

THE BULLETIN OF THE



USER GROUP

**+ CAS-TI**

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You are invited to visit one or another website:

Dr. Richard Morris: SingSurf mathematical visualization program

<https://github.com/RichardMorris/SingSurf>

You can find many Mathematics and Mathematics Education papers:

(Examples: Towards Automated Discovery of Geometrical Theorems in GeoGebra, ChatGPT may excel in States Medical Licensing Examination but falters in basic Linear Algebra)

<https://www.researchgate.net>

Visit Steve Arnold's great website:

<https://compasstech.com.au/>

Original from Carl Friedrich Gauss (in English): General investigations of curved surfaces of 1827 and 1825

<https://archive.org/details/cu31924001557226/page/n1/mode/2up>

Browse in the whole Digital Library:

<https://archive.org/details/texts?tab=collection>

The Electronic Library of Mathematics (ELibM):

<https://www.emis.de/ELibM.html>

Download a textbook on Classical Differential Geometry from:

<https://www.math.ucla.edu/~petersen/DGnotes.pdf>

Materials for Nspire and Python (in French):

<https://www.chireux.fr/mp/info/>

Maths et Géographie : 12 vidéos de Jean-Jacques Dahan :

<http://revue.sesamath.net/spip.php?breve405>

Find alternatives for MATHEMATICA:

<https://alternativeto.net/software/mathematica/?license=free>

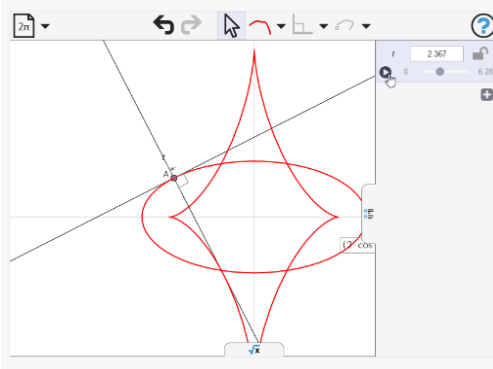
Materialien zu den Vorträgen der ÖMG-Fortbildungstagung für Lehrkräfte vom 14. April 2023 (German):

[https://www.oemg.ac.at/DK/LFT\\_Vortraege/2023/index.html](https://www.oemg.ac.at/DK/LFT_Vortraege/2023/index.html)

Try Phil Todd's GXWeb

A free tool (Geometry Expressions):

<https://geometryexpressions.com/gxweb/>



Liebe DUG-Mitglieder,

Ich habe in meiner letzten DUG Info die Verspätung des DNL130 angekündigt. Meine Vorbereitung und die Teilnahme an der ACA2023, sowie private Gründe haben alles verzögert. Die Reise nach und von Polen (mit dem Auto) war ohne Probleme. ACA war sehr gut organisiert und Ihr könnt das Programm der Educational Session auf Seite 24 finden. Es gab insgesamt neun Sessions mit unterschiedlich vielen Teilnehmern aus 27 Ländern.

Ich bemühe mich, die Vorträge unserer Gruppe in geschriebener Form für den DNL zu erhalten. Einige Zusagen dafür habe ich bereits.

Mit diesem DNL enden die Beiträge von Michel Beaudin (Exercises) und Felipe de Jesús Martínez Vargas (Geometrical Characteristics). Ein zeitliches Zusammentreffen brachte der Aufsatz von Roland Schröder und ein Artikel im „Spektrum der Wissenschaft“.

Ich möchte Euch auch dieses Mal auf die Links auf der ersten Seite und auf Seite 23 hinweisen. Heute habe ich noch alle Links getestet – sie funktionieren. Gerade heute noch habe ich eine Mail mit den Links zu Python empfangen (siehe unten).

Die Aufgaben ab Seite 32 liegen schon sehr lange auf meinem Stapel. Ich denke aber, dass sie noch immer aktuell sind.

Mit den besten Grüßen und Wünschen  
für den verbleibenden Sommer  
Euer Josef

Dear DUG-Members

I announced the delay of DNL130 in my last DUG info. My preparation and participation in ACA2023 as well as private reasons delayed everything. The trip to and from Poland (by car) was without problems. ACA was very well organized and you can find the Educational Session program on page 24. There were nine sessions in total with different numbers of participants from 27 countries.

I try to get our group's presentations in written form for the DNL. I already have some commitments for this.

With this DNL the contributions of Michel Beaudin (Exercises) and Felipe de Jesús Martínez Vargas (Geometrical Characteristics) end. A coincidence in time brought the contribution of Roland Schröder and an article in the "Spektrum der Wissenschaft".

I would like to point out to you the links on the first page and on page 23. Today I tested all links - they work. Just today I received a mail with links to Python (see below).

The tasks starting on page 32 have been on my pile of papers for a very long time. But I think they are still up to date.

With best regards and wishes  
for the remaining summer,

Yours Josef

Python for employability:

<https://www.mathscareers.org.uk/python-for-a-level-maths-undergraduate-maths-and-employability/>

Jupyter notebook covering the A-Level syllabus:

[https://drstephenlynch.github.io/webpages/Python\\_for\\_A\\_Level\\_Mathematics\\_and\\_Beyond.html](https://drstephenlynch.github.io/webpages/Python_for_A_Level_Mathematics_and_Beyond.html)

Beyond A-Level Maths:

[https://drstephenlynch.github.io/webpages/DSAP\\_Jupyter\\_Notebook.html](https://drstephenlynch.github.io/webpages/DSAP_Jupyter_Notebook.html)

The *DERIVE-NEWSLETTER* is the Bulletin of the *DERIVE & CAS-TI User Group*. It is published at least four times a year with a content of 40 pages minimum. The goals of the *DNL* are to enable the exchange of experiences made with *DERIVE*, *TI-CAS* and other CAS as well to create a group to discuss the possibilities of new methodical and didactical manners in teaching mathematics.

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### Contributions:

Please send all contributions to the Editor. Non-English speakers are encouraged to write their contributions in English to reinforce the international touch of the *DNL*. It must be said, though, that non-English articles will be warmly welcomed nonetheless. Your contributions will be edited but not assessed. By submitting articles, the author gives his consent for reprinting it in the *DNL*. The more contributions you will send, the more lively and richer in contents the *DERIVE & CAS-TI Newsletter* will be.

Next issue:

September 2023

### **Preview: Contributions waiting to be published**

Some simulations of Random Experiments, J. Böhm, AUT, Lorenz Kopp, GER  
Wonderful World of Pedal Curves, J. Böhm, AUT  
Simulating a Graphing Calculator in *DERIVE*, J. Böhm, AUT  
Logos of Companies as an Inspiration for Math Teaching  
Exciting Surfaces in the FAZ, BooleanPlots.mth, P. Schofield, UK  
Old traditional examples for a CAS – What's new? J. Böhm, AUT  
Mandelbrot and Newton with *DERIVE*, Roman Hašek, CZ  
Tutorials for the NSpireCAS, G. Herweyers, BEL  
Dirac Algebra, Clifford Algebra, Vector-Matrix-Extension, D. R. Lunsford, USA  
Another Approach to Taylor Series, D. Oertel, GER  
Charge in a Magnetic Field, H. Ludwig, GER  
More Applications of TI-Innovator™ Hub and TI-Innovator™ Rover  
Surfaces and their Duals, Cayley Symmetroid, J. Böhm, AUT  
A Collection of Special Problems, W. Alvermann, GER  
DERIVE Bugs? D. Welz, GER  
Tweening & Morphing with TI-NspireCX-II-T, J. Böhm, AUT  
The Gap between Poor and Rich, J. Böhm, AUT  
More functions from M. Myers and from Bhuvanesh's Mathtools-library  
TaxiCab Conics, Two alternate Approaches to Conics, R. Haas, USA  
QR-Code light, Random numbers following a given distribution  
Quartiles, Numeros Primos, F. de Jesús Martínez Vargas, Mexico  
153 is another Special Number, and others

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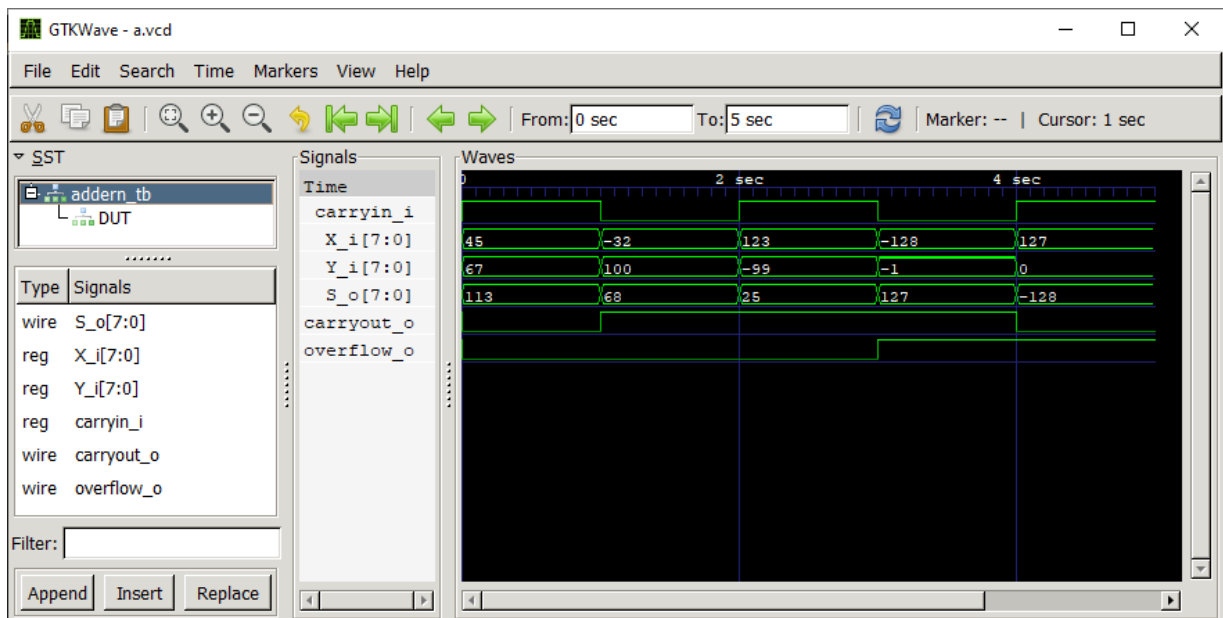
### Mail from Rick Nungester, USA

