THE DERIVE - NEWSLETTER #120

ISSN 1990-7079

THE BULLETIN OF THE DERIVE USER GROUP

+ CAS-TI

Contents:

December 2020

Some Recommended Links

Mathematik
alpha 2021

A website, full of very useful materials and programs (e.g. matlex.exe, which is a mathematical specialized dictionary for 26 languages = mathematisches Fachwörterbuch)

https://mathematikalpha.de/

Für unsere deutschsprachigen Mitglieder, die die mathem. Zeitschrift Alpha (aus der ehemalign DDR nicht kennen). Hier gibt es viele komplette Jahrgänge zum download.

https://mathematikalpha.de/alpha-pdf

Colleagues and Friends

I have spent the last 12 months delving into the inconsistencies being published and/or announced re Gravity Waves, Dark Matter, Dark Energy, Black Holes etc. with much consternation, since there seems to have been a lot of bad science fiction included as well as some fraudulent claims.

So, this third draft is worth bringing to the public's attention, in the hope that more time and money won't be wasted in some of those contentious areas of alleged research.

I have been publishing a few papers over the last year on Academia.edu so the link is included as well.

I hope that you will take the time to read right thru my paper and proffer me constructive feedback

Sincerely

David Halprin

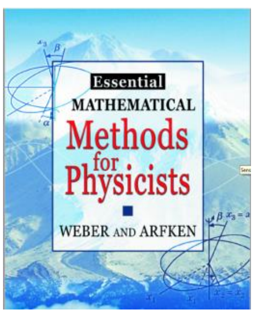
https://independent.academia.edu/DavidHalprin

New book:

Albrecht Beutelspacher, Null, unendlich und die wilde 13
Die wichtigsten Zahlen und ihre Geschichten, C.H.Beck, 2020

You can download a complete book (911 pages):

https://www.academia.edu/8545356/Essential_Mathematical_Methods_for_Physicists



Dear DU-Members,

I am quite sure, that we will not forget this extraordinary year 2020 for a long while. We all had to get used to a „New Normality“. The elder among us, being retired, were affected in another way than the younger ones – in terms of health and economy.

I can imagine – probably not really correctly – how challenging teaching, but also learning and studying must be under these circumstances. Many thanks and great appreciation to all of you – from my retirement's chair.

This DNL refers to materials directing to times of TI-89, TI-92 and Voyage 200. I tried to customize functions for Nspire and DERIVE, because mathematics and programming techniques behind look very interesting. I hope that the DUG-community will share my estimation.

Jürgen Wagner comes back to the eternal young topic of Turtle Graphics. His programs use the DRAW-features of the latest Nspire version and offer an introduction to the so important but not always easy to follow issue of recursive thinking. Maybe that you will try to apply chasing the turtle up a Lindenmayerbranch or -tree (DNLs 25, 51, 52)?

Recently I was invited to attend a webinar organized by TI-Germany. A collection of activities connected with the Herrnhuter Star was presented. Frank Liebner, he lives in Herrnhut (Oberlausitz, Germany, former DDR) provided an impressive virtual guided tour through history and town. Many thanks to all presenters. My participation inspired me to further investigate the geometry of the Herrnhuter Star. And, moreover, this fits excellent to Christmas time.

My wife and I wish you and your families a Merry Christmas and a Happy, and - what else? - a Healthy New Year 2021.

The attached file shall accompany you. The pictures are from a time just before “C”:

Josef

Liebe DUG-Mitglieder,

dieses merkwürdige Jahr 2020 werden wir wohl lange nicht vergessen. Wir alle mussten uns sehr rasch an eine neue „Normalität“ gewöhnen. Uns ältere, die wir schon im Ruhestand sind, hat es anders betroffen, als die jüngeren unter uns – sowohl in gesundheitlicher als auch in wirtschaftlicher Hinsicht.

Ich kann mir vorstellen – wahrscheinlich aber nicht wirklich – wie herausfordernd das Unterrichten, aber auch das Lernen bzw. Studieren jetzt sein muss. Vielen Dank und Anerkennung von meiner Pensionistenbank an Euch alle.

Dieser DNL bezieht sich in zwei Beiträgen auf die Zeit des TI-89, TI-92 und Voyage 200. Ich habe Funktionen für TI-Nspire und DERIVE aktualisiert, weil ich sowohl deren mathematischen Inhalt als auch die Programmiertechnik interessant finde. Ich hoffe, dass die DUG-Gemeinde meine Einschätzung teilen kann.

Jürgen Wagner hat das ewig junge Thema der Turtle-Grafik aufgegriffen. Seine Programme nützen die DRAW-Möglichkeiten des Nspire und bieten auch einen Einstieg in das so wichtige aber nicht immer einfache rekursive Denken. Vielleicht versuchen Sie einmal, die Schildkröte auf die Lindenmayerzweige und -bäume (DNLs 25, 51, 52) zu jagen?

Kürzlich war ich eingeladen, an einem TI-Webinar teilzunehmen. Da wurde eine Sammlung von mathematischen Aktivitäten rund um den Herrnhuter Stern vorgestellt. Frank Liebner, er stammt selbst aus Herrnhut (Oberlausitz, ehem. DDR) gab eine eindrucksvolle virtuelle Führung durch Ort und Geschichte. Vielen Dank an alle Beteiligten. Die Teilnahme hat mich zur weiteren Beschäftigung mit dem Herrnhuter Stern angeregt. Außerdem passt das gut zu Weihnachten.

Meine Frau und ich wünschen Euch und Euren Familien Frohe Weihnachten und - was sonst? - ein Glückliches und vor Allem Gesundes Neues Jahr 2021.

Das Anhängsel möge Euch begleiten. Es ist gerade noch vor „C“ entstanden.

Josef

The *DERIVE-NEWSLETTER* is the Bulletin of the *DERIVE* & CAS-*TI User Group*. It is published at least four times a year with a content of 40 pages minimum. The goals of the *DNL* are to enable the exchange of experiences made with *DERIVE, TI*-CAS and other CAS as well to create a group to discuss the possibilities of new methodical and didactical manners in teaching mathematics.

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Contributions:
Please send all contributions to the Editor. Non-English speakers are encouraged to write their contributions in English to reinforce the international touch of the *DNL*. It must be said, though, that non-English articles will be warmly welcomed nonetheless. Your contributions will be edited but not assessed. By submitting articles, the author gives his consent for reprinting it in the *DNL*. The more contributions you will send, the more lively and richer in contents the *DERIVE* & CAS-*TI Newsletter* will be.

Next issue: March 2021

Preview: Contributions waiting to be published

Some simulations of Random Experiments, J. Böhm, AUT, Lorenz Kopp, GER
Wonderful World of Pedal Curves, J. Böhm, AUT
Simulating a Graphing Calculator in *DERIVE*, J. Böhm, AUT
Cubics, Quartics – Interesting features, T. Koller & J. Böhm, AUT
Logos of Companies as an Inspiration for Math Teaching
Exciting Surfaces in the FAZ
BooleanPlots.mth, P. Schofield, UK
Old traditional examples for a CAS – What´s new? J. Böhm, AUT
Mandelbrot and Newton with *DERIVE*, Roman Hašek, CZ
Tutorials for the NSpireCAS, G. Herweyers, BEL
Dirac Algebra, Clifford Algebra, Vector-Matrix-Extension, D. R. Lunsford, USA
Another Approach to Taylor Series, D. Oertel, GER
Statistics of Shuffling Cards, Charge in a Magnetic Field, H. Ludwig, GER
More Applications of TI-Innovator™ Hub and TI-Innovator™ Rover
Surfaces and their Duals, Cayley Symmetroid, J. Böhm, AUT
Investigations of Lottery Game Outcomes, W Pröpper, GER
A Collection of Special Problems, W. Alvermann, GER
DERIVE Bugs? D. Welz, GER
Tweening & Morphing with TI-NspireCX-II-T, J. Böhm. AUT
The Gap between Poor and Rich, J. Böhm, AUT
Final Exam, part 2 and solutions
More functions from Matthew and from Bhuvanesh's Mathtools-library
QR-Code light, 153 is another Special Number, and others

Impressum:
Medieninhaber: *DERIVE* User Group, A-3042 Würmla, D´Lust 1, AUSTRIA
Richtung: Fachzeitschrift
Herausgeber: Mag. Josef Böhm

Mail from Francisco Marcelo Fernández, Argentina

Dear Josef,

I hope you are going through this pandemonium satisfactorily. It seems that I do not understand how SUBST actually works. I am attaching a Derive file that clearly shows my inquiry.

Best regards,

Marcelo

If I can do this substitution

$$\#1: \quad \mathrm{SUBST}\left(\frac{1}{e^{2\cdot z}+1},\ e^{2\cdot z}+1,\ \xi\right)$$

$$\#2: \quad \frac{1}{\xi}$$

and this one

$$\#3: \quad \mathrm{SUBST}\left(\frac{1}{(e^{2\cdot z}+1)^2},\ e^{2\cdot z}+1,\ \xi\right)$$

$$\#4: \quad \frac{1}{\xi^2}$$

why I cannot do this?

$$\#5: \quad \mathrm{SUBST}\left(\frac{1}{(e^{2\cdot z}+1)^2},\ (e^{2\cdot z}+1)^2,\ \xi\right)$$

$$\#6: \quad \frac{1}{(e^{2\cdot z}+1)^2}$$

Dear Marcelo,

try this "little" trick:

$$\mathrm{SUBST}\left(\frac{1}{(e^{2\cdot z}+1)^2},\ e^{2\cdot z}+1,\ \sqrt{\xi}\right) = \frac{1}{\xi}$$

$$\frac{1}{(e^{2\cdot z}+1)^2}$$

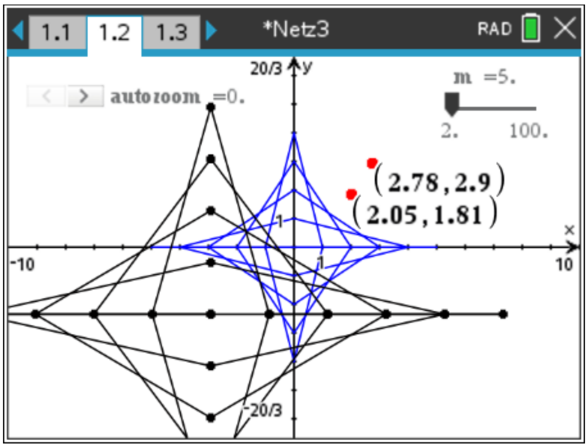
or: Highlight e^(2z+1)^2; then Ctrl+T → substitute for subexpression: ξ, voila!

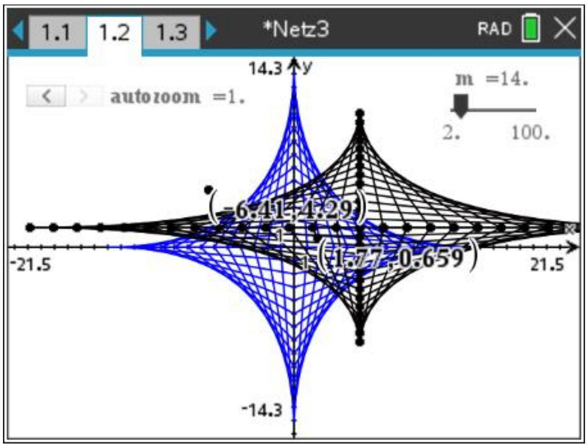
$$\frac{1}{\xi}$$

From Sebastian Rauh

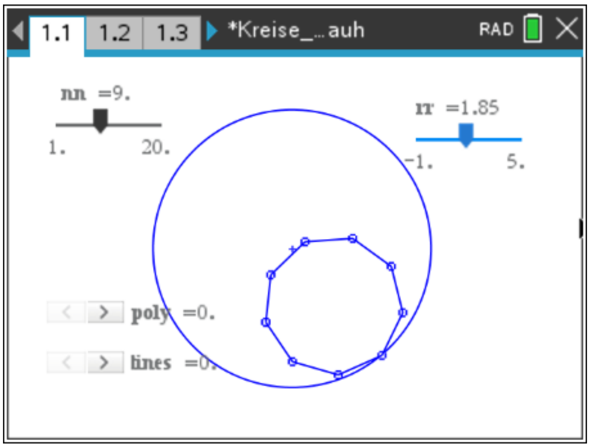
Hello Josef,
Thanks for DNL119,
Really great, exactly that with the projection matrices. I have searched for this. Here is a little test. Just move the points with the coordinates indicated.

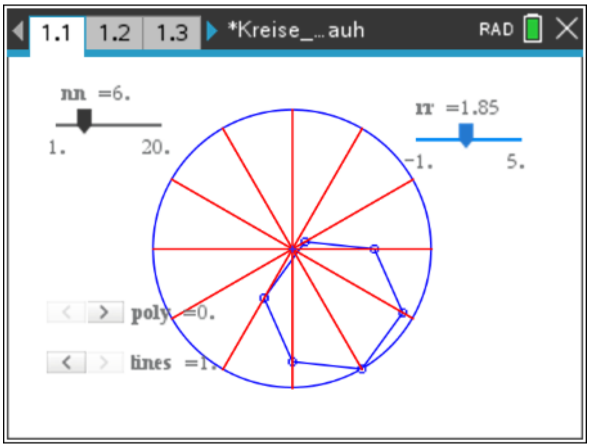
Regards, Sebastian





Once more from Sebastian: Play around with circles and sliders …





Sebastian again:

calculate [[0,1][1,1]]^n (for n=1,2,3,4,5..)

Sebastian

Josef

Very interesting, nice to proof.

This is much easier but it might be a nice exercise for students to practice matrix multiplication (paper & pen, of course) and proof by induction:

[1,0;1,1]^n for n = 1,2,3,…

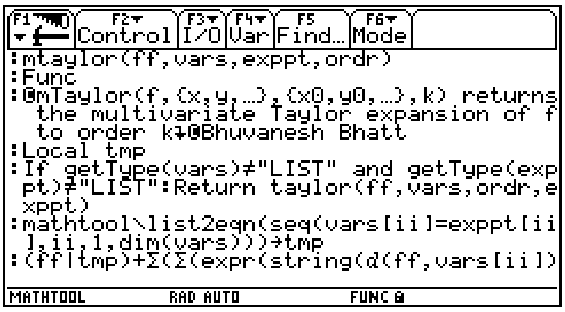
I came across this because at first I made a typing error, following your invitation ☺.

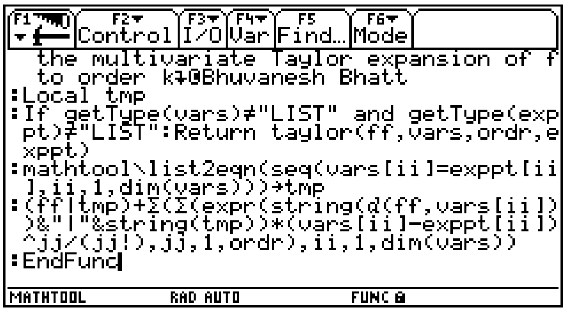
A Bug with mTaylor?

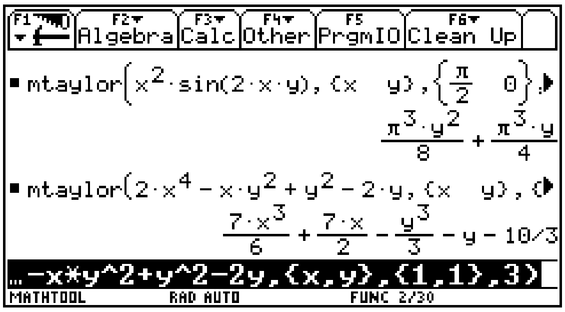
Josef Böhm, Würmla, Austria

Working with Bhuvanesh Bhatt's great mathtool-library a came across strange results, which seem to be caused by a bug. The description of function *mtaylor* reads as follows:

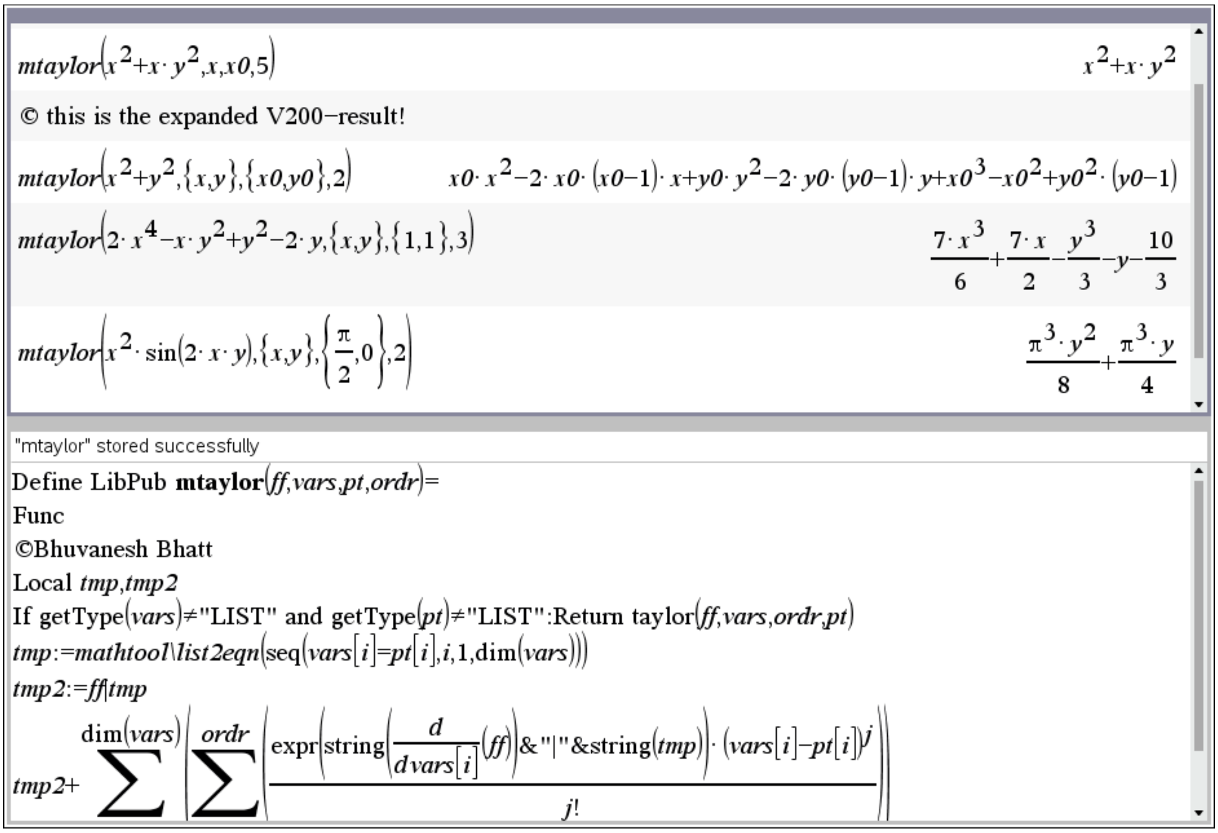
mTaylor(f(x,y,…),{x,y,…},{x0,y0,…},k) returns the multivariate Taylor expansion of f(x,y,…) to order k







Transferred to TI-NspireCAS:



I doublechecked the results given above (which are the same as given with TI-92) and found out that they are obviously wrong.

