

**THE DERIVE - NEWSLETTER #129**

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**THE BULLETIN OF THE**



**USER GROUP**

**+ CAS-TI**

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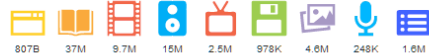
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**March 2023**

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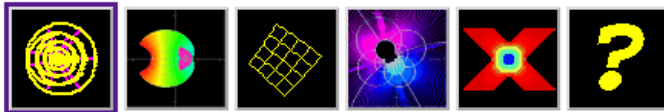




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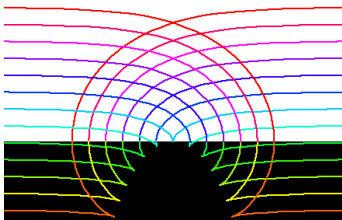
## Graphics for Complex Analysis

<https://www-users.cse.umn.edu/~arnold/complex.html>



## Visual Dictionary of Special Plane Curves

[http://xahlee.info/SpecialPlaneCurves\\_dir/specialPlaneCurves.html](http://xahlee.info/SpecialPlaneCurves_dir/specialPlaneCurves.html)



## Free Math Textbooks

[http://xahlee.info/math/math\\_books.html](http://xahlee.info/math/math_books.html)

## Project Euler

<https://projecteuler.net/>

<https://projekteuler.de/problems>



## FREE Math Help

<https://www.mathgoodies.com/>



## Sing Muse on the Hypotenuse

Influences of Poetry and Rhetoric on the Formation of Greek Mathematics

<https://apostolosdoxiadis.com/wp-content/uploads/2013/07/Sing-Muse-of-the-Hypotenuse1.pdf>

Dear DUG-Members,

Best regards from Easter Sunday 2023. I have just finished this newsletter last week and am happy that I can send it – after once more proof reading – to Walter Wegscheider to set it on our website and make it ready for download.

This was a tough time for us, not only because of the contributions for this DNL but because of private reasons. My wife Noor had a bad accident when we were skiing. She lost the slope in the dense fog, fell unhappily and broke her upper arm. Now she is on the best way to recover.

Ich hatte eine Kontrolluntersuchung wegen meines Lymphoms. It was determined that it had grown again and that I need treatment again, which has started last week.

Please note the links on the opposite page. Archive.org provides thousands of media. You are invited to visit the other websites, too.

We have one more contribution from Michel Beaudin, presenting several exercises. There are some solutions and you might try to find the solutions of the other exercises on you own. I must admit that it is long ago for me treating similar problems. As a teacher for Secondary II students I had nothing to do with sine integral etc. It is a good opportunity for me – and I hope for some of you, too – to call back memories.

Benno Grabinger demonstrates in his contribution not only, that penalty shootout is not really fair but also provides a way how to make it fairer. He uses recursive programming and simulation with TI-NspireCAS. I was able to transfer his code for DERIVE. By the way we can learn about the Thue-Morse sequence. The German version of his contribution is included in MTH129.zip. Don't blame Benno if there are mistakes in the translation, because it was me – supported by DeepL.

DeepL helped also translating Felipe's DERIVE contribution with a mechanic's stuff. I thought it might be nice to have some Spanish in our newsletter. This article is to be continued and finished in our next issue. The TI-Nspire version is also given.

I hope that you enjoy "Plotting Binary Signals". The DERIVE code was not so easy for me and I wonder if there is more elegant one. However, it works – and surprisingly enough – at least for me – it works for TI-Nspire, too. The article offers a good opportunity to introduce Polish notation of logic expressions.

And there we have again a paper from Wolfgang Alvermann. He is always looking for interesting problems that are suitable for school. His problem demonstrates once more that most problems – and even easy ones – having been solved open additional questions or additional ways to the solution. (Original version is in MTH129.zip).

This is what we as teachers should present our students and encourage them to find their own questions (and answers, if possible, of course). But even posing new questions or generalizing the problem is a very mathematical process and – in my opinion – much more valuable than applying any fixed rule – which can be done by the calculator.

In order to have an even number of pages a put some springtime pictures from our garden on the last page.

With my best regards

Yours Josef

The *DERIVE-NEWSLETTER* is the Bulletin of the *DERIVE & CAS-TI User Group*. It is published at least four times a year with a content of 40 pages minimum. The goals of the *DNL* are to enable the exchange of experiences made with *DERIVE*, *TI-CAS* and other CAS as well to create a group to discuss the possibilities of new methodical and didactical manners in teaching mathematics.

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### Contributions:

Please send all contributions to the Editor. Non-English speakers are encouraged to write their contributions in English to reinforce the international touch of the *DNL*. It must be said, though, that non-English articles will be warmly welcomed nonetheless. Your contributions will be edited but not assessed. By submitting articles, the author gives his consent for reprinting it in the *DNL*. The more contributions you will send, the more lively and richer in contents the *DERIVE & CAS-TI Newsletter* will be.

Next issue:

June 2023

### **Preview: Contributions waiting to be published**

Some simulations of Random Experiments, J. Böhm, AUT, Lorenz Kopp, GER  
Wonderful World of Pedal Curves, J. Böhm, AUT  
Simulating a Graphing Calculator in *DERIVE*, J. Böhm, AUT  
Cubics, Quartics – Interesting features, T. Koller & J. Böhm, AUT  
Logos of Companies as an Inspiration for Math Teaching  
Exciting Surfaces in the FAZ, BooleanPlots.mth, P. Schofield, UK  
Old traditional examples for a CAS – What's new? J. Böhm, AUT  
Mandelbrot and Newton with *DERIVE*, Roman Hašek, CZ  
Tutorials for the NSpireCAS, G. Herweyers, BEL  
Dirac Algebra, Clifford Algebra, Vector-Matrix-Extension, D. R. Lunsford, USA  
Another Approach to Taylor Series, D. Oertel, GER  
Charge in a Magnetic Field, H. Ludwig, GER  
More Applications of TI-Innovator™ Hub and TI-Innovator™ Rover  
Surfaces and their Duals, Cayley Symmetroid, J. Böhm, AUT  
A Collection of Special Problems, W. Alvermann, GER  
DERIVE Bugs? D. Welz, GER  
Tweening & Morphing with TI-NspireCX-II-T, J. Böhm, AUT  
The Gap between Poor and Rich, J. Böhm, AUT  
More functions from M. Myers and from Bhuvanesh's Mathtools-library  
TaxiCab Conics, Two alternate Approaches to Conics, R. Haas, USA  
QR-Code light, Random numbers following a given distribution  
Properties of Polygons, Quartiles, F. de Jesús Martínez Vargas, Mexico  
153 is another Special Number, and others

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Here we continue Michel Beaudin's list of exercises connected with his Résumé 1 presented in earlier DNLs.

### Exercise list for Summary 1 – Part 3

**Problem 5** Application of first order differential equations: RL circuit.

The following exercise is a good way to review several concepts. Consider the DE of an RL circuit:

$$L \frac{di}{dt} + Ri = E(t), \quad i(0) = 0.$$

- a) Showing your calculations, solve this first-order linear DE.
- b) Show, when the source  $E(t)$  is constant, say  $E(t) = E_0$ , that the DE is, in addition to being linear, separable. Solve it this way and conclude that the current tends to  $\frac{E_0}{R}$  when  $t$  tends to infinity.

For the rest of the problem, we will assume  $L = 1\text{H}$  and  $R = 5\Omega$ . The source  $E(t)$  (in volts V) will take different forms. Find  $i(t)$  and sketch its graph (for  $t \geq 0$ ) for each of the following situations.

- c)  $E(t) = 25$  (here, you can of course use b))
- d)  $E(t) = \begin{cases} 25 & \text{for } 0 < t < 2 \\ 0 & \text{else} \end{cases}$
- e)  $E(t) = \begin{cases} 25 & \text{for } 0 < t < 2 \\ 0 & \text{for } 2 < t < 4 \\ 25 & \text{for } 4 < t < 6 \\ 0 & \text{for } t > 6 \end{cases}$
- f)  $E(t) = \begin{cases} 25 & \text{for } 0 < t < 2 \\ 0 & \text{for } 2 < t < 4 \end{cases} \quad E(t+4) = E(t)$  (so periodic source, period 4).

- a) Let's solve this first order linear DE. We rewrite the DE as

$$\frac{di}{dt} + \frac{R}{L}i = \frac{1}{L}E(t).$$

An integrating factor is then  $\mu(t) = e^{Rt/L}$ . So

$$\frac{d}{dt}(\mu(t)i(t)) = \frac{1}{L}\mu(t)E(t) \Rightarrow \mu(t)i(t) - \mu(0)i(0) = \frac{1}{L} \int_0^t \mu(\tau)E(\tau) d\tau.$$

Thus

$$i(t) = \frac{e^{-Rt/L}}{L} \int_0^t e^{R\tau/L} E(\tau) d\tau = \frac{1}{L} \int_0^t e^{-R(t-\tau)/L} E(\tau) d\tau = E(t) * \frac{e^{-Rt/L}}{L}.$$

The last equality follows from the definition of convolution: This is to be expected since the initial condition is zero and the impulse response is

$$\mathfrak{F}^{-1} \left[ \frac{1}{sL + R} \right] = \frac{1}{L} \mathfrak{F}^{-1} \left[ \frac{1}{L + R/L} \right] = \frac{1}{L} e^{-Rt/L}.$$

b) In effect, we return to the original DE which becomes  $L \frac{di}{dt} + Ri = E_0$ ,  $i(0) = 0$ . We separate

$$\frac{di}{E_0 - Ri} = \frac{1}{L} dt \Rightarrow \int \frac{1}{E_0 - Ri} di = \frac{1}{L} \int 1 dt + C \Rightarrow -\frac{1}{R} \ln(E_0 - Ri) = \frac{t}{L} + C.$$

Note that we could also have taken as primitive  $-\frac{1}{R} \ln(Ri - E_0)$ . Let's continue by using  $K_1, K_2$  to denote constants from manipulations:

We have  $\ln(E_0 - Ri) = -\frac{Rt}{L} + K_1 \Rightarrow E_0 - Ri = e^{-Rt/L} \cdot K_2$ . But  $i(0) = 0$  force  $K_2 = E_0$  and then

$i(t) = \frac{E_0}{R} (1 - e^{-Rt/L})$ . This tends to  $\frac{E_0}{R}$  when  $t$  tends to infinity.

For the rest of the problem, we assume  $L = 1\text{H}$  and  $R = 5\Omega$ . The source  $E(t)$  (in volts V) will take different forms. Let us find  $i(t)$  and sketch its graph (for  $t \geq 0$ ) for each of the following situations. We will give the calculation details for question e) which generalizes questions c) and d). For question f), we will use convolution.

c)  $E(t) = 25$ : then we find  $i(t) = 5(1 - e^{-5t})u(t)$ . The respective graph is shown in Figure 1.

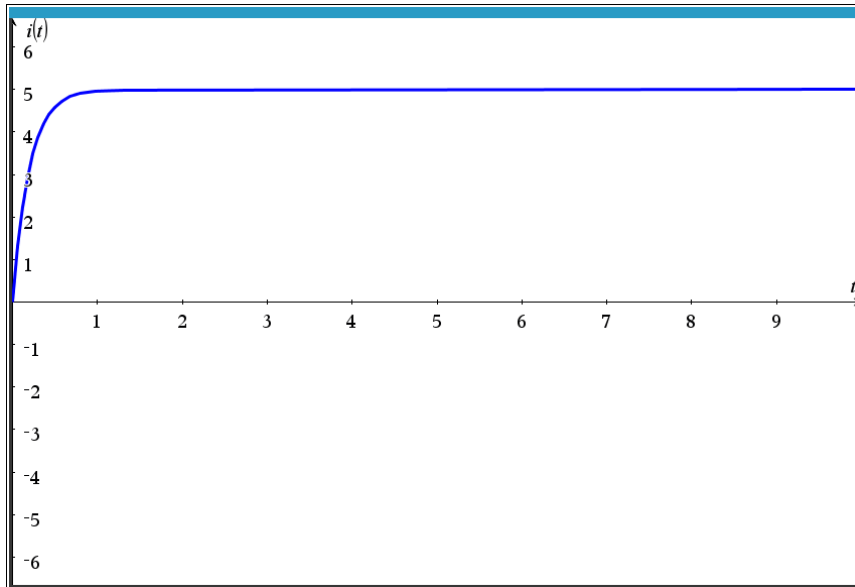


Figure 1

d)  $E(t) = \begin{cases} 25 & \text{for } 0 < t < 2 \\ 0 & \text{else} \end{cases}$  then we receive  $i(t) = \begin{cases} 5(1 - e^{-5t}) & \text{for } 0 \leq t < 2 \\ 5(e^{-5(t-2)} - e^{-5t}) & \text{for } t \geq 2 \end{cases}$

And the graph for  $i(t)$  is given on the next page.:

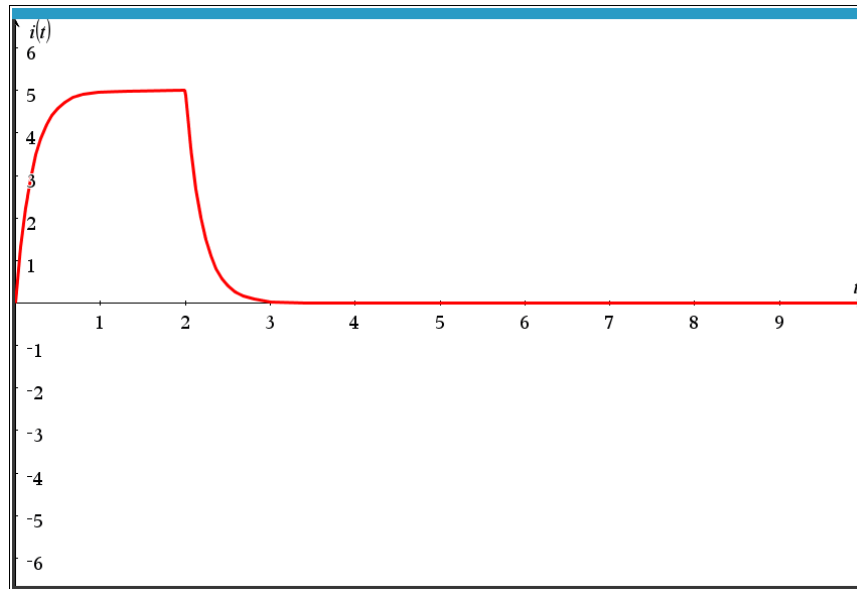


Figure 2

$$e) \quad E(t) = \begin{cases} 25 & \text{for } 0 < t < 2 \\ 0 & \text{for } 2 < t < 4 \\ 25 & \text{for } 4 < t < 6 \\ 0 & \text{for } t > 6 \end{cases}$$

Let us show that we find

$$i(t) = \begin{cases} 5(1 - e^{-5t}) & \text{for } 0 \leq t < 2 \\ 5(e^{-5(t-2)} - e^{-5t}) & \text{for } 2 \leq t < 4 \\ 5(1 - e^{-5t} + e^{-5(t-2)} - e^{-5(t-4)}) & \text{for } 4 \leq t < 6 \\ 5(e^{-5(t-6)} - e^{-5(t-4)} + e^{-5(t-2)} - e^{-5t}) & \text{for } t \geq 6 \end{cases}$$

Indeed, we have  $E(t) = 25(u(t) - u(t-2) + u(t-4) - u(t-6))$ .