Josef,

I teach Digital Logic at a local community college and just finished doing so for the 5th time in mid-March. Digital signal timing diagrams are a part of the course. [This is the textbook](#) and one of the CAD tools we use is [GTKWave](#). I attached a couple images from GTKWave that I use in the course. I have to draw timing diagrams on handouts and test papers, and typically use ASCII or Excel -- "cell borders" make nice looking lines but the manual process is tedious.

To think that my old friend Derive can do the job so well! I will be using it in the future to generate the diagrams I pass on to students. Thank you so much for your efforts with Binary Signals!

Rick



### Mail from jaca98@interia.pl

ERROR Derive6 LaplaceTransforms.dfw ☹

Problem 1: Compare the results:

$$\text{Laplace}\left(\frac{4+t}{2}, t, s\right) = \frac{\int_0^{\infty} e^{-s \cdot t} \cdot (t+4) dt}{2}$$

$$\text{Laplace}\left(2 + \frac{t}{2}, t, s\right) = \frac{2}{s} + \frac{1}{2 \cdot s}$$

Problem 2: Compare the results:

$$\text{Laplace}\left(\left(2 + \frac{t}{2}\right) \cdot \text{SIN}(2 \cdot t), t, s\right) = \frac{2}{s^2 + 4}$$

$$\text{Laplace}(2 \cdot \text{SIN}(2 \cdot t), t, s) + \text{Laplace}\left(\frac{t}{2} \cdot \text{SIN}(2 \cdot t), t, s\right) = \frac{2 \cdot (2 \cdot s^2 + s + 8)}{(s^2 + 4)^2}$$

**DUG:** This is what I answered:

Dear colleague in Poland,

yes, I believe that in Terence Etchell's utility might be a bug.

But you can even convince his function to return the correct solution:

### Problem 1

**Apply EXPAND first, and it will work:**

$$\#9: \text{Laplace}\left(\text{EXPAND}\left(\frac{4 + t}{2}\right), t, s\right) = \frac{2}{s} + \frac{1}{2 \cdot s^2}$$

Same trick does not work for Problem 2.

I'd recommend to use rather the built-in function LAPLACE(u) with parameters  $t$  and  $s$  by default:

#1:  $s \in \text{Rea1}(0, \infty)$

$$\#2: \text{LAPLACE}\left(2 + \frac{t}{2}, t, s\right) = \frac{4 \cdot s + 1}{2 \cdot s^2}$$

$$\#3: \text{LAPLACE}\left(\frac{4 + t}{2}, t, s\right) = \frac{4 \cdot s + 1}{2 \cdot s^2}$$

$$\#4: \text{LAPLACE}(2 \cdot \text{SIN}(2 \cdot t)) + \text{LAPLACE}\left(\frac{t}{2} \cdot \text{SIN}(2 \cdot t)\right) = \frac{2 \cdot (2 \cdot s^2 + s + 8)}{(s^2 + 4)^2}$$

$$\#5: \text{LAPLACE}\left(\left(2 + \frac{t}{2}\right) \cdot \text{SIN}(2 \cdot t)\right) = \frac{2 \cdot (2 \cdot s^2 + s + 8)}{(s^2 + 4)^2}$$

$$\#6: \text{LAPLACE}\left(\frac{4 + t}{2} \cdot \text{SIN}(2 \cdot t)\right) = \frac{2 \cdot (2 \cdot s^2 + s + 8)}{(s^2 + 4)^2}$$

This is how TI-Nspire answers using Michel Beaudin's (Canada) great library:

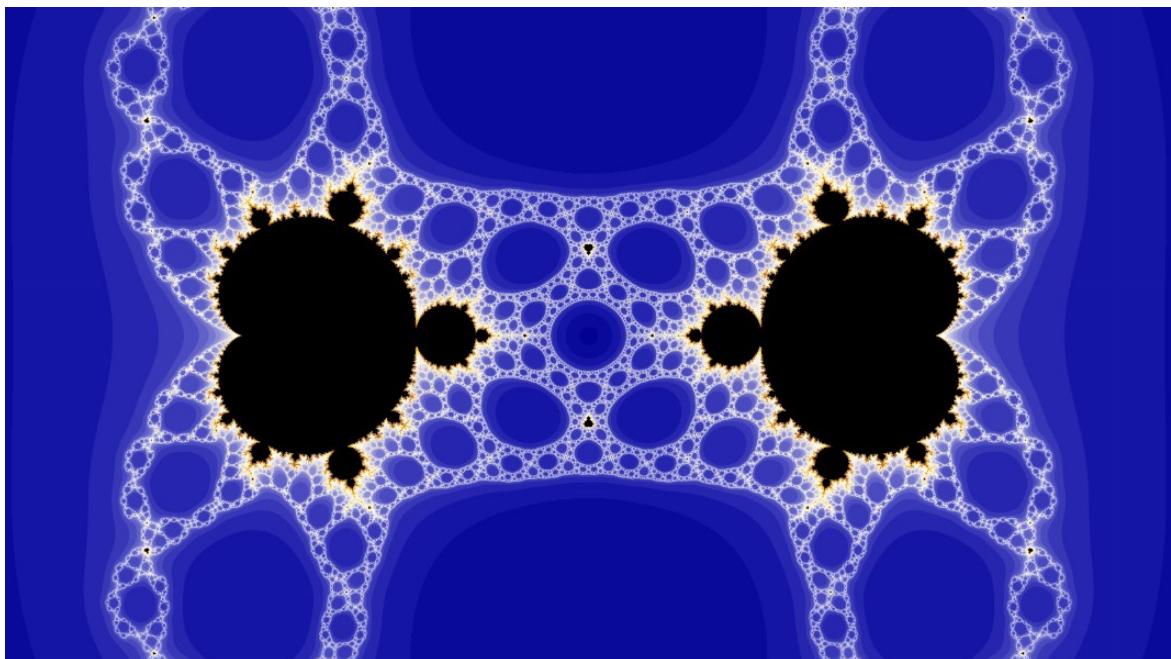
$\text{ets\_specfunc}\backslash\text{laplace}\left(2+\frac{t}{2}\right)$	$\frac{2}{s} + \frac{1}{2 \cdot s^2}$
$\text{ets\_specfunc}\backslash\text{laplace}\left(\frac{4+t}{2}\right)$	$\frac{2}{s} + \frac{1}{2 \cdot s^2}$
$\text{ets\_specfunc}\backslash\text{laplace}\left(\left(2+\frac{t}{2}\right) \cdot \sin(2 \cdot t)\right)$	$\frac{2 \cdot (2 \cdot s^2 + s + 8)}{(s^2 + 4)^2}$
$\text{ets\_specfunc}\backslash\text{laplace}(2 \cdot \sin(2 \cdot t)) + \text{ets\_specfunc}\backslash\text{laplace}\left(\frac{t}{2} \cdot \sin(2 \cdot t)\right)$	$\frac{2 \cdot (2 \cdot s^2 + s + 8)}{(s^2 + 4)^2}$

See the output of WMaxima:

```
(%i1)  laplace((4+t)/2,t,s);
        4   1
        s  + s^2
(%o1)  -----
        2

(%i2)  laplace((2+t/2)*sin(2*t),t,s);
(%o2)  4      2 s
        s^2 + 4  + (s^2 + 4)^2
```

Hope this helps.  
Josef



One of R. Devaney's "Mandelpinkis"

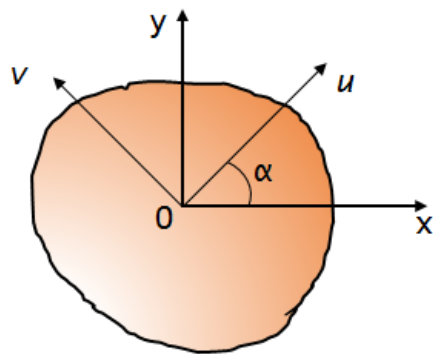
(Article in "Spektrum der Wissenschaft" 7/23)