DERIVE doesn't provide any function for Multivariate Taylor expansion. So, I tried other CA-systems:

These are the results provided by *Maxima*:

```
(%i1)     taylor(2*x^4-x*y^2+y^2-2*y,[x,y],[1,1],2);
(%o1)/T/      -2 (y-1)+7 (x-1)+(12 (x-1)^2-2 (y-1) (x-1))+...

(%i2)     ratexpand(taylor(2*x^4-x*y^2+y^2-2*y,[x,y],[1,1],2));
(%o2)     -2 (x-1) (y-1)-2 (y-1)+7 (x-1)+12 (x-1)^2

(%i3)     expand(taylor(2*x^4-x*y^2+y^2-2*y,[x,y],[1,1],2));
(%o3)     -2 x y+12 x^2-15 x+5

(%i4)     expand(taylor(2*x^4-x*y^2+y^2-2*y,[x,y],[1,1],3));
(%o4)     -x y^2+y^2-2 y+8 x^3-12 x^2+8 x-2

(%i5)     expand(taylor(x^2*sin(x*y),[x,y],[pi/2,0],1));
(%o5)     %pi^3 y/8

(%i6)     expand(taylor(x^2*sin(x*y),[x,y],[pi/2,0],3));
(%o6)     -%pi^5 y^3/192 + 3 %pi x^2 y/2 - 3 %pi^2 x y/4 + %pi^3 y/8

(%i7)     expand(taylor(x^3+2*x^2*y+y^2*z+x*z+z*y^4,[x,y,z],[1,1,1],1));
(%o7)     3 z+8 y+8 x-13

(%i8)     expand(taylor(x^3+2*x^2*y+y^2*z+x*z+z*y^4,[x,y,z],[1,1,1],3));
(%o8)     7 y^2 z-8 y z+x z+3 z+4 y^3-12 y^2+2 x^2 y+12 y+x^3-4
```

It's not surprising that *MATHEMATICA* gives the same result!!

```
Normal[Series[2 x^4 - xy^2 + y^2 - 2 y, {x, 1, 3}, {y, 1, 3}]]
1 + 8 (-1 + x) + 12 (-1 + x)^2 + 8 (-1 + x)^3 - xy^2 + (-1 + y)^2

Expand [%]
-2 + 8 x - 12 x^2 + 8 x^3 - xy^2 - 2 y + y^2
```

Question: What's wrong with Bhuvanesh's function?

Inspecting the last row of the Nspire program code we can see, that only the powers of (vars[i]-pt[i]) have been considered. So we miss the mixed partial derivatives and the mixed products (vars[i]-pt[i])*(vars[k]-pt[k]) with $i \neq k$.

So the correct calculation of mtaylor(f(x,y),{x,y},{x0,y0},2) should read:

$$f(x_0,y_0)+f_{x(x_0,y_0)}(x-x_0)+f_{y(x_0,y_0)}(y-y_0)+$$
$$\frac{1}{2!}\left(f_{xx(x_0,y_0)}(x-x_0)^2+2f_{xy(x_0,y_0)}(x-x_0)(y-y_0)+f_{yy(x_0,y_0)}(y-y_0)^2\right)$$

As Multivariate Taylor Expansion is not implemented in DERIVE, I tried to produce an own one.

This is my home-made DERIVE function for multivariate Taylor expansion (but for only 2 variables!).

$$\text{mt\_part}(f\_, u, v, u0, v0, k) := \text{SUBST}(f\_, [u, v], [u0, v0]) + \sum_{i=0}^{k} \frac{1}{k!}\cdot \text{SUBST}\left(\text{IF}\left(i = 0, \left(\frac{d}{du}\right)^k f\_, \text{IF}\left(i = k, \left(\frac{d}{dv}\right)^k f\_, \text{COMB}(k, i)\cdot\left(\frac{d}{dv}\right)^i \left(\frac{d}{du}\right)^{k-i} f\_\right)\right), [u, v], [u0, v0]\right)\cdot(u - u0)^{k-i}\cdot(v - v0)^i$$

$$\text{mtaylor}(f\_, u, v, u0, v0, n) := \text{SUBST}(f\_, [u, v], [u0, v0]) + \sum_{k=1}^{n} \text{mt\_part}(f\_, u, v, u0, v0, k)$$

$$f := 2\cdot x^4 - x\cdot y^2 + y^2 - 2\cdot y$$

$$\text{mtaylor}\left(f, x, y, -\frac{1}{2}, 1, 1\right) = -2\cdot x + y - \frac{19}{8}$$

$$\text{mtaylor}\left(f, x, y, -\frac{1}{2}, 1, 2\right) = \frac{24\cdot x^2 + 8\cdot x\cdot(3 - 2\cdot y) + 12\cdot y^2 - 24\cdot y + 7}{8}$$

$$\text{mtaylor}\left(f, x, y, -\frac{1}{2}, 1, 3\right) = -\frac{32\cdot x^3 + 24\cdot x^2 + 8\cdot x\cdot(y^2 + 1) - 8\cdot y^2 + 16\cdot y + 1}{8}$$

The results are correct, as I do hope.

It took me some thoughts and attempts to generalize for more than two variables. See how I did:

Alas, I am not quite sure if I found the most elegant way. I need two auxiliary functions. The first one generates all variations of *n* elements to classes of *k* elements with repetition.

```
perms(n, k, m, i, j, k_, l, s, ps) :=
  Prog
    m := VECTOR(z, z, Σ(10^u, u, 0, k - 1), Σ(n·10^u, u, 0, k - 1))
    i := DIM(m)
    j := DIM(m↓1)
    k_ := 1
    ps := []
    Loop
      If k_ > i exit
      l := 1
      s := 1
      Loop
        If l > j exit
        If m↓k_↓l = 0 ∨ m↓k_↓l > n
           s := 0
        l :+ 1
      If s = 1
         ps := APPEND(ps, [m↓k_])
      k_ :+ 1
    VECTOR(VECTOR(p↓i_, i_, DIM(p)), p, ps)
```

I need it for finding all mixed partial derivatives. When we have 3 variables and we want to find the 2nd order Taylor expansion it needs f_{xx}, f_{xy}, f_{xz}, f_{yx}, …, f_{zy}, f_{zz} which can be expressed by the rows in the matrix given as perms(3,2). The numbers represent the variables 1 through 3 (how they are given in the list of variables *vs*).

The partial derivative of function *f* (with variables given in list *vs*) with respect to list *mx* evaluated for point *invs* is defined as:

```
pd(f, vs, mx, invs, i, d) :=
  Prog
    i := 1
    d := f
    Loop
      If i > DIM(mx) exit
      d := ∂(d, vs↓(mx↓[i]))
      i :+ 1
    SUBST(d, vs, invs)
```

```
       3  2  2      2        3
pd(4·x ·y ·z  + 3·y ·x + z·y , [x, y, z], [1, 2, 3], [1, 1, 1]) = 48

       3  2  2      2        3
pd(4·x ·y ·z  + 3·y ·x + z·y , [x, y, z], [1, 2, 2], [1, 1, 1]) = 30

       3  2  2      2        3                                       2
pd(4·x ·y ·z  + 3·y ·x + z·y , [x, y, z], [1, 2, 3], [x, y, z]) = 48·x ·y·z

       3  2  2      2        3                                          2  2
pd(4·x ·y ·z  + 3·y ·x + z·y , [x, y, z], [1, 2, 2], [x, y, z]) = 6·(4·x ·z  + 1)
```

The first two expressions give $f_{xyz}(x = y = z = 1)$ and $f_{xyy}(x = y = z = 1)$. The next two ones are the general “mixed” partial derivates.

```
multitaylor(f, n, vs, invs, ps, mt, i) :=
    mt := SUBST(f, vs, invs)
    i := 1
    Loop
      If i > n exit
      ps := perms(DIM(vs), i)
      mt := mt + 1/i!·∑(pd(f, vs, ps↓m, invs)·∏(vs↓ps↓m↓j - invs↓ps↓m↓j, j, 1,
           DIM(ps↓i)), m, 1, DIM(ps))
      i :+ 1
    EXPAND(mt)
```

First results compared with *Maxima*-answers are promising:

```
                3  2  2      2        3
multitaylor(4·x ·y ·z  + 3·y ·x + z·y , 1, [x, y, z], [1, 1, 1])

15·x + 17·y + 9·z - 33

                3  2  2      2        3
multitaylor(4·x ·y ·z  + 3·y ·x + z·y , 2, [x, y, z], [1, 1, 1])

    2                                     2                       2
12·x  + 30·x·y + 24·x·z - 63·x + 10·y  + 19·y·z - 52·y + 4·z  - 42·z + 66
```

```
expand(taylor(4*x^3*y^2*z^2+3*y^2*x+z*y^3,[x,y,z],[1,1,1],1));
```

$9\,z+17\,y+15\,x-33$

```
expand(taylor(4*x^3*y^2*z^2+3*y^2*x+z*y^3,[x,y,z],[1,1,1],2));
```

$4\,z^2+19\,y\,z+24\,x\,z-42\,z+10\,y^2+30\,x\,y-52\,y+12\,x^2-63\,x+66$

$$\text{multitaylor}\left(x^2\cdot \text{SIN}(x\cdot y),\ 1,\ [x,\ y],\ \left[\frac{\pi}{2},\ 0\right]\right) = \frac{\pi^3\cdot y}{8}$$

$$\text{multitaylor}\left(x^2\cdot \text{SIN}(x\cdot y),\ 3,\ [x,\ y],\ \left[\frac{\pi}{2},\ 0\right]\right) = \frac{3\cdot\pi\cdot x^2\cdot y}{2} - \frac{3\cdot\pi^2\cdot x\cdot y}{4} - \frac{\pi^5\cdot y^3}{192} + \frac{\pi^3\cdot y}{8}$$

```
expand(taylor(x^2*sin(x*y),[x,y],[pi/2,0],1));
```

$$\frac{\pi^3 y}{8}$$

```
expand(taylor(x^2*sin(x*y),[x,y],[pi/2,0],3));
```

$$-\frac{\pi^5 y^3}{192}+\frac{3\pi x^2 y}{2}-\frac{3\pi^2 x y}{4}+\frac{\pi^3 y}{8}$$

Finally, please compare:

$$\text{multitaylor}(x^3\cdot y^4\cdot w^5 + 3\cdot x\cdot z\cdot w - 5\cdot w^3\cdot x^5\cdot z^4,\ 3,\ [x,\ y,\ z,\ w],\ [1,\ 2,\ 3,\ 4])$$

$$2155\cdot w^3 + 6420\cdot w^2\cdot x + 20480\cdot w^2\cdot y - 6480\cdot w^2\cdot z - 48420\cdot w^2 - 132960\cdot w\cdot x^2 + 122880\cdot w\cdot x\cdot y - 129597\cdot w\cdot x\cdot z + 321840\cdot w\cdot x + 30720\cdot w\cdot y^2 - 368640\cdot w\cdot y - 12960\cdot w\cdot z^2 + 233280\cdot w\cdot z + 46800\cdot w - 242816\cdot x^3 + 98304\cdot x^2\cdot y - 345600\cdot x^2\cdot z + 1890432\cdot x^2 + 73728\cdot x\cdot y^2 - 884736\cdot x\cdot y - 86400\cdot x\cdot z^2 + 1555200\cdot x\cdot z - 3620160\cdot x + 8192\cdot y^3 - 221184\cdot y^2 + 1474560\cdot y - 3840\cdot z^3 + 155520\cdot z^2 - 1555200\cdot z + 1573440$$

```
(%i3)  expand(taylor(x^3*y^4*w^5+3*x*z*w-5*w^3*x^5*z^4,[x,y,z,w],[1,2,3,4],3));
```

(%o3) $-3840\,z^3-86400\,x\,z^2-12960\,w\,z^2+155520\,z^2-345600\,x^2\,z-129597\,w\,x\,z+1555200\,x\,z-6480\,w^2\,z+233280\,w\,z-1555200\,z+8192\,y^3+73728\,x\,y^2+30720\,w\,y^2-221184\,y^2+98304\,x^2\,y+122880\,w\,x\,y-884736\,x\,y+20480\,w^2\,y-368640\,w\,y+1474560\,y-242816\,x^3-132960\,w\,x^2+1890432\,x^2+6420\,w^2\,x+321840\,w\,x-3620160\,x+2155\,w^3-48420\,w^2+46800\,w+1573440$

It works, indeed!!

I'd like to close the circle. I started with Bhuvanesh`s TI-92 function which proved to be not correct. Then I tried – hopefully successful – to produce the multivariate Taylor expansion using DERIVE and the results were justified by Maxima and MATHEMATICA.

So I will turn back to the TI-92 (TI-89 and Voyage 200) version with an intermediate stop producing a TI-Nspire version, too. Some programming steps are easier (e.g. providing the for-endfor-loop) and others are more complicated (e.g. extracting digits out of a number).

Overall - I repeat myself – I am not quite sure that my procedure is the most elegant one.

multitaylor() with TI-Nspire:

$$f:=4\cdot x^3\cdot y^2\cdot z^2+3\cdot y^2\cdot x+z\cdot y^3 \qquad 4\cdot x^3\cdot y^2\cdot z^2+3\cdot x\cdot y^2+y^3\cdot z$$

$perms(2,3)$ {"111","112","121","122","211","212","221","22▸

$$pd(f,\{x,y,z\},\text{"123"},\{1,1,1\}) \qquad 48$$

$$\text{expand}(multitaylor(f,1,\{x,y,z\},\{1,1,1\})) \qquad 15\cdot x+17\cdot y+9\cdot z-33$$

$$\text{expand}(multitaylor(f,2,\{x,y,z\},\{1,1,1\}))$$

perms 1/15

Define LibPub **perms**(n,k)=

Func

Local m,i,j,s,ps,st

$$m:=\text{seq}\left(z,z,\sum_{u=0}^{k-1}\left(10^u\right),\sum_{u=0}^{k-1}\left(n\cdot 10^u\right)\right)$$

$ps:=\{\square\}$

For $i,1,\dim(m)$

$s:=1$

$st:=\text{string}(m[i])$

multitaylor 1/9

Define LibPub **multitaylor**$(f,n,vs,invs)$=

Func

Local ps,mt,i

$mt:=f$

For $i,1,\dim(vs)$

$$mt:=\lim_{vs[i]\to invs[i]}(mt)$$

EndFor

For $i,1,n$

$ps:=mathtool\backslash perms(\dim(vs),i)$

pd 1/9

Define LibPub **pd**$(f,vs,mx,invs)$=

Func

Local i,d,va

$d:=f$

For $i,1,\dim(mx)$

$va:=vs[\text{expr}(\text{mid}(mx,i,1))]$

$$d:=\frac{d}{dva}(d)$$

EndFor

For $i,1,\dim(invs)$

$$f:=4\cdot x^3\cdot y^2\cdot z^2+3\cdot y^2\cdot x+z\cdot y^3 \qquad 4\cdot x^3\cdot y^2\cdot z^2+3\cdot x\cdot y^2+y^3\cdot z$$

$perms(2,3)$ {"111","112","121","122","211","212","221","222"}

$$pd(f,\{x,y,z\},\text{"123"},\{1,1,1\}) \qquad 48$$

$$\text{expand}(multitaylor(f,1,\{x,y,z\},\{1,1,1\})) \qquad 15\cdot x+17\cdot y+9\cdot z-33$$

$$\text{expand}(multitaylor(f,2,\{x,y,z\},\{1,1,1\})) \qquad 12\cdot x^2+30\cdot x\cdot y+24\cdot x\cdot z-63\cdot x+10\cdot y^2+19\cdot y\cdot z-52\cdot y+4\cdot z^2-42\cdot z+66$$

$$\text{expand}(multitaylor(f,2,\{x,y,z\},\{x0,y0,z0\}))$$

$$12\cdot x0\cdot y0^2\cdot z0^2\cdot x^2+24\cdot x0^2\cdot y0\cdot z0^2\cdot x\cdot y+6\cdot y0\cdot x\cdot y+24\cdot x0^2\cdot y0^2\cdot z0\cdot x\cdot z-60\cdot x0^2\cdot y0^2\cdot z0^2\cdot x-3\cdot y0^2\cdot x+4▸$$

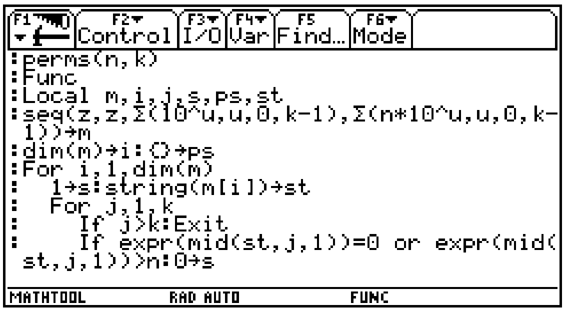
$$multitaylor\left(x^2\cdot\sin(x\cdot y),1,\{x,y\},\left\{\frac{\pi}{2},0\right\}\right) \qquad \frac{\pi^3\cdot y}{8}$$

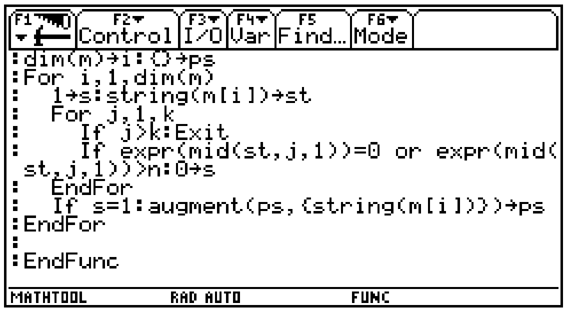
$$\text{expand}(multitaylor(f,1,\{x,y,z\},\{1,2,3\})) \qquad 444\cdot x+192\cdot y+104\cdot z-960$$

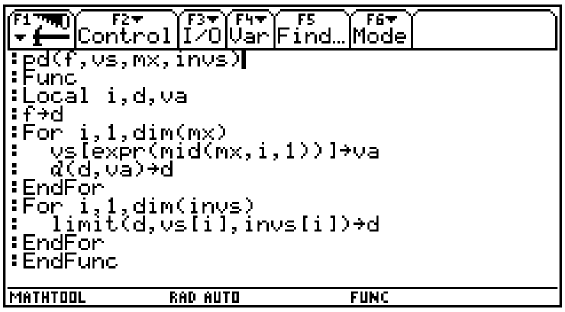
$$\text{expand}(multitaylor(f,1,\{z,x,y\},\{3,1,2\})) \qquad 444\cdot x+192\cdot y+104\cdot z-960$$

$$multitaylor\left(x^2\cdot\sin(x\cdot y),3,\{x,y\},\left\{\frac{\pi}{2},0\right\}\right) \qquad \frac{3\cdot\pi\cdot x^2\cdot y}{2}-\frac{3\cdot\pi^2\cdot x\cdot y}{4}-\frac{\pi^5\cdot y^3}{192}+\frac{\pi^3\cdot y}{8}$$

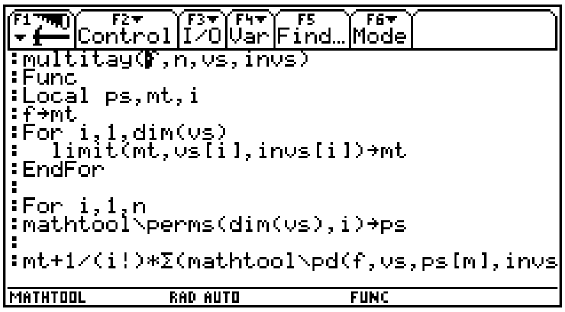
Turning back to the TI-92/Voyage 200 I start with the auxiliary function perms(n,k):