The Laplace transform is applied to the DE  $\frac{di}{dt} + 5i = E(t)$ ,  $i(0) = 0$ . Denoting by  $I(s)$  the transform of  $i(t)$ , we have by the properties of the Laplace transform table

$$(s+5)I(s) = 25 \left( \frac{1}{s} - \frac{e^{-2s}}{s} + \frac{e^{-4s}}{s} - \frac{e^{-6s}}{s} \right) \Leftrightarrow I(s) = \frac{25}{s(s+5)} \sum_{k=0}^3 (-1)^k e^{-2ks}$$

Or  $\frac{25}{s(s+5)} = \frac{5}{s} - \frac{5}{s+5} \Leftrightarrow 5 - 5e^{-5t} = 5(1 - e^{-5t})$  and  $e^{-as}F(s) \leftrightarrow f(t-a)u(t-a)$ . Hence

$$i(t) = 5 \sum_{k=0}^3 (-1)^k (1 - e^{-5(t-2k)}) u(t-2k).$$

This sum is the same as the expression in the brace above. Here is the graph of the current:

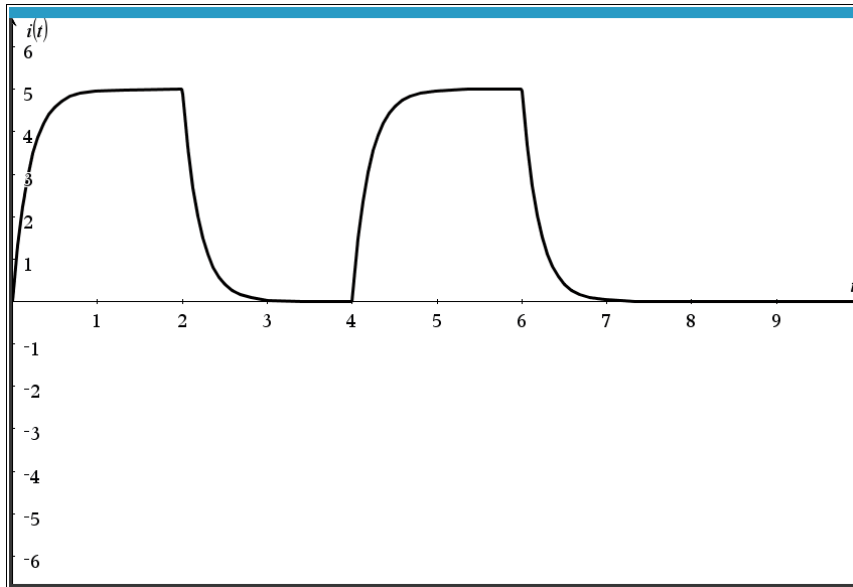


Figure 3

$$f) \quad E(t) = \begin{cases} 25 & \text{for } 0 < t < 2 \\ 0 & \text{for } 2 < t < 4 \end{cases} \quad E(t+4) = E(t) \quad (\text{thus, a periodic source of period 4}).$$

We therefore expect to have an infinite series as an answer. For graphics, this will cause a problem, unless we restrict ourselves to a partial sum. This can be avoided with the computer since we can use convolution which can be plotted even if the integral does not simplify!

The source being periodic, we use the modulo trick to define it. This trick is as follows: let  $g$  be a function initially defined on the interval  $[a, b]$ . To extend  $g$  periodically outside of the interval with a period equal to  $b - a$ , we do the following composition:  $g(\text{mod}(x - a, b - a) + a)$  where "mod" is the modulo function of the system. In this case, the source will be defined as follows. First, we set

$$E_1(t) = \begin{cases} 25 & \text{for } 0 < t < 2 \\ 0 & \text{for } 2 < t < 4 \end{cases}$$

Then we pose  $E(t) = E_1(\text{mod}(t, 4))$  since  $a = 0$  and  $b = 4$ . Figure 4 shows an Nspire session where we defined the source, then used the convolution definition to obtain the current in integral form (not simplified by Nspire). This then allows the current graph to be plotted (Figure 5). This graph will be slow to appear even on the computer!

Since the DE is  $\frac{di}{dt} + 5i = E(t)$ ,  $i(0) = 0$ , the application of the Laplace transform, posing  $I(s)$  and  $\hat{E}(s)$

the transforms of  $i(t)$  and  $E(t)$  respectively then gives  $sI(s) + 5I(s) = \hat{E}(s)$ , which implies that

$I(s) = \frac{1}{s+5} \cdot \hat{E}(s) \leftrightarrow i(t) = e^{-5t} * E(t)$ . Let's set  $h(t) = e^{-5t}$  (which has been stored in memory even though it is not shown in Figure 4).

$e1(t) := \begin{cases} 25, 0 < t < 2 \\ 0, 2 < t < 4 \end{cases}$	Done
$e(t) := e1(\text{mod}(t, 4))$	Done
$\int_0^t (e(\tau) \cdot h(t-\tau)) d\tau$	$\int_0^t \left( e^{5 \cdot \tau} \cdot \begin{cases} 25, 0 < \text{mod}(\tau, 4) < 2 \\ 0, 2 < \text{mod}(\tau, 4) < 4 \end{cases} \right) d\tau \cdot e^{-5 \cdot t}$

Figure 4



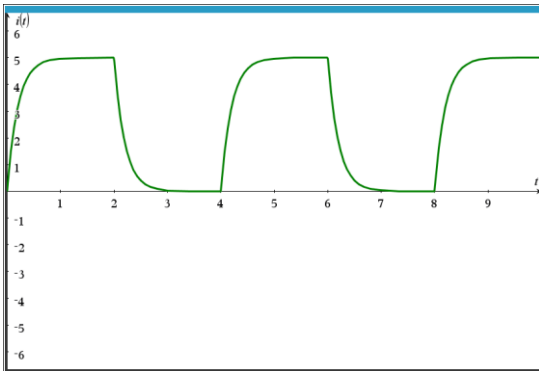
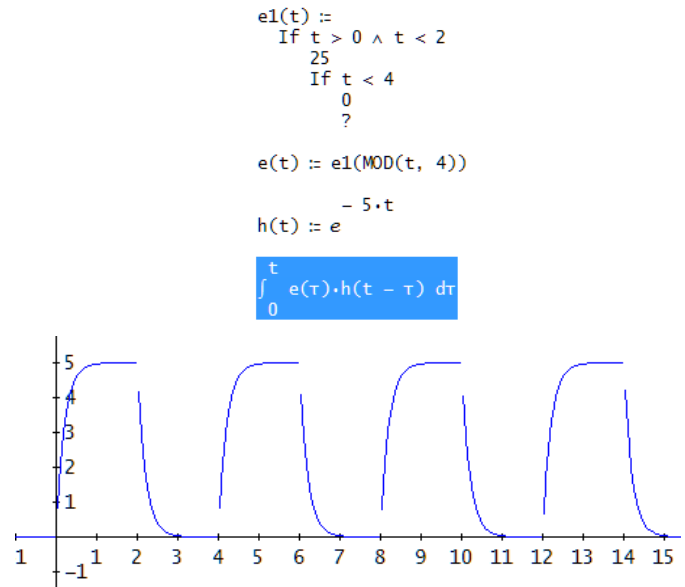


Figure 5

With TI-Nspire (above) and DERIVE (right)



### Problem 6 Special functions

Let the following 3 functions be defined by integrals: the *error function*, the *integral sine function* and the *integral cosine function*:

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt, \quad \operatorname{Si}(x) = \int_0^x \frac{\sin t}{t} dt, \quad \operatorname{Ci}(x) = \operatorname{Ln}(x) + \gamma + \int_0^x \frac{\cos(t) - 1}{t} dt.$$

Here,  $\gamma$  is the Euler constant:  $\gamma \equiv \lim_{n \rightarrow \infty} \left( \sum_{k=1}^n \frac{1}{k} - \ln(n) \right) \approx 0.5772156649$ . We can see that

$$\frac{d}{dx} \operatorname{Ci}(x) = \frac{\cos x}{x} \quad (x \neq 0).$$

- a) Plot their graphs and try to find out (and demonstrate if possible) the value of the limit to infinity of each.

- b) Repeat the following steps to prove that  $\int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$ . Let  $a > 0$  for the following disks and

squares:

$$D_a = \{(x, y) : x^2 + y^2 \leq a^2\}, \quad C_a = \{(x, y) : 0 \leq x \leq a, 0 \leq y \leq a\}$$

i) Show that  $\left( \int_0^a e^{-x^2} dx \right)^2 = \iint_{C_a} e^{-(x^2+y^2)} dx dy$

ii) Show that  $\iint_{D_a} e^{-(x^2+y^2)} dx dy = \frac{\pi}{4} (1 - e^{-a^2})$  (use polar coordinates!)

iii) Show that  $\lim_{a \rightarrow \infty} \iint_{D_a} e^{-(x^2+y^2)} dx dy = \lim_{a \rightarrow \infty} \iint_{C_a} e^{-(x^2+y^2)} dx dy$

c) Find: i)  $\frac{d}{dx} \text{Si}(x)$       ii)  $\frac{d}{dx} \text{erf}(x)$       iii)  $\frac{d}{dx}(x \text{erf}(x))$   
 iv)  $\frac{d}{dx} \text{erf}(\sqrt{x})$       v)  $\sqrt{\frac{2}{\pi}} \int_0^x e^{-t^2/2} dt$       vi)  $\sqrt{\frac{2}{\pi}} \int_{x_1}^{x_2} e^{-t^2/2} dt$

d) Let  $S_n(x) = \sum_{k=1}^n \frac{\sin((2k-1)x)}{2k-1}$ . Try to convince yourself (the proof will come later) that the sequence  $S_n\left(\frac{\pi}{2n}\right)$  seems to tend towards the value  $\frac{1}{2} \text{Si}(\pi)$  when  $n$  tends to infinity.

- e) Find the exact value of the following integral (and check your answer by evaluating your answer and the integral numerically)

$$\int_2^{\pi} \frac{\sin x}{x^2 - 1} dx$$

If necessary, use the Si and Ci functions defined at the beginning of the exercise.

### Problem 6e)

Note that this is not an improper integral since the expression to be integrated is continuous over the interval  $[2, \pi]$ . The area under the curve seems to be 0.130119 :

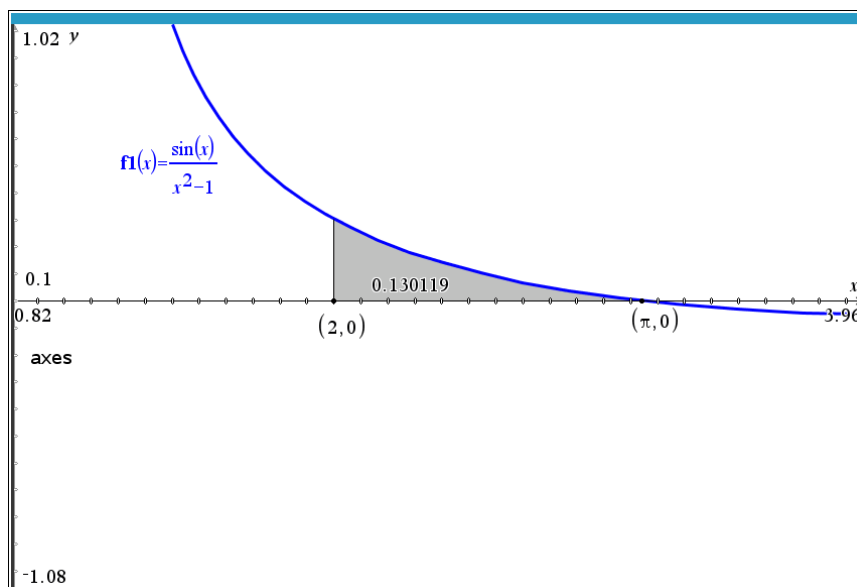


Figure 6

To find this value in exact form, special functions such as the *error function*, the *integral sine function* and the *integral cosine function* can be used (see their definitions above).

Maple gives the exact value of  $I = \int_2^{\pi} \frac{\sin x}{x^2 - 1} dx$  without problems.

The reader can check that if we increase the precision to "20 digits", then the approximate value becomes 0.13011886023663081572. Figure 7 shows a screen capture with the default precision of 10 significant digits:

```
f :=  $\frac{\sin(x)}{x^2 - 1}$ ;
                                     f :=  $\frac{\sin(x)}{x^2 - 1}$ 
int(f, x = 2 .. Pi);

$$\frac{\text{Si}(3) \cos(1)}{2} - \frac{\text{Ci}(3) \sin(1)}{2} - \frac{\text{Si}(1) \cos(1)}{2} - \frac{\text{Ci}(1) \sin(1)}{2}$$


$$+ \frac{\text{Si}(-\pi - 1) \cos(1)}{2} + \frac{\text{Ci}(\pi + 1) \sin(1)}{2}$$


$$- \frac{\text{Si}(-\pi + 1) \cos(1)}{2} + \frac{\text{Ci}(\pi - 1) \sin(1)}{2}$$

evalf(%);
0.1301188601
```

Figure 7

Let's show how to get the exact value: we have  $\frac{\sin x}{x^2 - 1} = \frac{1}{2} \sin(x) \left( \frac{1}{x-1} - \frac{1}{x+1} \right)$ . We will therefore have

to transform the  $\sin(x)$  factor into terms of  $\sin(x-1)$  and  $\sin(x+1)$ . We use trigonometric identities:  $\sin(a \pm b) = \sin(a)\cos(b) \pm \cos(a)\sin(b)$ . So

$$\sin(x) = \sin((x-1) + 1) = \sin(x-1)\cos(1) + \cos(x-1)\sin(1)$$

$$\sin(x) = \sin((x+1) - 1) = \sin(x+1)\cos(1) - \cos(x+1)\sin(1)$$

This allows to write

$$\begin{aligned} \frac{\sin x}{x^2 - 1} &= \frac{1}{2} \left( \frac{\sin(x)}{x-1} - \frac{\sin(x)}{x+1} \right) = \\ &= \frac{1}{2} \left( \frac{\sin(x-1)\cos(1) + \cos(x-1)\sin(1)}{x-1} \right) - \frac{1}{2} \left( \frac{\sin(x+1)\cos(1) - \cos(x+1)\sin(1)}{x+1} \right) \end{aligned}$$

The linearity of the integral and the change of variables  $y = x - 1$  (resp.  $y = x + 1$ ) means that the integral is

$$I = \frac{1}{2} \cos(1) \int_1^{\pi-1} \frac{\sin(y)}{y} dy + \frac{1}{2} \sin(1) \int_1^{\pi-1} \frac{\cos(y)}{y} dy - \frac{1}{2} \cos(1) \int_3^{\pi+1} \frac{\sin(y)}{y} dy + \frac{1}{2} \sin(1) \int_3^{\pi+1} \frac{\cos(y)}{y} dy$$

Or  $\int_1^{\pi-1} \frac{\sin y}{y} dy = \int_1^0 \frac{\sin y}{y} dy + \int_0^{\pi-1} \frac{\sin y}{y} dy = \text{Si}(\pi-1) - \text{Si}(1)$ . We do the same for the other three integrals

and obtain (since the terms  $\text{Ln}(x) + \gamma$  in  $\text{Ci}(x)$  simplify)

$$I = \frac{1}{2} \cos(1)(\text{Si}(\pi-1) - \text{Si}(1)) + \frac{1}{2} \sin(1)(\text{Ci}(\pi-1) - \text{Ci}(1)) \\ - \frac{1}{2} \cos(1)(\text{Si}(\pi+1) - \text{Si}(3)) + \frac{1}{2} \sin(1)(\text{Ci}(\pi+1) - \text{Ci}(3))$$

This answer is the same as that given by Maple in figure 7 since the Si function is odd. Nspire confirms (we have defined the functions Si and Ci in the kit\_ets\_mb library but they are not implemented in the software!).

```

si(x):=kit_ets_mb\si(x) • Done
ci(x):=kit_ets_mb\ci(x) • Done
Nous augmentons l'affichage à "Float 12" pour voir plus de décimales et recopions la réponse
trouvée à la main:
1/2 • cos(1) • (si(π-1)-si(1))+1/2 • sin(1) • (ci(π-1)-ci(1))-1/2 • cos(1) • (si(π+1)-si(3))+1/2 • sin(1) • (ci(π+1)
-ci(3))
• 0.130118860237
L'intégrateur de Nspire confirme.
∫ sin(x)
x^2-1 dx • 0.130118860237
Son intégrateur numérique aussi:
nInt(sin(x)
x^2-1 ,x,2,π) • 0.130118860237

```

Figure 8

*DERIVE* and the sine integral:

$$\int_2^{\pi} \frac{\text{SIN}(x)}{x^2-1} dx = \frac{\int_2^{\pi} \frac{\text{SIN}(x)}{x-1} dx}{2} - \frac{\int_2^{\pi} \frac{\text{SIN}(x)}{x+1} dx}{2}$$

$$\int_2^{\pi} \frac{\text{SIN}(x)}{x^2-1} dx = 0.1301188602$$

$$\frac{1}{2} \cdot (\cos(1) \cdot (\text{SI}(\pi-1) - \text{SI}(1) - \text{SI}(\pi+1) + \text{SI}(3))) + \text{SIN}(1) \cdot (\text{CI}(\pi-1) - \text{CI}(1) + \text{CI}(\pi+1) - \text{CI}(3)))$$

0.1301188602

## Penalty Shootout in Football

Benno Grabinger, Germany

At the last European Football Championship 20/21, some matches were only decided by a penalty shoot-out. Especially in memory remains the dramatic penalty shoot-out in the final match Italy vs England, which Italy won 3:2. In the end the lack of nerve of the last English shooter, who was only 19 years old, was



decisive. Of great importance, and this will be shown in the following, was also the fact that Italy could begin the shoot-out. In the case of a penalty shoot-out in the usual order ABABAB... of teams A and B, team A has an advantage over team B. This advantage will be calculated here under certain model assumptions. Furthermore, a fairer penalty shoot-out system will be proposed. First of all, the rules for the usual penalty shoot-out at the end of a football match are given.

### Rules for the penalty shoot-out

The referee decides on which goal the shots are to be taken. The referee tosses a coin and the winner of the toss gets to choose if his team takes the first shot or whether he prefers to let the opposing team go first.

Both teams now alternate shots.