Will maybe investigated in a later Newsletter, Josef

CARACTERÍSTICAS GEOMÉTRICAS DE LAS SECCIONES PLANASGEOMETRICAL CHARACTERISTICS OF PLANE SECTIONS

## Part 2

- **Angulo principal** **Principal Angle**



$$\text{Ang\_princ}(\text{polygon}) := \begin{bmatrix} \text{Angulo principal} \\ \text{Principal angle} \\ [\alpha =, \alpha(\text{polygon})] \end{bmatrix}$$

$$\text{Ang\_princ}(\text{pol1}) = \begin{bmatrix} \text{Angulo principal} \\ \text{Principal angle} \\ [\alpha =, 0.5739712003] \end{bmatrix}$$

v

- **Momentos de inercia principal**
- **Principal moments of inertia**

$$\text{Mom\_inert\_princ}(\text{pol1}) = \begin{bmatrix} \text{Momentos de inercia principal} \\ \text{Principal moments of Inertia} \\ \begin{bmatrix} I_u = 0.007381268274 \\ I_v = 0.02661873172 \end{bmatrix} \end{bmatrix}$$

- **Momentos de inercia principales respecto al centroide**
- **Principal moments of inertia wrt to the centroid**

$$\text{Mom\_inert\_princ\_centr}(\text{pol1}) = \begin{bmatrix} \text{Momentos de inercia principal respecto al centroide} \\ \text{Principal moments of Inertia wrt Centroid} \\ \begin{bmatrix} I_{uc} = 0.007539756829 \\ I_{vc} = 0.00096024317 \end{bmatrix} \end{bmatrix}$$

- **Radios de giro con relación al eje principal de inercia**
- **Radii of gyration in relation to the main axis of inertia**

$$\text{Rad\_princ\_gyr}(\text{pol1}) = \begin{bmatrix} \text{Radio de giro principales} \\ \text{Radii of principal gyration} \\ \begin{bmatrix} i_u = 0.2480132435 \\ i_v = 0.4709806411 \end{bmatrix} \end{bmatrix}$$



- **Radio de giro principales respecto al centroide**
- **Principal radii of gyration with respect to the centroid**

$$\text{Rad\_gyr\_centr}(\text{pol1}) = \begin{bmatrix} \text{Radio de giro principales respecto al centroide} \\ \text{Principal radii of gyr. wrt centroid} \\ \begin{bmatrix} i_{uc} = 0.250661738 \\ i_{vc} = 0.0894540464 \end{bmatrix} \end{bmatrix}$$

- **Módulo de la sección elástica**
- **Modulus of the elastic section**

Es la relación del momento de inercia respecto al eje dado a la distancia hasta el punto más alejado de la sección transversal.

It is the ratio of the moment of inertia with respect to the given axis to the distance to the farthest point of the cross section.

$$\text{Mod\_elastic}(\text{pol1}) = \begin{bmatrix} \text{Módulo de la sección elástica} \\ \text{Modulus of the elastic section} \\ \begin{bmatrix} W_{el\_top} = 0.01866666666 \\ W_{el\_bot} = 0.01866666666 \\ W_{el\_left} = 0.008285714285 \\ W_{el\_right} = 0.008285714285 \end{bmatrix} \end{bmatrix}$$

- **Módulo polar de la sección**
- **Polar modulus of the section**

Es la relación del momento de inercia polar a la distancia desde el polo hasta el punto más alejado de la sección.

It is the ratio of the polar moment of inertia to the distance from the pole to the farthest point of the section.

$$\text{Mod\_pol\_sect}(\text{pol1}) = \begin{bmatrix} \text{Módulo polar de la sección} \\ \text{Polar modulus of the section} \\ [W_p =, 0.07375635565] \end{bmatrix}$$

## ● RESUMEN SUMMARY

RESUMEN\_orig(pol1)

```

[
    [ Perímetro = 2.6 ]
    [ Área = 0.12 ]
    Centro de gravedad
    [ xc = 0.35 ]
    [ yc = 0.3 ]
    Momento estático
    [ Mx = 0.036 ]
    [ My = 0.042 ]
    Momentos de inercia
    [ Ix = 0.0164 ]
    [ Iy = 0.0176 ]
    [ Ixy = 0.0096 ]
]

```

summary(pol1)

```

[
    [ Perimeter = 2.6 ]
    [ Area = 0.12 ]
    [Center of Gravity, [xc =, 0.35, yc =, 0.3]]
    [Static Moment, [Mx =, 0.036, My =, 0.042]]
    [Inertial Moment, [Ix =, 0.0164, Iy =, 0.0176, Ixy =, 0.0096]]
    [Centroidal Moments of Inertia, [Ixc =, 0.0056, Iyc =, 0.0029, Ixcyc =, -0.003]]
    [Polar moment of Inertia, [Ip =, 0.034]]
    [Polar moment of centroidal Inertia, [Ip_xy =, 0.0085]]
    [Polar radius of Inertia, [rp_xy =, 0.2661453237]]
    [Rotation radii, [rx =, 0.3696845502, ry =, 0.3829708431]]
    [Rot radii wrt centr., [rxc =, 0.2160246899, ryc =, 0.1554563175]]
    [Principal angle, [α (Radian) =, 0.5739712003, α (Degree) =, 32.88612734]]
    [Principal moments of Inertia, [Iu =, 0.007381268274, Iv =, 0.02661873172]]
    [Princ. moments of Inertia wrt Centroid, [Iuc =, 0.007539756829, Ivc =, 0.00096024317]]
    [Princ. Radii of gyration wrt Centr., [iuc =, 0.250661738, ivc =, 0.0894540464]]
    [Radii of principal gyration, [iu =, 0.2480132435, iv =, 0.4709806411]]
    [Modulus of elastic section, [
        W_e1_top = 0.01866666666 W_e1_bot = 0.01866666666
        W_e1_left = 0.008285714285 W_e1_right = 0.008285714285
    ]]
    [Polar modulus of the elastic section, [Wp =, 0.07375635565]]
]

```

## Representación gráfica Graphic representation

- Dibuja polígono referido al centroide
- Draw polygon referred to centroid

```

Translate(P, Ux, Uy, P_) :=
  Prog
#106:   "use coordenadas homogeneas"
        P_ := INSERT(VECTOR(1, x, DIM(P)), P', 3)'
        DELETE((P_·[1, 0, 0; 0, 1, 0; Ux, Uy, 1])', 3)'

#107: Polygon_c(polygon) := Translate(polygon, - Cx(polygon), - Cy(polygon))

```

- Elipse de inercia Inertial ellipse

```

#108: Rot(θ) :=  $\begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}$ 

```

```

ellipse(p, q, θ, pts_, k, a_, b_, out_) :=
  Prog
  a_ := p/2
  b_ := q/2
  out_ := []
  pts_ := 40
  k := 0
#109: Loop
  out_ := INSERT([-a_ + p/pts_·k, b_·√(1 - ((-a_ + p/pts_·k)/a_)^2)], out_, k + 1)
  If k = pts_
    exit
  k := + 1
  out_ := APPEND(out_, REVERSE(out_·[1, 0; 0, -1]))
  out_·Rot(θ)

```

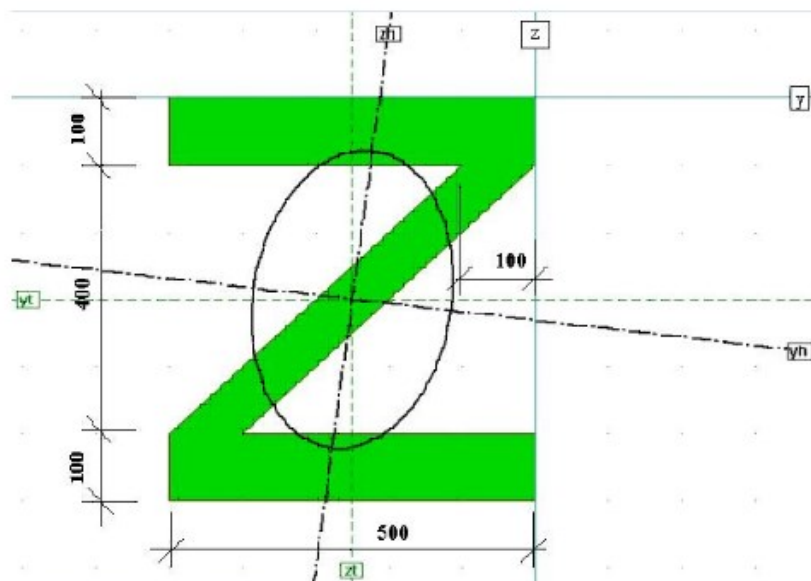
```

ellipsoid(polygon, elip1, elip2, elip3, elip4, elip5) :=
  Prog
  elip1 := APPEND(ellipse(2·iv(polygon), 2·iu(polygon), 0), [[0, 0]])
  elip2 := APPEND(elip1, [[0, iu(polygon)]])
#110:  elip3 := APPEND(elip2, [[0, -iu(polygon)]])
  elip4 := APPEND(elip3, [[0, 0]])
  elip5 := APPEND(elip4, [[iv(polygon), 0]])
  elip5

```

### EJEMPLO

### EXAMPLE



**Průřezové charakteristiky**

Souřadnice těžiště:

$$y_T = -250.0 \text{ mm}, \quad z_T = -300.0 \text{ mm}$$

Průřezová plocha:  $A = 1.400\text{E}+05 \text{ mm}^2$ Obvod průřezu:  $O = 3331.4 \text{ mm}$ 

