot h^{5}\cdot\left(\mathbf{e}^{x}-1\right)^{2}}}$$

Why?

$$\frac{d}{dx}\left(em(\lambda)|\lambda=\frac{c\cdot h}{k\cdot t\cdot x}\right) \qquad \boxed{\frac{-k^{5}\cdot x^{4}\cdot\left((x-5)\cdot \mathbf{e}^{x}+5\right)\cdot t^{5}}{c^{3}\cdot h^{4}\cdot\left(\mathbf{e}^{x}-1\right)^{2}}}$$

$$\frac{\dfrac{-k^{5}\cdot x^{4}\cdot\left((x-5)\cdot \mathbf{e}^{x}+5\right)\cdot t^{5}}{c^{3}\cdot h^{4}\cdot\left(\mathbf{e}^{x}-1\right)^{2}}}{\dfrac{k^{6}\cdot t^{6}\cdot x^{6}\cdot\left((x-5)\cdot \mathbf{e}^{x}+5\right)}{c^{4}\cdot h^{5}\cdot\left(\mathbf{e}^{x}-1\right)^{2}}} \qquad \boxed{\frac{-c\cdot h}{k\cdot t\cdot x^{2}}}$$

$$\lambda(x):=\frac{c\cdot h}{k\cdot t\cdot x} \qquad \text{Done}$$

$$\frac{d}{dx}(\lambda(x)) \qquad \frac{-c\cdot h}{k\cdot x^{2}\cdot t}$$

A commonly used testing strategy is to build the quotient or the difference of the two expressions:

What is the meaning of this quotient?

If we remember that λ is a function of x we can calculate the inner derivative of the function λ(x) and we see that the quotient of the derivatives of the two ways is the inner derivative.

Students are experimenting with the sum operator

On the way to the definite integral a teacher explained the students the use of the sum operator. Next day the students showed him some of their results:

$$\sum_{a=0}^{\sqrt{2}}\left(1-\frac{a^{2}}{2}\right)-\sum_{a=\sqrt{2}}^{2.5}\left(1-\frac{a^{2}}{2}\right) \qquad 3.23917$$

approx

$$\sum_{a=0}^{\sqrt{2}}\left(1-\frac{a^{2}}{2}\right)-\sum_{a=\sqrt{2}}^{\frac{5}{2}}\left(1-\frac{a^{2}}{2}\right) \qquad \frac{7\cdot\sqrt{2}}{6}+\frac{15}{8}$$

exact

exact

$$\frac{7\cdot\sqrt{2}}{6}+\frac{15}{8.} \qquad 3.52492$$

$$\int_{0}^{\sqrt{2}}\left(1-\frac{x^{2}}{2}\right)dx-\int_{\sqrt{2}}^{2.5}\left(1-\frac{x^{2}}{2}\right)dx \qquad 1.98978$$

This way of using sums is only "senseful" if 3.23917 = 3.52492

$\sum_{x=0}^{2}(x)$	3	
$\sum_{x=0}^{\sqrt{2}}(x)$	$\frac{\sqrt{2}}{2}+1$	exact
$\frac{\sqrt{2}}{2.}+1$	1.70711	
$\sum_{x=0}^{\sqrt{2.}}(x)$	$1.$	approx
$\sum_{x=0.}^{\sqrt{2}}(x)$	$1.$	approx

Is 1.70711.. really equals 1?

⇨ Not only the visualization is dangerous we also cannot trust CAS.

Teachers must be prepared that students like to experiment with their technology tools and specially to look for wrong results.

How would you explain your students these results?

Michel Beaudin's answer concerning the sum problem:

1. Introduction

Usually, when we see the summation symbol, the index goes from a lower *integer* bound to an upper *integer* bound. For example,

$$\sum_{i=2}^{5}(2\cdot i+1) \blacktriangleright 32$$ because 5 + 7 + 9 + 11 = 32.

More generally, we know (and so does a CAS) that $\sum_{i=1}^{n}(i) \blacktriangleright \frac{n\cdot(n+1)}{2}$

Nspire is also able to compute the general sum $\sum_{i=a}^{b}(i) \blacktriangleright \frac{-a^2}{2}+\frac{a}{2}+\frac{b^2}{2}+\frac{b}{2}$

We will soon understand how this result can be found.