1. Whoever has scored the most goals after 5 rounds is the winner of the penalty shoot-out.
2. The penalty shoot-out may be ended prematurely. This is the case when one team has scored more goals than the other team can score with the penalties which are still to be executed.  
Example: If the score is 3:0 after 3 rounds, then the second team is still entitled to 2 penalties, but cannot use them to make up for the other team's lead.
3. If the score is tied after 5 rounds, the penalty shoot-out continues, until one team has scored one more goal than the opponent with the same number of shots.

### Model assumptions for a penalty shoot-out

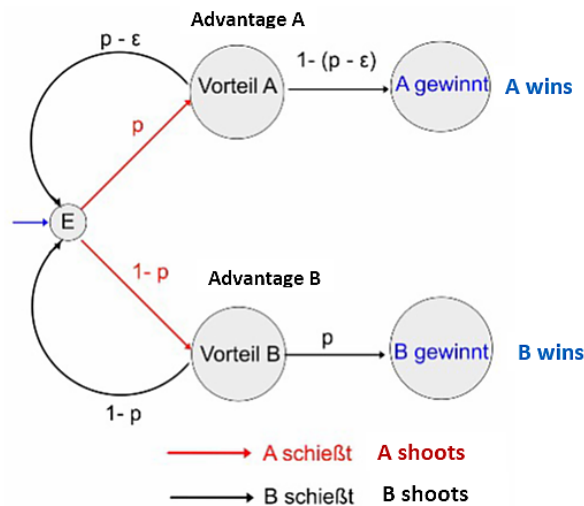
There are good and not so good penalty shooters, i.e. for every shooter there is an individual probability of scoring a penalty kick. In addition, the form of the day also influences the probability of a hit. Because a general statement is to be made here these individual hit probabilities cannot be taken into account. Therefore, an average score probability  $p$  is assumed for each shooter. Statistics<sup>[1]</sup> show that the probability  $p$  is lower than for penalties during regular playing time, and is approximately about 75%. Furthermore, it is known that the probability of a shooter scoring decreases when the opposing team is in front. The influence of the pressure on the player then reduces the probability of a score by an average of  $\varepsilon = 5\%$ . The following two assumptions apply to the model considered here.

1. Score probability  $p$  (average 75%)
2. If the opposing team is ahead, then the probability of scoring is  $p - \varepsilon$ . ( $\varepsilon$  is 5% on average).

The order of team A and then team B each executing one shot is referred to here as one *round*. The respective score of the penalty shoot-out is designated by  $a:b$ , where  $a$  is the number of goals scored by team A and  $b$  is the number of goals scored by team B.

The probability that A wins when  $nr = 5$  and  $a = b$  (rule 3) can be easily calculated. For shorthand, this probability is denoted  $w(p, \varepsilon)$ . To calculate  $w(p, \varepsilon)$ , consider a random walk on the following graph:

[1] <https://www.dfb.de/news/detail/11-fakten-zum-elfmeter-80237/>



In the situation  $nr = 5$  and  $a = b$  the start is in node E (deuce). Team A shoots and scores a goal with probability  $p$ . The random walk is then in knot "Advantage A", because if team B misses now, A has won. If team A does not score (probability  $1 - p$ ), then the odds are in the node "Advantage B". If team A has scored, it is one point ahead of team B. Team B then only scores with probability  $p - \varepsilon$ . In this case the random walk is in node E again and everything starts from the beginning. This happens with the probability  $p(p - \varepsilon)$ .

Accordingly, one can also end up in E again with the probability  $(1 - p)(1 - p)$ , namely if A and B both fail to hit. From E, team A can win directly, i.e. without landing in E again, with the probability  $p(1 - (p - \varepsilon))$ .

Therefore, for the probability  $w(p, \varepsilon)$  holds:

$$w(p, \varepsilon) = p(p - \varepsilon)w(p, \varepsilon) + (1 - p)(1 - p)w(p, \varepsilon) + p(1 - (p - \varepsilon))$$

Solving for  $w$  gives:  $w(p, \varepsilon) = \frac{p(1 - (p - \varepsilon))}{1 - p^2 + p\varepsilon - 1 + 2p - p^2}$ .

which is simplified:  $w(p, \varepsilon) = \frac{1 - (p - \varepsilon)}{\varepsilon + 2(1 - p)}$  The probability that team B wins is then equal to

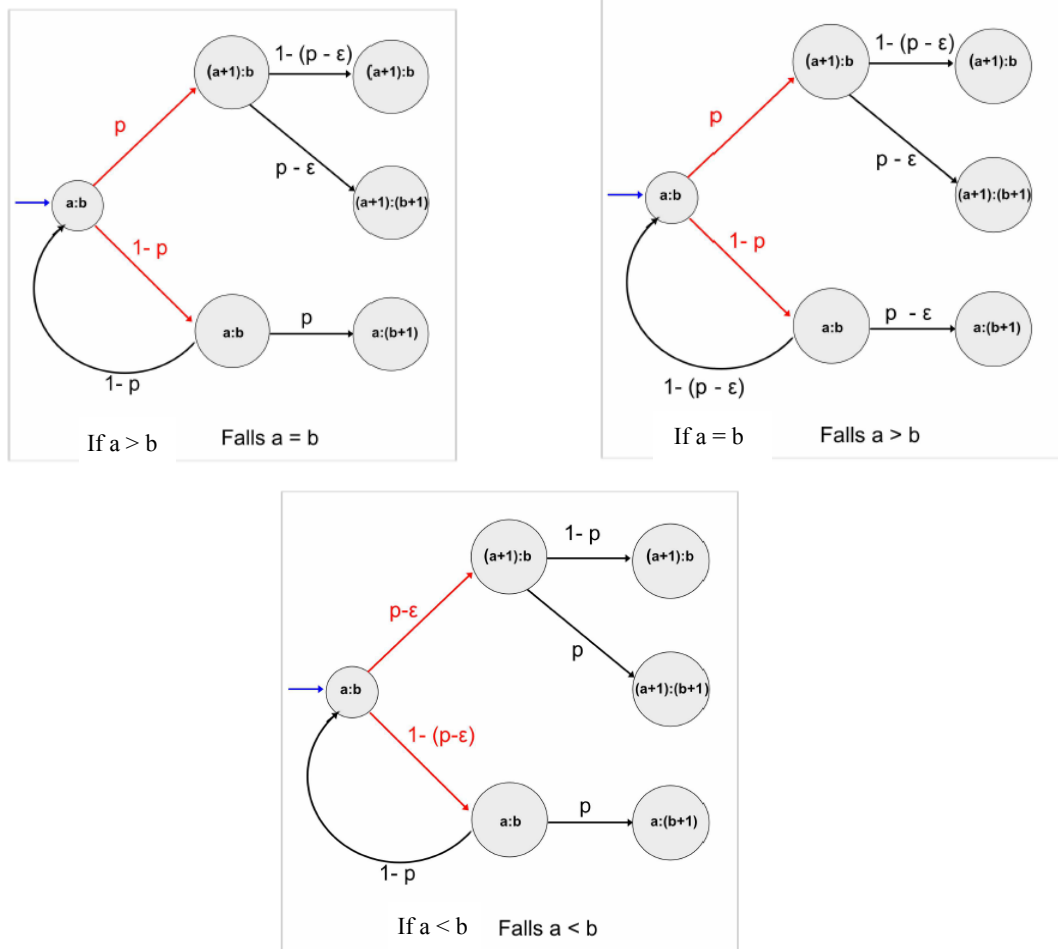
$$v(p, \varepsilon) = 1 - w(p, \varepsilon) = 1 - \frac{1 - (p - \varepsilon)}{\varepsilon + 2(1 - p)} = \frac{\varepsilon + 2(1 - p) - 1 + (p - \varepsilon)}{\varepsilon + 2(1 - p)} = \frac{1 - p}{\varepsilon + 2(1 - p)}$$

$w$  and  $v$  are calculated for special values  $p = 75\%$  and  $\varepsilon = 5\%$ :

© Team A wins	
$w(p, \varepsilon) := \frac{1 - (p - \varepsilon)}{\varepsilon + 2 \cdot (1 - p)}$	Done
$w(0.75, 0.05)$	0.545455
© Team B wins	
$v(p, \varepsilon) := 1 - w(p, \varepsilon)$	Done
$v(0.75, 0.05)$	0.454545
© Difference	
$w(0.75, 0.05) - v(0.75, 0.05)$	0.090909

In this situation, team A thus has an advantage of about 9%.

We now look at the penalty shoot-out from the beginning and ask about the probability of team A winning. Because of the many possibilities of how the penalty shootout can take place, it does not make sense to create a probability tree with branches representing these possibilities. It is more comfortable to consider a function  $g(n,a,b,p,\epsilon)$  which gives the probability of team A winning, if  $nr$  rounds have been played and the score is  $a:b$ . With these designations,  $g(0,0,0,0.75,0.05)$  would then provide the winning probability for A from the beginning. Function  $g$  can be constructed recursively. However, it must be noted that depending on whether  $a > b$ ,  $a = b$  or  $a < b$  applies to the score, function  $g$  looks different, i.e. a threefold case distinction is necessary. The probabilities required in this context can be seen in the following three figures below.



If one adds the termination conditions of the recursion to these three cases, then we can define the following function:

$$g(n,a,b,p,\epsilon) = \begin{cases} 1, & n=5 \text{ and } a > b \\ 0, & n=5 \text{ and } a < b \\ w(p,\epsilon), & n=5 \text{ and } a = b \\ (1-p) \cdot (1-p-\epsilon) \cdot g(n+1,a,b,p,\epsilon) + p \cdot (p-\epsilon) \cdot g(n+1,a+1,b+1,p,\epsilon) + (1-p) \cdot (p-\epsilon) \cdot g(n+1,a+1,b,p,\epsilon) + (1-p) \cdot p \cdot g(n+1,a,b+1,p,\epsilon), & a > b \\ (1-p)^2 \cdot g(n+1,a,b,p,\epsilon) + p \cdot (p-\epsilon) \cdot g(n+1,a+1,b+1,p,\epsilon) + (1-p) \cdot (p-\epsilon) \cdot g(n+1,a+1,b,p,\epsilon) + (1-p) \cdot p \cdot g(n+1,a,b+1,p,\epsilon), & a = b, \text{Else} \\ (1-p-\epsilon) \cdot (1-p) \cdot g(n+1,a,b,p,\epsilon) + p \cdot (p-\epsilon) \cdot g(n+1,a+1,b+1,p,\epsilon) + (p-\epsilon) \cdot (1-p) \cdot g(n+1,a+1,b,p,\epsilon) + (1-p) \cdot p \cdot g(n+1,a,b+1,p,\epsilon), & a < b \end{cases}$$

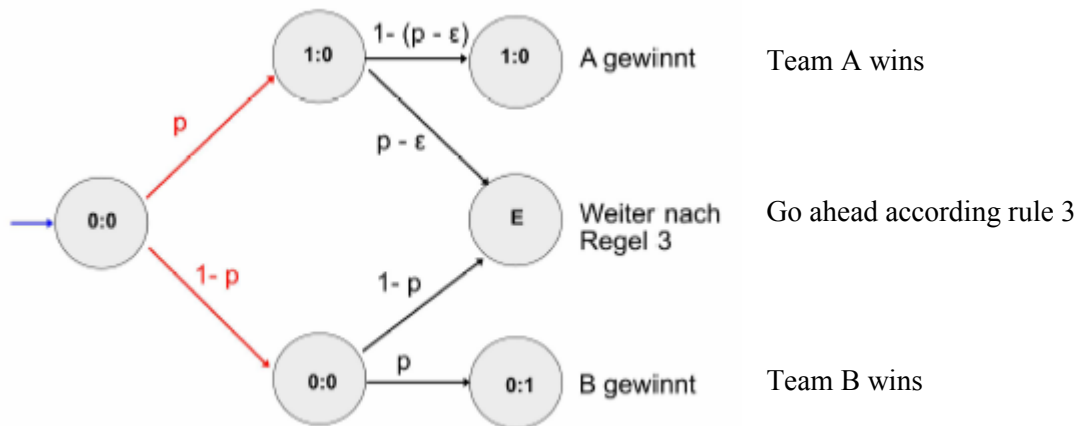
Done

Calculation of  $g(0,0,0,0.75,0.05)$  delivers a surprise:

$g(0,0,0,0.75,0.05)$ 

0.545455

At first glance it may seem surprising that the probability of A winning the entire penalty shoot-out is as great as the probability of winning a penalty shoot-out only according to Rule 3. This somewhat confusing situation can be clarified by considering only one round instead of the usual 5 rounds, which can have the results 1:0 (A wins), 0:1 (B wins) or E. E stands for Deuce (in German: Einstand), which can be expressed as 0:0 or 1:1. If E occurs, then follows the previously considered penalty shoot-out according to rule 3 follows. The score 1:0 occurs with the probability  $p \cdot (1 - (p - \epsilon))$ . The score E occurs with the probability  $p \cdot (p - \epsilon) + (1 - p) \cdot (1 - p)$ .



The probability that A wins this penalty shootout is then given by:

$$\begin{aligned}
 P(A) &= p \cdot (1 - (p - \epsilon)) + [p \cdot (p - \epsilon) + (1 - p)^2] w(p, \epsilon) \\
 &= p \cdot (1 - (p - \epsilon)) + [p \cdot (p - \epsilon) + (1 - p)^2] \frac{1 - (p - \epsilon)}{\epsilon + 2(1 - p)} \\
 &= (1 - (p - \epsilon)) \cdot \left[ p + \frac{(1 - p)^2 + p(p - \epsilon)}{\epsilon + 2(1 - p)} \right] \\
 &= (1 - (p - \epsilon)) \frac{p\epsilon + 2p(1 - p) + (1 - p)^2 + p(p - \epsilon)}{\epsilon + 2(1 - p)} \\
 &= \frac{1 - (p - \epsilon)}{\epsilon + 2(1 - p)} = w(p, \epsilon).
 \end{aligned}$$

If you look more closely, you will see that the procedure of the round works in exactly the same way as shooting according to rule 3. Whoever is in the lead after the shots of A and B wins. At deuce, everything repeats itself.

What happens when you play two rounds? The last round is replaced by the penalty shoot-out according to rule 3. Then you are in exactly the same situation as before with only one round. So, you continue and finally come to the conclusion that team A wins in a penalty shoot-out over 5 rounds, with a possible subsequent shootout according to rule 3, with probability  $w$ . If you only think about the probability of winning, then you can do without the five rounds and start right away with the penalty shootout according to rule 3. For the teams' winning probabilities, this would have the same effect as the usual penalty shootout.



Why then is this not done in practice? One can only speculate. Perhaps the inventors of the penalty shootout were not aware of this when it was introduced. Perhaps, however, they want to keep the incentive for the spectators high, by allowing more shots on goal.

For the values  $p = 0.75$  and  $\varepsilon = 0.05$ ,  $w(p, \varepsilon)$  has already been calculated and shown, that team A has an advantage of about 9%.

General rule is:  $w(p, \varepsilon) - v(p, \varepsilon) = \frac{1-(p-\varepsilon)}{\varepsilon+2(1-p)} - \frac{1-p}{\varepsilon+2(1-p)} = \frac{\varepsilon}{\varepsilon+2(1-p)} \geq 0$  for  $0 \leq p \leq 1$  and  $0 \leq \varepsilon \leq p$ .

Consequently, penalty shoot-outs in the traditional way are always unfair, regardless of the values for  $p$  and  $\varepsilon$ .

Can the rules for the penalty shoot-out be changed to eliminate this injustice? In principle, this is possible, but it is a certain challenge for spectators and referees as well.

You play the first round as usual in the order A B. The advantage that A had in the first round is compensated by the fact that B starts the second round, i.e. the first and second rounds are then A B B A. Any advantage for A or B in this sequence is again balanced by reversing again the order: A B B A B A A B.

How do you continue this sequence? The construction principle of the sequence is: Each letter of the initial word is "inverted". The resulting new word is appended to the original word. If one continues this process as desired, one obtains the so-called *Thue-Morse sequence*<sup>[2]</sup>.

The description of the sequence becomes easier if you replace A by 0 and B by 1.

Then the inversion is mathematically easily possible by  $x = x - 1$ ,  $x \in \{0,1\}$ .