Souřadnice těžiště (měřené od okraje obdélníkové obálky průřezu):

$$y_T = 250.0 \text{ mm}, \quad z_T = 300.0 \text{ mm}$$

Momenty setrvačnosti:

$$I_y = 6.967\text{E}+09 \text{ mm}^4, \quad I_z = 2.650\text{E}+09 \text{ mm}^4, \quad I_{yz} = 5.333\text{E}+08 \text{ mm}^4$$

Sklon hlavních centrálních os:  $\beta_1 = -7.1^\circ$ 

Poloměry setrvačnosti:

$$i_y = 2.215\text{E}+02 \text{ mm}, \quad i_z = 1.376\text{E}+02 \text{ mm}$$

Hlavní momenty setrvačnosti:

$$I_{yh} = 6.933\text{E}+09 \text{ mm}^4, \quad I_{zh} = 2.584\text{E}+09 \text{ mm}^4$$

Hlavní poloměry setrvačnosti:

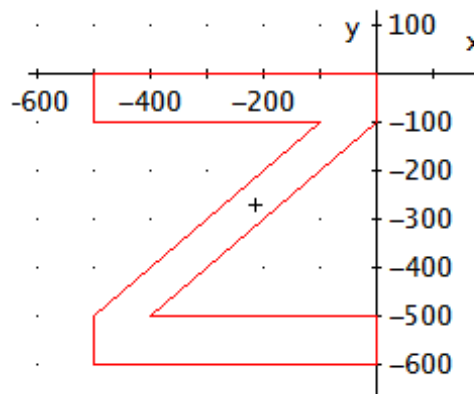
$$i_{yh} = 2.223\text{E}+02 \text{ mm}, \quad i_{zh} = 1.358\text{E}+02 \text{ mm}$$

Moment tuhosti v prostém kroucení:  $I_k = 1.009\text{E}+09 \text{ mm}^4$ Polární moment setrvačnosti:  $I_p = 9.517\text{E}+09 \text{ mm}^4$ Polární poloměr setrvačnosti:  $i_p = 2.607\text{E}+02 \text{ mm}$ 

```

perfil_Z :=
  [ 0 0
  -500 0
  -500 -100
  -100 -100
  -500 -500
  -500 -600
  0 -600
  0 -500
  -400 -500
  0 -100
  0 0 ]

```



```

Translate(P, Ux, Uy, P_) :=

```

```

  Prog

```

```

  "use coordenadas homogeneas"

```

```

  P_ := INSERT(VECTOR(1, x, DIM(P)), P', 3)'

```

```

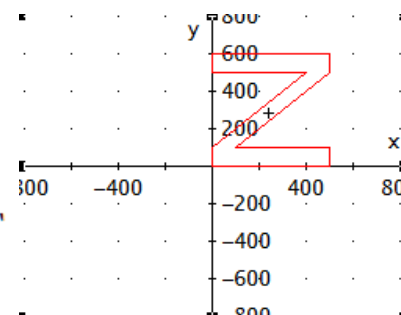
  DELETE((P_·[1, 0, 0; 0, 1, 0; Ux, Uy, 1])', 3)'

```

```

polygon_tr := Translate(perfil_Z, 500, 600)

```

**Polígono referido al centroide****Polygon referred to the centroid**

```

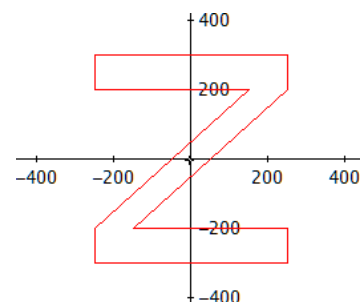
#118: Polygon_c(polygon) := Translate(polygon, - Cx(polygon), - Cy(polygon))

```

```

#119: Polygon_c(polygon_tr)

```



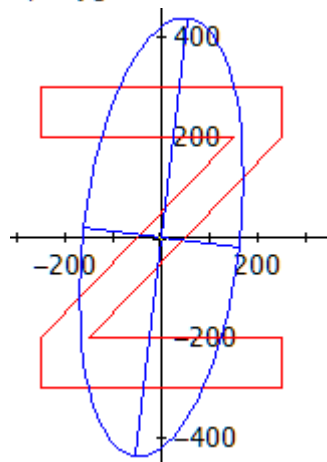
summary(polygon\_tr)

$$\begin{aligned}
 & \left[ \begin{array}{l} \text{Perimeter} = 3331.370849 \\ \text{Area} = 1.4 \cdot 10^5 \end{array} \right] \\
 & \text{[Center of Gravity, [xc =, 250, yc =, 300]]} \\
 & \text{[Static Moment, [Mx =, } 4.2 \cdot 10^7 \text{, My =, } 3.5 \cdot 10^7 \text{]]} \\
 & \text{[Inertial Moment, [Ix =, } 1.946666666 \cdot 10^{10} \text{, Iy =, } 1.14 \cdot 10^{10} \text{, Ixy =, } 1.103333333 \cdot 10^{10} \text{]]} \\
 & \text{[Centroidal Moments of Inertia, [Ixc =, } 6.866666666 \cdot 10^9 \text{, Iyc =, } 2.65 \cdot 10^9 \text{, Ixcyc =, } 5.333333333 \cdot 10^8 \text{]]} \\
 & \text{[Polar moment of Inertia, [Ip =, } 3.086666666 \cdot 10^{10} \text{]]} \\
 & \text{[Polar moment of centroidal Inertia, [Ip_xy =, } 9.516666666 \cdot 10^9 \text{]]} \\
 & \text{[Polar radius of Inertia, [rp_xy =, 260.7224395]]} \\
 & \text{[Rotation radii, [rx =, 372.8908942, ry =, 285.3569193]]} \\
 & \text{[Rot radii wrt centr., [rxc =, 221.4669705, ryc =, 137.5811448]]} \\
 & \text{[Principal angle, [\alpha (Radian) =, -0.1238833799, \alpha (Degree) =, -7.097994825]]} \\
 & \text{[Principal moments of Inertia, [Iu =, } 2.718076804 \cdot 10^{10} \text{, Iv =, } 3.685898625 \cdot 10^9 \text{]]} \\
 & \text{[Princ. moments of Inertia wrt Centroid, [Iuc =, } 6.933077890 \cdot 10^9 \text{, Ivc =, } 2.583588776 \cdot 10^9 \text{]]} \\
 & \text{[Princ. Radii of gyration wrt Centr., [iuc =, 222.5353565, ivc =, 135.8462570]]} \\
 & \text{[Radii of principal gyration, [iu =, 440.6226766, iv =, 162.2585816]]} \\
 & \text{[Modulus of elastic section, [ } \begin{array}{l} \text{W\_el\_top} = 2.288888888 \cdot 10^7 \quad \text{W\_el\_bot} = 2.288888888 \cdot 10^7 \\ \text{W\_el\_left} = 1.06 \cdot 10^7 \quad \text{W\_el\_right} = 1.06 \cdot 10^7 \end{array} \text{]]} \\
 & \text{[Polar modulus of the elastic section, [Wp =, } 7.904143387 \cdot 10^7 \text{]]}
 \end{aligned}$$

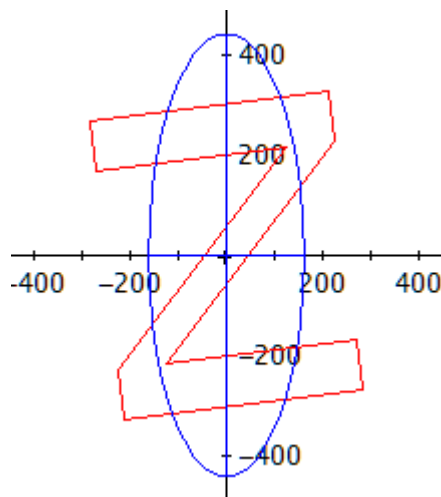
**Orientación sobre ejes principales de inercia.**

**Orientation on main axes of inertia.**

ellipsoid(polygon\_tr)·Rot(α(polygon\_tr))



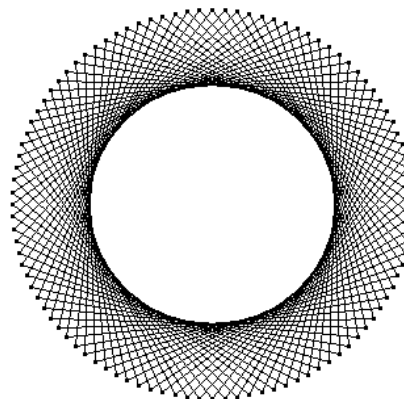
```
[Polygon_c(polygon_tr)•Rot(- α(polygon_tr)), RE(ellipsoid(polygon_tr))]
```



## Zahlenfolgen auf einem aufgewickelten Zahlenstrahl

Roland Schröder, Deutschland

Zur Einführung in das Thema wird eine unendliche arithmetische Folge mit beliebigem Anfangsglied und der konstanten Differenz  $d = 29$  auf einem Kreis mit  $k = 100$  Punkten (äquidistant verteilt) dargestellt (rechts), wobei die Punkte in einem Drehsinn von 1 bis 100 durchnummeriert und je zwei aufeinanderfolgende Glieder der Folge mittels einer Sehne verbunden sind.