In fact the CAS needs to be able to compute a sum of the form

$$\sum_{i=a}^{b}(f(i)) = f(a)+f(a+1)+f(a+2)+\ldots+f(b)$$

2. Some examples

Now let us explain how the following results returned by Nspire were obtained:

Example 1: here the lower and upper limits are rational numbers or integers. Rational numbers can be entered in exact or approximate forms:

$$\sum_{i=2}^{13/2}(i) \blacktriangleright 20 \qquad \sum_{i=2}^{6.6}(i) \blacktriangleright 20.$$

Explanation for example 1: no matter you type 13/2 or 6.5, because 6 is the last integer before 6.5, we then have

$2 + 3 + 4 + 5 + 6 = 20.$

Example 2: one (or both) bound is a non rational number.

$$\sum_{i=\sqrt{3}}^{7}(i) \blacktriangleright 6\cdot\sqrt{3}+15$$

The last value approximates to $\text{approx}\left(\frac{\sqrt{3}}{2}+\frac{53}{2}\right) \blacktriangleright 27.366$

while $\displaystyle\sum_{i=\sqrt{3.}}^{7}(i) \blacktriangleright 25.3923$

Explanation for example 2: because $\sqrt{3}$ is not a rational number and because there is no floating point entry, Nspire seems to use the formula

$$\sum_{i=a}^{b}(i) \blacktriangleright \frac{-a^2}{2}+\frac{a}{2}+\frac{b^2}{2}+\frac{b}{2}$$

because $\frac{-a^2}{2}+\frac{a}{2}+\frac{b^2}{2}+\frac{b}{2}|a=\sqrt{3}$ and $b=7 \blacktriangleright \frac{\sqrt{3}}{2}+\frac{53}{2}$

But in the second case, we have $\sqrt{3.} \blacktriangleright 1.73205$

so Nspire computes the sum

$1.73205+2.73205+3.73205+4.73205+5.73205+6.73205 \blacktriangleright 25.3923$

The next page is a calculus page containing an example found in Helmut's paper.

$$\sum_{a=0}^{\sqrt{2}}\left(1-\frac{a^2}{2}\right)-\sum_{a=\sqrt{2}}^{2.5}\left(1-\frac{a^2}{2}\right) \qquad 3.23917$$

$$\sum_{a=0}^{\sqrt{2}}\left(1-\frac{a^2}{2}\right)-\sum_{a=\sqrt{2}}^{\frac{5}{2}}\left(1-\frac{a^2}{2}\right) \qquad \frac{7\cdot\sqrt{2}}{6}+\frac{15}{8}$$

$$\left(\frac{7\cdot\sqrt{2}}{6}+\frac{15}{8}\right)\blacktriangleright\text{Decimal} \qquad 3.52492$$

$$helmut(a1,b1,a2,b2):=\sum_{a=a1}^{b1}\left(1-\frac{a^2}{2}\right)-\sum_{a=a2}^{b2}\left(1-\frac{a^2}{2}\right) \qquad Done$$

$$helmut\left(0,\sqrt{2},\sqrt{2},\frac{5}{2}\right) \qquad \frac{7\cdot\sqrt{2}}{6}+\frac{15}{8}$$

$$helmut(0,\sqrt{2},\sqrt{2},2.5) \qquad 3.23917$$

$$helmut(a1,b1,a2,b2) \qquad \frac{a1^3}{6}-\frac{a1^2}{4}-\frac{11\cdot a1}{12}-\frac{a2^3}{6}+\frac{a2^2}{4}+\frac{11\cdot a2}{12}-\frac{b1^3}{6}-\frac{b1^2}{4}+\frac{11\cdot b1}{12}+\frac{b2^3}{6}+\frac{b2^2}{4}-\frac{11\cdot b2}{12}$$

3. Finding an antidifference

Definition: an *antidifference* F of a function f is a function F such that $F(n+1) - F(n) = f(n)$.

So if we need to compute the sum

$$\sum_{i=a}^{b}(f(i))$$

that is if we need to compute

$f(a)+f(a+1)+f(a+2)+\ldots+f(b)$

and if we can find an antidifference F of f, then

$f(a)+f(a+1)+f(a+2)+\ldots+f(b)=$

$(F(a+1)-F(a))+(F(a+2)-F(a+1))+(F(a+3)-F(a+2))+\ldots+(F(b+1)-F(b))$

$=F(b+1)-F(a)$

In other words,

$$\sum_{i=a}^{b}(f(i))=F(b+1)-F(a)$$

We will be able to find an antidifference if we are able to solve the *difference equation*
$F(n+1) - F(n) = f(n)$.