<pre> Define <b>invert</b>(list)= Func   © Inversion of elements of a list Local m,i m:={ } For i,1,dim(list)   m:=augment(m,{1-list[i]})   © 1-x gives the dual complement EndFor Return m EndFunc </pre>	<pre> Define <b>t</b>(i)= Func   © gives the first 2<sup>i</sup> elements of the   © Thue-Morse sequence If i=0 Then   {0} Else   Return augment(t(i-1),invert(t(i-1))) EndIf EndFunc </pre>
--	--

$t(0)$	{0}
$t(1)$	{0,1}
$t(2)$	{0,1,1,0}
$t(3)$	{0,1,1,0,1,0,0,1}
$t(4)$	{0,1,1,0,1,0,0,1,1,0,0,1,0,1,1,0}
$t(5)$	{0,1,1,0,1,0,0,1,1,0,0,1,0,1,1,0,1,0,0,1,0,1,1,0,0,1,1,0,1,0,0,1}

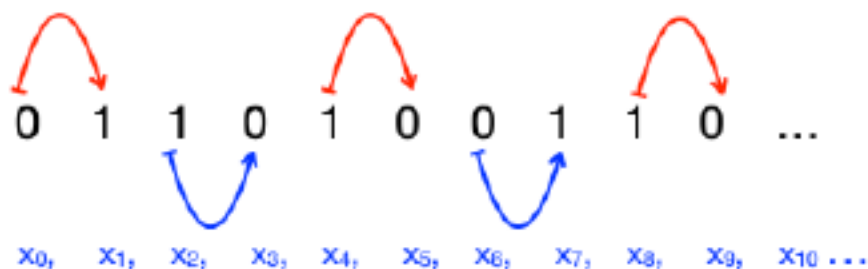
<sup>[2]</sup> <https://thatsmaths.com/2019/02/21/multiple-discoveries-of-the-thue-morse-sequence/>  
[https://en.wikipedia.org/wiki/Thue%E2%80%93Morse\\_sequence](https://en.wikipedia.org/wiki/Thue%E2%80%93Morse_sequence)

In order to recognize further properties of the sequence, one deletes all elements with an odd place number:

$$\begin{array}{cccccccccccc}
 x_0 & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 & \dots \\
 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & \dots \\
 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & \dots \\
 x_0 & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 & \dots
 \end{array}$$

The resulting sequence then matches the original sequence. This shows, that  $x_2 = x_1$ ,  $x_4 = x_2$ ,  $x_6 = x_3, \dots$  holds, i.e. it is  $x_{2n} = x_n$ .

Next, one follows the parity changes in the sequence. It can be seen that these take place on positions with an odd number of place:



So, we have:  $x_{2n+1} = 1 - x_{2n}$

Summary:

$$\begin{array}{l}
 x_0 = 0 \\
 x_{2n} = x_n \\
 x_{2n+1} = 1 - x_{2n} = 1 - x_n
 \end{array}$$

From this, one can set up a recursion formula for calculating the sequence:

$$x_k = \begin{cases} 0 & k = 0 \\ x_{\frac{k}{2}} & k \text{ even, } k > 0 \\ x_{\frac{k-1}{2}} & k \text{ odd, } k > 0 \end{cases}$$

```

x
0/12
Define x(k)=
Func
© Returns the k-th element of the Thue-Morse sequence.
© Counting starts with 0.
If k=0 Then
0
Else
If mod(k,2)=0 Then
x(k/2)
Else
1-x(k/2)
© 1-x is the inverse of x
EndIf
EndIf
EndFunc

```

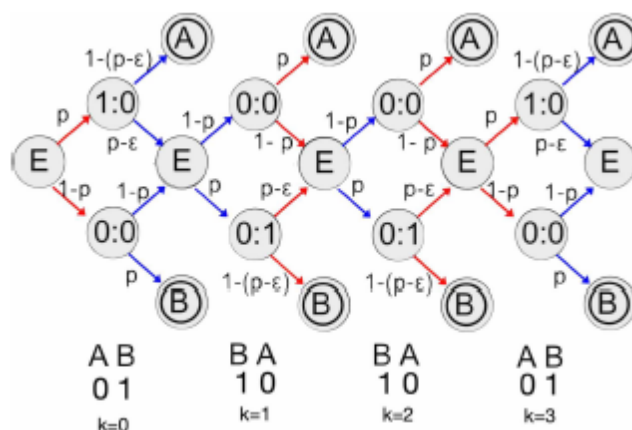
```

seq(x(n),n,0,31)
{0,1,1,0,1,0,0,1,1,0,0,1,0,1,1,0,1,0,0,1,0,1,1,0,0,1,0,1,1,0,0,1}

```

In the following function  $x(k)$  is used to calculate the winning probabilities of teams A and B when shooting in the Thue-Morse order. Before this calculation can be performed, terms for these probabilities of winning are set up.

The probability  $pa(k)$  that team A has won after  $k$  rounds or sooner if shooting is done in the Thue-Morse order is calculated. For this purpose, the following illustration is used to form these probabilities. The first round has the number  $k=0$ .



Etc.

$$pa(0) = p \cdot (1 - (p - \varepsilon))$$

$$pa(1) = pa(0) + [p \cdot (p - \varepsilon) + (1 - p)^2] \cdot (1 - p) \cdot p$$

$$pa(2) = pa(1) + [p \cdot (p - \varepsilon) + (1 - p)^2]^2 \cdot (1 - p) \cdot p$$

$$pa(3) = pa(2) + [p \cdot (p - \varepsilon) + (1 - p)^2]^3 \cdot p \cdot (1 - (p - \varepsilon))$$

In summary:

$$pa(k) = \begin{cases} pa(k-1) + [p(p-\varepsilon) + (1-p)^2]^k \cdot p \cdot (1-(p-\varepsilon)) & \text{falls AB} & \text{if AB} \\ pa(k-1) + [p(p-\varepsilon) + (1-p)^2]^k \cdot (1-p) \cdot p & \text{falls BA} & \text{if BA} \end{cases}$$

For comparison, consider the probabilities  $pb(k)$ , which indicate the probability of team B winning after  $k$  rounds or earlier. The result is:

$$pb(k) = \begin{cases} pb(k-1) + [p(p-\varepsilon) + (1-p)^2]^k \cdot (1-p) \cdot p & \text{falls AB} & \text{if AB} \\ pb(k-1) + [p(p-\varepsilon) + (1-p)^2]^k \cdot p \cdot (1-(p-\varepsilon)) & \text{falls BA} & \text{if BA} \end{cases}$$

The following functions take over the calculation of the values of  $pa$  and  $pb$ :

```

round_ 1/2
Define round_(k)=
Func
© First round has number k
Return {x(2·k),x(2·k+1)}
EndFunc

```

```

pa 1/1
Define pa(k)=
Func
    p · (1-(p-ε)),          k=0
Return {pa(k-1) + [p · (p-ε) + (1-p)2]k · p · (1-(p-ε)), round_(k)={0,1}
    pa(k-1) + [p · (p-ε) + (1-p)2]k · (1-p) · p,   round_(k)={1,0}
EndFunc

```

```

pb 1/1
Define pb(k)=
Func
    (1-p) · p,          k=0
Return {pb(k-1) + [p · (p-ε) + (1-p)2]k · (1-p) · p,   round_(k)={0,1}
    pb(k-1) + [p · (p-ε) + (1-p)2]k · p · (1-(p-ε)), round_(k)={1,0}
EndFunc

```

$\text{seq}(pa(k),k,1,8) p=0.75 \text{ and } \varepsilon=0.05$ $\{0.335156,0.399873,0.445498,0.467836,0.483584,0.492836,0.497365,0.500026\}$
$\text{seq}(pb(k),k,1,8) p=0.75 \text{ and } \varepsilon=0.05$ $\{0.319688,0.397348,0.435369,0.462174,0.475297,0.483007,0.488442,0.491636\}$
$\text{seq}(pa(k),k,1,8)-\text{seq}(pb(k),k,1,8) p=0.75 \text{ and } \varepsilon=0.05$ $\{0.015469,0.002525,0.01013,0.005662,0.008287,0.009829,0.008923,0.008391\}$

The last figure shows that the differences in winning probabilities for teams A and B are negligible. This type of penalty shootout is fair in contrast to conventional practice.

It is a nice challenge to transfer Benno's function to DERIVE (Josef):

```
#1: [Notation := Decimal, NotationDigits := 6]
```

```
#2: w(p, ε) :=  $\frac{1 - (p - \varepsilon)}{\varepsilon + 2 \cdot (1 - p)}$ 
```

```
shoot(n, p, ε, i, wina, winb, za, zb) :=
  Prog
```

```
  dummy := RANDOM(0)
```

```
  wina := 0
```

```
  winb := 0
```

```
  i := 1
```

```
  Loop
```

```
    If i > n exit
```

```
#3:    za := RANDOM(1)
```

```
    zb := RANDOM(1)
```

```
    If za ≤ p ∧ zb ≥ p - ε
```

```
      wina := wina + 1
```

```
    If za > p ∧ zb < p
```

```
      winb := winb + 1
```

```
    i := i + 1
```

```
  ["A wins", "B wins"; wina/(wina + winb), winb/(wina + winb)]
```

```
#4: w(0.75, 0.05) = 0.5454
```

```
#5: shoot(10000, 0.75, 0.05) =  $\begin{bmatrix} \text{A wins} & \text{B wins} \\ 0.5504 & 0.4495 \end{bmatrix}$ 
```

```
#6: shoot(100000, 0.75, 0.05) =  $\begin{bmatrix} \text{A wins} & \text{B wins} \\ 0.5458 & 0.4541 \end{bmatrix}$ 
```

```
#7: w(0.6, 0.15) = 0.5789
```

```
#8: shoot(100000, 0.6, 0.15) =  $\begin{bmatrix} \text{A wins} & \text{B wins} \\ 0.5811 & 0.4188 \end{bmatrix}$ 
```

The *Thue-Morse* sequence::

```

mt(k) :=
  If k = 0
    0
#9:    If MOD(k, 2) = 0
        mt(k/2)
        1 - mt((k - 1)/2)

#10:  VECTOR(mt(n), n, 0, 31)

#11:  [0, 1, 1, 0, 1, 0, 0, 1, 1, 0, 0, 1, 0, 1, 1, 0, 1, 0, 0, 1, 0, 1, 1, 0,
        0, 1, 1, 0, 1, 0, 0, 1]

#12:  round_(k) := [mt(2·k), mt(2·k + 1)]

pa(p, ε, k) :=
  If k = 0
    p·(1 - (p - ε))
#13:  If round_(k) = [0, 1]
        pa(p, ε, k - 1) + (p·(p - ε) + (1 - p)^2)^k·p·(1 - (p - ε))
        pa(p, ε, k - 1) + (p·(p - ε) + (1 - p)^2)^k·p·(1 - p)

pb(p, ε, k) :=
  If k = 0
    p·(1 - p)
#14:  If round_(k) = [0, 1]
        pb(p, ε, k - 1) + (p·(p - ε) + (1 - p)^2)^k·p·(1 - p)
        pb(p, ε, k - 1) + (p·(p - ε) + (1 - p)^2)^k·p·(1 - (p - ε))

#15:  VECTOR(pa(0.75, 0.05, k), k, 1, 8)

#16:  [0.335156, 0.399873, 0.445498, 0.467835, 0.483583, 0.492835, 0.497365,
        0.500026]

#17:  VECTOR(pb(0.75, 0.05, k), k, 1, 8)

#18:  [0.319687, 0.397347, 0.435368, 0.462173, 0.475296, 0.483006, 0.488442,
        0.491635]

#19:  VECTOR(pa(0.75, 0.05, k) - pb(0.75, 0.05, k), k, 1, 8)

#20:  [0.0154687, 0.00252539, 0.0101296, 0.00566213, 0.00828677, 0.00982875,
        0.00892284, 0.00839062]

We can vary the values of the parameters p, ε, and k:

#21:  VECTOR(pa(0.6, 0.15, k) - pb(0.6, 0.15, k), k, 1, 8)

#22:  [0.0513, 0.034659, 0.0418146, 0.0387377, 0.0400607, 0.0406297, 0.0403850,
        0.0402798]

```

## Plotting Binary Signals

Josef Böhm, Würmla

Every now and then I browse through my large collection of mathematics papers. So, I recently came across two pages from a very old Mathcad paper: Logic in Mathcad, see below:

$A := (0 \ 1 \ 1 \ 0 \ 1 \ 1 \ 1 \ 0 \ 0 \ 0 \ 0 \ 1 \ 1 \ 1 \ 1)^T$  These vectors are entered as matrices  
 having one row and 15 columns so as to  
 $B := (1 \ 1 \ 1 \ 1 \ 0 \ 0 \ 1 \ 0 \ 1 \ 0 \ 0 \ 1 \ 0 \ 1 \ 1)^T$  save on space. The superscript T means  
 that the transpose of the vectors is used.  
 $C := (0 \ 1 \ 0 \ 0 \ 1 \ 1 \ 1 \ 0 \ 1 \ 1 \ 0 \ 1 \ 1 \ 0 \ 1)^T$  This makes them one-column vectors.  
 $m := 0.. \text{länge}(A) - 1$

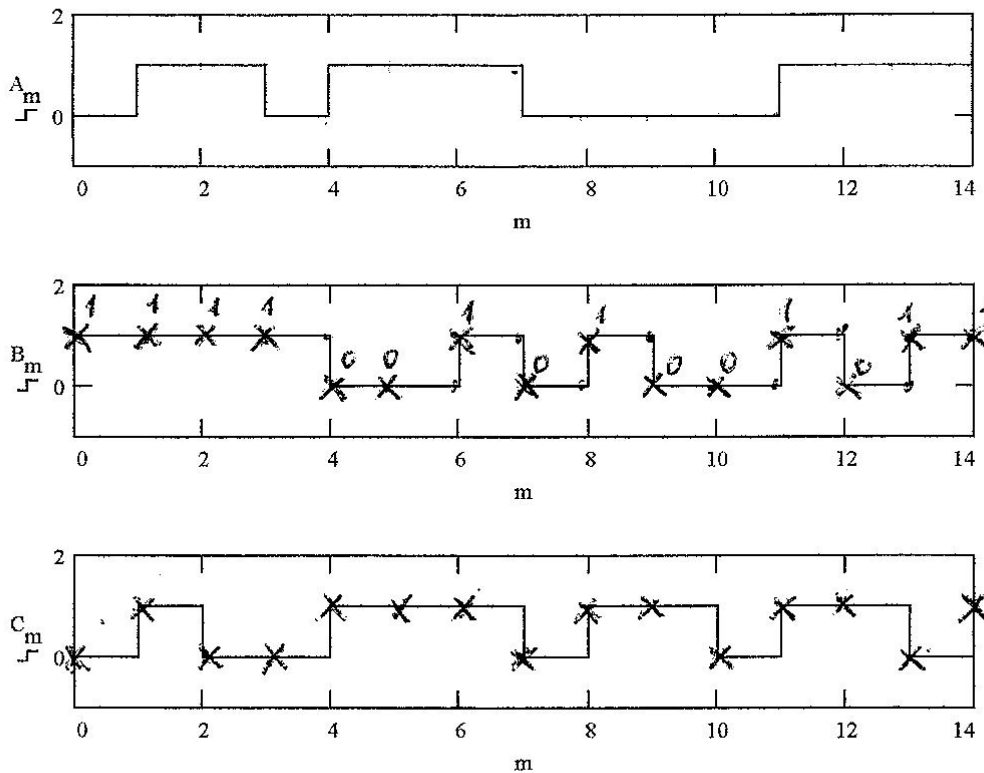


Figure 1: Plot of the signals which will be used in the subsequent examples.

$D := \text{AND}(A, \text{AND}(B, C))$

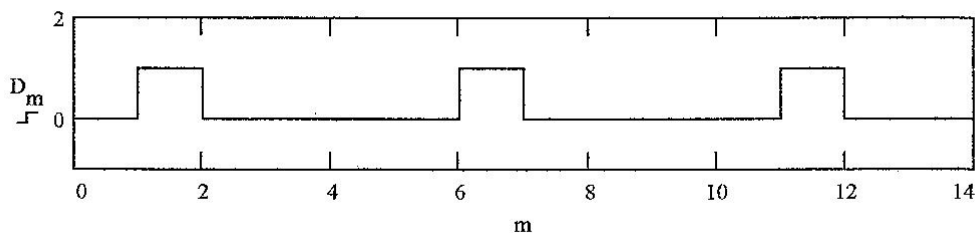
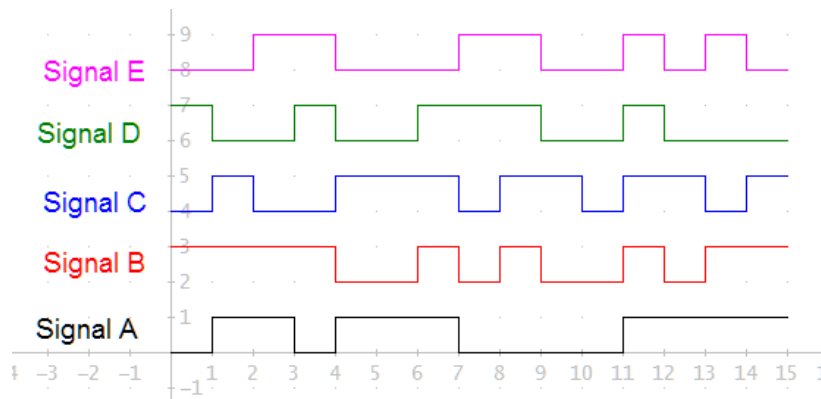


Figure 2: Plot of the  $D = A \text{ AND } (B \text{ AND } C)$ .