Der DERIVE-Befehl für diese Darstellung im rechtwinkligen Koordinatensystem lautet:

```
VECTOR([cos(pi/50 * 29n), sin(pi/50 * 29n)], n, 0, 100).
```

Zur Darstellung geometrischer Folgen findet man im DERIVE-Newsletter #93 unter dem Titel „Light in the Coffee Cup“ einige Anregungen. In diesem Artikel wurde allerdings implizit eine Forderung an die Anzahl der Punkte auf dem Kreis gestellt, welche die Vielfalt der Darstellungen stark einengt. Nach Betrachtung des sehenswerten Videos unter dem Link <https://www.youtube.com/watch?v=qhbuKbxJsk8&list=PLmNp3NTX4KXIR-wOcyI3lzJC2eJj-kVsO2&index=2>

stellt man fest, dass eine Forderung an die Anzahl der Punkte auf dem Kreis die Darstellungsmöglichkeiten einschränkt. Lässt man jede Punktezahl auf dem Kreis zu und beginnt die Darstellung geometrischer Folgen mit jeder der Zahlen auf dem Kreis, so wird dieser Ansatz mit einer Formenvielfalt in den Darstellungen belohnt.

Für einen Startpunkt mit der Nummer  $a$  auf einem Kreis mit  $k$  Punkten lassen sich die Glieder einer geometrischen Folge mit dem konstanten Faktor  $q$  und dem Anfangsglied  $a$  darstellen durch den DERIVE-Ausdruck:

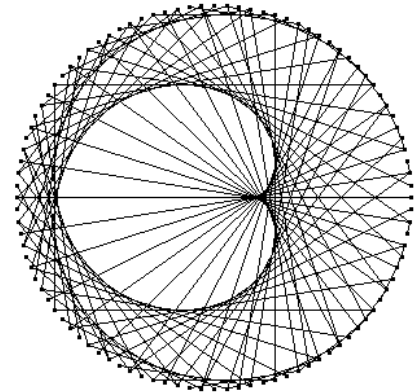
$$\#1: \left[ \cos\left(\frac{2 \cdot \pi}{k} \cdot \text{MOD}(a \cdot q^n, k)\right), \sin\left(\frac{2 \cdot \pi}{k} \cdot \text{MOD}(a \cdot q^n, k)\right) \right]$$

Nach der Festlegung von  $q$  und  $k$  durch Zahlenwerte führt der Ausdruck

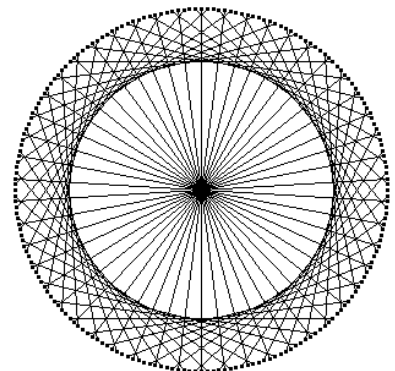
$$\#2: \text{VECTOR}(\text{VECTOR}(\#1, n, 1, 2), a, 0, k)$$

zu einer Darstellung, für die hier einige Beispiele gegeben werden.

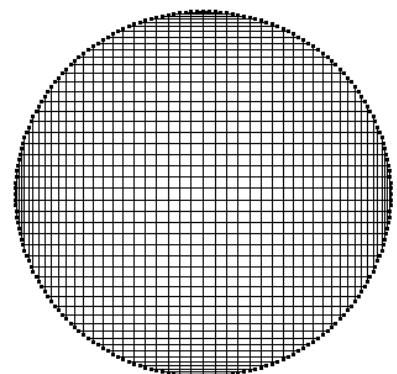
Für  $k = 200$  und  $q = 34$  ergibt sich dieser Graph:



Für  $k = 200$  und  $q = 51$  ergibt sich dieser Graph:



Und für  $k = 200$  und  $q = 99$  findet man diesen Graph:

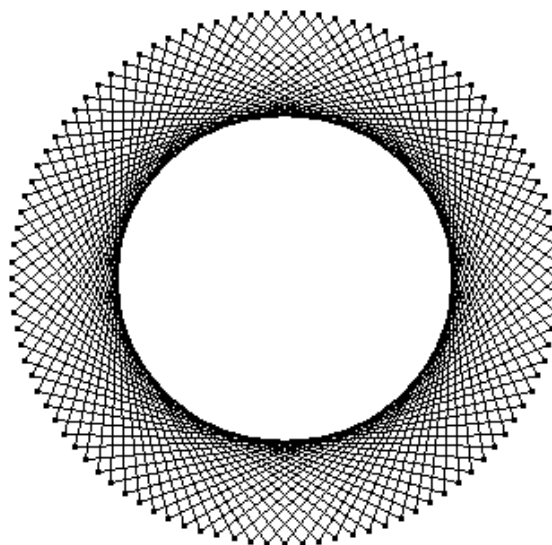


Die statischen Bilder dieser Graphen sind auch im Video ([Link](#)) zu sehen, aber weitere Graphen erscheinen in sehr rascher Folge. Wer ein wenig bei einzelnen Graphen seines Interesses verweilen möchte, dem sei Ausdruck #2 zum Experimentieren empfohlen.

## Number Sequences on a coiled Number Line

Roland Schröder, Germany

To introduce the topic, an infinite arithmetic sequence with any initial element and the constant difference  $d = 29$  is represented on a circle with  $k = 100$  points (equidistantly distributed), where the points are numbered in a direction of rotation from 1 to 100 and two consecutive elements of the sequence are connected by a chord (right picture).



The DERIVE command for this representation in the Cartesian coordinate system is as follows:

```
VECTOR([cos(pi/50 * 29n), sin(pi/50 * 29n)], n, 0, 100).
```

For the representation of geometric sequences one can find some suggestions in DERIVE Newsletter #93 under the title "Light in the Coffee Cup". In this article, however, a requirement was implicitly made on the number of points on the circle, which strongly restricts the variety of representations.

After watching the video at the link given below, which is worth seeing, you realize that a requirement for the number of points on the circle limits the display options.

<https://www.youtube.com/watch?v=qhbuKbxJsk8&list=PLmNp3NTX4KXIR-wOcyI3lzJC2eJj-kVsO2&index=2>

If one allows any number of points on the circle and starts the representation of geometric sequences with any of the numbers on the circle, this approach is rewarded with a variety of shapes in the representations. For a starting point with number  $a$  on a circle with  $k$  points, the members of a geometric sequence with the constant factor  $q$  and the initial element  $a$  can be represented by the DERIVE expression:

$$\#1: \left[ \cos\left(\frac{2 \cdot \pi}{k} \cdot \text{MOD}(a \cdot q^n, k)\right), \sin\left(\frac{2 \cdot \pi}{k} \cdot \text{MOD}(a \cdot q^n, k)\right) \right]$$

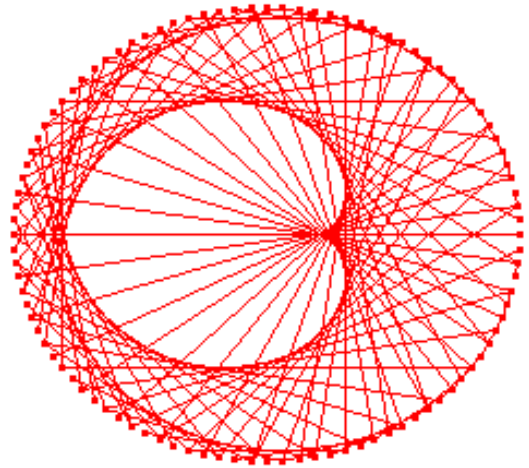
After defining  $q$  and  $k$  by numerical values, the expression

$$\#2: \text{vector}(\text{vector}(\#1, n, 1, 2), a, 0, k)$$

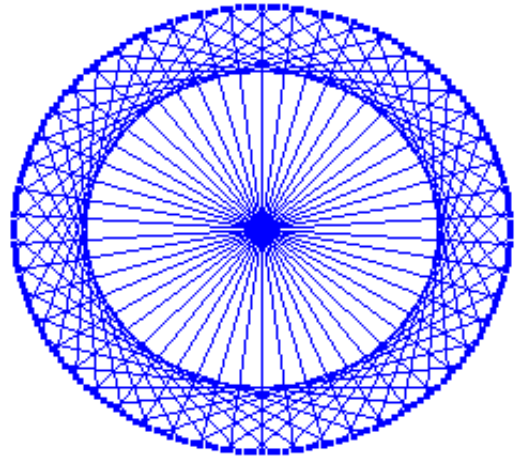
leads to representations for which some examples are given on the next page.



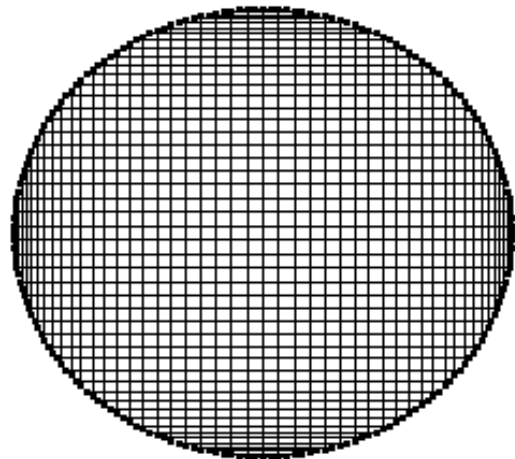
For  $k = 200$  and  $q = 34$  this graph is obtained:



Take  $k = 200$  and  $q = 51$ :



Now for  $k = 200$  and  $q = 99$ :



The static images of these graphs can also be seen in the video ([link](#)), but more graphs appear in very rapid succession. For those who want to linger a little on individual graphs of their interest, experimenting with expression #2 is recommended.