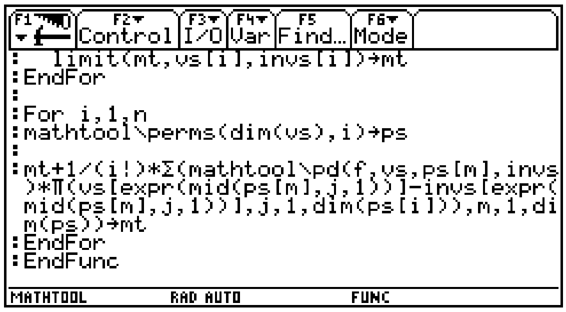




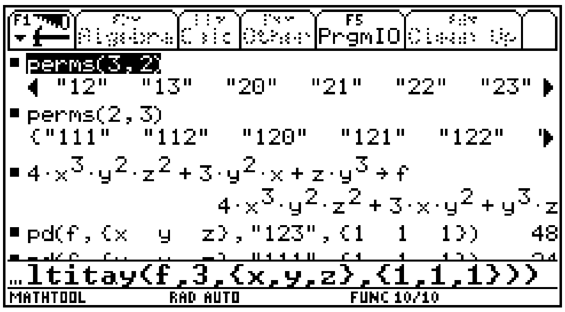


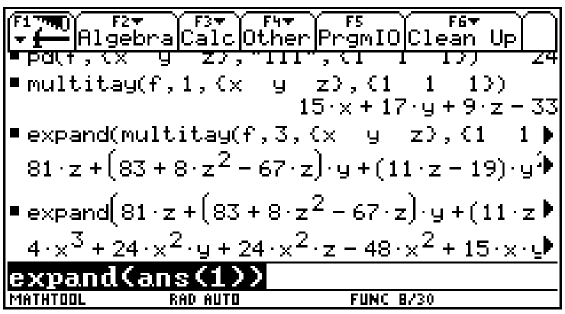
The Multivariate Taylor Expansion:

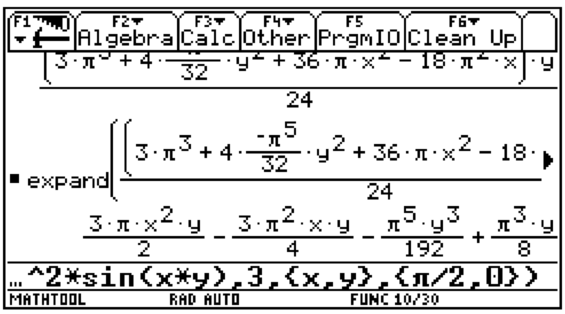




Examples follow:







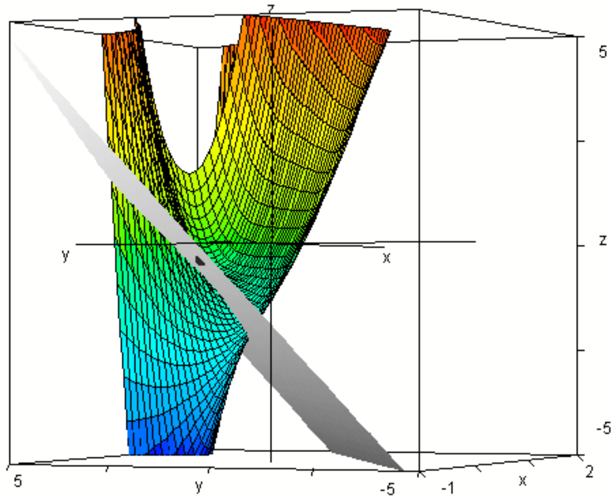
$$fu(x, y) := 2 \cdot x^4 - x \cdot y^2 + y^2 - 2 \cdot y$$

Point on the surface [-1/2, 1, fu(-1/2,1)]

$$\left(pt := \left[-\frac{1}{1}, fu\left(-\frac{1}{2}, 1\right)\right]\right) = pt := \left[-1, -\frac{3}{8}\right]$$

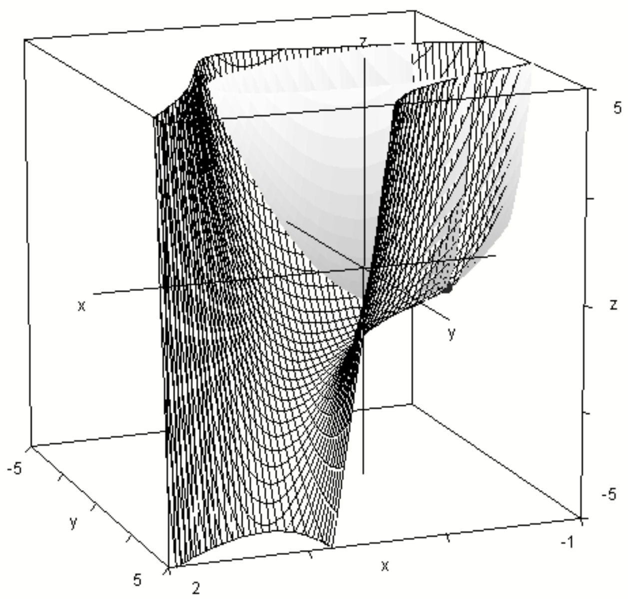
$$\mathrm{multitaylor}\left(fu(x, y), 1, [x, y], \left[-\frac{1}{2}, 1\right]\right) = -2 \cdot x + y - \frac{19}{8}$$

Surface and its tangent plane:



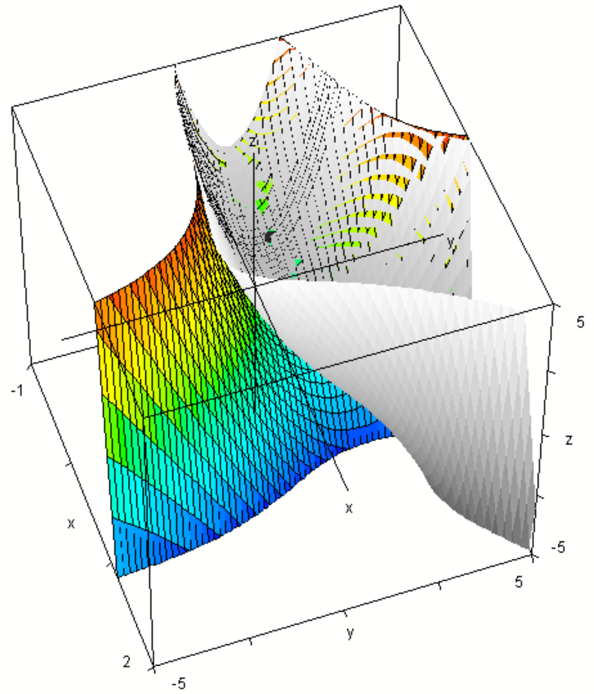
$$\mathrm{multitaylor}\left(fu(x, y), 2, [x, y], \left[-\frac{1}{2}, 1\right]\right) = 3 \cdot x^2 - 2 \cdot x \cdot y + 3 \cdot x + \frac{3 \cdot y^2}{2} - 3 \cdot y + \frac{7}{8}$$

Surface and 2nd order Taylor Expansion

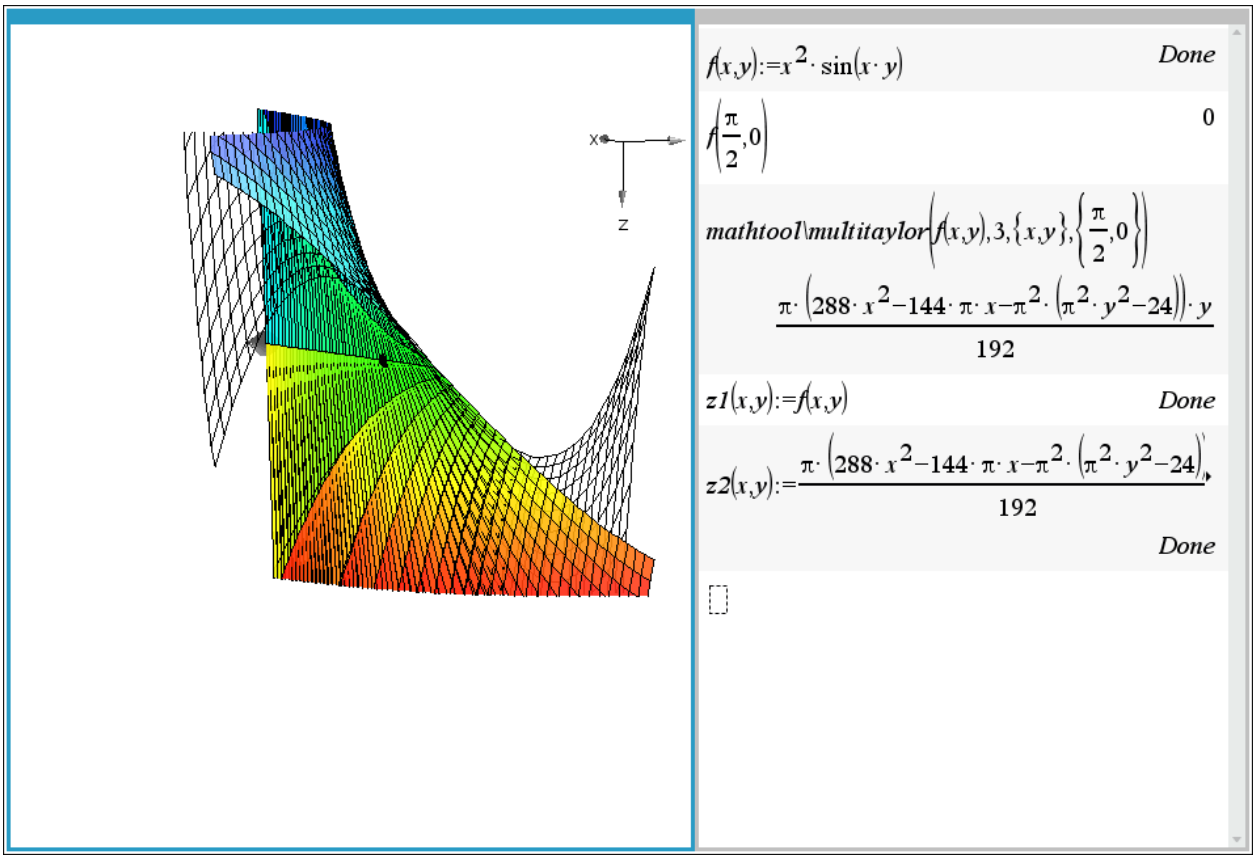


$$\text{multitaylor}\left(\text{fu}(x,\ y),\ 3,\ [x,\ y],\ \left[-\frac{1}{2},\ 1\right]\right) = -4\cdot x^3 - 3\cdot x^2 - x\cdot y^2 - x + y^2 - 2\cdot y - \frac{1}{8}$$

Surface and 3rd order Taylor Expansion



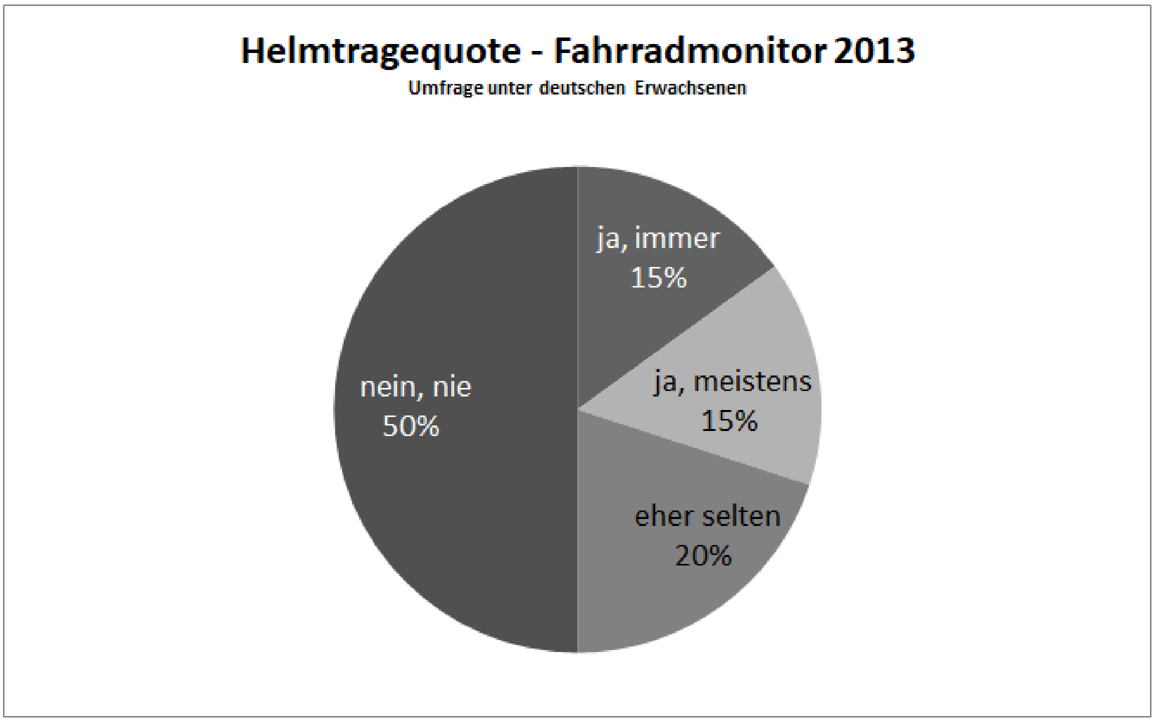
Another example – but now on the Nspire-screen (surface + Taylor expansion + point)



Wolfgang Alvermann showed in DNL#118 his mathematics final exam from 1968. We compare this with a nowadays final exam. It is a central exam for secondary vocational school from Germany. After the first – compulsory – part, and the first block of the second – electoral – part I proceed with Blocks Stochastics and one task of Block Analytic Geometry. Josef

Task 2A

The last publication of an article concerning “I wear a cycle helmet” was several years ago. The editorial office of a specialist journal researches current data in order to publish just in time before vacation season a two-part article on this issue. The editor has found the results of a survey from 2013:



Quelle: https://ich-trag-helm.de/wp-content/uploads/2016/01/Fahrradmonitor.png (Abruf 06.06.2018).

(no, never: 50%, yes, always: 15%, yes, mostly: 15%, rather rarely: 20%

Additionally he found out that every year 2.88 Mio. adults undertake a bicycle tour. The route Passau-Vienna is traveled by 100 000 adults every year. The circuit around Lake Constance is completed by 12 000 adults and the Baltic Sea route is usually traveled by 5 000 cyclists.

a) For the first part of his article the editor needs various information:

Based on the results of the 2013-survey determine the number of adults which can be expected to wear always a helmet when they are traveling by bicycle.

Calculate the probability that

- at least 2 600 Baltic Sea cyclist never wear a helmet.
- at most 1 800 cyclists around Lake Constance are wearing always a helmet.
- more than 15 000 but less than 25 000 cyclists, touring Passau-Vienna are wearing a helmet most of the time. (12 points)

b) The editor would like to get more recent data for the second part of his article and completes the first part with a survey on the topic “I wear a helmet”. All adults, who made bike tour in fall 2018 were asked to participate in an online-survey. A total of 1 500 adults took part in the survey. 900 persons indicated that they always – even at very short distances – are wearing a helmet and 375 persons indicated that they never wear a helmet.

For the second part of the article these data shall be generalized and be compared with the data from 2013. Investigate with a security probability of 90% in each case how the “I always wear a helmet”-behavior and the “I never wear a helmet”-behavior have changed since 2013.

(12 points)

Task 2B

Company *RoRa* offers rolled turf for various purposes. *RoRa* wants to develop a new lawn mixture for the European football championship 2020 which is to be held in twelve European Cities. It should meet the special needs of football, which is characterized by a high stability and shear strength, by uniform growth as well as good regeneration behavior. 20 mixtures have been developed for this purpose and tested by series of measuring. Growth experiments were carried out for the both most promising mixtures in a glass house.

a) In one test the length of the blades was measured.
One week after the first lawn cutting the length of the blades was measured at 20 points (all data are given in mm).

Rom	29; 31; 24; 21; 25; 32; 29; 27; 28; 32; 28; 30; 27; 24; 22; 28; 31; 21; 25; 26
Wembley	30; 24; 26; 25; 23; 28; 25; 26; 25; 30; 23; 29; 27; 24; 30; 28; 25; 25; 27; 26

Measurement results of mixture *Rom* were evaluated; there are a dispersion interval $[\overline{x}-\sigma;\ \overline{x}+\sigma]=[23.64;30.36]$ and a boxplot available (see appendix).

Find out the 1σ-dispersion interval and draw the respective boxplot for mixture *Wembley* on the appendix. Decide which of both mixtures is more suitable in terms of the requirement “uniform growth”.

(10 points)

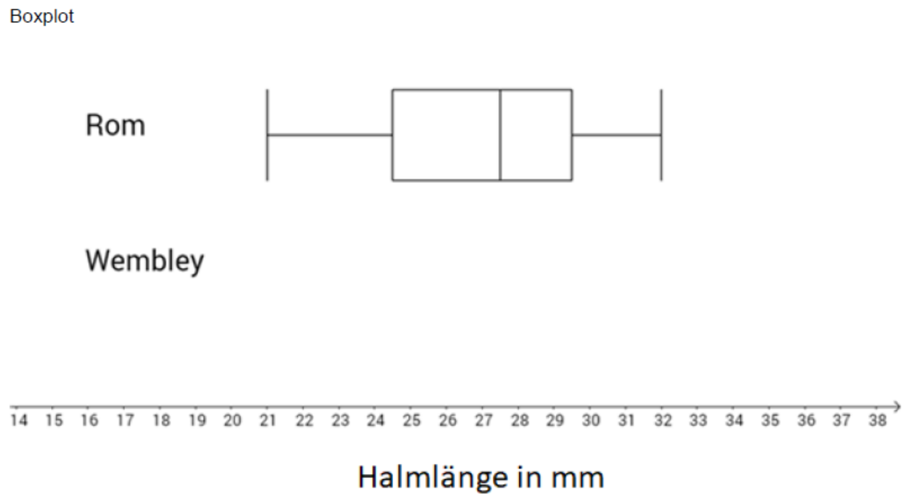
b) For reviewing stability and shear strength a new semi-automatic testing machine is used. When wrongly used, the incorrect measurement is indicated by a warning light. The producer of the machine states from experience that on the average three out of five measurements are incorrect if the operators are untrained. There will be no failure for a trained staff of operators.
Calculate the number of measures which must be performed by an untrained person, to gain a correct measurement result with a security of at least 95%.

For application of a lawn mixture at the European championship fast regeneration of the lawn is a requirement. Experiments testing the regeneration behavior of the mixtures showed that 700 out of 800 blades of mixture *Wembley* could recover within three days. For the mixture *Rom* a confidence interval $CI_{95\%}=[0.8034;0.8487]$ could be found.

Determine with a security of 95% the proportion of regenerated blades in mixture *Wembley*.

Interpret the results with regard to the regeneration behavior for both mixtures in the sense of the task.

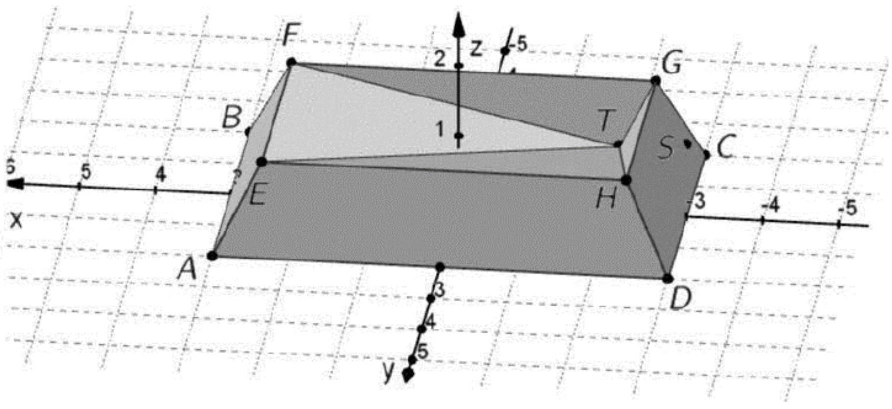
(14 points)



Task 3A

A local artist is to create a concrete sculpture on a public square in Hanover. For this purpose, he has prepared two drawings (figures 1 and 2), which shall form the base for calculating the consumption of material.