I had the idea to produce a DERIVE – and subsequently a Nspire – tool to create similar plots. We will see that we can address several competencies (I must admit that I really dislike this word connected with maths education!) which the students should achieve – besides the challenge programming the respective function(s).

I marked the bits in signal B above and was not really happy with this representation because I wondered how a zero at the end of an encoded message should be represented in case of “10” as the last two bits.

I defined the binary encoded messages as strings and wanted to get plots of the signals as shown:



In the first step I convert the strings into a sequence of bits, which is done by function  $mv(message)$ .

I show the DERIVE procedure for this first step, followed by the TI-Nspire procedure with the signals defined on a Notes-page.

```
a := 011011100001111
b := 111100101001011
c := 010011101101101
d := 100100111001000
e := 001100011001010

mv(x_) := VECTOR(CODES_TO_NAME(NAME_TO_CODES(x_)), i, DIM(x_))

mv(a) = [0, 1, 1, 0, 1, 1, 1, 0, 0, 0, 0, 1, 1, 1, 1]
```

TI-Nspire:

Given signals:

```
a:="011011100001111" ▶ 011011100001111
b:="111100101001011" ▶ 111100101001011
c:="010011101101101" ▶ 010011101101101
d:="100100111001000" ▶ 100100111001000
e:="001100011001010" ▶ 001100011001010
mv(a) ▶ { 0,1,1,0,1,1,1,0,0,0,0,1,1,1,1 }
mv(b) ▶ { 1,1,1,1,0,0,1,0,1,0,0,1,0,1,1 }
mv(e) ▶ { 0,0,1,1,0,0,0,1,1,0,0,1,0,0,1 }
```

```
Define mv(x_)=
Func
seq(expr(mid(x_,i,1)),i,1,dim(x_))
EndFunc
```



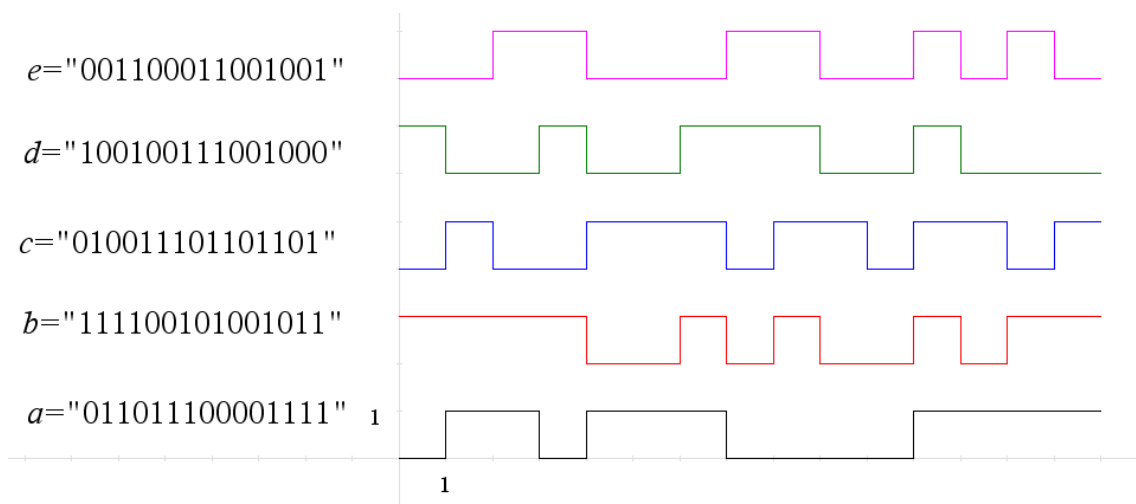
Next step was to produce a program which results in the plot of a given signal. I started with DERIVE which makes plotting easier. One can enter the signal together with the base level and gets immediately the plot in the 2D-Plot Window. (Base level 0 is given by default.)

With TI-Nspire this is not so comfortable. The output of my program are the lists of the coordinates of the plot, which must be used to define a scatter diagram in the Graphs-application. The way how this is done is shown in the screen shot below together with the respective Nspire program followed by the plots of the given signals.

signal(a)  
 signal(b, 2)  
 signal(c, 4)  
 signal(d, 6)  
 signal(e, 8)

<pre> signal(a,0) Scatter sequences in sx and sy Done sxa:=sx:sya:=sy {0,0,1,1,1,0,0,1,1,1,1,0,0,0,0,1,1,1,1}         </pre>	<pre> signal 8/26 Define signal(s,l)= Prgm Local vs,i,lbit,nbit,i vs:=mv(s) i:=0 sx:={}:sy:={} If vs[1]=0 Then sx:={i,i+1}:sy:={l,l} Else sx:={i,i+1}:sy:={l+1,l+1} EndIf While dim(vs)&gt;1 lbit:=vs[1] vs:=right(vs,dim(vs)-1) i:=i+1 nbit:=vs[1] If lbit nbit=1 Then sx:=augment(sx,{i+1}): sy:=augment(sy,{l+1}) ElseIf lbit=1 and nbit=0 Then sx:=augment(sx,{i,i+1}): sy:=augment(sy,{l,l}) ElseIf lbit=0 and nbit=0 Then sx:=augment(sx,{i+1}): sy:=augment(sy,{l}) ElseIf lbit=0 and nbit=1 Then sx:=augment(sx,{i,i+1}): sy:=augment(sy,{l+1,l+1}) EndIf lbit:=nbit EndWhile Disp "Scatter sequences in sx and sy" EndPrgm         </pre>
<pre> signal(b,2) Scatter sequences in sx and sy Done sxb:=sx:syb:=sy {3,3,3,3,3,2,2,3,3,2,2,3,3,2,2,3,3,2,2,3,3,3}         </pre>	
<pre> signal(c,4) Scatter sequences in sx and sy Done sxc:=sx:syc:=sy {4,4,5,5,4,4,4,5,5,5,4,4,5,5,5,4,4,5,5,4,4,5,5}         </pre>	
<pre> signal(d,6) Scatter sequences in sx and sy Done sxd:=sx:syd:=sy {7,7,6,6,6,7,7,6,6,6,7,7,6,6,6,7,7,6,6,6,6}         </pre>	
<pre> signal(e,8) Scatter sequences in sx and sy Done sxe:=sx:sye:=sy {8,8,8,9,9,9,8,8,8,9,9,9,8,8,8,9,9,8,8,9,9,8,8}         </pre>	

The plot of the given signals on the TI-Nspire Graphs Application:



This is the DERIVE code:

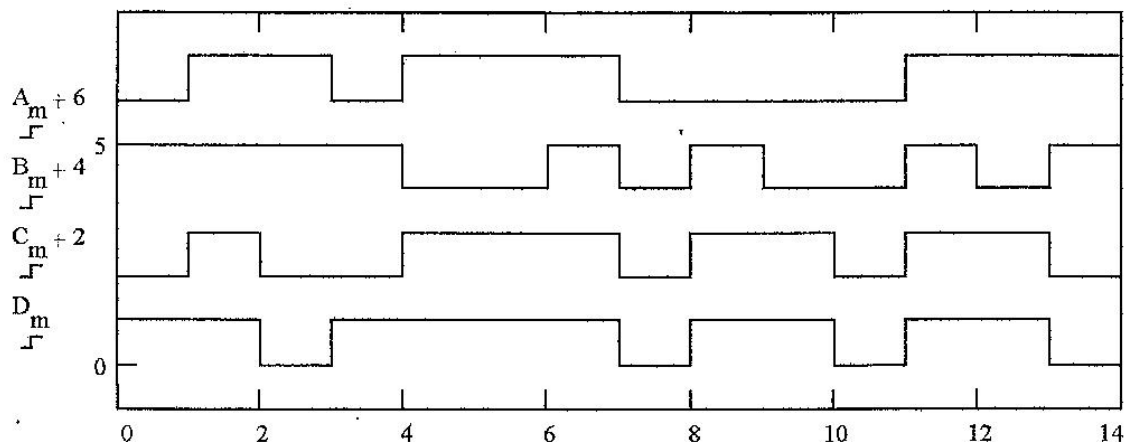
```

signal(x, l := 0, vx, sig, lbit, nbit, i) :=
  Prog
    vx := mv(x)
    i := 0
    sig := IF(vx↓1 = 0, [i, l; i + 1, l], [i, l + 1; i + 1, l + 1])
  Loop
    lbit := FIRST(vx)
    vx := REST(vx)
    If DIM(vx) = 0
      RETURN sig
    i := i + 1
    nbit := FIRST(vx)
    sig := IF(lbit·nbit = 1, APPEND(sig, [[i + 1, l + 1]]),
             IF(lbit = 1 ∧ nbit = 0, APPEND(sig, [[i, l]], [[i + 1, l]]),
             IF(lbit = 0 ∧ nbit = 0, APPEND(sig, [[i + 1, l]]),
             IF(lbit = 0 ∧ nbit = 1,
                APPEND(sig, [[i, l + 1]], [[i + 1, l + 1]]))))))
    lbit := nbit

```

There were some examples in the paper mentioned above demonstrating logical combinations of the A, B, and C signals. This is the last one of them:

$D := \text{OR}(C, \text{XOR}(A, \text{OR}(C, B)))$



We see that Polish notation is used. This might be a good occasion to introduce this to our students. Since I did not find an easy way to apply the Boolean operators on the signals, I had the idea to produce “Boolean functions” in order to apply Polish notation, too.

So, I produced functions for all Boolean operations for DERIVE and simultaneously for TI-Nspire, too.

```

and_(x, y, bo) :=
  Prog
    bo := ""
    [x := mv(x), y := mv(y)]
  Loop
    If DIM(x) = 0
      RETURN bo
    bo := APPEND(bo, FIRST(x) ∧ FIRST(y))
    [x := REST(x), y := REST(y)]

```

$\text{and\_}(a, b) = 011000100001011$

<code>or_(a,b)</code>	"111111101001111"
<code>and_(a,b)</code>	"011000100001011"
<code>or_(a,b)</code>	"111111101001111"
<code>not_(a)</code>	"100100011110000"
<code>xor_(a,b)</code>	"100111001000100"
<code>nand_(a,b)</code>	"100111011110100"

```

"or_" stored successfully
Define or_(x,y)=
Func
Local bo,i
bo:=""
for i,1,dim(x)
  bo:=bo&string(x[i] or y[i])
EndFor
EndFunc

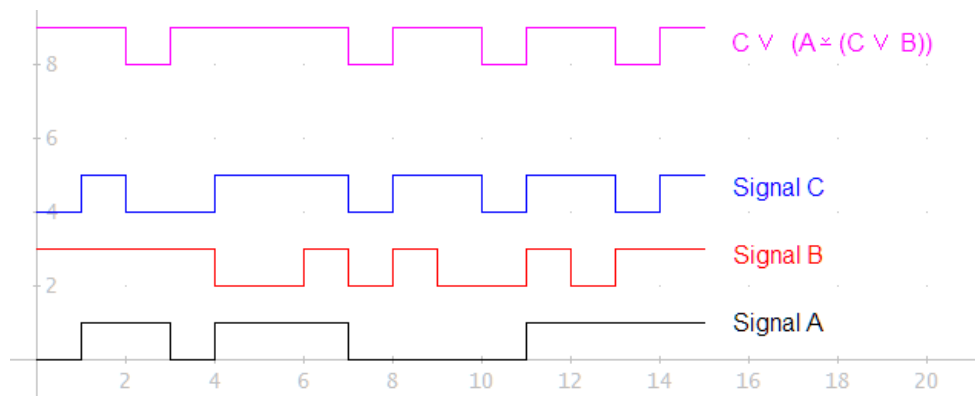
```

Then I tried them on the logic composition which I found in the paper:

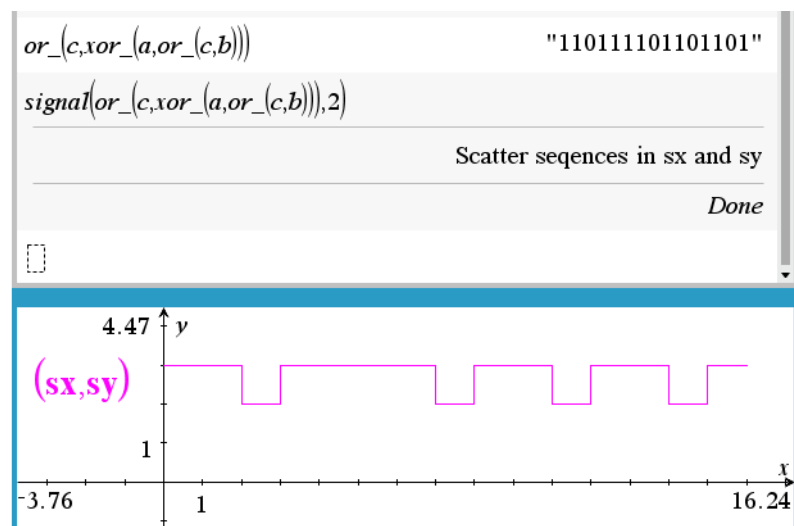
```
or_(c, xor_(a, or_(c, b)))
```

```
110111101101101
```

```
signal(or_(c, xor_(a, or_(c, b))), 8)
```



The same procedure performed in the TI-Nspire environment:



We can use this tool to demonstrate Boolean identities and accompany truth tables. See a very short example:

You can see the truth table, followed by the plots of the respective signals

[a :=, b :=]

TRUTH\_TABLE(a, b,  $\neg(a \wedge b)$ ,  $\neg a \vee \neg b$ )

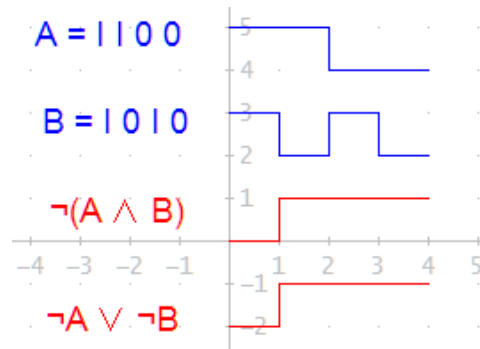
a	b	$\neg(a \wedge b)$	$\neg a \vee \neg b$
true	true	false	false
true	false	true	true
false	true	true	true
false	false	true	true

signal(1100, 4)

signal(1010, 2)

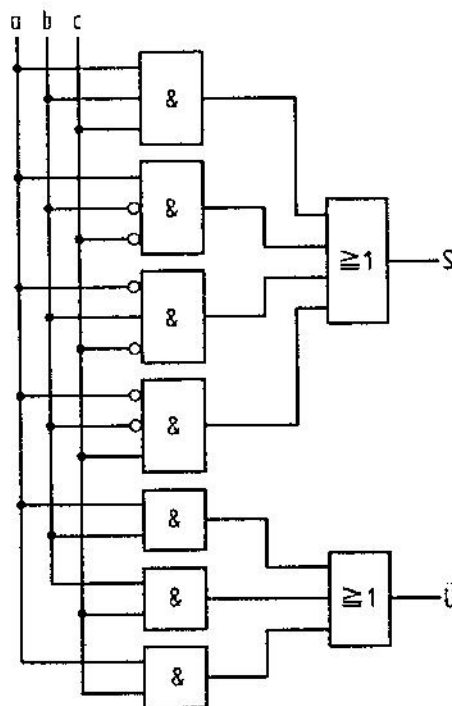
signal(not\_(and\_(1010, 1100)))

signal(or\_(not\_(1010), not\_(1100)), -2)



In the same map of very old papers I found another one about “Logic Operators”. The exercises were to write BASIC programs for “Automatic creation of truth tables for digital circuits”. A more complex exercise showed the diagram of a Full Adder together with the expected program output.

### Volladdierer



$$S = (a \wedge b \wedge c) \vee (a \wedge \bar{b} \wedge \bar{c}) \vee (\bar{a} \wedge b \wedge \bar{c}) \vee (\bar{a} \wedge \bar{b} \wedge c)$$

$$U = (a \wedge b \wedge c) \vee (a \wedge b \wedge \bar{c}) \vee (a \wedge \bar{b} \wedge c) \vee (\bar{a} \wedge b \wedge c)$$

& is the circuit symbol for AND,  $\geq$  is the symbol for OR, the small  $\circ$  is the negation; S is the output of the sum, and U is the output of the carry (for “Übertrag” in German).

```
>RUN
WIEVIEL EINGANGSVARIABLE ? 3
WIEVIEL AUSGANGSVARIABLE ? 2

A B C      Q1 Q2
0 0 0      0 0
0 0 1      1 0
0 1 0      1 0
0 1 1      0 1
1 0 0      1 0
1 0 1      0 1
1 1 0      0 1
1 1 1      1 1
READY
```

It might be not so easy for students to transfer the logic operation in polish notation (another “competence?”).