It was a nice coincidence that when I received Roland's contribution, I remembered an article which appeared in the German edition of the Scientific American (Spektrum der Wissenschaft 7/22) entitled

## DAS EINMALEINS IM KREIS

(The Multiplication Table in a Circle)

The author (Christoph Pöppe) refers to a paper written by Simon Plouffe:

La forme de  $b^n \bmod p$ , The Shape of  $b^n \bmod p$  (2020).

Links for download of these papers (pdfs) are given at the end of this article.

BTW, you might remember that we had a great contribution provided by Simon Plouffe in DNLs 115, 116 and 117: "π, the primes and the Lambert W-function".

Plouffe's approach is similar to Roland's one, but it is not quite the same – and he works with much larger numbers in order to receive impressive pictures. Additionally, he colors the segments according to their length.

All what I wanted to do was, trying to reproduce some of his pictures and experimenting with own parameters. What I could not add, was the coloring. So, all my graphs have only one color.

Plouffe starts presenting  $10^n \bmod 17$  on the circle:

$$\#1: \text{circ}(k) := \text{VECTOR}\left(\left[\cos\left(\frac{2 \cdot \pi}{k} \cdot a\right), \sin\left(\frac{2 \cdot \pi}{k} \cdot a\right)\right], a, 0, k\right)$$

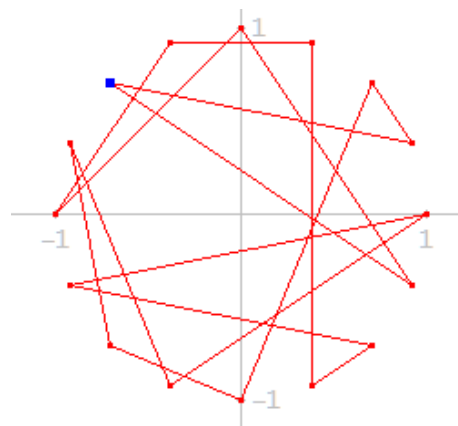
$$\#2: \text{seq1} := \text{VECTOR}(\text{MOD}(10^n, 17), n, 0, 16)$$

$$\#3: \text{seq1} := [1, 10, 15, 14, 4, 6, 9, 5, 16, 7, 2, 3, 13, 11, 8, 12, 1]$$

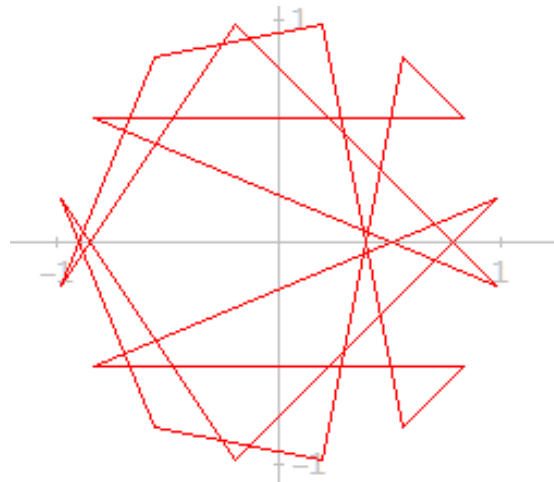
$$\#4: \text{VECTOR}(\text{circ}(16))_i, i, \text{seq1}$$

$\text{circ}(k)$  provides  $k+1$  equally distributed points on a circle of radius 1.  $\text{seq1}$  gives the order how to visit the points on the circle and connecting them performs the VECTOR command.

The blue point is point #10. (check it, please!)



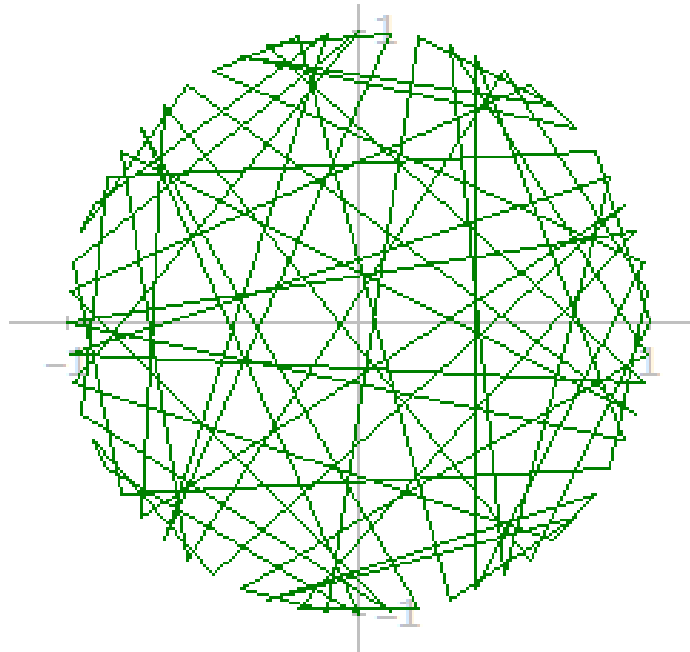
Since I wanted to have the graph symmetric wrt the  $x$ -axis I rotated the figure by  $\pi/16$  and plotted #4 once more:



Plouffe investigates the form of  $b^n \bmod p$ .  $b$  is the “base” and  $p$  is a prime

$$\text{circ2}(b, p) := \text{VECTOR} \left( \left[ \cos \left( \frac{2 \cdot \pi}{p-1} \cdot \text{MOD}(b^k, p) \right), \sin \left( \frac{2 \cdot \pi}{p-1} \cdot \text{MOD}(b^k, p) \right) \right], k, 0, p-1 \right)$$

`circ2(10, 61)`



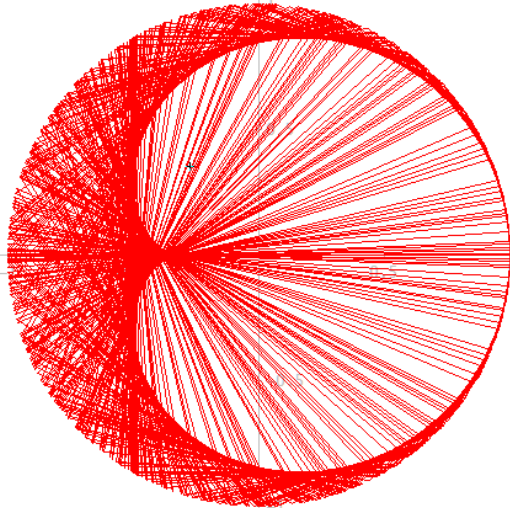
Plouffe’s special interest was to find an explanation for the connection between the choice of  $b$  and  $p$  and the number of spikes appearing in the pattern created by choosing larger primes.

I don’t want to copy parts of his paper here. You are invited to download it from one of the websites given below.

What I will do is only trying to reproduce some of his amazing patterns (unfortunately only one colored) with DERIVE and the TI-NspireCAS.

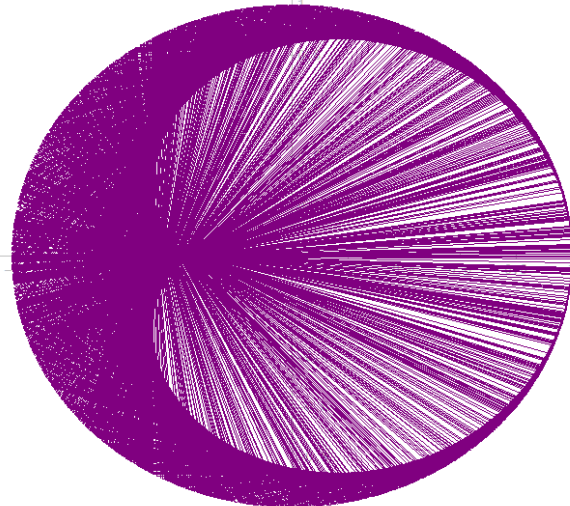
It is interesting that  $b = 2$  will give the cardioid for different primes  $p$ :