The sculpture consists of a symmetrical truncated pyramid with a volume of 23.434 m^3, from which a crooked pyramid shall be carved out downwards with its base in the rectangle *E, F, G* and *H* and its top *T*.



A(3|2|0), *C*(-3|-2|0), *D*(-3|2|0), *E*(2.4|,1.6|,1.2), *F*(2.4|-1.6|1.2), *H*(-2.4|1.6|1.2), *T*(-2|-1|0.5) (all measures in m)

Figure 1

a) Coordinates of points *B* and *G* are missed in figure 1. What are their coordinates?

The triangular surface with the smallest inclination from the horizontal should be covered with tiles. Give reasons that this is the triangular plane given by points *E*, *F* and *T*.

Calculate the number of square tiles with edge length 15 cm keeping in mind that there is an offcut of 10%.

The sculpture is to be erected on a foundation. This must be strengthened if the mass after having carved out the pyramid exceeds 45 000kg. The concrete used has a density of 2000 kg/m^3. The weight of the tiles can be neglected.

Investigate if strengthening is necessary?

(14 points)

b) To avoid that water can remain in the sculpture a drain pipe is to be built in, which should run in the extension of points *E* and *T* and should exit in point *S* lying in the side face described by points *C*, *D*, *G* and *H* (figures 1 and 2).

Show that the equation for the plan containing this side face (*CDGH*) is *e*: $-1.2x + 0.6z = 3.6$.

Calculate the coordinates of the exit point *S* and the length of the pipe.

(10 points)

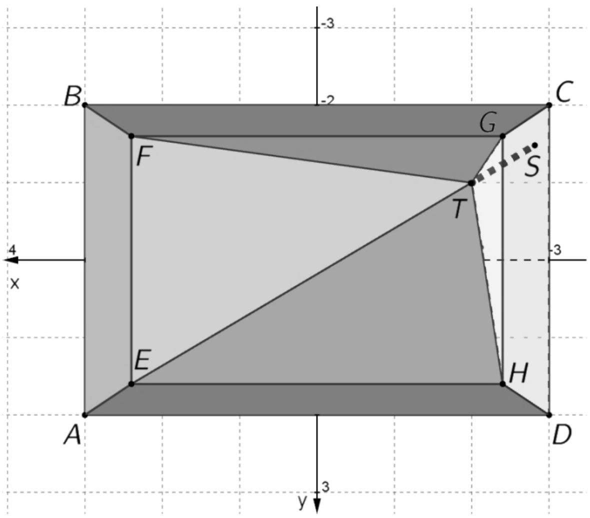


Figure 2: Top view of the sculpture together with the visible part of the drain pipe

I needed some space for other contributions, so Task 3B will be presented in the next DNL together with all solutions. Many thanks for your understanding, Josef.

Next contribution is the second part of Matthew Myers' collection of TI-89 functions. His Part Two consists of three sections: 2 Math Functions, 3 Number Theory, 4 Congruences.

Here we start with the Math Functions. I reprint three original TI-89/92/V200 programs and some TI-89 screen shots to give an impression how much efforts are necessary to develop the programs/functions if you are bound on the TI-89 handheld like Matthew.

TI-89 Keystroke Functions, Part Two

by
Matthew A. Myers, U.S.A.

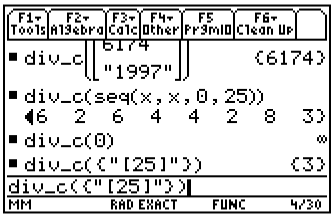
2 Math Functions

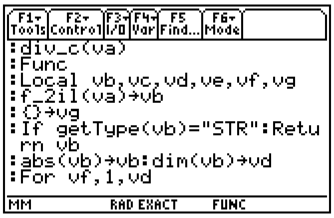
div_c(A) **ord_p**(A,B) **syn_div**(A,B)
divisors(A) **pow_mod**(A,B,C)

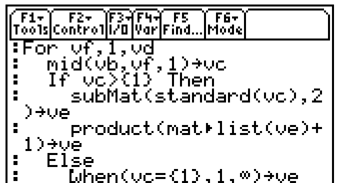
The **div_c**() function (τ in Stewart, p. 62, but d in Hardy & Wright, p. 239) counts the number of divisors of the integer, and the **divisors**() function provides a LIST of them. The **ord_p**() function finds the ordinality of a number (often denoted ORD_p), and is related to divisibility since it is the largest exponent of a particular prime divisor of the integer. Going into higher math, the **pow_mod**() function finds the modulus of a number raised to a power, and the **syn_div**() function performs synthetic division. Both Apostol (pp. 38-39) and Hardy & Wright (p. 239) discuss divisors, their count, and other related topics. Clark has more information about ORD_p, while synthetic division is covered by Larson, Hostetler, & Munn (p. 356-358), and by Tussy, Gustafson, & Koenig (p. 532-537).

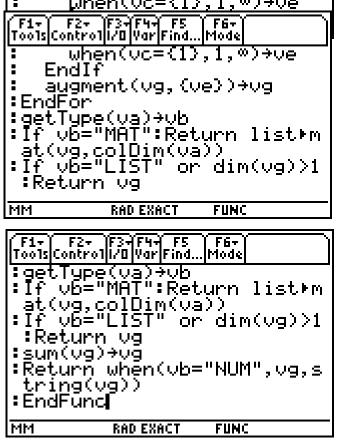
div_c(INTs)

Requires: **f_2i**(), **f_2il**(), **f_2l**(), and **standard**() functions. Returns a count of all positive integer divisors of each INT. Programming logic: Take the absolute value of INT, and if it is greater than 1, convert it to standard form, add 1 to all exponents, and take the product of these numbers.

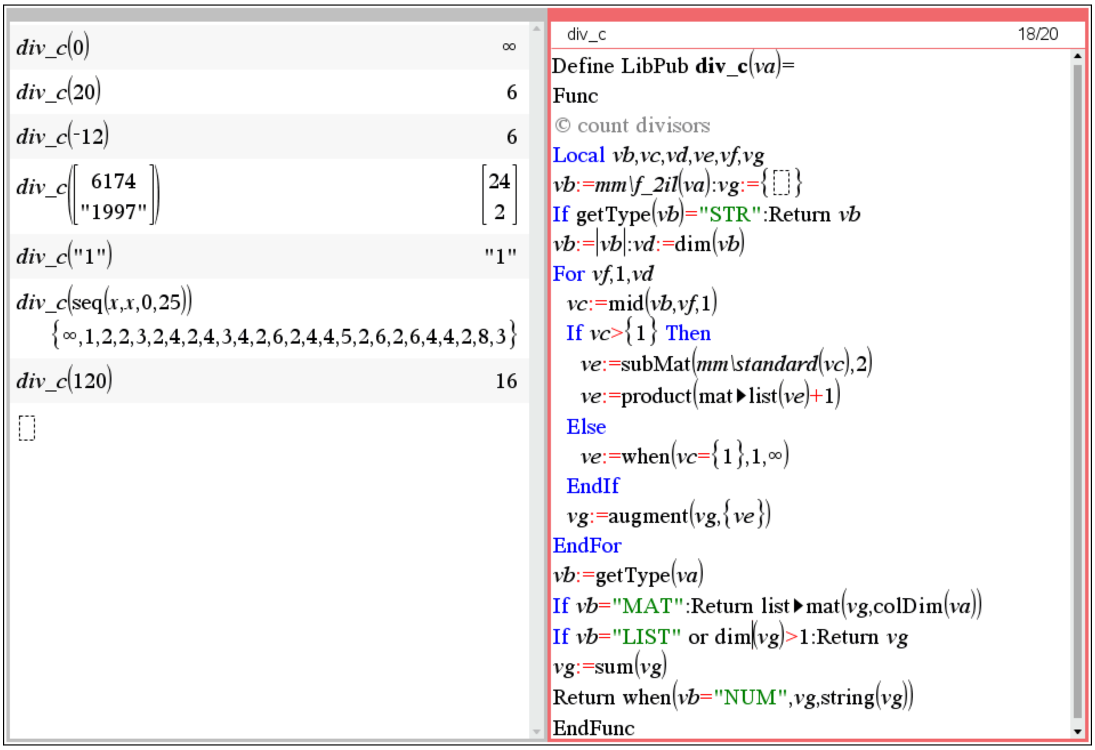








```
div_c(va)
Func
Local vb,vc,vd,ve,vf,vg
mm\f_2il(va)→vb
{}→vg
If getType(vb)="STR":Return vb
abs(vb)→vb:dim(vb)→vd
For vf,1,vd
  mid(vb,vf,1)→vc
  If vc>{1} Then
    subMat(mm\standard(vc),2)→ve
    product(mat▸list(ve)+1)→ve
  Else
    when(vc={1},1,∞)→ve
  EndIf
  augment(vg,{ve})→vg
EndFor
getType(va)→vb
If vb="MAT":Return list▸mat(vg,colDim(va))
If vb="LIST" or dim(vg)>1:Return vg
sum(vg)→vg
Return when(vb="NUM",vg,string(vg))
EndFunc
```



```
div_c(0)                         ∞
div_c(20)                        6
div_c(-12)                       6
div_c([6174;"1997"])             [24;2]
div_c("1")                       "1"
div_c(seq(x,x,0,25))
  {∞,1,2,2,3,2,4,2,4,3,4,2,6,2,4,4,5,2,6,2,6,4,4,2,8,3}
div_c(120)                       16
```

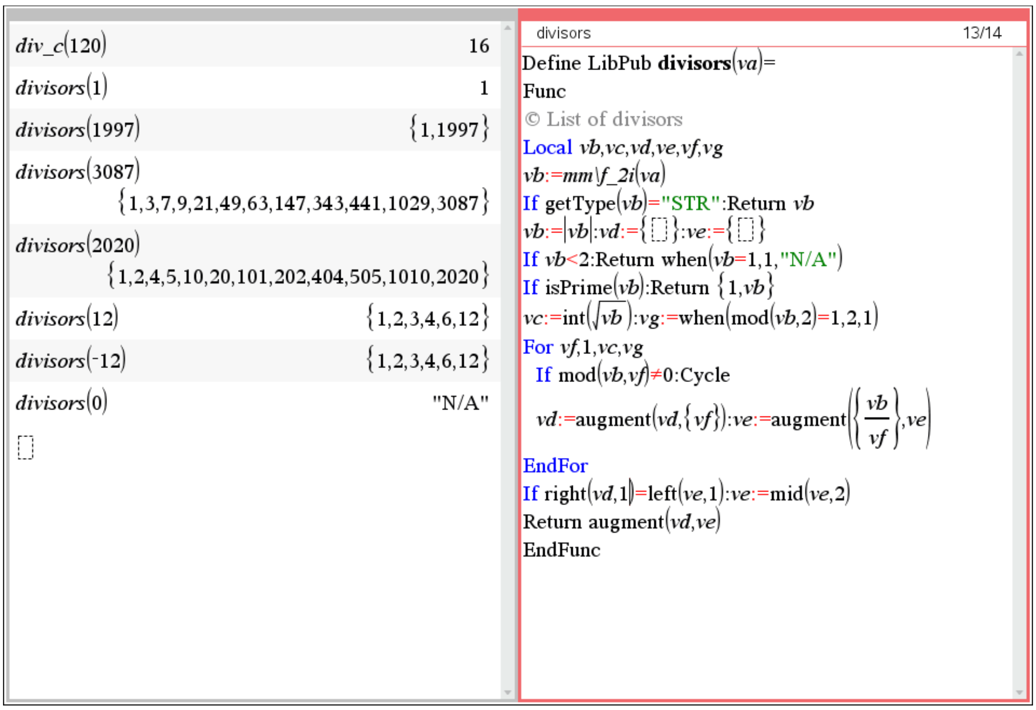
```
div_c                                              18/20
Define LibPub div_c(va)=
Func
© count divisors
Local vb,vc,vd,ve,vf,vg
vb:=mm\f_2il(va):vg:={}
If getType(vb)="STR":Return vb
vb:=|vb|:vd:=dim(vb)
For vf,1,vd
  vc:=mid(vb,vf,1)
  If vc>{1} Then
    ve:=subMat(mm\standard(vc),2)
    ve:=product(mat▸list(ve)+1)
  Else
    ve:=when(vc={1},1,∞)
  EndIf
  vg:=augment(vg,{ve})
EndFor
vb:=getType(va)
If vb="MAT":Return list▸mat(vg,colDim(va))
If vb="LIST" or dim(vg)>1:Return vg
vg:=sum(vg)
Return when(vb="NUM",vg,string(vg))
EndFunc
```

divisors(INT)

Requires: **f_2i**() function.

Returns a LIST of all positive integer divisors of INT, in ascending order.

Programming logic: To avoid sorting, brute force testing of most integers ≤ √(abs(INT)) for divisibility is used, then each divisor D, and INT/D, are appended to lists joined together afterwards. If INT is prime then 1 and INT are the only divisors. If INT is odd, even integers are skipped for divisor testing.



```
div_c(120)                       16
divisors(1)                      1
divisors(1997)                   {1,1997}
divisors(3087)
  {1,3,7,9,21,49,63,147,343,441,1029,3087}
divisors(2020)
  {1,2,4,5,10,20,101,202,404,505,1010,2020}
divisors(12)                     {1,2,3,4,6,12}
divisors(-12)                    {1,2,3,4,6,12}
divisors(0)                      "N/A"
```

```
divisors                                           13/14
Define LibPub divisors(va)=
Func
© List of divisors
Local vb,vc,vd,ve,vf,vg
vb:=mm\f_2i(va)
If getType(vb)="STR":Return vb
vb:=|vb|:vd:={}:ve:={}
If vb<2:Return when(vb=1,1,"N/A")
If isPrime(vb):Return {1,vb}
vc:=int(√(vb)):vg:=when(mod(vb,2)=1,2,1)
For vf,1,vc,vg
  If mod(vb,vf)≠0:Cycle
  vd:=augment(vd,{vf}):ve:=augment({vb/vf},ve)
EndFor
If right(vd,1)=left(ve,1):ve:=mid(ve,2)
Return augment(vd,ve)
EndFunc
```

```
DIVISORS(0) = [0, 1]

div_c(0) = 2

DIVISORS(12) = [1, 2, 3, 4, 6, 12]

DIVISORS(-12) = [-12, -6, -4, -3, -2, -1, 1, 2, 3, 4, 6, 12]

DIVISORS(1997) = [1, 1997]

div_c(n) := DIM(DIVISORS(n))

div_c(1997) = 2

DIVISORS(2020) = [1, 2, 4, 5, 10, 20, 101, 202, 404, 505, 1010, 2020]

div_c(2020) = 12
```

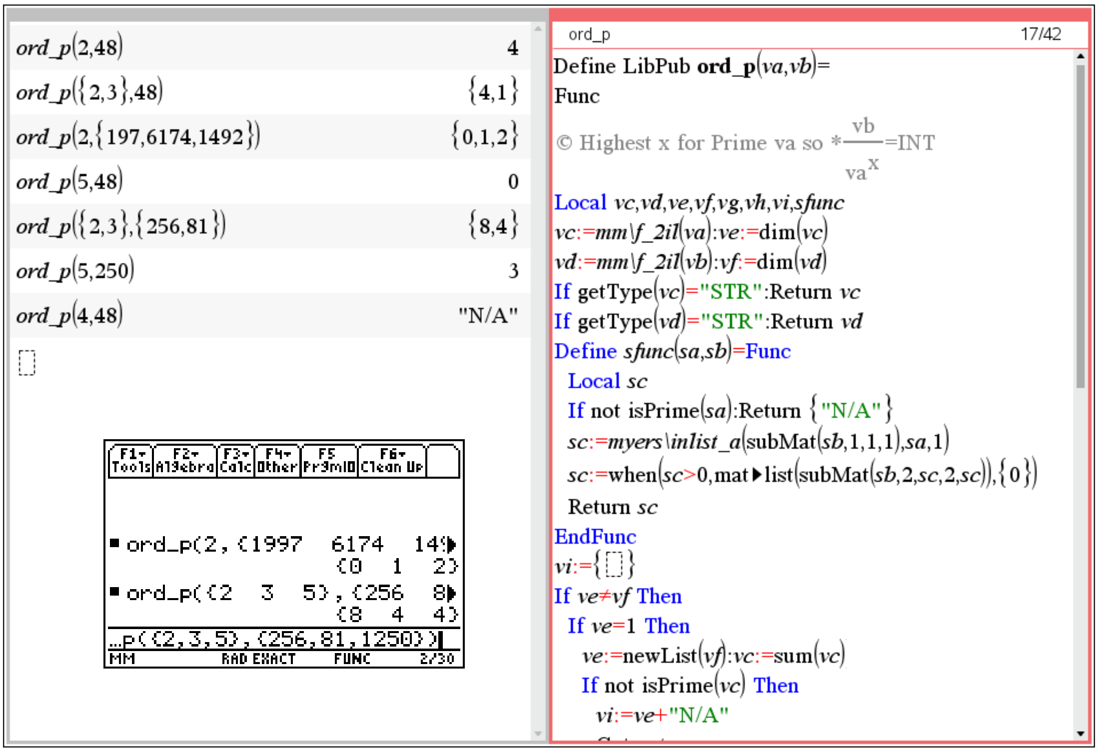
Other results with DERIVE for 0 and negative arguments.

Maxima and MATHEMATICA give the same results for positive and negative arguments. Both of them don't give a result for DIVISORS(0).

ord_p(Prime_NUMs_A,INTs_B)

Requires: **f_2i**(), **f_2il**(), **f_2l**(), **inlist_a**(), and **standard**() functions.
Returns Prime_NUM_A's exponent when INT_B is in standard form.
Programming logic: Test Prime_NUM_A for primality, convert INT_B to standard form, find the exponent of the matching prime.