So, let me try to demonstrate this circuit using my signal plots:

$$\text{sum\_}(p, q, r) := (p \wedge q \wedge r) \vee (p \wedge \neg q \wedge \neg r) \vee (\neg p \wedge q \wedge \neg r) \vee (\neg p \wedge \neg q \wedge r)$$

$$\text{carry\_}(p, q, r) := (p \wedge q \wedge r) \vee (p \wedge q \wedge \neg r) \vee (p \wedge \neg q \wedge r) \vee (\neg p \wedge q \wedge r)$$

TRUTH\_TABLE(p, q, r, sum\_(p, q, r), carry\_(p, q, r))

p	q	r	$(\neg p \wedge ((\neg q \wedge r) \vee (q \wedge \neg r))) \vee (p \wedge ((\neg q \wedge \neg r) \vee (q \wedge r)))$	$(p \vee (q \wedge r)) \wedge (q \vee r)$
true	true	true	true	true
true	true	false	false	true
true	false	true	false	true
true	false	false	true	false
false	true	true	false	true
false	true	false	true	false
false	false	true	true	false
false	false	false	false	false

You must read the last column in reverse order, because TRUTH\_TABLE starts with [1,1,1] and ends with [0,0,0]. It is remarkable that DERIVE simplifies the Boolean expressions for sum\_ and carry\_.

turnm(mat) := VECTOR(mat, i, DIM(mat), 1, -1)

turnm(TRUTH\_TABLE(p, q, r, sum\_(p, q, r), carry\_(p, q, r)))

false	false	false	false	false
false	false	true	true	false
false	true	false	true	false
false	true	true	false	true

With TI-Nspire we can make use of the respective function provided in Bhuvanesh Bhatt's great library mathtool.tns:

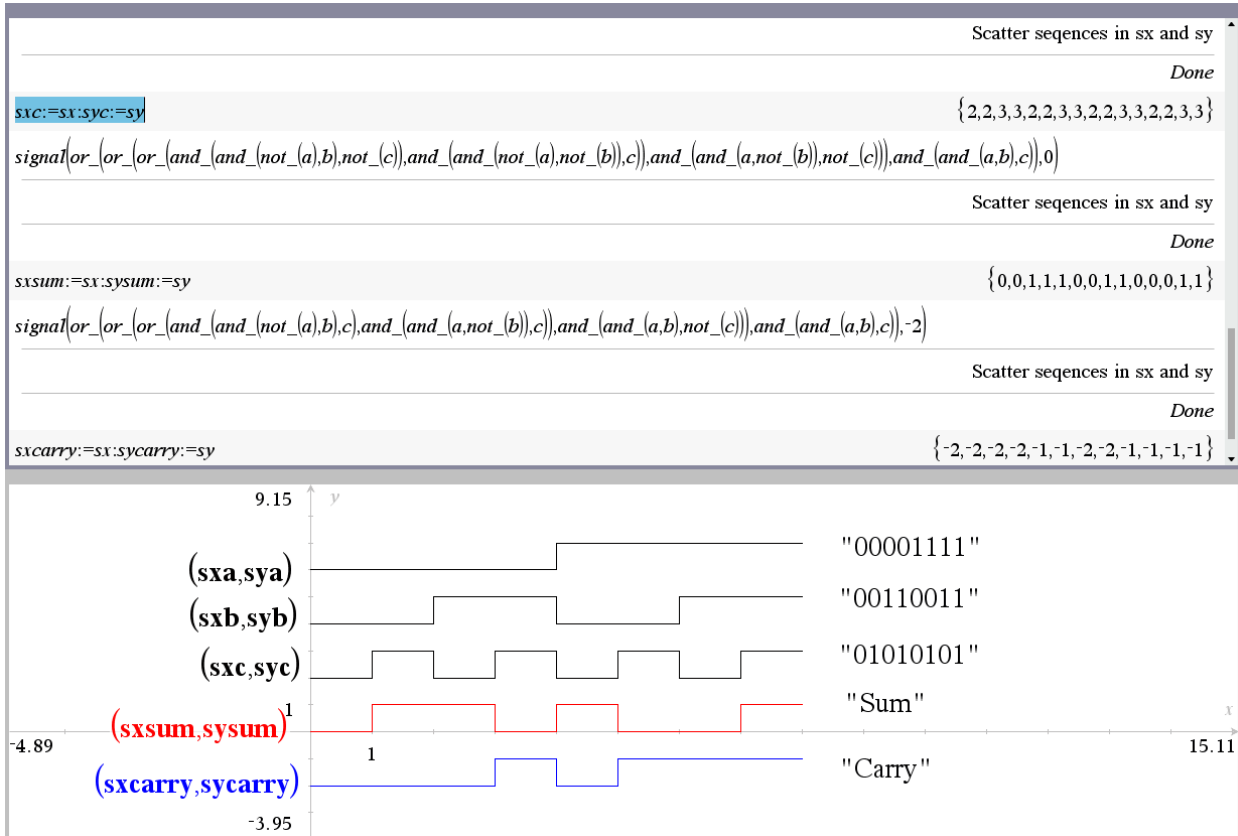
**mathtool\truthtbl(p and q and r or p and not q and not r or not p and q and not r or not p and not q and r)**

r	p	q	not r and not p and q or not r and p and not q or r and not p and not q or r and p and q
true	true	true	true
true	true	false	false
true	false	true	false
true	false	false	true
false	true	true	false
false	true	false	true
false	false	true	true
false	false	false	false

**mathtool\truthtbl(p and q and r or p and not q and r or not p and q and r or p and q and not r)**

r	p	q	not r and p and q or r and not p and q or r and p
true	true	true	true
true	true	false	true
true	false	true	true
true	false	false	false
false	true	true	true
false	true	false	false
false	false	true	false
false	false	false	false

We can perform demonstrating the full adder with TI-Nspire. All signals must be stored as two lists for the  $x$ - and  $y$ -coordinates of the graphs given as scatter diagrams.



truth\_t() (function truth\_table.tns from DNL#98/2015)

truth\_t()

Enter variables: a,b,c

Enter Boolean expression: (a and b and c) or (a and not b and not c) or (not a and not b and c) or (not a and b and not c)

a	b	c	"(a and b and c) or (a and not b and not c) or (not a and not b and c) or (not a and b and not c)"
true	true	true	true
true	true	false	false
true	false	true	false
true	false	false	true
false	true	true	false
false	true	false	true
false	false	true	true
false	false	false	false

## CARACTERÍSTICAS GEOMÉTRICAS DE LAS SECCIONES PLANAS

### GEOMETRICAL CHARACTERISTICS OF PLANE SECTIONS

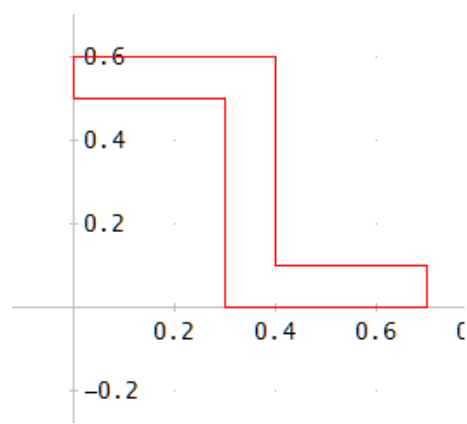
Martínez Vargas Felipe de Jesús, Mexico

#### Polígonos con áreas discontinuas – Polygons with discontinuous areas

- Si tenemos múltiples polígonos se presentarán los puntos de sus vértices en el mismo sentido.
- Cerrar cada polígono coincidiendo el primer vértice con el último.

If we have multiple polygons, the points of their vertices will be presented in the same direction. Close each polygon coinciding the first vertex with the last one.

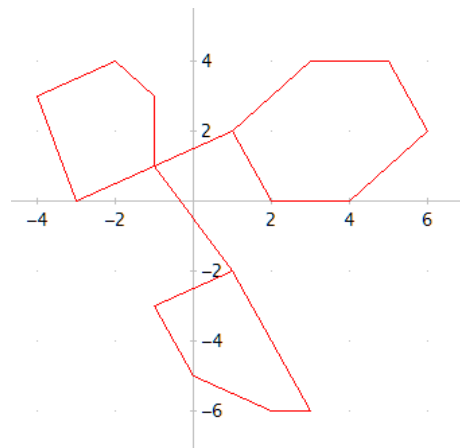
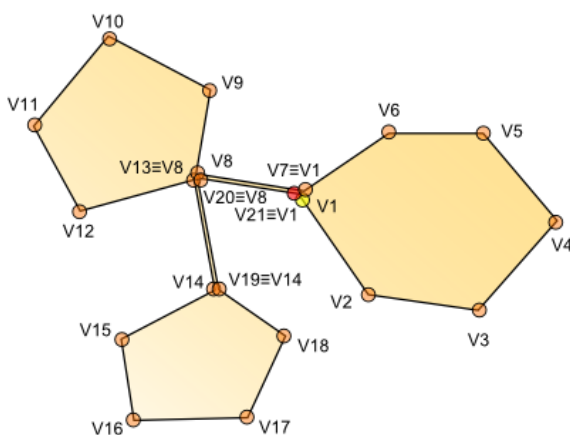
```
#1: pol1 :=
[ 0.7  0
  0.7  0.1
  0.4  0.1
  0.4  0.6
  0  0.6
  0  0.5
  0.3  0.5
  0.3  0
  0.7  0 ]
```



#### Polígonos con áreas discontinuas – Polygons with discontinuous areas

- Si tenemos múltiples polígonos se presentarán los puntos de sus vértices en el mismo sentido.
- Cerrar cada polígono coincidiendo el primer vértice con el último.

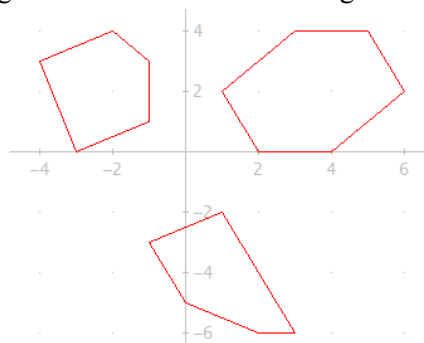
If we have multiple polygons, the points of their vertices will be presented in the same direction. Close each polygon coinciding the first vertex with the last one.



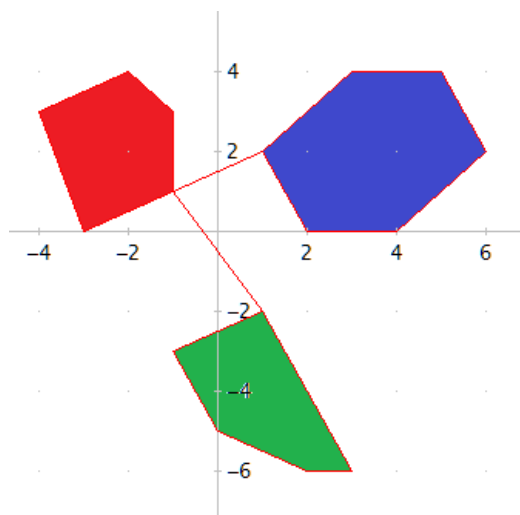
```
pol2' = [ 1  2  4  6  5  3  1  -1  -1  -2  -4  -3  -1  1  -1  0  2  3  1  -1  1 ]
         [ 2  0  0  2  4  4  2  1  3  4  3  0  1  -2  -3  -5  -6  -6  -2  1  2 ]
```

I present pol2 (consisting of three separated polygons) in transposed form in order to save space. Felipe added a small auxiliary function which enables plotting the parts without connecting segments first and composing them into the format which makes the figure suitable for the following functions.

$$\text{pol21} := \begin{bmatrix} 1 & 2 \\ 2 & 0 \\ 4 & 0 \\ 6 & 2 \\ 5 & 4 \\ 3 & 4 \\ 1 & 2 \end{bmatrix}, \text{pol22} := \begin{bmatrix} -1 & 1 \\ -1 & 3 \\ -2 & 4 \\ -4 & 3 \\ -3 & 0 \\ -1 & 1 \end{bmatrix}, \text{pol23} := \begin{bmatrix} 1 & -2 \\ -1 & -3 \\ 0 & -5 \\ 2 & -6 \\ 3 & -6 \\ 1 & -2 \end{bmatrix}$$



```
POLY_COMPOUND(v, n, i, out_) :=
  Prog
  n := DIM(v)
  i := 1
  out_ := []
  Loop
  out_ := APPEND(out_, [FIRST(v_i)])
  If i = n - 1
    exit
  i := i + 1
  out_ := APPEND(out_, REVERSE(out_))
```



```
POLY_COMPOUND([pol21, pol22, pol23])
```

gives pol2 from above.

How to produce the colored plot?

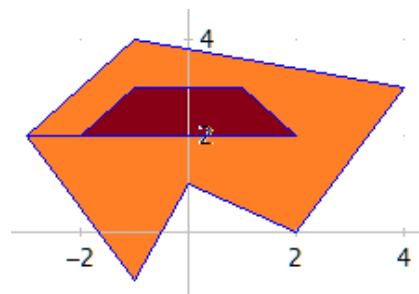
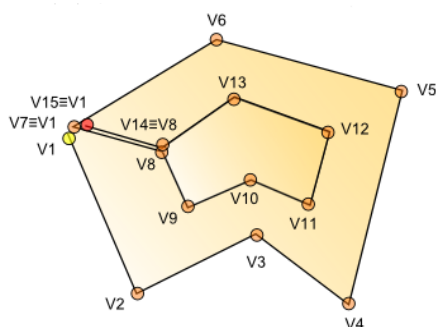
See DNL#63 from 2006:

*Tips from Tania: Thick Lines and Coloured Plots.*

### Polígonos con áreas superpuestas – Polygons with overlapping areas

- Para múltiples polígonos se puede dibujar áreas superpuestas, donde sus vértices deben de tener el mismo sentido.
- Cerrar cada polígono coincidiendo el primer vértice con el último.

For multiple polygons you can draw overlapping areas, where their vertices must have the same direction. Close each polygon coinciding the first vertex with the last one.



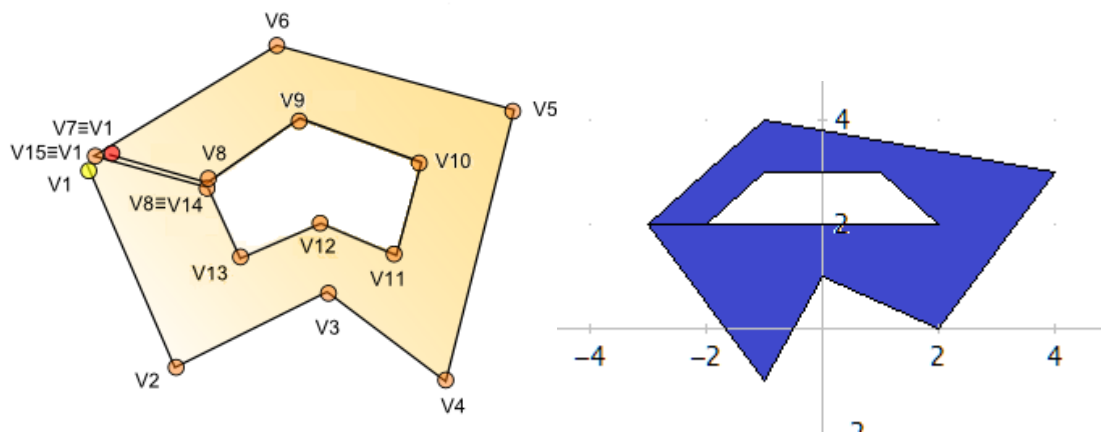
$$\text{pol3}' = \begin{bmatrix} -3 & -1 & 0 & 2 & 4 & -1 & -3 & -2 & 2 & 1 & -1 & -2 & -3 \\ 2 & -1 & 1 & 0 & 3 & 4 & 2 & 2 & 2 & 3 & 3 & 2 & 2 \end{bmatrix}$$



## Polígonos con áreas huecas

### Polygons with hollow areas

- Para múltiples polígonos se puede dibujar áreas huecas dentro del polígono.
- Los vértices deben darse en sentido contrario a las manecillas del reloj y la sección hueca en sentido de las manecillas.
- Cerrar cada polígono coincidiendo el primer vértice con el último.
- For multiple polygons you can draw hollow areas within the polygon.
- The vertices should be counterclockwise and the hollow section should be clockwise.
- Close each polygon coinciding the first vertex with the last one.



$$\text{po141} := \begin{bmatrix} -3 & 2 \\ -1 & -1 \\ 0 & 1 \\ 2 & 0 \\ 4 & 3 \\ -1 & 4 \\ -3 & 2 \end{bmatrix}, \quad \text{po142} := \begin{bmatrix} -2 & 2 \\ -1 & 3 \\ 1 & 3 \\ 2 & 2 \\ -2 & 2 \end{bmatrix}$$

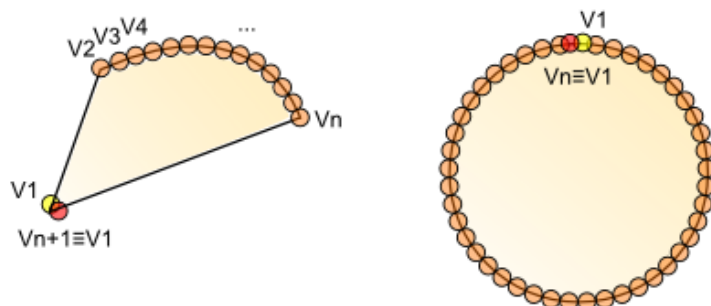
```
po14 := POLY_COMPOUND([po141, po142])
```

## Polígonos con lados curvos

### Polygons with curved sides

Usando una cantidad adecuada de puntos se puede dibujar lados curvados (y por lo tanto figuras genéricas).