It is easy to solve a *linear* difference equation of the form
$y(n+1) = p(n)\cdot y(n) + q(n),\ \ y(xo) = yo$
where p and q are given expressions of the variable n. The solution found in ***Derive*** can easily be exported to Nspire:

$$\textbf{lin1\_difference}(p,q,x,xo,yo) := \prod_{x=xo}^{x-1}(p)\cdot\left(yo+\sum_{x=xo}^{x-1}\left(\frac{q}{\prod_{x=xo}^{x}(p)}\right)\right) \blacktriangleright \textit{Done}$$

Example: let's use this function with the simple recurrence equation
$y(n+1)=2\cdot y(n)+n,\ y(1)=2$

$\textbf{lin1\_difference}(2,n,n,1,2) \blacktriangleright 2\cdot 2^{n}-n-1$

And let's check the answer:

$\textbf{rep}(n):=2\cdot 2^{n}-n-1 \blacktriangleright \textit{Done}$
$\textbf{rep}(1) \blacktriangleright 2$
$\textbf{rep}(n+1)-2\cdot \textbf{rep}(n)-n \blacktriangleright 0$

Then it becomes easy to define a function for providing an antidifference:

$\textbf{lin1\_difference}(1,f,x,0,0) \rightarrow \textbf{antidif}(f,x) \blacktriangleright \textit{Done}$

Let's do some examples:

$$\textbf{antidif}(i,i) \blacktriangleright \frac{i\cdot(i-1)}{2}$$

$$\textbf{antidif}\left(1-\frac{a^2}{2},a\right) \blacktriangleright \frac{{}^-a\cdot\left(2\cdot a^2-3\cdot a-11\right)}{12}$$

Sometimes, the system is not able to find a closed form:

$$\textbf{antidif}\left(\frac{1}{n^2},n\right) \blacktriangleright \sum_{n=0}^{n-1}\left(\frac{1}{n^2}\right)$$

My (Josef's) comment and contribution for a possible discussion:

The "Sum-problem" was fully resolved by Michel's answer.

I must say, that I don't completely share your (Helmut's) opinion, that NspireCAS behaves wrong.
I think that the reason is a – too – slack use of the concept of a function.

Nspire and DERIVE as well are performing exactly what the user is demanding (example 2). This can easily be seen following the DERIVE-treatment.

Problem #1:

#1: $f(x) := \dfrac{80\cdot x}{x^2 + k^2}$

#2: $fa(x) := \dfrac{d}{dx} f(x)$

#3: $fb(x) := \dfrac{d}{dx} fa(x)$

#4: $\text{SOLUTIONS}(fa(x) = 0,\ x) = [k,\ -k,\ \infty,\ -\infty]$

#5: $[fb(k),\ fb(-k)] = \left[\dfrac{80}{k^3},\ \dfrac{40}{k^3}\right]$

That's wrong, because CAS substitutes k for x in #3
"as demanded" by the user who does not understand
the concept of a function!

But:
I simplify #3:

#6: $fb(x) := \dfrac{160\cdot x\cdot(x^2 - 3\cdot k^2)}{(x^2 + k^2)^3}$

Now we redefine fb(x) and try again:

#7: $[fb(k),\ fb(-k)] = \left[-\dfrac{40}{k^3},\ \dfrac{40}{k^3}\right]$

GeoGebra does – not intended by the user (in my opinion) - not store the function fb(x) as given, but evaluates immediately.

fb(x):=Ableitung(f(x)) is – again in my opinion – not a complete function – automatically the derivative wrt x.

Taking the function as it is, the fb(k) should be Ableitung(f(k))!!

It is the teacher's task to demonstrate the correct use of the function concept.

A similar problem appears with the radiation formula.
Introducing x as a substitution we should not forget that x is a function of lambda.
So, what's about the chain rule?

#1: $$\mathrm{em}(\lambda) := \frac{c^2\cdot h}{\lambda^5}\cdot\frac{1}{e^{c\cdot h/(k\cdot\lambda\cdot t)}-1}$$

#2: $$\mathrm{dem}\lambda := \frac{d}{d\lambda}\,\mathrm{em}(\lambda)$$

#3: $$\mathrm{dem}\lambda := \frac{c^2\cdot h\cdot(e^{c\cdot h/(k\cdot t\cdot\lambda)}\cdot(c\cdot h-5\cdot k\cdot t\cdot\lambda)+5\cdot k\cdot t\cdot\lambda)}{k\cdot t\cdot\lambda^7\cdot(e^{c\cdot h/(k\cdot t\cdot\lambda)}-1)^2}$$

#4: SOLVE(demλ = 0, λ)

#5: $$e^{c\cdot h/(k\cdot t\cdot\lambda)}\cdot(5\cdot k\cdot t\cdot\lambda-c\cdot h)-5\cdot k\cdot t\cdot\lambda=0 \vee \lambda=0 \vee \frac{c^2\cdot h}{k\cdot t}=0$$

λ = 0 is impossible

#6: $$e^{c\cdot h/(k\cdot t\cdot\lambda)}\cdot(5\cdot k\cdot t\cdot\lambda-c\cdot h)-5\cdot k\cdot t\cdot\lambda=0$$