Examples with TI-Nspire and DERIVE:



DERIVE code:

```
ord_p(p, n, st, posp) :=
  Prog
    st := FACTORS(n)`
    posp := POSITION(p, st↓1)
    If posp = false
       RETURN false
    st↓2↓posp
```

$$\text{FACTORS(3960)}' = \begin{bmatrix} 2 & 3 & 5 & 11 \\ 3 & 2 & 1 & 1 \end{bmatrix}$$

```
ord_p(5, 3960) = 1

ord_p(2, 3960) = 3

ord_p(11, 3960) = 1

ord_p(7, 3960) = false
```

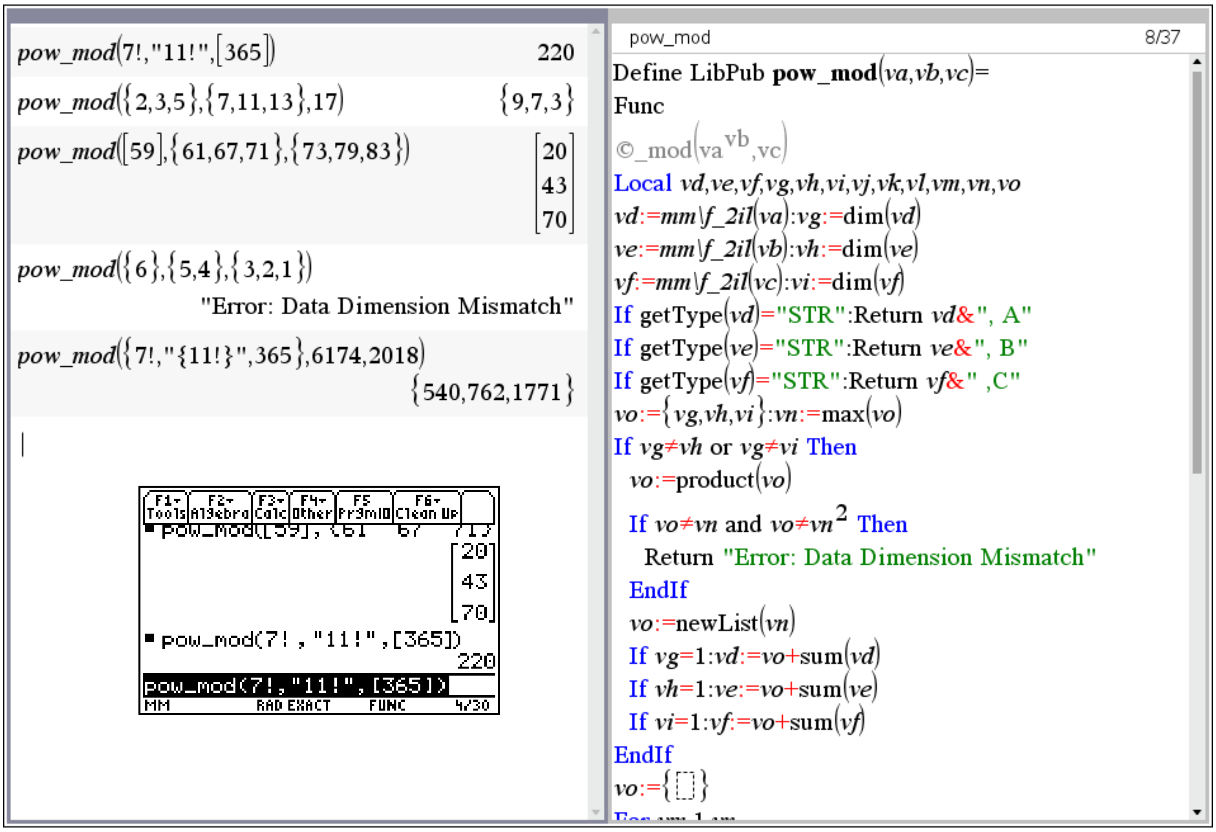
pow_mod(INTs_A,INTs_B,INTs_C)

Requires: **f_2i**(), **f_2il**(), and **f_2l**() functions.

Returns $\text{INT\_A}^{\text{INT\_B}}$ mod INT_C, the same as the TI-89 function mod(INT_A^INT_B,INT_C). The difference in this function compared to mod(), is the ability to work precisely with larger values that would cause the mod() function to round off (like when $10^{615} \le$ INT_A^INT_B).

Programming logic: To avoid INT_A^INT_B becoming rounded off in value, as mod() does, the function processes one digit of INT_B at a time, using congruence laws.

Examples and TI-Nspire:



The TI-89 code:

```
pow_mod(va,vb,vc)
Func© mod(va^vb,vc)
Local vd,ve,vf,vg,vh,vi,vj,vk,vl,vm,vn,vo
mm\f_2il(va)→vd:dim(vd)→vg
mm\f_2il(vb)→ve:dim(ve)→vh
mm\f_2il(vc)→vf:dim(vf)→vi
If getType(vd)="STR":Return vd&", A"
If getType(ve)="STR":Return ve&", B"
If getType(vf)="STR":Return vf&", C"
{vg,vh,vi}→vo:max(vo)→vn
If vg□vh or vg□vi Then
  product(vo)→vo
  If vo□vn and vo□vn^2 Then
    Return "Error: Data Dimension Mismatch"
  EndIf
  newList(vn)→vo
  If vg=1:vo+sum(vd)→vd
  If vh=1:vo+sum(ve)→ve
  If vi=1:vo+sum(vf)→vf
EndIf
{}→vo
For vm,1,vn
  mid(vd,vm,1)→vg
  sum(mid(ve,vm,1))→vh
  If vh≤Ø Then
    augment(vo,vg^vh)→vo
    Cycle
  EndIf
  string(vh)→vh
  mid(vf,vm,1)→vi:dim(vh)→vk:1→vl
  For vj,vk,1,-1
    mod(vl*vg^expr(mid(vh,vj,1)),vi)→vl
    mod(vg^1Ø,vi)→vg
  EndFor
  augment(vo,vl)→vo
EndFor
getType(va)→vd
If vd="MAT":Return list▸mat(vo,colDim(va))
If vd="LIST" or dim(vo)>1:Return vo
sum(vo)→vo
Return when(vd="NUM",vo,string(vo))
EndFunc
```

Examples realized with DERIVE:

$$\mathrm{POWER\_MOD}(7!,\ 11!,\ 365) = 220$$

$$\mathrm{VECTOR}\left(\mathrm{POWER\_MOD}(v_1,\ v_2,\ 17),\ v,\ \begin{bmatrix} 2 & 7 \\ 3 & 11 \\ 13 & 17 \end{bmatrix}\right) = [9,\ 7,\ 13]$$

$$\mathrm{VECTOR}\left(\mathrm{POWER\_MOD}(59,\ v_1,\ v_2),\ v,\ \begin{bmatrix} 61 & 73 \\ 67 & 79 \\ 71 & 83 \end{bmatrix}\right) = [20,\ 43,\ 70]$$

$$\mathrm{VECTOR}(\mathrm{POWER\_MOD}(x,\ 6174,\ 2018),\ x,\ [7!,\ 11!,\ 365]) = [540,\ 762,\ 1771]$$

syn_div(Special_A,ANY_B)

Requires: **f_2l**() and **f_2s**() functions.

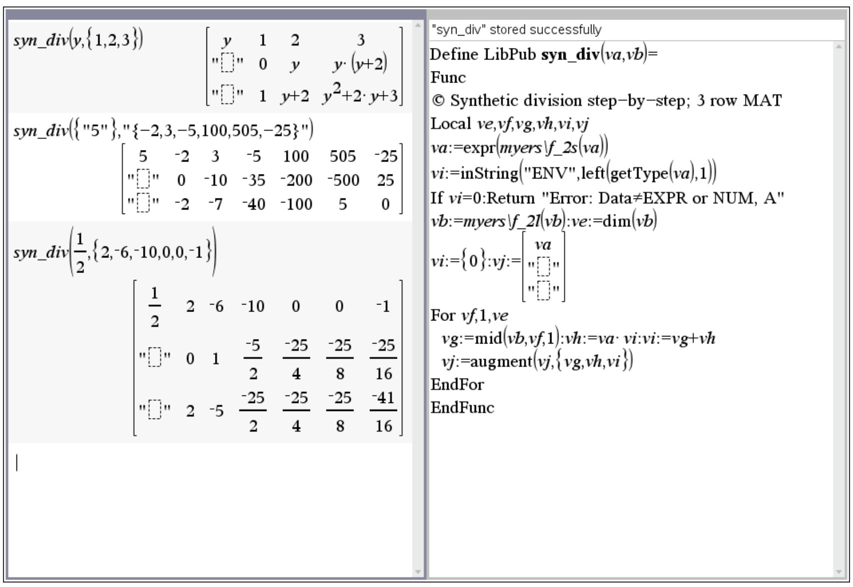
Input: Special_A must be convertible to an EXPR, NUM, or VAR data type, and must be the "k" value from a binomial of the form x - k. ANY_B is a LIST of N+1 coefficients of an Nth degree polynomial. Example: $9x^3 + 8x + 7$ should be: {9,0,8,7}.

Returns a step-by-step solved synthetic division problem as a 3 row MAT. This is one method that can be used to determine factors of a polynomial (when the bottom right number in the MAT is 0, then x – ANY_B is a factor).

Programming logic: Assumes ANY_B is a LIST of N+1 coefficients of an Nth degree polynomial. It divides ANY_B by Special_A to find the remainder, which is displayed as the bottom matrix item on the right. This is done by first putting Special_A as the leftmost item in the first row followed by the items of ANY_B. Then the second and third rows are filled in from left to right by alternately adding the first two items in a column to obtain the third item (skipping the first column, and using 0 as the second column's second row's item), and multiplying Special_A by the columns bottom item to obtain the next column's second row item.

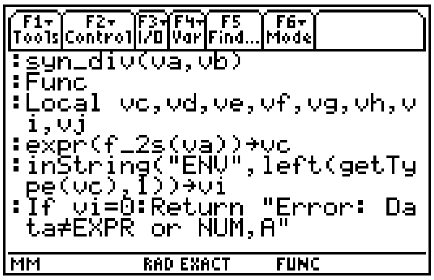
```
syn_div(va,vb)
Func© Synthetic division step-by-step;3 row MAT
Local vc,vd,ve,vf,vg,vh,vi,vj
expr(mm\f_2s(va))→vc
inString("ENV",left(getType(vc),1))→vi
If vi=0:Return "Error: Data□EXPR or NUM,A"
mm\f_2l(vb)→vd:dim(vd)→ve
{0}→vi:[[vc][""][""]]→vj
For vf,1,ve
  mid(vd,vf,1)→vg
  vc*vi→vh:vg+vh→vi
  augment(vj,{vg,vh,vi})→vj
EndFor
Return vj
EndFunc
```

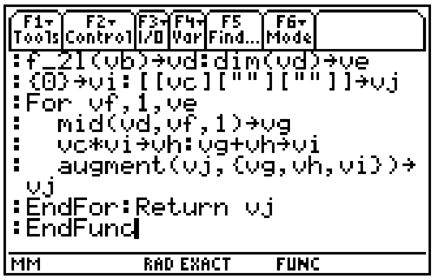
See Examples and TI-Nspire



syn_div is also known as Horner's Method or Horner's Scheme.

```
        horner(va, cfs, co, sch, i) :=
          Prog
            sch := [va; " "; " "]
            co := [cfs↓1; 0; cfs↓1]
            sch := APPEND_COLUMNS(sch, co)
            "RETURN sch"
#1:         i := 2
            Loop
              co := [cfs↓i; co↓3↓1·va; cfs↓i + co↓3↓1·va]
              sch := APPEND_COLUMNS(sch, co)
              i :+ 1
              If i > DIM(cfs) exit
            sch
```



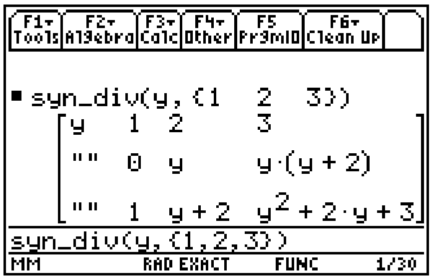


#2: horner(5, [-2, 3, -5, 100, 505, -25])

#3: $$\begin{bmatrix} 5 & -2 & 3 & -5 & 100 & 505 & -25 \\ & 0 & -10 & -35 & -200 & -500 & 25 \\ & -2 & -7 & -40 & -100 & 5 & 0 \end{bmatrix}$$

#4: horner(y, [1, 2, 3])

#5: $$\begin{bmatrix} y & 1 & 2 & 3 \\ & 0 & y & y\cdot(y+2) \\ & 1 & y+2 & y^2+2\cdot y+3 \end{bmatrix}$$



#6: $$\text{horner}\left(\frac{1}{2}, [2, -6, -10, 0, 0, -1]\right)$$

#7: $$\begin{bmatrix} \frac{1}{2} & 2 & -6 & -10 & 0 & 0 & -1 \\ & 0 & 1 & -\frac{5}{2} & -\frac{25}{4} & -\frac{25}{8} & -\frac{25}{16} \\ & 2 & -5 & -\frac{25}{2} & -\frac{25}{4} & -\frac{25}{8} & -\frac{41}{16} \end{bmatrix}$$

+

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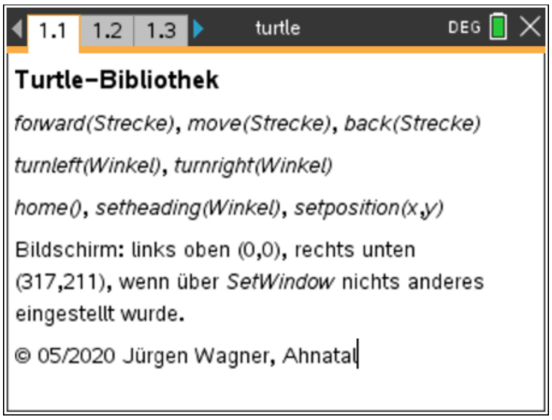
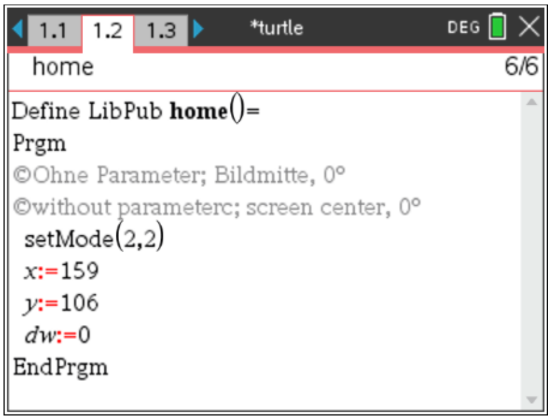
Programming language LOGO and its TURTLE is a topic which has not disappeared in our minds. It is used as introduction in programming for young people. LOGO and the turtle are welcome means to demonstrate and to apply recursive procedures. You can find earlier articles on this topic in DNLs 25, 38 and 57. We don't have seen turtle graphics on the TI-Nspire screen until now. But this will change with Jürgens' contribution.
It is necessary to store turtle.tns in your MyLib-folder, because it is a library file, needed for all other programs. Jürgen Wagner applies the newly implemented TI-Nspire draw-commands.

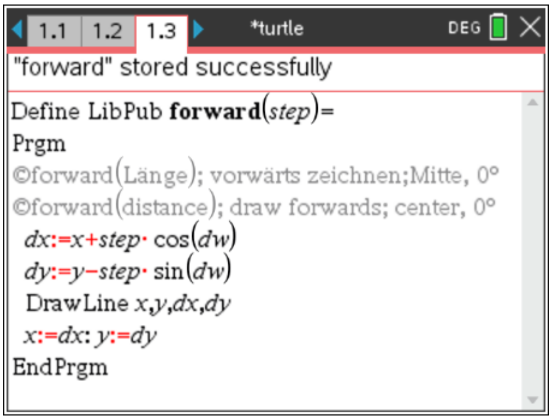
Turtle Graphics

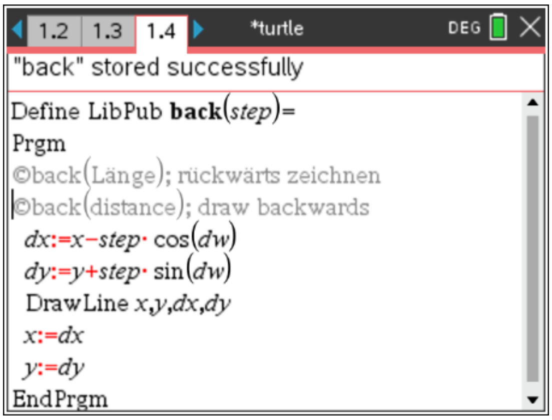
Jürgen Wagner, Ahnatal, Germany

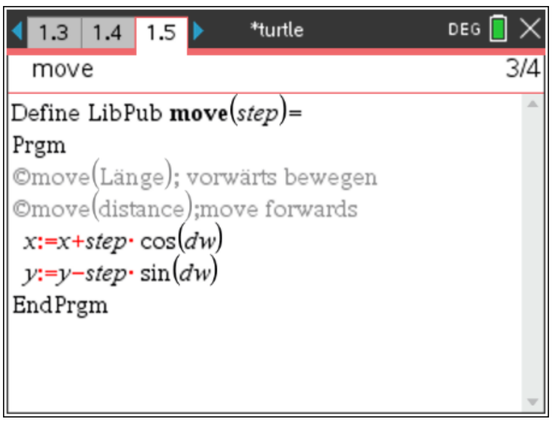
We start presenting the tools provided in the library turtle.tns:

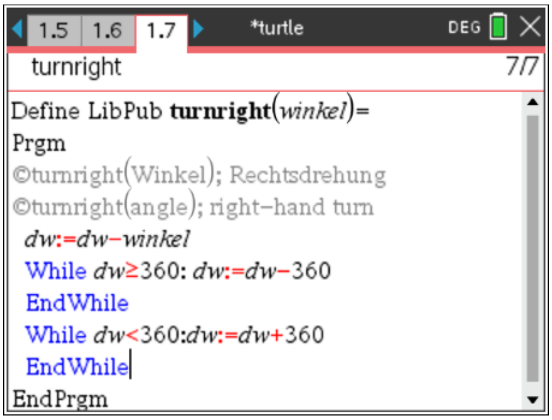


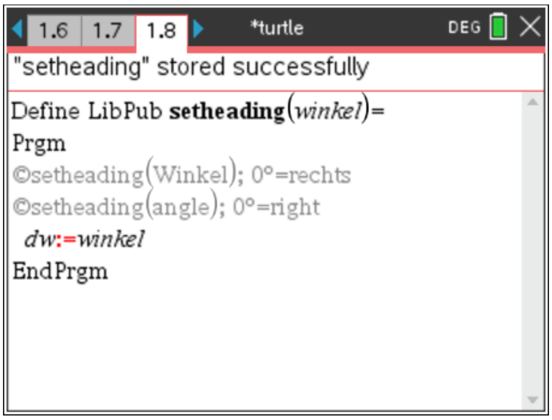


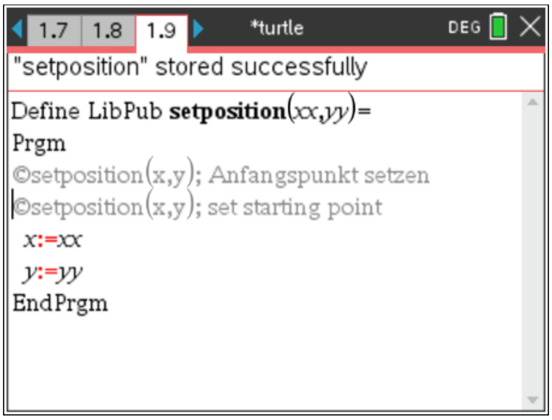


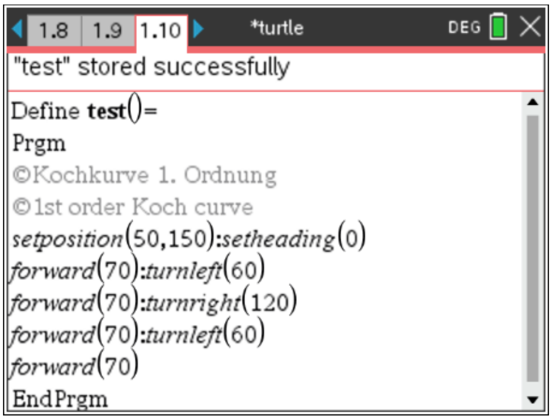


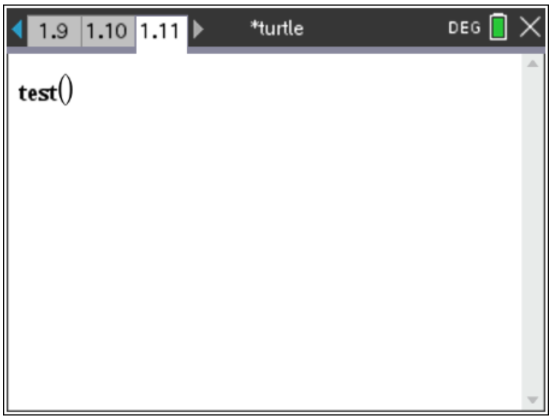


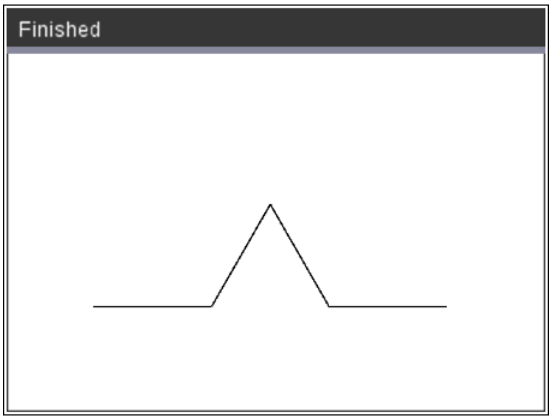




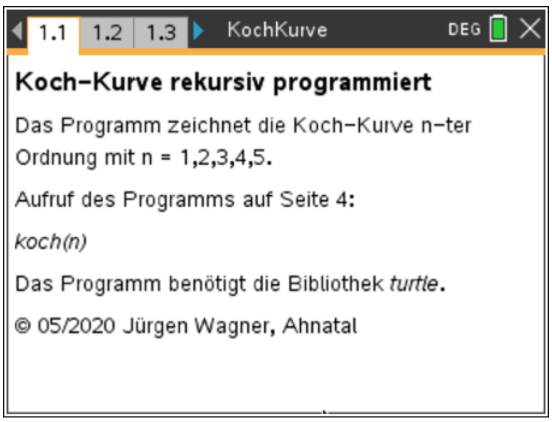


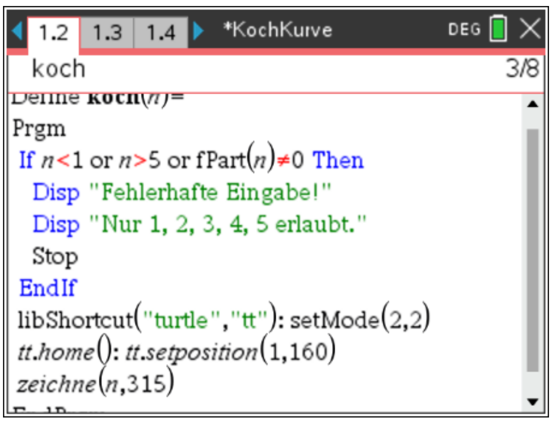


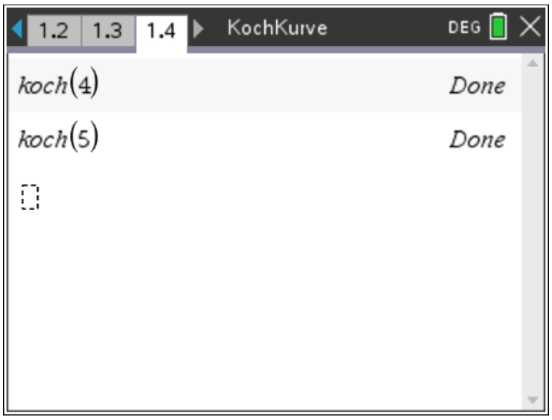


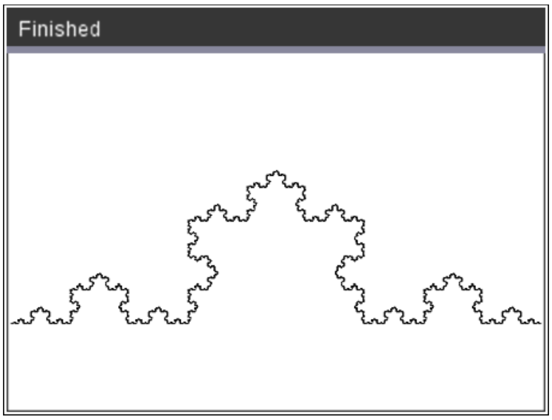


First application is – what else, the Koch- or Snowflake Curve:

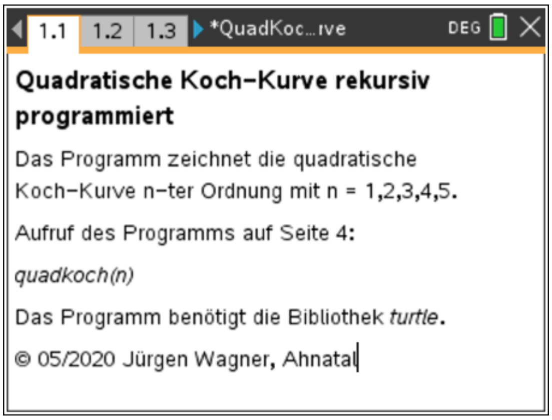


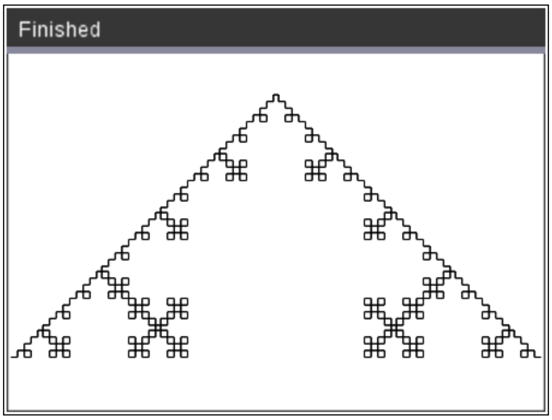




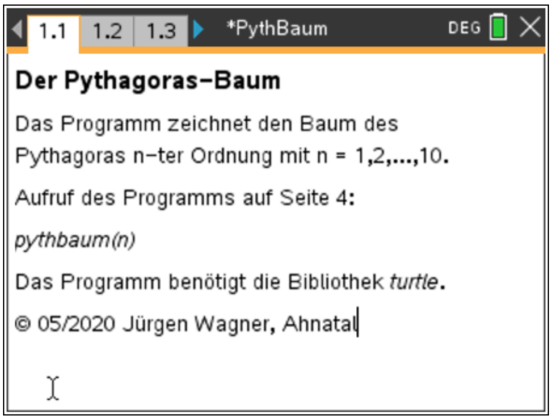


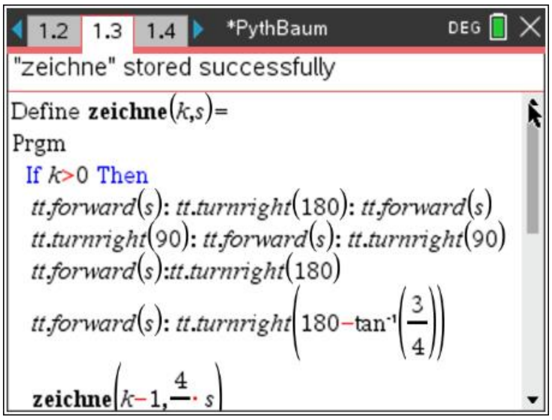
The quadratic Koch curve:

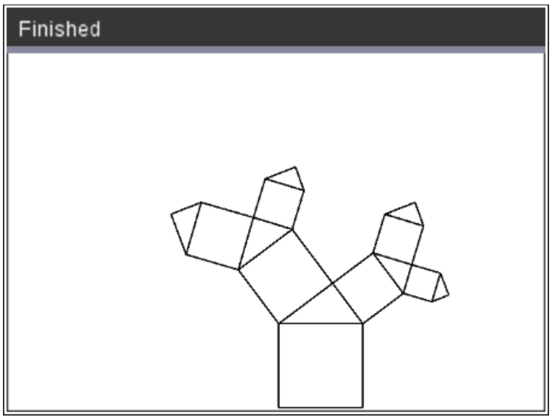


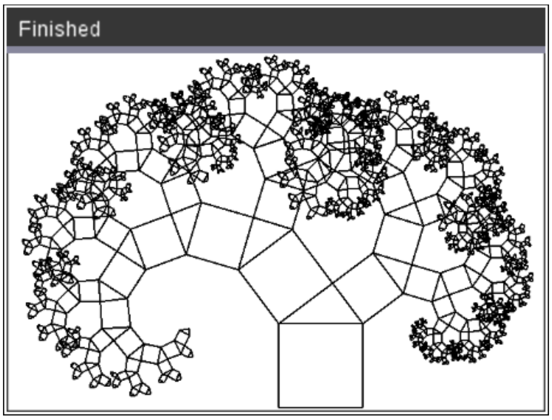


The Pythagoras Tree with pythbaum(3) and pythbaum(11):

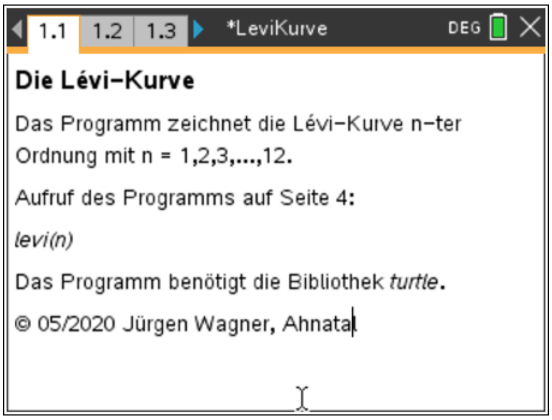


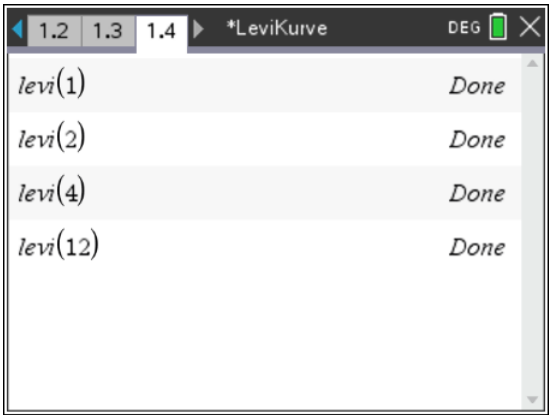


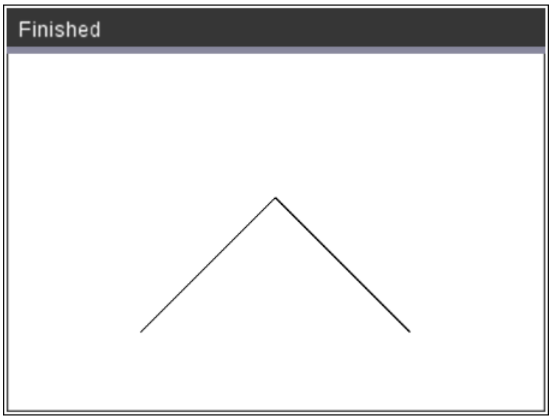


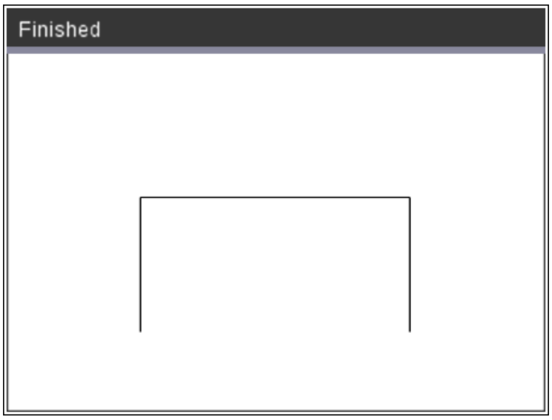


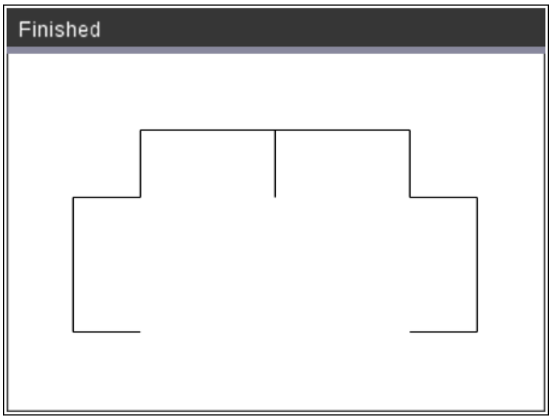
Then the Lévi C curve with some iteration steps:

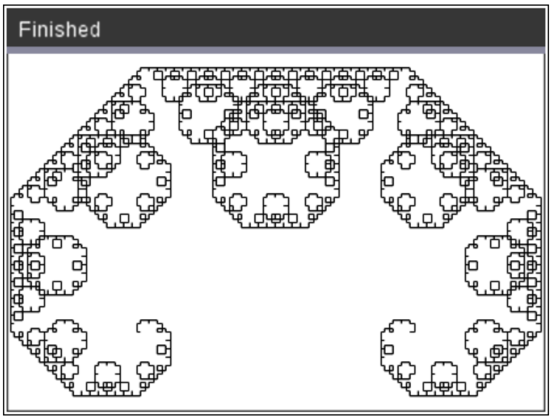




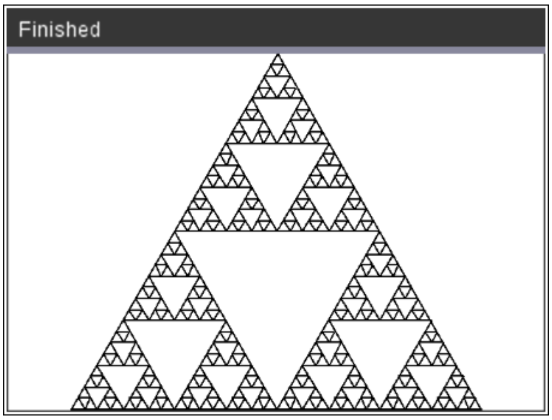


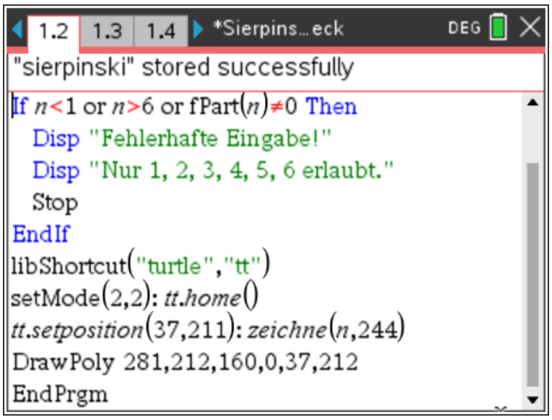






Finally, we will not miss the Sierpinski triangle:





Two programs for pythbaum

```
Define pythbaum(n)=
Prgm
: If n<1 or n>10 or fPart(n)≠0 Then
:   Disp "Fehlerhafte Eingabe!"
:   Disp "Nur 1, 2, 3, ... ,10 erlaubt."
:   Stop
: EndIf
: libShortcut("turtle","tt")
: setMode(2,2)
: tt.home()
: tt.setposition(160,210)
: zeichne(n,50)
:EndPrgm

Define zeichne(k,s)=
Prgm
:  If k>0 Then
:   tt.forward(s)
:   tt.turnright(180)
:   tt.forward(s)
:   tt.turnright(90)
:   tt.forward(s)
:   tt.turnright(90)
:   tt.forward(s)
:   tt.turnright(180)
:   tt.forward(s)
:   tt.turnright(180−tan⁻¹(((3)/(4))))
:   zeichne(k−1,((4)/(5))*s)
:   tt.turnright(90)
:   zeichne(k−1,((3)/(5))*s)
:   tt.turnright(tan⁻¹(((3)/(4))))
:   tt.forward(s)
:   tt.turnleft(90)
:  Else
:   tt.forward(s)
: EndIf
:EndPrgm
```

Two programs for levi

```
Define levi(n)=
Prgm
: If n<1 or n>12 or fPart(n)≠0 Then
:   Disp "Fehlerhafte Eingabe!"
:   Disp "Nur 1, 2, 3, ... , 12 erlaubt."
:   Stop
: EndIf
: libShortcut("turtle","tt")
: setMode(2,2)
: tt.home()
: tt.setposition(78,165)
: zeichne(n,160)
:EndPrgm

Define zeichne(k,s)=
Prgm
:  If k>0 Then
:   tt.turnleft(45)
:   zeichne(k−1,((√(2))/(2))*s)
:   tt.turnright(90)
:   zeichne(k−1,((√(2))/(2))*s)
:   tt.turnleft(45)
:  Else
:   tt.forward(s)
: EndIf
:EndPrgm
```

The Herrnhuter Star

Josef Böhm, Würmla, Österreich

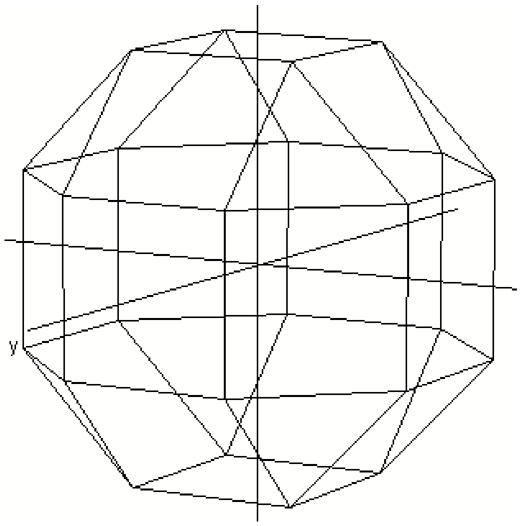
The famous Herrnhuter Star is an interesting figure, not only because it looks so beautiful, but also because it offers a lot of inspiring mathematical activities.

From the point of a geometrical view this star is based on a *Rhombocuboctahedron*. This polyhedron consists of 18 squares and 8 equilateral triangles with all edges of equal length. The solid can be created from six regular octagons and looks like given below. (All figures to be presented in the following are drawn with edge length a = 3).

The next DERIVE command gives a regular octagon lying in a horizontal plane in a distance of 1.5 below the *xy*-plane. (The value for r gives an edge length of a = 3.)

$$r := \frac{3\cdot\sqrt{(2\cdot\sqrt{2} + 4)}}{2}$$

$$m1 := \mathrm{VECTOR}\left(\left[r\cdot\mathrm{COS}\left(\frac{\pi}{4}\cdot k + \frac{\pi}{8}\right),\ r\cdot\mathrm{SIN}\left(\frac{\pi}{4}\cdot k + \frac{\pi}{8}\right),\ -\frac{3}{2}\right],\ k,\ 0,\ 8\right)$$



The other five octahedrons are produced in a similar way. Parallel octahedrons have the distance a = 3.