Using an adequate number of points, curved sides (and therefore generic figures) can be drawn.



**PROPIEDADES DE LA SECCIÓN REPRESENTATIVA –  
PROPERTIES OF THE CROSS-SECTION**

• **Perímetro Perimeter**

```

P(polygon, x, y, n) :=
  Prog
  x := polygon COL 1
  y := polygon COL 2
#18: n := DIM(polygon)
      Σ(√((x↓(i + 1) - x↓i)^2 + (y↓(i + 1) - y↓i)^2), i, 1, n - 1)

ΔP(polygon, x, y, j, k, n, sWise, oWise, links_) :=
  Prog
  [x := polygon COL 1, y := polygon COL 2, n := DIM(polygon)]
  links_ := 0
  j := 1
  Loop
  sWise := oWise := 0
  k := j + 2
  Loop
  If (x↓j = x↓k ∧ y↓j = y↓k) ∧ (x↓(j + 1) = x↓(k - 1) ∧ y↓(j + 1) = y↓(k - 1))
    oWise := oWise + 1
  If (x↓j = x↓(k - 1) ∧ y↓j = y↓(k - 1)) ∧ (x↓(j + 1) = x↓k ∧ y↓(j + 1) = y↓k)
#19:   sWise := sWise + 1
      0
  If k = n
    exit
    k := + 1
  If oWise > 0
    links_ := links_ + √((x↓(j + 1) - x↓j)^2 + (y↓(j + 1) - y↓j)^2) · (2 · (oWise - sWise))
    0
  If j = n - 2
    exit
    j := + 1
  links_

```

• **Área Area**

```

A(polygon, x, y, n) :=
  Prog
  x := polygon COL 1
  y := polygon COL 2
#20: n := DIM(polygon)
      If n < 3
        0
        1/2 · Σ(x↓i · y↓(i + 1) - x↓(i + 1) · y↓i, i, 1, n - 1)

#21: per_ar(pol) := [ Perímetro/Perimeter = P(pol) - ΔP(pol) ]
                   [ Área/Area = |A(pol)| ]

#22: per_ar(pol1) = [ Perímetro/Perimeter = 2.6 ]
                   [ Área/Area = 0.12 ]

#23: per_ar(pol2) = [ Perímetro/Perimeter = √10 + 9·√5 + 5·√2 + 7 ]
                   [ Área/Area = 30 ]

#24: per_ar(POLY_COMPOUND([pol21, pol22, pol23]))

#25: [ Perímetro/Perimeter = √10 + 9·√5 + 5·√2 + 7 ]
      [ Área/Area = 30 ]

#26: [per_ar(pol21), per_ar(pol22), per_ar(pol23)]

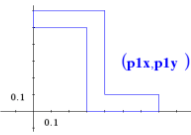
```

For TI-Nspire : propiades .tns:

<pre>dperim 9/16 Define dperim(pol)= Func Local x,y,n,j,k,swise,owise,links_ x:=mat▶list(pol[1]);y:=mat▶list(pol[2]);n:=dim(pol)[2];links_:=0 For j,1,n-2   swise:=0;owise:=0   For k,j+2,n     If x[j]=x[k] and y[j]=y[k] and x[j+1]=x[k-1] and y[j+1]=y[k-1] Then       owise:=owise+1     Else       If x[j]=x[k-1] and y[j]=y[k-1] and x[j+1]=x[k] and y[j+1]=y[k] Then         swise:=swise+1       EndIf;EndIf;EndFor   If owise&gt;0 Then</pre>	<pre>per 4/4 Define per(pol)= Func Local x,y x:=mat▶list(pol[1]) y:=mat▶list(pol[2]) dim(pol)[2]-1 <math display="block">\sum_{i=1}^{dim(pol)[2]-1} \left( \sqrt{(x[i+1]-x[i])^2+(y[i+1]-y[i])^2} \right)</math> EndFunc</pre>
<pre>per_ar 1/1 Define per_ar(pol)= Func ["Perimeter =" per(pol)-dperim(pol)] ["Area =" area(pol)] EndFunc</pre>	<pre>area 2/7 Define area(pol)= Func Local x,y,n x:=mat▶list(pol[1]);y:=mat▶list(pol[2]);n:=dim(pol)[2] If n&lt;3 Then   0 Else   <math display="block">\frac{1}{2} \cdot \sum_{i=1}^{dim(pol)[2]-1} \left( x[i] \cdot y[i+1] - x[i+1] \cdot y[i] \right)</math> EndIf EndFunc</pre>

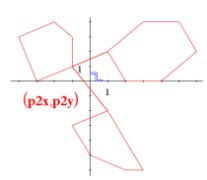
We need two lists in order to plot the polygons as a scatter diagram.

```
pol1:=⎣0.7 0.7 0.4 0.4 0 0 0.3 0.3 0.7⎤
      ⎣0 0.1 0.1 0.6 0.6 0.5 0.5 0 0⎥
p1x:=mat▶list(pol1[1]) p1y:=mat▶list(pol1[2])
```



```
per_ar(pol1) ▶ ["Perimeter =" 2.6]
              ["Area =" 0.12]
```

```
pol2:=⎣1 2 4 6 5 3 1 -1 -1 -2 -4 -3 -1 1 -1 0 2 3 1 -1 1⎤
      ⎣2 0 0 2 4 4 2 1 3 4 3 0 1 -2 -3 -5 -6 -6 -2 1 2⎥
p2x:=mat▶list(pol2[1]) p2y:=mat▶list(pol2[2])
```

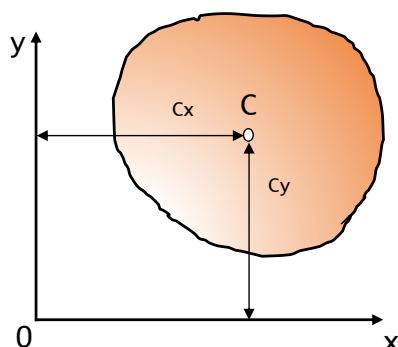


```
per_ar(pol2) ▶ ["Perimeter =" 5·√5+(√2+4)·√5+5·√2+7]
              ["Area =" 30]
```

```
pol4:=⎣-3 -1 0 2 4 -1 -3 -2 -1 1 2 -2 -3⎤
      ⎣2 -1 1 0 3 4 2 2 3 3 2 2 2⎥
```

```
per_ar(pol4) ▶ ["Perimeter =" (√13+4)·(√2+2)+2·√5-2]
              ["Area =" 31/2]
```

## Centroide Centroid



Es el punto C donde se considera concentrada el área total de una figura, donde se supone está ubicado el centro geométrico del cuerpo plano y homogéneo.

Las coordenadas del centroide del área total son Cx y Cy.

It is the point C where the total area of a figure is considered to be concentrated, where the geometric center of the flat and homogeneous body is supposed to be located.

The coordinates of the centroid of the total area are Cx and Cy.

```
Cy(polygon, x, y, n) :=
Prog
[x := polygon COL 1, y := polygon COL 2]
n := DIM(polygon)
1/(6*A(polygon))*Σ((y↓i + y↓(i + 1))*(x↓i*y↓(i + 1) - x↓(i + 1)*y↓i), i, 1, n - 1)
```

```
Cx(polygon, x, y, n) :=
Prog
[x := polygon COL 1, y := polygon COL 2]
n := DIM(polygon)
1/(6*A(polygon))*Σ((x↓i + x↓(i + 1))*(x↓i*y↓(i + 1) - x↓(i + 1)*y↓i), i, 1, n - 1)
```

```
Centr_grav(polygon) := [ Centro de gravedad/Center of Gravity ]
                        [ xc = Cx(polygon) ]
                        [ yc = Cy(polygon) ]
```

```
Centr_grav(pol1) = [ Centro de gravedad/Center of Gravity ]
                   [ xc = 0.35 ]
                   [ yc = 0.3 ]
```

## Momento estático de área Static moment of area

El primer momento de área (también momento estático o de primer orden) es una magnitud geométrica que se define para un área plana.

The first moment of area (also static or first order moment) is a geometric quantity that is defined for a plane area.

```
My(polygon, x, y, n) :=
Prog
[x := polygon COL 1, y := polygon COL 2]
n := DIM(polygon)
ABS(A(polygon))*Cx(polygon)
```

```
Mx(polygon, x, y, n) :=
Prog
[x := polygon COL 1, y := polygon COL 2]
n := DIM(polygon)
ABS(A(polygon))*Cy(polygon)
```

$$\text{Mom\_est}(\text{polygon}) := \begin{bmatrix} \text{Momento estático/Static Moment} \\ \begin{bmatrix} M_x = M_x(\text{polygon}) \\ M_y = M_y(\text{polygon}) \end{bmatrix} \end{bmatrix}$$

$$\text{Mom\_est}(\text{pol1}) = \begin{bmatrix} \text{Momento estático/Static Moment} \\ \begin{bmatrix} M_x = 0.036 \\ M_y = 0.042 \end{bmatrix} \end{bmatrix}$$

## Momento de inercia de la figura plana **Moment of inertia of a plane figure**

Propiedad geométrica de la sección transversal de elementos estructurales

Geometric property of the cross-section of structural elements

```
Jxx(polygon, x, y, n, X_, Y_) :=
  Prog
  [x := polygon COL 1, y := polygon COL 2]
  n := DIM(polygon)
  1/12 * Σ((x↓i * y↓(i + 1) - x↓(i + 1) * y↓i) * (y↓i^2 + y↓i * y↓(i + 1) + y↓(i + 1)^2), i, 1, n - 1)

Jyy(polygon, x, y, n, X_, Y_) :=
  Prog
  [x := polygon COL 1, y := polygon COL 2]
  n := DIM(polygon)
  1/12 * Σ((x↓i * y↓(i + 1) - x↓(i + 1) * y↓i) * (x↓i^2 + x↓i * x↓(i + 1) + x↓(i + 1)^2), i, 1, n - 1)

Ix(polygon) := |Jxx(polygon)|
Iy(polygon) := |Jyy(polygon)|
```

## Momento de inercia centrífugo **Moment of centrifugal inertia**

Según sea la posición de los ejes, positivos o negativos o igual a cero. Los ejes respecto a los cuales el momento de inercia centrífugo es igual a cero se denomina *ejes de inercia principal*. Los ejes principales que pasan a través del centro de gravedad de la sección se denominan *ejes principales centrales*.

Depending on the position of the axes, positive or negative or equal to zero. The axes with respect to which the centrifugal moment of inertia is equal to zero are called *principal axes of inertia*. The principal axes passing through the center of gravity of the section are called *central principal axes*.

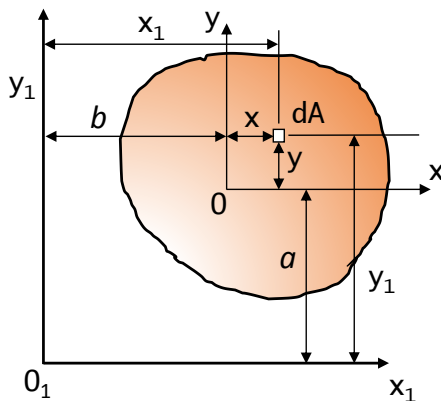
```
Jxy(polygon, x, y, n) :=
  Prog
  [x := polygon COL 1, y := polygon COL 2]
  n := DIM(polygon)
  1/24 * Σ((x↓i * y↓(i + 1) - x↓(i + 1) * y↓i) * (x↓i * y↓(i + 1) + 2 * x↓i * y↓i + 2 * x↓(i + 1) * y↓(i + 1) + x↓(i + 1) * y↓i), i, 1, n - 1)

Ixy(polygon) := Jxy(polygon)
```

$$\text{Mom\_inert}(\text{polygon}) := \begin{bmatrix} \text{Momentos de inercia/Intertial Moment} \\ \begin{bmatrix} I_x = I_x(\text{polygon}) \\ I_y = I_y(\text{polygon}) \\ I_{xy} = I_{xy}(\text{polygon}) \end{bmatrix} \end{bmatrix}$$

$$\text{Mom\_inert}(\text{pol1}) = \begin{bmatrix} \text{Momentos de inercia/Intertial Moment} \\ \begin{bmatrix} I_x = 0.0164 \\ I_y = 0.0176 \\ I_{xy} = 0.0096 \end{bmatrix} \end{bmatrix}$$

### Momentos de inercia respecto a los ejes paralelos Moments of inertia with respect to parallel axes



El momento de inercia de un área con respecto a un eje es igual al momento de inercia del área con respecto a un eje paralelo que atraviesa su centroide, más el producto del área y el cuadrado de la distancia perpendicular entre los ejes.

The moment of inertia of an area with respect to an axis is equal to the moment of inertia of the area with respect to a parallel axis passing through its centroid, plus the product of the area and the square of the perpendicular distance between the axes.

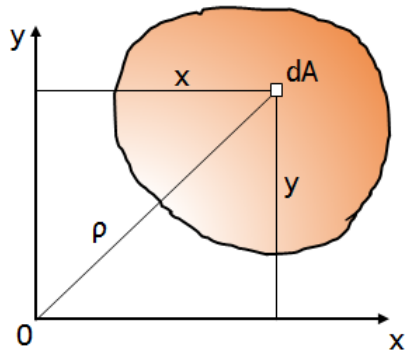
$$\text{Mom\_inert\_centr}(\text{polygon}) := \begin{bmatrix} \text{Momentos de inercia centroidal} \\ \text{Centroidal moments of inertia} \\ \begin{bmatrix} I_{xc} = I_{xc}(\text{polygon}) \\ I_{yc} = I_{yc}(\text{polygon}) \\ I_{xcyc} = I_{xcyc}(\text{polygon}) \end{bmatrix} \end{bmatrix}$$

$$\text{Mom\_inert\_centr}(\text{pol1}) = \begin{bmatrix} \text{Momentos de inercia centroidal} \\ \text{Centroidal moments of inertia} \\ \begin{bmatrix} I_{xc} = 0.0056 \\ I_{yc} = 0.0029 \\ I_{xcyc} = -0.003 \end{bmatrix} \end{bmatrix}$$

### Momento polar de inercia Polar moment of inertia

Es una cantidad utilizada para predecir habilidad para resistir la torsión del objeto, en los objetos (o segmentos de los objetos) con un invariante circular de sección transversal y sin deformaciones importantes o fuera del plano de deformaciones.

It is a quantity used to predict ability to resist object torsion, in objects (or segments of objects) with an invariant circular cross-section and without major deformations or out-of-plane deformations.



$$I_p(\text{polygon}) := I_x(\text{polygon}) + I_y(\text{polygon})$$

$$\text{Mom\_pol}(\text{polygon}) := \begin{bmatrix} \text{Momento polar de inercia} \\ \text{Polar moment of inertia} \\ [I_p =, I_p(\text{polygon})] \end{bmatrix}$$

$$\text{Mom\_pol}(\text{pol1}) = \begin{bmatrix} \text{Momento polar de inercia} \\ \text{Polar moment of inertia} \\ [I_p =, 0.034] \end{bmatrix}$$

**Momento polar de inercia respecto a los ejes paralelos** **Polar moment of inertia with respect to parallel axes**

$$I_{p_{xy}}(\text{polygon}) := I_{xc}(\text{polygon}) + I_{yc}(\text{polygon})$$

$$\text{Mom\_pol\_centr}(\text{polygon}) := \begin{bmatrix} \text{Momento polar de inercia centroidal} \\ \text{Polar moment of centroidal inertia} \\ [I_{p_{xy}} =, I_{p_{xy}}(\text{polygon})] \end{bmatrix}$$

$$\text{Mom\_pol\_centr}(\text{pol1}) = \begin{bmatrix} \text{Momento polar de inercia centroidal} \\ \text{Polar moment of centroidal inertia} \\ [I_{p_{xy}} =, 0.0085] \end{bmatrix}$$

**Radio polar de inercia** **Polar radius of inertia**

$$r_{p_{xy}}(\text{polygon}) := \sqrt{\frac{I_{p_{xy}}(\text{polygon})}{|A(\text{polygon})|}}$$

$$\text{Rad\_pol}(\text{polygon}) := \begin{bmatrix} \text{Radios polar de inercia} \\ \text{Polar radius of inertia} \\ [r_{p_{xy}} =, r_{p_{xy}}(\text{polygon})] \end{bmatrix}$$

$$\text{Rad\_pol}(\text{pol1}) = \begin{bmatrix} \text{Radios polar de inercia} \\ \text{Polar radius of inertia} \\ [r_{p_{xy}} =, 0.2661453237] \end{bmatrix}$$

**Radios de giro** **Rotation radii**

Describe la forma en la cual el área transversal se distribuye alrededor de su eje centroidal. Concretamente es el valor medio cuadrático de distancia de los puntos de la sección respecto a un eje que pasa por el centro de la misma.