`circ2(2, 1009)`



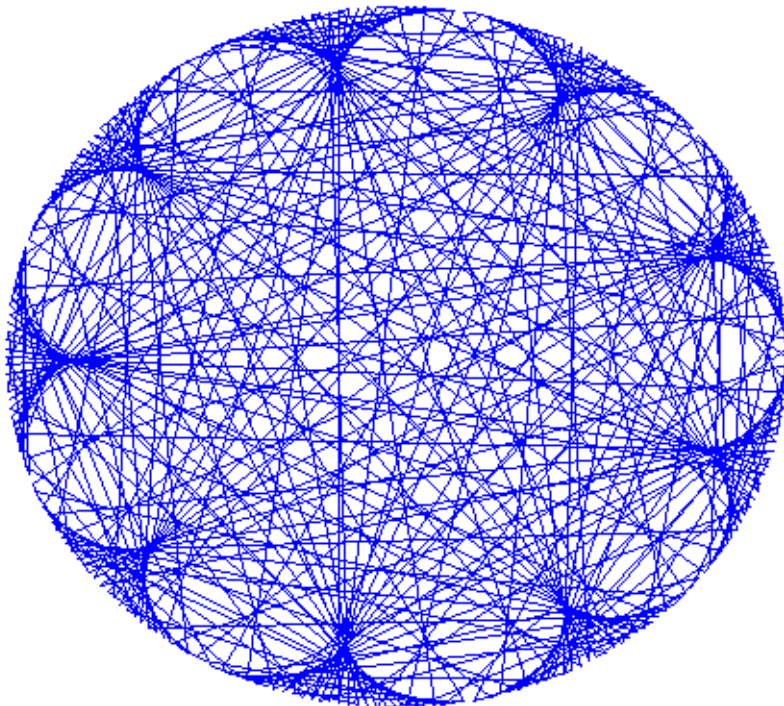
`NEXT_PRIME(3500) = 3511`

`circ2(2, 3511)`



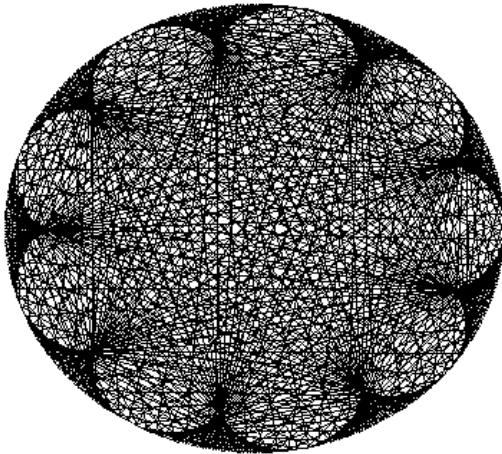
I will start other combinations ( $b, p$  with  $b = 10$  and  $p = 257$ ):

`circ2(10, 257)`

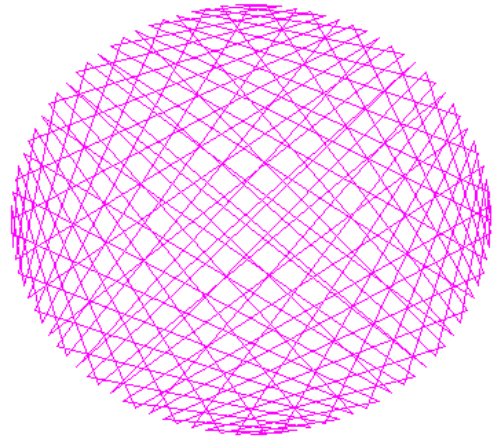


There are nine spikes. The explanation and the formula giving the result 9 for the number of spikes can be found in Plouffe's paper.

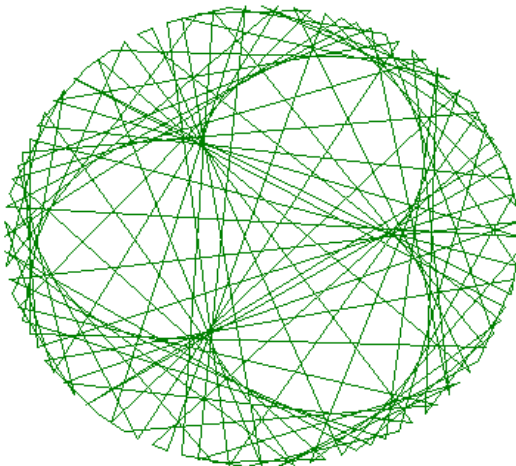
circ2(10, 503)



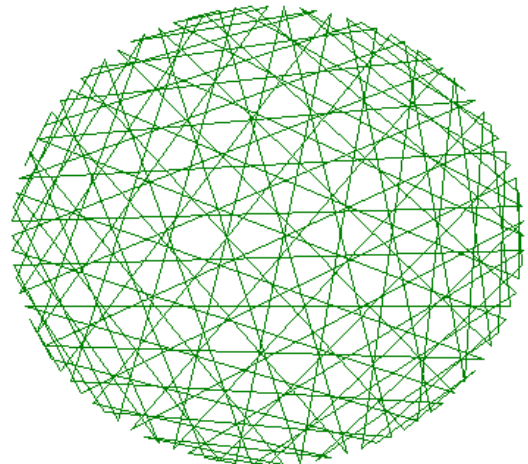
circ2(98, 101)



circ2(53, 101)

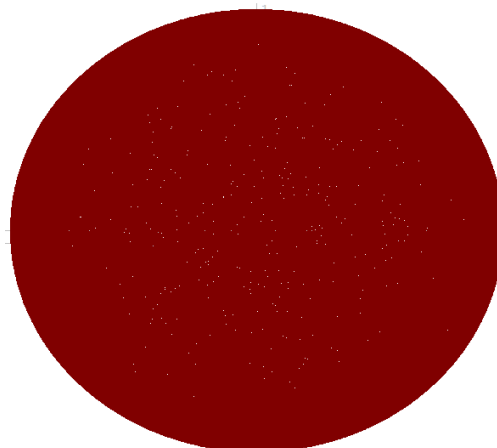


circ2(75, 101, 6)

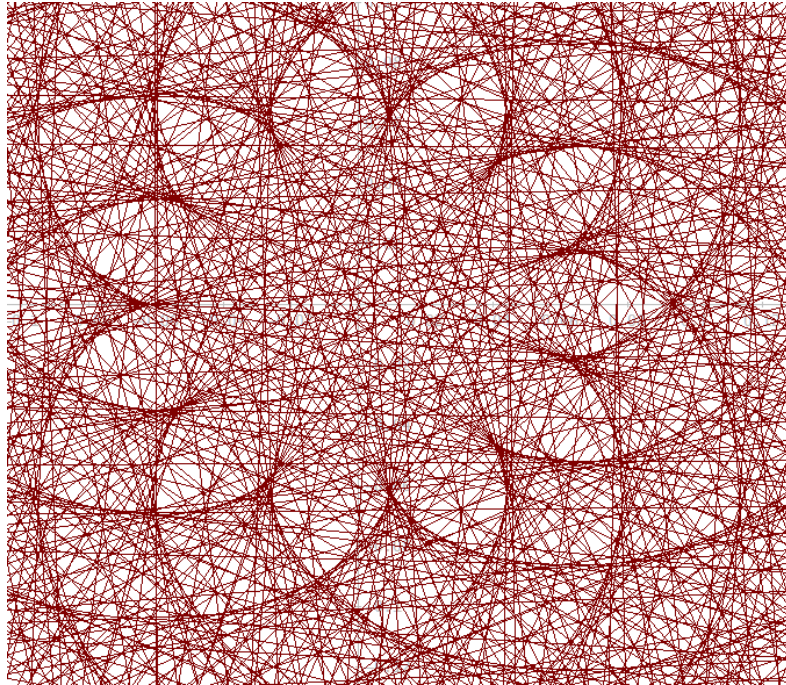


Have a plot with more than 5000 segments. It's remarkable what we can achieve with good old DERIVE, isn't it?

circ2(53, 5003)

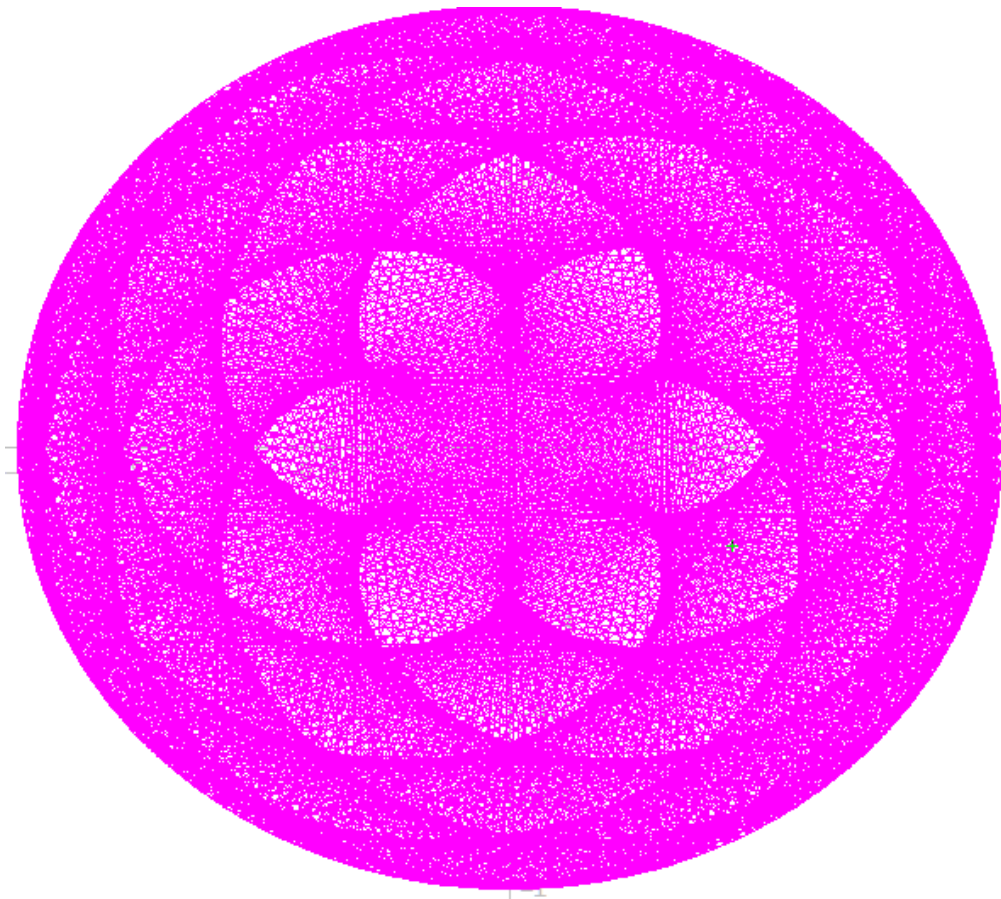


Do you think, this is only a dark brown spot? Zoom in and be surprised!