Then all 26 vertices are defined and combined to squares and triangles taking care of the orientation – this for later purposes.

$$\left[p1 := m1_1,\ p2 := m1_2,\ p3 := m1_3,\ p4 := m1_4,\ p5 := m1_5,\ p6 := m1_6,\ p7 := m1_7,\ p8 := m1_8\right]$$

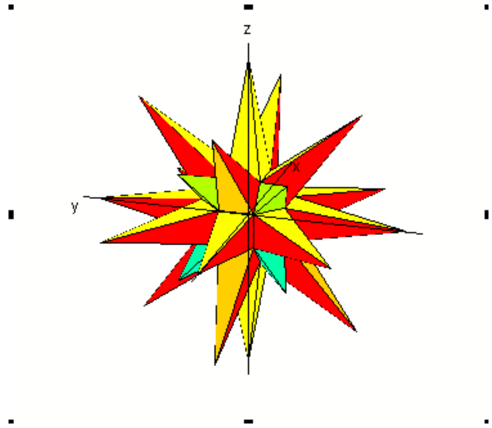
```
[sq1 := [p1, p2, q2, q1], sq2 := [p2, p3, q3, q2], sq3 := [p3, p4, q4, q3], ....

[tr1 := [s7, p2, p1], tr2 := [s6, p4, p3], tr3 := [r6, p6, p5], ....
```

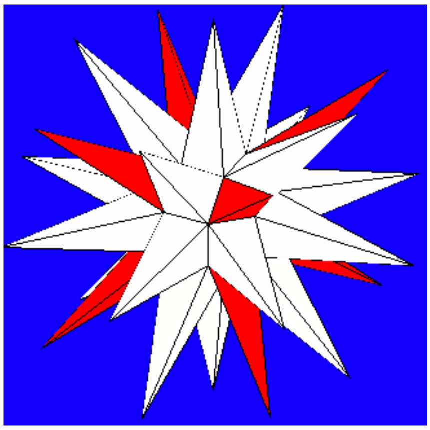
Next two functions generate pyramids above the squares and triangles with variable altitudes h and hh. When the plots will be performed sliders will be introduced to vary the height of the spikes.

```
pyrs(sq_, h, m, n, s) :=
  Prog
    m := (sq_↓1 + sq_↓3)/2
    n := (sq_↓2 - sq_↓1) × (sq_↓3 - sq_↓1)
    n := n/ABS(n)
    s := m + h·n
    [s, sq_↓1; s, sq_↓2; s, sq_↓3; s, sq_↓4; s, sq_↓1]

pyrtr(tr_, hh, m, n, s) :=
  Prog
    m := (tr_↓1 + tr_↓2 + tr_↓3)/3
    n := (tr_↓2 - tr_↓1) × (tr_↓3 - tr_↓1)
    n := n/ABS(n)
    s := m + hh·n
    [s, tr_↓1; s, tr_↓2; s, tr_↓3; s, tr_↓1]
```

Next two expressions can immediately be plotted after having installed slider bars for *h* and *hh*.

```
VECTOR(pyrs(s_), s_, squares)

VECTOR(pyrtr(t_), t_, triangles)
```

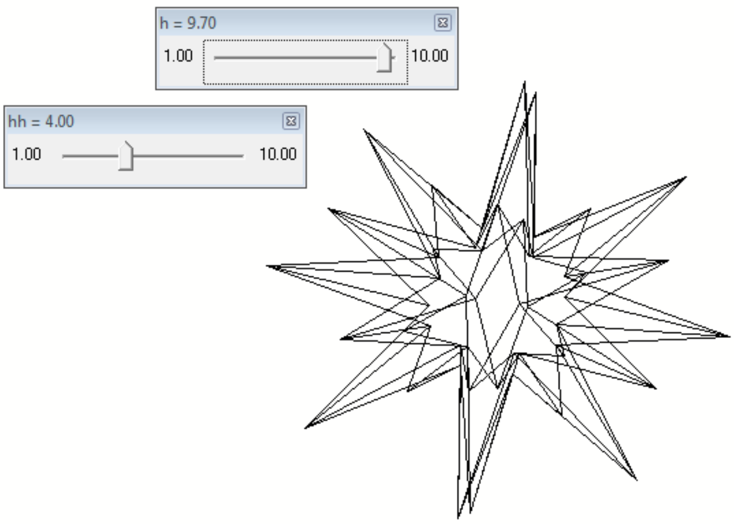




You will be rewarded with beautiful pictures for your efforts.

The stars can be modified in their appearance what concerns the length of the spikes and then turned around by using the mouse or the keyboard.

If you want to see the wire frame model, then you have to change the above functions slightly – and then you will receive:



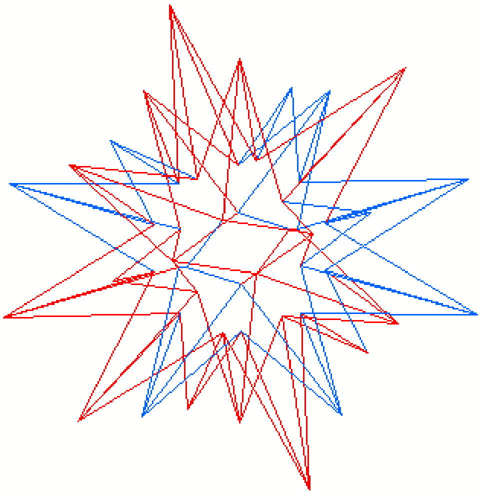
It is a good exercise to use the symmetries of the polyhedron to create the whole body from one four-sided and one three-sided pyramid applying rotation matrices. For example:

$$\text{VECTOR}\left(\left[\text{pyrs\_w(sq2, 8)}\cdot\text{ROTATE\_X}\left(\frac{k\cdot\pi}{4}\right)\right],\ k,\ 0,\ 7\right)$$

$$\text{VECTOR}\left(\left[\text{pyrtr\_w(tr1, 4)}\cdot\text{ROTATE\_Z}\left(\frac{k\cdot\pi}{2}\right)\right],\ k,\ 0,\ 3\right)$$

$$\text{VECTOR}\left(\left[\text{pyrtr\_w(tr1, 4)}\cdot\text{ROTATE\_X}\left(-\ \frac{\pi}{2}\right)\cdot\text{ROTATE\_Z}\left(\frac{k\cdot\pi}{2}\right)\right],\ k,\ 0,\ 3\right)$$

The first expression gives the result of eight rotations of the square-based pyramids around the *x*-axis by 45°, the three-sided pyramids are rotated by 90° around the *z*-axis. Additional rotations are necessary to achieve a plot like this:



Four-sided spikes in red, three-sided ones in blue.

Now the mathematical part will start:

A table found in Wikipedia shall serve as reference:

Größen eines Rhombenkuboktaeders mit Kantenlänge *a*	
Volumen **≈ 8,71 a³**	$V = \frac{2}{3}a^3\,(6+5\sqrt{2})$
Oberflächeninhalt **≈ 21,46 a²**	$A_O = 2\,a^2\,(9+\sqrt{3})$
Umkugelradius **≈ 1,4 a**	$R = \frac{a}{2}\sqrt{5+2\sqrt{2}}$
Kantenkugelradius **≈ 1,31 a**	$r = \frac{a}{2}\sqrt{4+2\sqrt{2}}$
Flächenwinkel (Quadrat–Quadrat) = 135°	$\cos\,\alpha_1 = -\frac{1}{2}\sqrt{2}$
Flächenwinkel (Quadrat–Trigon) ≈ 144° 44′ 8″	$\cos\,\alpha_2 = -\sqrt{\frac{2}{3}}$
Eckenraumwinkel **≈ 1,108 π**	$\Omega = 2\pi - \arccos\left(-\frac{2}{3}\sqrt{2}\right)$

The surface is easy work to calculate. The volume is to be composed from 18 four-sided and 8 three-sided pyramids, all of them inward facing with their vertex in the center (= origin). Bases are squares and equilateral triangles with sides *a*. Then the altitudes of the pyramids are needed and their side edges = distance of a base vertex of the center of the solid.

The side edge *s* is given as:

$$s := \frac{a\cdot\sqrt{(2\cdot\sqrt{2}+5)}}{2}$$

Then the altitudes can be calculated …

$$\left(h4 := \sqrt{\left(\left(\frac{a\cdot\sqrt{(2\cdot\sqrt{2}+5)}}{2}\right)^2 - \left(\frac{a\cdot\sqrt{2}}{2}\right)^2\right)}\right) = h4 := \frac{(\sqrt{2}+1)\cdot|a|}{2}$$

$$\left(h3 := \sqrt{\left(\left(\frac{a\cdot\sqrt{(2\cdot\sqrt{2}+5)}}{2}\right)^2 - \left(\frac{a\cdot\sqrt{3}}{3}\right)^2\right)}\right) = h3 := \frac{\sqrt{3}\cdot(\sqrt{2}+3)\cdot|a|}{6}$$

… and finally, the total volume:

$$\frac{18\cdot a^2\cdot a\cdot\left(\frac{\sqrt{2}}{2}+\frac{1}{2}\right)}{3} + \frac{\frac{8\cdot a^2}{4}\cdot\sqrt{3}\cdot a\cdot\left(\frac{\sqrt{6}}{6}+\frac{\sqrt{3}}{2}\right)}{3} = \frac{2\cdot a^3\cdot(5\cdot\sqrt{2}+6)}{3}$$

The radius of the circumscribed sphere is the distance of a vertex to the center (= side edge of the inwards oriented pyramids *s*) and the radius of an "edge sphere" is the distance of an edge to the center:

$$\left\|\left[a\cdot\left(\frac{\sqrt{2}}{2}+\frac{1}{2}\right), \frac{a}{2}, \frac{a}{2}\right]\right\| = \frac{\sqrt{(2\cdot\sqrt{2}+5)}\cdot|a|}{2}$$

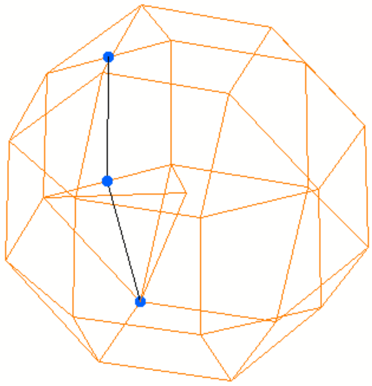
$$\left\|\left[a\cdot\left(\frac{\sqrt{2}}{2}+\frac{1}{2}\right), 0, \frac{a}{2}\right]\right\| = \frac{\sqrt{(2\cdot\sqrt{2}+4)}\cdot|a|}{2}$$

The plane angle between two neighboring squares is the exterior angle of a regular octagon. For calculating the angle between square and triangle one needs three points: the vertex of a triangle S7 and the midpoints oft wo sides (see sketch):

$$\mathrm{ARCCOS}\left(\frac{\left(s7-\frac{p2+p1}{2}\right)\cdot\left(\frac{q2+q1}{2}-\frac{p2+p1}{2}\right)}{\left|s7-\frac{p2+p1}{2}\right|\cdot\left|\frac{q2+q1}{2}-\frac{p2+p1}{2}\right|}\right)$$

$$\mathrm{ARCCOS}\left(-\frac{\sqrt{6}}{3}\right)$$

144.7356103



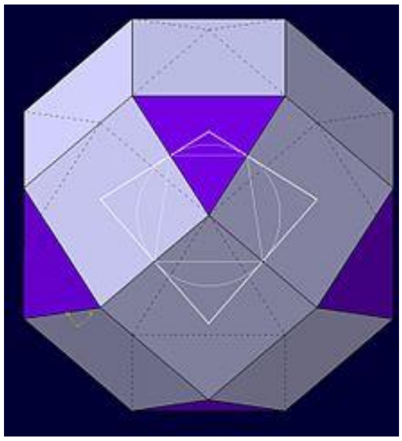
That' fine, but now it will become really exciting:

In Wikipedia and in Weisstein's Concise Encyclopedia as well, I found the body which is dual to the Rhombiocuboctahedron: the *Deltoidalicositetrahedron* (pretty name, isn't it?). It consists of 24 side faces (corresponding to the 24 vertices of the RhCO), which all are deltoids with same sides – but with the different orientation.

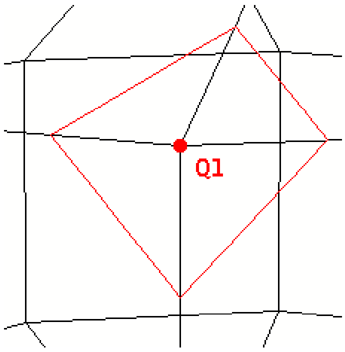
These deltoids are generated as follows: Connect the midpoints of four edges meeting in one vertex point giving an isosceles trapezium with three sides of equal length.

The circumcircle of this trapezium is the incircle of a deltoid. All deltoids produced in this way form the requested solid.

(See the illustration from Wikipedia.)



To support this, vertex Q1 is plotted together with edges to P1, Q2, S2 and Q8 and their respective midpoints. They – points T1-T4 form the trapezium.



The lengths of the sides are calculated and result as $a\sqrt{2}/2$, $a\sqrt{2}/2$, $a\sqrt{2}/2$ and $a/2$ with a being the edge length of the initial rhombicuboctahedron. All plots are performed with a = 3. The relationship trapezium-deltoid is investigated in the plane. The results obtained shall be transferred to the 3D-spatial figure.

Draw the trapezium with a = 3 in the plane and calculate the measures for the deltoid connected with it.

T1 = right base point is clear: [a√2/4]. Point T2 on the top right is calculated. Its x-coordinate is $a/4$. So, a circle with its center in T4 and radius $a\sqrt{2}/2$ is to be intersected with $x = a/4$.

$$\mathrm{SOLVE}\left(x = \frac{a}{4} \wedge \left(x - \frac{a\cdot\sqrt{2}}{4}\right)^2 + y^2 = \left(\frac{a\cdot\sqrt{2}}{2}\right)^2,\ [x,\ y]\right)$$

$$\left(x = \frac{a}{4} \wedge y = -\frac{a\cdot\sqrt{(2\cdot\sqrt{2} + 5)}}{4}\right) \vee \left(x = \frac{a}{4} \wedge y = \frac{a\cdot\sqrt{(2\cdot\sqrt{2} + 5)}}{4}\right)$$

These are the general vertices of the trapezium:

$$\left[t1\_ := \left[\frac{\sqrt{2}\cdot a}{4},\ 0\right],\ t4\_ := \left[-\frac{\sqrt{2}\cdot a}{4},\ 0\right],\ t2\_ := \left[\frac{a}{4},\ \frac{a\cdot\sqrt{(2\cdot\sqrt{2} + 5)}}{4}\right],\ t3\_ := \left[-\frac{a}{4},\ \frac{a\cdot\sqrt{(2\cdot\sqrt{2} + 5)}}{4}\right]\right]$$

The intersection point of the diagonal from top right to bottom left with the y-axis gives the intersection point of the diagonals DD. It will be used later.

$$dd := \left[0,\ \frac{a\cdot\sqrt{(14 - 8\cdot\sqrt{2})}}{4}\right]$$

The formulae for the circumcircle of a trapezium are:

Radius: $r=\dfrac{1}{4A}\sqrt{(ab+cd)(ac+bd)(ad+bc)}$

Area A: $A=\sqrt{s(s-a)(s-b)(s-c)(s-d)}$ with $s=\dfrac{a+b+c+d}{2}$

This is easy work supported by CAS and leads to the radius $r\_$ of the circumcircle:

$$r\_ := \frac{\sqrt{(34\cdot\sqrt{2}+204)}\cdot|a|}{34}$$

Its center is the intersection point of the perpendicular bisector of a side and the y-axis:

$$\left[0,\ \frac{a\cdot\sqrt{(136\cdot\sqrt{2}+238)}}{68}\right]$$

The circumcircle can now be plotted (with a = 3, of course).:

$$x^2+\left(y-\frac{a\cdot\sqrt{(136\cdot\sqrt{2}+238)}}{68}\right)^2=r\_^2$$

The vertices of the deltoid are the intersection points of the tangents to the circumcircle of the trapezium in its vertices. Two of these tangents are presented:

$$ta1 := \frac{x\cdot a\cdot\sqrt{2}}{4}+\left(y-\frac{a\cdot\sqrt{(136\cdot\sqrt{2}+238)}}{68}\right)\cdot\left(0-\frac{a\cdot\sqrt{(136\cdot\sqrt{2}+238)}}{68}\right)=r\_^2$$

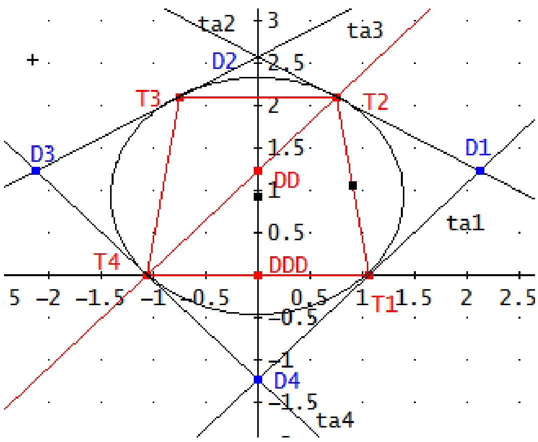
$$ta4 := -\frac{x\cdot a\cdot\sqrt{2}}{4}+\left(y-\frac{a\cdot\sqrt{(136\cdot\sqrt{2}+238)}}{68}\right)\cdot\left(0-\frac{a\cdot\sqrt{(136\cdot\sqrt{2}+238)}}{68}\right)=r\_^2$$

Their intersection point (here D4) is one vertex of the deltoid.

$$(\mathrm{SOLUTIONS}(ta1 \wedge ta4,\ [x,\ y]))_1=\left[0,\ -\frac{a\cdot\sqrt{(14-8\cdot\sqrt{2})}}{4}\right]$$

$$d4 := \left[0,\ -\frac{a\cdot\sqrt{(14-8\cdot\sqrt{2})}}{4}\right]$$

All this can – and should – be plotted in the 2D-plot window accompanying the calculation. Any errors in thinking or calculation can immediately be detected and corrected.



We proceed calculating the measures of the deltoid. Another Wikipedia (German) table serves as reference again. (In this table is *a* the longer side of the deltoid.)

It can be seen that D1, D3 and the intersection point of the diagonals have the same *y*-value.

With the deltoid points already known it is easy work to calculate the lengths of the sides and diagonals of the deltoid.

Sides *a* and *b*, and both diagonals *e* and *f*:

(Here *a* is always the edge of the RhCO.)

Größen des Drachenvierecks	
Seitenverhältnis	$b = \frac{1}{7}(4+\sqrt{2})\,a$
Flächeninhalt	$A = \frac{a^2}{14}\sqrt{61+38\sqrt{2}}$
Inkreisradius	$r = \frac{a}{2}\sqrt{\frac{7+4\sqrt{2}}{17}}$
1. Diagonale	$e = \frac{a}{2}\sqrt{4+2\sqrt{2}}$
2. Diagonale	$f = \frac{a}{7}\sqrt{46+15\sqrt{2}}$
Spitze Winkel (3) ≈ 81° 34′ 44″	$\cos\alpha = \frac{1}{4}(2-\sqrt{2})$
Stumpfer Winkel (1) ≈ 115° 15′ 47″	$\cos\beta = -\frac{1}{8}(2+\sqrt{2})$

$$(b\_d := |d2 - d1|) = b\_d := \frac{2\cdot\sqrt{(10-\sqrt{2})}\cdot|a|}{7}$$

$$(a\_d := |d4 - d1|) = a\_d := \sqrt{(4 - 2\cdot\sqrt{2})}\cdot|a|$$

$$(e\_d := |d1 - d3|) = e\_d := \sqrt{2}\cdot|a|$$

$$(f\_d := |d2 - d4|) = f\_d := \frac{2\cdot\sqrt{(31-8\cdot\sqrt{2})}\cdot|a|}{7}$$

It can be shown that these results agree completely with those given in the table.

The English Wikipedia table does not provide these tables. Here we find only the relationships of both deltoid sides *a* and *b* to the diagonals *e* and *f*. (e.g. for *a*):

Wikipedia: e_d = a_d/2*√(4+2√2) und f_d = a_d/7*√(46+15√2)

$$\frac{a\_d}{2}\cdot\sqrt{(4+2\cdot\sqrt{2})} = \sqrt{2}\cdot|a|$$

$$\frac{a\_d}{7}\cdot\sqrt{(46+15\cdot\sqrt{2})} = \frac{2\cdot\sqrt{(31-8\cdot\sqrt{2})}\cdot|a|}{7}$$

Later we will need the distances of the midpoint of the trapezium base to the vertices of the deltoid:

$$|d2 - dd| = \frac{\sqrt{(62-16\cdot\sqrt{2})}\cdot|a|}{14} \quad |d4 - dd| = \frac{\sqrt{(14-8\cdot\sqrt{2})}\cdot|a|}{2}$$

The area of the deltoid is still missed. It is calculated applying the classical formula using both diagonals:

$$\frac{f\_d\cdot e\_d}{2} = \frac{a^2\cdot\sqrt{(62-16\cdot\sqrt{2})}}{7}$$

The Wikipedia table presents the area as a function of side *a*:

$$\frac{a\_d^2}{14}\cdot\sqrt{(61 + 38\cdot\sqrt{2})} = \frac{a^2\cdot\sqrt{(62 - 16\cdot\sqrt{2})}}{7}$$

At the end of the table are the angles of the deltoid.
(I did not calculate the "Eckenraumwinkel = spatial angle in the vertices):

$$\mathrm{ARCCOS}\left(\frac{(d4 - d1)\cdot(d4 - d3)}{|d4 - d1|\cdot|d4 - d3|}\right) = 81.57894188$$

$$\mathrm{ARCCOS}\left(\frac{(d2 - d1)\cdot(d2 - d3)}{|d2 - d1|\cdot|d2 - d3|}\right) = 115.2631743$$

All values fit and they are to be transferred into space in order to construct the first deltoid around the trapezium.