Substitution - but, x is a function of λ

#7: $$\frac{c\cdot h}{k\cdot t\cdot\lambda}=x$$

#8: $$\mathrm{SOLVE}\left(\frac{c\cdot h}{k\cdot t\cdot\lambda}=x,\ \lambda\right)=\left(\lambda=\frac{c\cdot h}{k\cdot t\cdot x}\right)$$

#9: $$\mathrm{em}\left(\frac{c\cdot h}{k\cdot t\cdot x}\right)=\frac{k^5\cdot t^5\cdot x^5}{c^3\cdot h^4\cdot(e^x-1)}$$

Before differentiating, remember: x = x(λ) ... don't forget the chain rule!!

#10: $$\left(\frac{d}{dx}\,\frac{k^5\cdot t^5\cdot x^5}{c^3\cdot h^4\cdot(e^x-1)}\right)\cdot\frac{d}{d\lambda}\,\frac{c\cdot h}{k\cdot t\cdot\lambda}$$

#11: $$\left(-\frac{k^5\cdot t^5\cdot x^4\cdot(e^x\cdot(x-5)+5)}{c^3\cdot h^4\cdot(e^x-1)^2}\right)\cdot\frac{d}{d\lambda}\,\frac{c\cdot h}{k\cdot t\cdot\lambda}$$

#12: $$\left(-\frac{k^5\cdot t^5\cdot x^4\cdot(e^x\cdot(x-5)+5)}{c^3\cdot h^4\cdot(e^x-1)^2}\right)\cdot\left(-\frac{c\cdot h}{k\cdot t\cdot\lambda^2}\right)$$

#13: $$\left(-\frac{k^5\cdot t^5\cdot x^4\cdot(e^x\cdot(x-5)+5)}{c^3\cdot h^4\cdot(e^x-1)^2}\right)\cdot\left(-\frac{c\cdot h}{k\cdot t\cdot\lambda^2}\right)=0$$

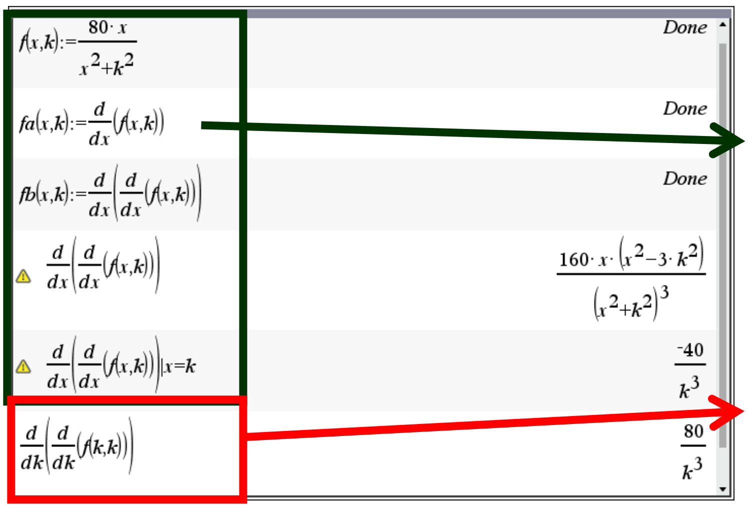
Another question: can we be sure that the “maximum value” is really a maximum?
The 2nd derivative cannot give a satisfying answer if we don’t know the signs of the parameters.

Maybe that I am wrong in my ideas and conclusions, but I wanted to – as I said on top of my mail – to continue the discussion.
You will forgive for using my first love DERIVE ☺.
Josef

And here is Helmut’s answer on my comments:

Hello Josef,
this is a fundamental problem for me, because of this mistake – and in my opinion it is a mistake – my didactic principle for using CAS is destroyed namely the development of new language elements of the mathematical language in order to operate with their names instead of complex – in sense of complicated - expressions. This is one of the great advantages of CAS which confirms W. Dörfler’s thesis, that technology does not only support cognition but becomes part of cognition. This is for me one of the strongest arguments to use CAS. If the only argument for the use of CAS would be to outsource complex calculations to the tool, then I would be an opponent to use CAS in teaching.

My reason why the TI-Nspire sequence is wrong:



According to the order of the definitions of the functions f, fa, fb, I’d expect that CAS starts differentiating f wrt x (see definition of fa) followed by differentiating the result wrt to x (definition of fb) and not till then replaces x by k in this function fb(x,k).

Nspire replaces x by k at first and then differentiates with respect to k. And this is in contrary to the logical order of the definitions of fa and fb

It would be great if we could have other reactions on Helmut’s lecture and on his “Right or Wrong” examples.