Describes the way in which the cross-sectional area is distributed around its centroidal axis. Specifically, it is the root mean square value of the distance of the points of the section with respect to an axis passing through the center of the section.

$$\left[ rx(\text{polygon}) := \sqrt{\frac{Ix(\text{polygon})}{|A(\text{polygon})|}}, ry(\text{polygon}) := \sqrt{\frac{Iy(\text{polygon})}{|A(\text{polygon})|}} \right]$$

$$\text{Rot\_rad}(\text{polygon}) := \left[ \begin{array}{c} \text{Radios de giro/Rotation radii} \\ \left[ \begin{array}{c} rx = rx(\text{polygon}) \\ ry = ry(\text{polygon}) \end{array} \right] \end{array} \right]$$

$$\text{Rot\_rad}(\text{pol1}) = \left[ \begin{array}{c} \text{Radios de giro/Rotation radii} \\ \left[ \begin{array}{c} rx = 0.3696845502 \\ ry = 0.3829708431 \end{array} \right] \end{array} \right]$$

Radios de giro respecto al centríde **Rotation radii with respect to the centroid**

$$\left[ rxc(\text{polygon}) := \sqrt{\frac{Ixc(\text{polygon})}{|A(\text{polygon})|}}, ryc(\text{polygon}) := \sqrt{\frac{Iyc(\text{polygon})}{|A(\text{polygon})|}} \right]$$

$$\text{Rot\_rad\_centr}(\text{polygon}) := \left[ \begin{array}{c} \text{Radios de giro centrídales} \\ \text{Rot radii wrt centr.} \\ \left[ \begin{array}{c} rxc = rxc(\text{polygon}) \\ ryc = ryc(\text{polygon}) \end{array} \right] \end{array} \right]$$

$$\text{Rot\_rad\_centr}(\text{pol1}) = \left[ \begin{array}{c} \text{Radios de giro centrídales} \\ \text{Rot radii wrt centr.} \\ \left[ \begin{array}{c} rxc = 0.2160246899 \\ ryc = 0.1554563175 \end{array} \right] \end{array} \right]$$

To be continued in the next DNL

Some websites for additional information

<https://derevofilm.ru/en/centrobezhnyi-moment-inercii-moment-inercii-dlya-chainikov/>

[https://en.wikipedia.org/wiki/Second\\_moment\\_of\\_area](https://en.wikipedia.org/wiki/Second_moment_of_area)

<https://apps.dtic.mil/dtic/tr/fulltext/u2/a183444.pdf>

<https://web.archive.org/web/20181003014106/https://pdfs.semanticscholar.org/fd18/036ba9b78174a2a161b184148028a43881a8.pdf>

[https://en.wikipedia.org/wiki/Second\\_polar\\_moment\\_of\\_area](https://en.wikipedia.org/wiki/Second_polar_moment_of_area)

A Russian website (in English):

<https://derevofilm.ru/en/centrobezhnyi-moment-inercii-moment-inercii-dlya-chainikov/>

Next page: The TI-Nspire collection of the functions: propiedades.tns





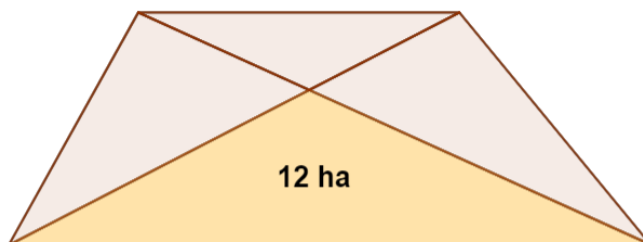
Wolfgang Alvermann visits regularly the website <https://www.spektrum.de/raetsel/> where one can find nice and sometimes really challenging mathematical puzzles. Here is one of them treated very extended using vectors. I tried to solve the problem in another way. It follows after Wolfgang's contribution. The original German version is contained in MTH129.zip.

### Wie groß ist das Landstück des Bauern?

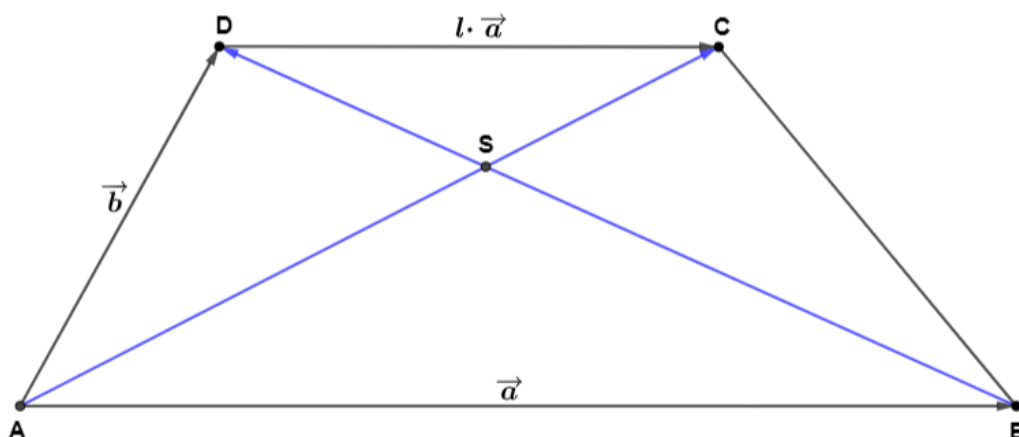
### What is the Size of the Farmer's Land?

<https://www.spektrum.de/raetsel/wie-gross-ist-das-landstueck-des-bauern/2101683>

"A farmer dies and leaves his four sons a trapezoidal piece of land. In his will he stipulated that it must be divided into four pieces by the two diagonals and that his eldest son should receive the largest piece of 12 hectares. What is the total land area if the diagonals intersect each other at a ratio of 2 : 1?"



The division ratio of the transversals in trapezoids - viewed vectorially:



Given are  $\vec{a}$  and  $\vec{b}$  and the factor  $l \neq 0$ .  $\vec{AC} = \vec{b} + l \cdot \vec{a}$  and  $\vec{BD} = -\vec{a} + \vec{b}$

The vector circulation ASBA results in the zero vector:

$$\vec{AS} + \vec{SB} + \vec{BA} = \vec{0} \quad \text{ting} \rightarrow \vec{a} \cdot (m \cdot l + n - 1) + \vec{b} \cdot (m - n) = \vec{0}$$

$$m \cdot (\vec{b} + l \cdot \vec{a}) - n \cdot (-\vec{a} + \vec{b}) - \vec{a} = \vec{0}$$

The zero vector can result only if the expressions in the parentheses become zero.

$$\left. \begin{matrix} m - n = 0 \\ m \cdot l + n - 1 = 0 \end{matrix} \right\} \rightarrow \begin{matrix} m = n \\ m \cdot (l + 1) = 1 \end{matrix} \quad l(m) = \frac{1}{m} - 1$$

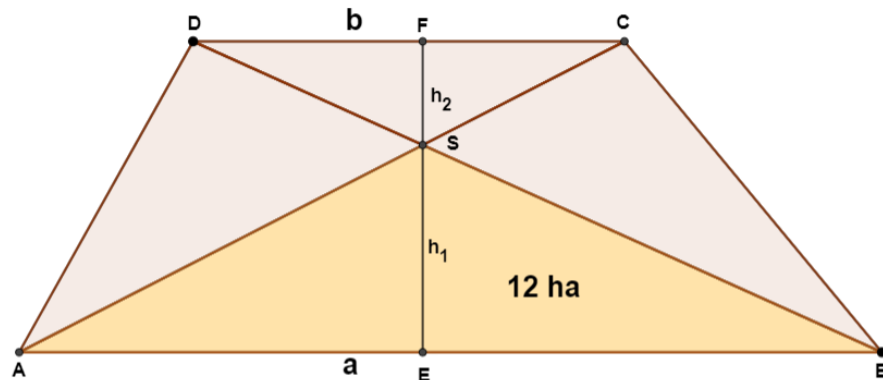
The two transversals always intersect each other in the ratio  $\frac{m}{1-m}$

For  $m = \frac{2}{3}$  we obtain  $l\left(\frac{2}{3}\right) = \frac{1}{2}$ , i.e.  $b = \frac{1}{2} \cdot a$ .

**Solution:**

The extended sketch; here applies:

$$\frac{\overline{SC}}{\overline{AS}} = \frac{\overline{SD}}{\overline{BS}} = \frac{1}{2}$$



Also:

The triangles ABS and CDS and AES and CFS as well are similar. From this follows:

$$\frac{h_2}{SC} = \frac{h_1}{AS} \Rightarrow \frac{h_2}{h_1} = \frac{1}{2} \quad \text{and} \quad \frac{h_2}{b} = \frac{h_1}{a} \rightarrow \boxed{\frac{h_2}{h_1} = \frac{b}{a} = \frac{1}{2}}$$

The formula for a trapezoid is:  $A_{Tr} = \frac{a+b}{2} \cdot h$

So, one gets  $A_{Tr} = \frac{a + \frac{a}{2}}{2} \cdot \left( h_1 + \frac{1}{2} \cdot h_1 \right)$

$$A_{Tr} = \frac{9}{8} \cdot a \cdot h_1$$

For the triangle ABS one obtains  $A_{ABS} = \frac{1}{2} \cdot a \cdot h_1 = 12 \text{ ha}$

$$\underline{a \cdot h_1 = 24 \text{ ha}}$$

and thus, the area of the trapezoid  $\boxed{A_{Tr} = 27 \text{ ha}}$

**Once again, the trapezoidal surface / points in general form**

$$\left. \begin{array}{l} \text{From } \triangle ABS = 12 \text{ ha follows with} \\ h_1 = \frac{24}{a} \\ h_2 = \frac{12}{a} \end{array} \right\} \begin{array}{l} A_{Tr} = \frac{a + \frac{a}{2}}{2} \cdot \frac{36}{a} \\ A_{Tr} = \frac{3\cancel{a}}{4} \cdot \frac{36}{\cancel{a}} \end{array} \rightarrow A_{Tr} = 27 \text{ ha}$$

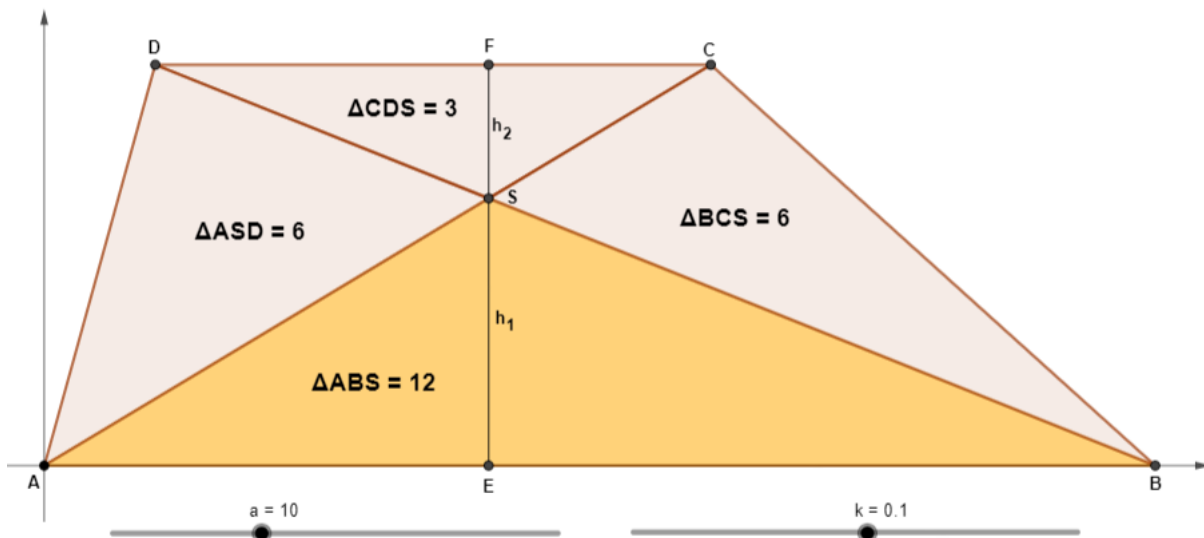
**The trapezoidal area is independent of a;** there are infinitely many trapezoids that satisfy the condition.

Points A, B, C, D, E, F and S are now to be determined as a function of a and a parameter  $k \in \mathbb{R}$ .

k moves the points D and C on  $y = \frac{36}{a}$  to the left and to the right, respectively.

The coordinates:

$$A = (0/0) \quad B = (a/0) \quad D = \left(k \cdot a / \frac{36}{a}\right) \quad C = \left(a \cdot \left(k + \frac{1}{2}\right) / \frac{36}{a}\right)$$



S lies on the line passing A and C with  $y = \frac{24}{a}$ .

$$\left. \begin{array}{l} y = \frac{72}{a^2 \cdot (2k+1)} x \\ \frac{24}{a} = \frac{72}{a^2 \cdot (2k+1)} x \end{array} \right\} \rightarrow x = \frac{a}{3} \cdot (2k+1) \quad S = \left(\frac{a}{3} \cdot (2k+1) / \frac{24}{a}\right)$$

This gives for E and F:  $E = \left(\frac{a}{3} \cdot (2k+1) / 0\right) \quad F = \left(\frac{a}{3} \cdot (2k+1) / \frac{36}{a}\right)$

With  $a = 10$  we obtain  $h_1 = 2.4$  und für  $h_2 = 1.2$ ; the remaining points of the figure result e.g. with  $k = 0$ :

$$\begin{array}{lll} B = (10/0) & C = \left(6 / \frac{18}{5}\right) & D = \left(1 / \frac{18}{5}\right) \\ E = (4/0) & F = \left(4 / \frac{18}{5}\right) & S = \left(4 / \frac{12}{5}\right) \end{array}$$

**The area ASD vectorially calculated:**

The area of the triangle ASD can be easily calculated with the formula  $A = \frac{1}{2} \cdot |\vec{v}_1 \times \vec{v}_2|$ .

$$\vec{v}_1 = \vec{AD} = \begin{pmatrix} k \cdot a \\ a \\ 0 \end{pmatrix} \quad \vec{v}_2 = \vec{AS} = \begin{pmatrix} \frac{a}{3} \cdot (2k+1) \\ 24 \\ a \\ 0 \end{pmatrix}$$

$$A_{ASD} = \frac{1}{2} \cdot \begin{vmatrix} k \cdot a & \frac{a}{3} \cdot (2k+1) \\ 36 & 24 \\ a & a \\ 0 & 0 \end{vmatrix} \rightarrow \begin{matrix} A_{ASD} = \frac{1}{2} \cdot \begin{pmatrix} 0 \\ 0 \\ -12 \end{pmatrix} \\ A_{ASD} = 6 \text{ ha} \end{matrix}$$

Independently of  $a$  and  $k$ , the area sizes always result in:

$$A_{ABS} = 12 \text{ ha}$$

$$A_{CDS} = 3 \text{ ha}$$

$$A_{ASD} = A_{BCS} = 6 \text{ ha}$$

I sent my solution using trigonometry to Wolfgang. I did not cheat looking for Heinrich Hemme's solution given on the website. It is a third approach to the solution – have a look!

This is my one:

Apply  $A = \frac{ab}{2} \cdot \sin \gamma$

$$\begin{matrix} AS = BS = 2t \\ DS = CS = t \end{matrix}$$

$$A_{\Pi} = \underbrace{\frac{4t^2}{2} \cdot \sin \gamma}_{12} + \underbrace{\frac{t^2}{2} \sin \gamma}_{\text{upper triangle}} + \underbrace{2 \cdot \frac{2t^2}{2} \sin(\pi - \gamma)}_{\text{left \& right } \Delta}$$

$$A_{\Pi} = \frac{9t^2}{2} \cdot \sin \alpha \quad \text{and} \quad \begin{matrix} 2t^2 \cdot \sin \alpha = 12 \\ t^2 \cdot \sin \alpha = 6 \end{matrix}$$

$$A_{\Pi} = \frac{9}{2} \cdot 6 = 27$$

By generalizing the problem Wolfgang did much more than only finding the area. So, this is another excellent example how a “easy” problem can lead to further investigations, e.g. are there more ways to find the solution? Thanks to Wolfgang for his inspiration! Josef.





**Springtime in Würmla**