As said above, the trapezium vertices are the midpoints of the edges starting in Q1. They are denoted with TT1_3, TT2_3, TT3_3 and TT4_3.(`[t1_3,t2_3,t3_3,t4_3,t1_3]` for plotting)

The midpoint DDD3 of the trapezium base in space is determined. And DD3 as intersection point of the diagonals (TT1_TT3_ ∩ TT2_TT4_) in parameter form:

$$ddd3 := \left[a\cdot\left(\frac{\sqrt{2}}{2} + \frac{1}{2}\right), \frac{a}{4}, \frac{a}{4}\right]$$

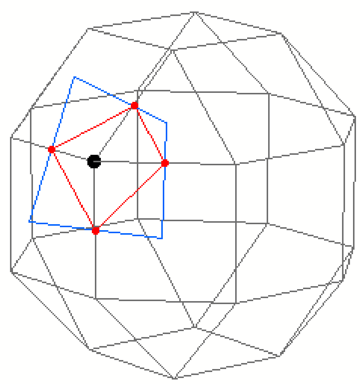
$$(dd3 := \mathrm{SUBST}(diag1\_, u, \sqrt{2} - 1)) = dd3 := \left[a, \frac{a}{2}, \frac{a}{2}\right]$$

The distance between DD and D2 is transferred to the diagonal in space:

$$D2\_3 := dd3 + \frac{\frac{dd3 - ddd3}{|dd3 - ddd3|}\cdot\sqrt{(62 - 16\cdot\sqrt{2})}\cdot|a|}{14}$$

$$D2\_3 := \left[a\cdot\left(\frac{\sqrt{2}}{7} + \frac{4}{7}\right), a\cdot\left(\frac{\sqrt{2}}{7} + \frac{4}{7}\right), a\cdot\left(\frac{\sqrt{2}}{7} + \frac{4}{7}\right)\right]$$

The remaining three points are constructed in a similar way and one obtains the first deltoid (*a* = 3!). The figure shows the initial trapezium together with the deltoid derived from it.



Having composed the first deltoid some spatial imagination might be helpful. Symmetries of the figure are applied and more deltoids are obtained by rotation around the z-axis – and simultaneously plotted.

.

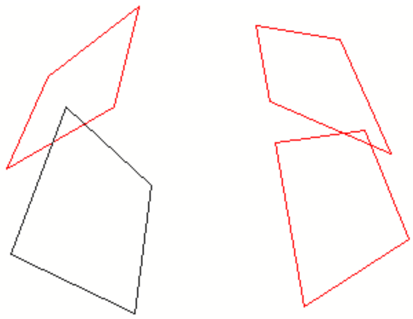
delt1 := [D1_3, D2_3, D3_3, D4_3, D1_3]

$$\text{delt1} := \begin{bmatrix} 3 & 3 & 0 \\ \dfrac{3\cdot\sqrt{2}}{7}+\dfrac{12}{7} & \dfrac{3\cdot\sqrt{2}}{7}+\dfrac{12}{7} & \dfrac{3\cdot\sqrt{2}}{7}+\dfrac{12}{7} \\ 3 & 0 & 3 \\ 3\cdot\sqrt{2} & 0 & 0 \\ 3 & 3 & 0 \end{bmatrix}$$

$$\text{VECTOR}\left(\left[\text{delt1}\cdot\text{ROTATE\_Z}\left(\frac{k\cdot\pi}{2}\right)\right],\ k,\ 4\right)$$

The right figure shows the first deltoid (black) together with the additional ones obtained by rotation (red).

Step by step we can watch the polyhedron growing and taking its final shape.

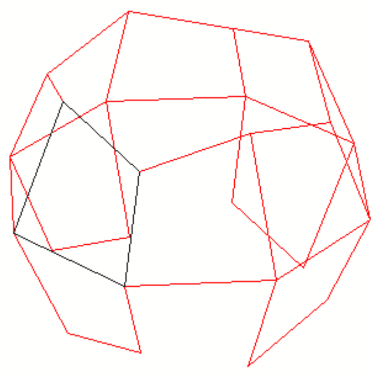


The surface consists of two families of deltoids with different orientation. So, it is a task to find out, which deltoids fit together stepwise when be rotated. This is some fiddling, but controlling by simultaneous plotting in the 3D-plot window it can be done without any problems. The second family of deltoids completes the wreath in the upper half of the figure.

$$\text{delt2} := \begin{bmatrix} 3\cdot\sqrt{2} & 0 & 0 \\ 3 & -3 & 0 \\ \dfrac{3\cdot\sqrt{2}}{7}+\dfrac{12}{7} & -\left(\dfrac{3\cdot\sqrt{2}}{7}+\dfrac{12}{7}\right) & \dfrac{3\cdot\sqrt{2}}{7}+\dfrac{12}{7} \\ 3 & 0 & 3 \\ 3\cdot\sqrt{2} & 0 & 0 \end{bmatrix}$$

$$\text{VECTOR}\left(\left[\text{delt2}\cdot\text{ROTATE\_Z}\left(\frac{k\cdot\pi}{2}\right)\right],\ k,\ 4\right)$$

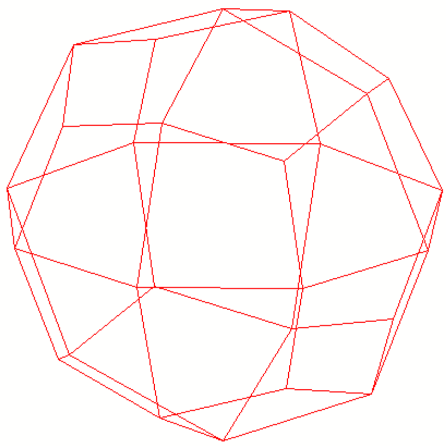
This is a picture of the state during building the polyhedron.



The next expression defines the bottom wreath and both following blocks complete the figure on top and bottom. Ready!

$$\left[\mathrm{VECTOR}\left(\left[\mathrm{delt3}\cdot\mathrm{ROTATE\_Z}\left(\frac{k\cdot\pi}{2}\right)\right],\ k,\ 4\right),\ \mathrm{VECTOR}\left(\left[\mathrm{delt4}\cdot\mathrm{ROTATE\_Z}\left(\frac{k\cdot\pi}{2}\right)\right],\ k,\ 4\right)\right]$$

$$\left[\left[\begin{array}{c} \mathrm{delt1}\cdot\mathrm{ROTATE\_Y}\left(\frac{\pi}{2}\right) \\ \mathrm{delt1}\cdot\mathrm{ROTATE\_Y}\left(\frac{3\cdot\pi}{2}\right) \\ \mathrm{delt2}\cdot\mathrm{ROTATE\_Y}\left(\frac{\pi}{2}\right) \\ \mathrm{delt2}\cdot\mathrm{ROTATE\_Y}\left(\frac{3\cdot\pi}{2}\right) \end{array}\right],\ \left[\begin{array}{c} \mathrm{delt3}\cdot\mathrm{ROTATE\_Y}\left(\frac{\pi}{2}\right) \\ \mathrm{delt3}\cdot\mathrm{ROTATE\_Y}\left(\frac{3\cdot\pi}{2}\right) \\ \mathrm{delt4}\cdot\mathrm{ROTATE\_Y}\left(\frac{\pi}{2}\right) \\ \mathrm{delt4}\cdot\mathrm{ROTATE\_Y}\left(\frac{3\cdot\pi}{2}\right) \end{array}\right]\right]$$



What is missed is the calculation of the measures of the solid – with a table from Wikipedia as a very welcome support. All measures are given as functions of the deltoid sides *a* and *b*. I remain expressing them as functions of the edge length *a* of the original rhombicuboctahedron.

Größen eines regelmäßigen Deltoidikositetraeders mit Kantenlänge *a* bzw. *b*	
Volumen **≈ 6,9a³ ≈ 14,91b³**	$V = \frac{2}{7}\,a^3\sqrt{292+206\sqrt{2}} = b^3\sqrt{122+71\sqrt{2}}$
Oberflächeninhalt **≈ 18,36a² ≈ 30,69b²**	$A_O = \frac{12}{7}\,a^2\sqrt{61+38\sqrt{2}} = 6\,b^2\sqrt{29-2\sqrt{2}}$
Inkugelradius	$\rho = a\sqrt{\frac{22+15\sqrt{2}}{34}} = \frac{b}{2}\sqrt{\frac{78+47\sqrt{2}}{17}}$
Kantenkugelradius	$r = \frac{a}{2}\,(1+\sqrt{2}) = \frac{b}{4}\,(2+3\sqrt{2})$
Flächenwinkel **≈ 138° 7′ 5″**	$\cos\,\alpha = -\frac{1}{17}\,(7+4\sqrt{2})$
3D-Kantenwinkel **= 135°**	$\cos\,\gamma = -\frac{1}{2}\sqrt{2}$

Calculation of the area of the surface is easy. I will demonstrate its conversion to a function of the deltoidal edges *a* (= *s_a*) and *b* (= *s_b*):

$$\frac{24\cdot e\_d\cdot f\_d}{2} = \frac{24\cdot a^2\cdot\sqrt{(62 - 16\cdot\sqrt{2})}}{7}$$

$$s\_a = a\_d$$

$$s\_a = \sqrt{(4 - 2\cdot\sqrt{2})}\cdot|a|$$

$$\text{SOLVE}(s\_a = \sqrt{(4 - 2\cdot\sqrt{2})}\cdot a,\ a) = \left[a = \frac{s\_a\cdot\sqrt{(\sqrt{2} + 2)}}{2}\right]$$

$$\text{SUBST}\left(\frac{24\cdot a^2\cdot\sqrt{(62 - 16\cdot\sqrt{2})}}{7},\ a,\ \frac{s\_a\cdot\sqrt{(\sqrt{2} + 2)}}{2}\right) = \frac{12\cdot s\_a^2\cdot\sqrt{(38\cdot\sqrt{2} + 61)}}{7}$$

$$b\_d = \frac{2\cdot\sqrt{(10 - \sqrt{2})}\cdot|a|}{7}$$

$$\text{SOLVE}\left(s\_b = \frac{2\cdot\sqrt{(10 - \sqrt{2})}\cdot a}{7},\ a\right) = \left[a = \frac{s\_b\cdot\sqrt{(2\cdot\sqrt{2} + 20)}}{4}\right]$$

$$\text{SUBST}\left(\frac{24\cdot a^2\cdot\sqrt{(62 - 16\cdot\sqrt{2})}}{7},\ a,\ \frac{s\_b\cdot\sqrt{(2\cdot\sqrt{2} + 20)}}{4}\right) = 6\cdot s\_b^2\cdot\sqrt{(29 - 2\cdot\sqrt{2})}$$

Finding the volume is a bit more complicated. The solid is composed of 24 inward facing pyramids with deltoids of equal area as base. Their altitude is the distance from the center (= origin) to one face (deltoid).

Das bietet eine schöne Gelegenheit der Anwendung der Vektorrechnung im R3. Man berechnet eine Flächenebene aus drei Punkten und über deren Normalvektor kommt man weiter zum Abstand.

CAS is a very welcome support. See the last steps of calculation. The volume is given as a function of edge length *a* and of deltoid side *s_a* as well.

$$\left[hd := \frac{|n\cdot[0,\ 0,\ 0] - 1|}{|n|}\right] = hd := \frac{\sqrt{(136\cdot\sqrt{2} + 238)}\cdot|a|}{17}$$

$$\frac{\frac{24\cdot e\_d\cdot f\_d}{2}\cdot 1}{3}\cdot hd = \frac{16\cdot(2\cdot\sqrt{2} + 1)\cdot|a|^3}{7}$$

$$\text{SUBST}\left(\frac{16\cdot(2\cdot\sqrt{2} + 1)\cdot a^3}{7},\ a,\ \frac{s\_a\cdot\sqrt{(\sqrt{2} + 2)}}{2}\right) = \frac{2\cdot s\_a^3\cdot\sqrt{(206\cdot\sqrt{2} + 292)}}{7}$$

The radius of the inscribed sphere equals the height *hd* of the pyramids.

Radius of the „edge sphere“ is the distance of a polyhedron edge to the center.

$$\left|\frac{[D3\_3 + D2\_3]}{2}\right| = \frac{\sqrt{(2\cdot\sqrt{2} + 4)}\cdot|a|}{2}$$

$$\text{SUBST}\left(\frac{\sqrt{(2\cdot\sqrt{2} + 4)}\cdot a}{2}, a, \frac{s\_a\cdot\sqrt{(\sqrt{2} + 2)}}{2}\right) = s\_a\cdot\left(\frac{\sqrt{2}}{2} + \frac{1}{2}\right)$$

$$\text{SUBST}\left(\frac{\sqrt{(2\cdot\sqrt{2} + 4)}\cdot a}{2}, a, \frac{s\_b\cdot\sqrt{(2\cdot\sqrt{2} + 20)}}{4}\right) = s\_b\cdot\left(\frac{3\cdot\sqrt{2}}{4} + \frac{1}{2}\right)$$

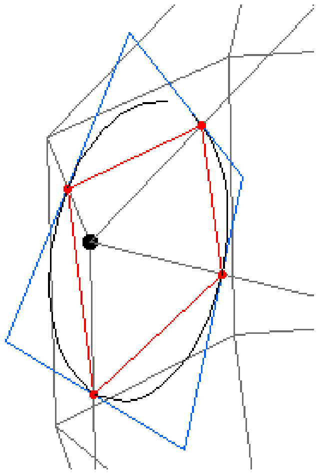
The angle between two face planes is the angle formed by the normal vectors oft wo neighboring deltoids. This is the last calculation step:

$$\text{COS}(\alpha) = \frac{\left[\frac{\sqrt{2}}{6}, \frac{1}{3} - \frac{\sqrt{2}}{6}, \frac{1}{3} - \frac{\sqrt{2}}{6}\right]\cdot\left[\frac{\sqrt{2}}{6}, \frac{\sqrt{2}}{6} - \frac{1}{3}, \frac{1}{3} - \frac{\sqrt{2}}{6}\right]}{\left\|\left[\frac{\sqrt{2}}{6}, \frac{1}{3} - \frac{\sqrt{2}}{6}, \frac{1}{3} - \frac{\sqrt{2}}{6}\right]\right\|\cdot\left\|\left[\frac{\sqrt{2}}{6}, \frac{\sqrt{2}}{6} - \frac{1}{3}, \frac{1}{3} - \frac{\sqrt{2}}{6}\right]\right\|}$$

$$\text{COS}(\alpha) = \frac{4\cdot\sqrt{2}}{17} + \frac{7}{17}$$

How one can see, all fits together. It is done. But I must admit that it took me as a whole some time and effort. The „small“ successes on the way always encouraged me to go on.

And such a project develops a certain momentum. So, I have set in my mind to additionally plot the circumcircle of the trapezium in the 3D plot window. I had success as the next figure is demonstrating.



This is a detail of the DERIVE 3D plot window. It is followed by detail of the parameter form of the circle:

$$\left[\frac{(\sqrt{2} - 2)\cdot\sqrt{(t\cdot(2\cdot\sqrt{2} + 5) - 6\cdot\sqrt{2} - 12)}\cdot\sqrt{(-t)}}{4} - \frac{\sqrt{2}\cdot t}{4} + \frac{3\cdot\sqrt{2}}{2} + \frac{3}{2},\right.$$

$$\left[\frac{\sqrt{2}\cdot((\sqrt{2} - 1)\cdot\sqrt{(t\cdot(2\cdot\sqrt{2} + 5) - 6\cdot\sqrt{2} - 12)}\cdot\sqrt{(-t)} - t + 3\cdot\sqrt{2} + 6)}{4}, - \frac{\sqrt{\ }}{\ }\right.$$

One can see the *x*-coordinate of the two branches of the conic.

It‘s quite possible that this can be done in an easier way, but “I did it My Way“.

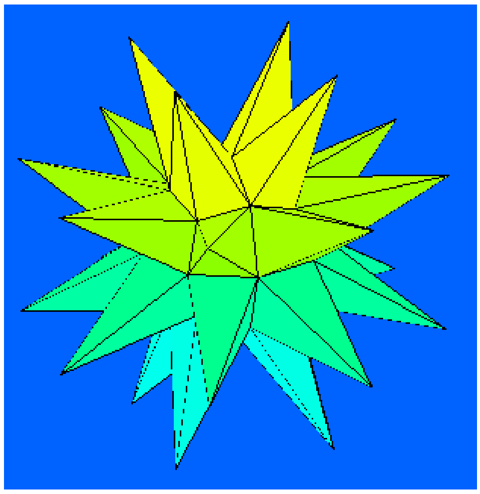
Now, when all is done, the question concerning „duality“ must be answered. In a polyhedron which is dual to another one, every vertex of the first one corresponds to a side face of the other one. Edges correspond to edges. How is this with our solid?

The rhombicuboctahedron has 26 side faces (18 squares and 8 equilateral triangles). This corresponds to 18 vertices with 4 edges of equal length starting and 8 vertices which are starting points of three equally long edges. The 24 vertices of the rhco are starting points where two triangle sides and two square sides meet correspond to 24 side faces (deltoids) with pairwise equally long and short sides.

The polyhedron dual to the deltoidal… is again a rhombicub…, of course. I wonder how the reverse construction could work?

Last comment: There is another idea (see momentum!!). Out from the deltoids spikes could grow – how would this star look like?

Short answer: Look at the star to the right!



My special thank goes to Hubert Langlotz, Frank Liebner, Sebastian Rauh and Dirk Ritschel. I participated a great T^3 webinar organized by them. I learned a lot about Herrnhut, the star and this event inspired me to this paper.

Some interesting websites:

https://en.wikipedia.org/wiki/Herrnhut

https://de.wikipedia.org/wiki/Rhombenkuboktaeder

https://mathworld.wolfram.com/SmallRhombicuboctahedron.html

http://www.mathematische-basteleien.de/rhombenkuboktaeder.htm

https://de.wikipedia.org/wiki/Deltoidalikositetraeder

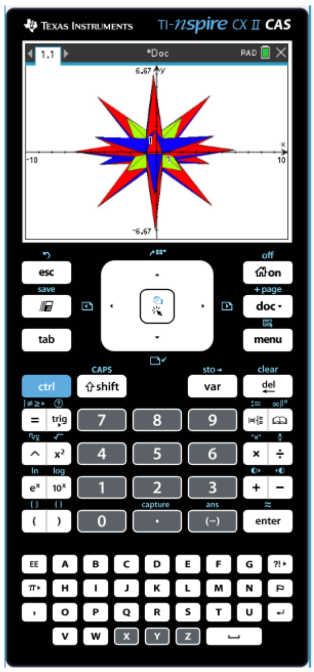
https://en.wikipedia.org/wiki/Deltoidal_icositetrahedron#Dimensions

https://mathworld.wolfram.com/DeltoidalIcositetrahedron.html

https://www.mineralienatlas.de/lexikon/index.php/Ikositetraeder

https://de.wikipedia.org/wiki/Polyeder#Dualit%C3%A4t

https://mathematikalpha.de/duale-polyeder



We cannot plot polyhedrons in the Nspire 3D graph window.
What we can do is, loading the DERIVE star as a background graph in our TI-Nspire.