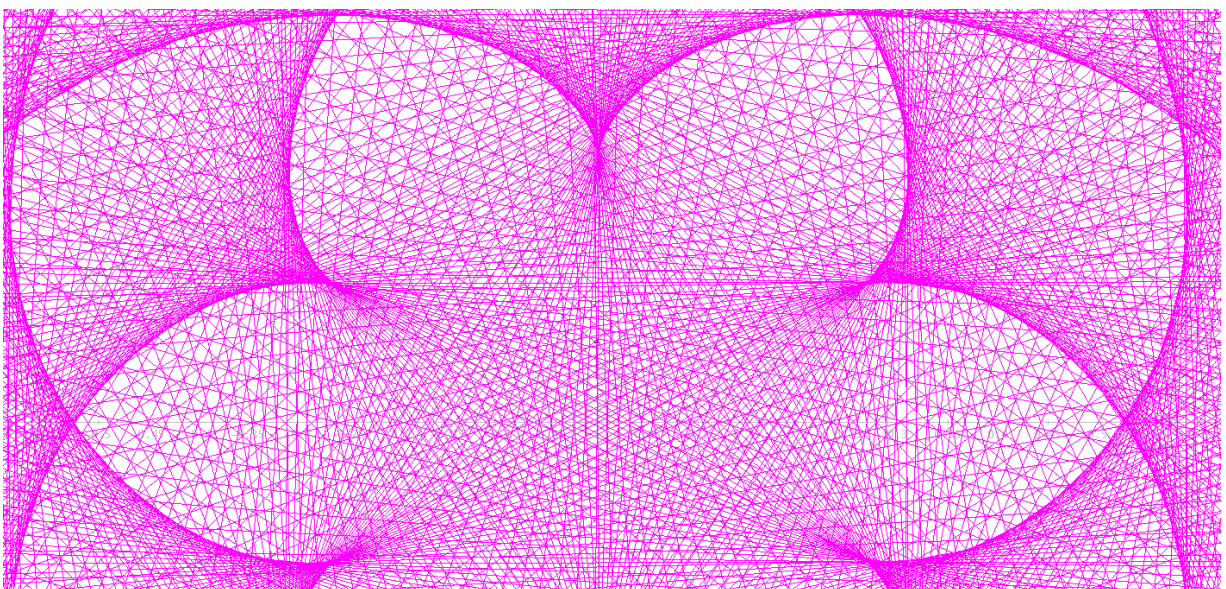
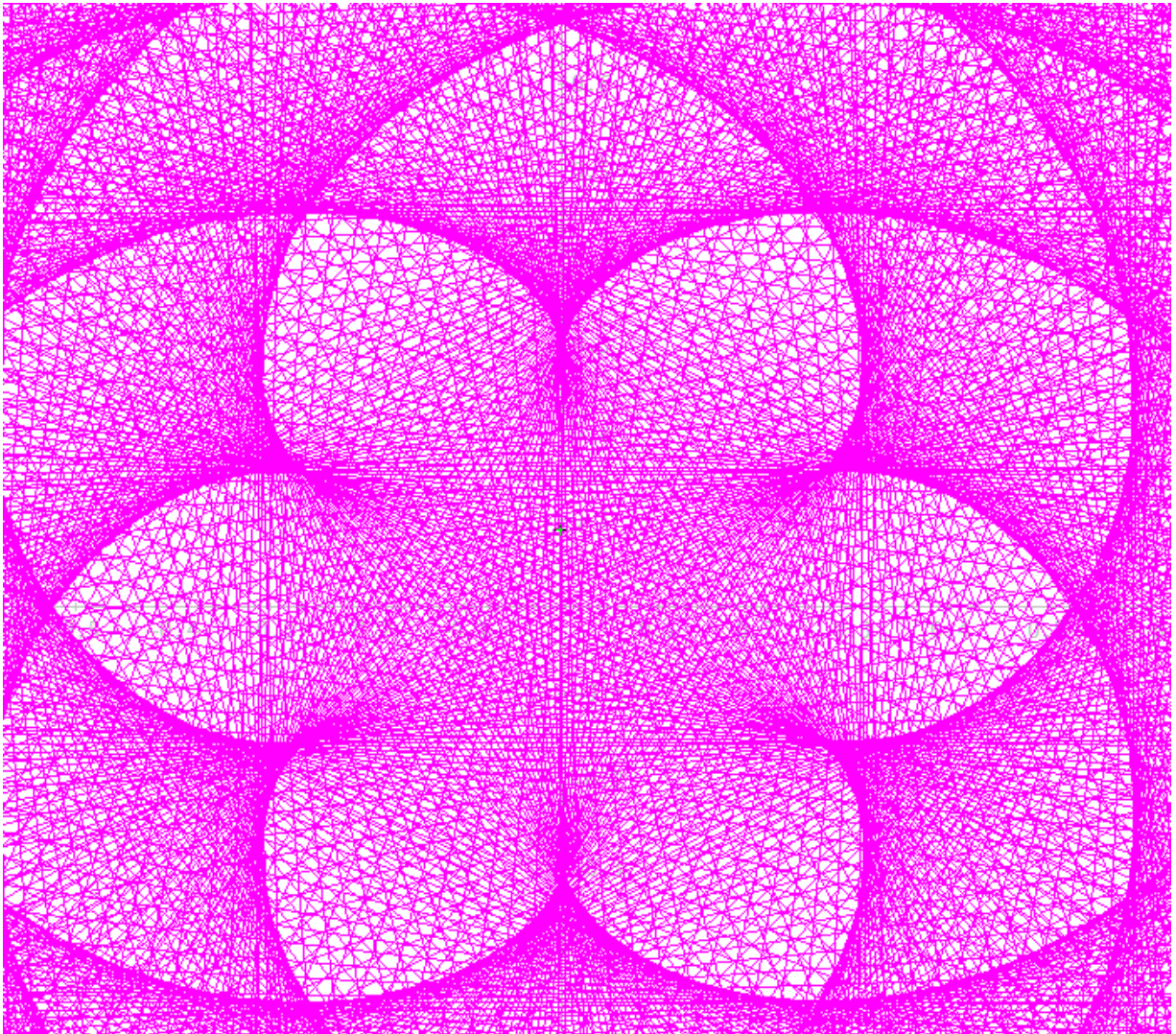


One last example with an amazing pattern and many spikes in different distances from the center:

`circ2(240, 1667)`



Next page opens a view into the interior of this figure:



Up to a certain size of the prime we can produce the graphs with TI-Nspire, too.

I recommend saving the plots, because it can happen that the software hangs up – which is my own experience experimenting with my function ... 😊.

The Multiplication Table on the TI-Nspire Screen.

File circls.tns

Just call  $cx(b,p)$  and  $cy(b,p)$  to generate two lists which can be plot as a scatter plot in the Graphs App.

$$circ2(b,p):=list\blacktriangleright mat\left( augment\left( seq\left( \cos\left( \frac{2\cdot\pi}{p-1}\cdot mod(b^k,p) \right), k, 0, p-1 \right), seq\left( \sin\left( \frac{2\cdot\pi}{p-1}\cdot mod(b^k,p) \right), k, 0, p-1 \right) \right), p \right)$$

► Done

$cx(b,p):=mat\blacktriangleright list(circ2(b,p)[1])$  ► Done     $cy(b,p):=mat\blacktriangleright list(circ2(b,p)[2])$  ► Done

Have a look on one list:

$cx(10,17)$

The first figure:

$p1x:=cx(10,17)$

$p1y:=cy(10,17)$

next example:

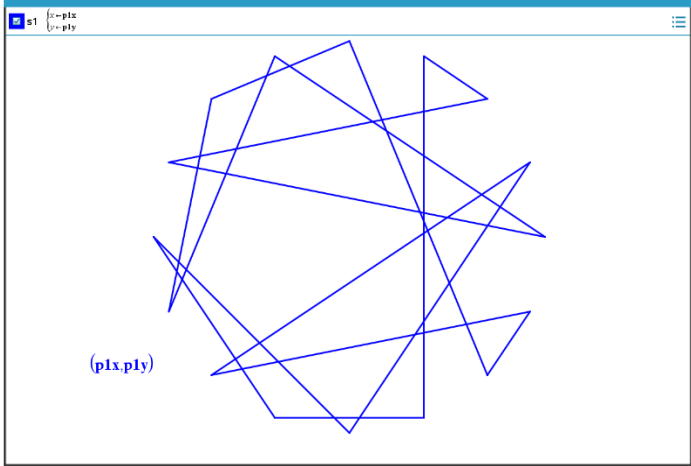
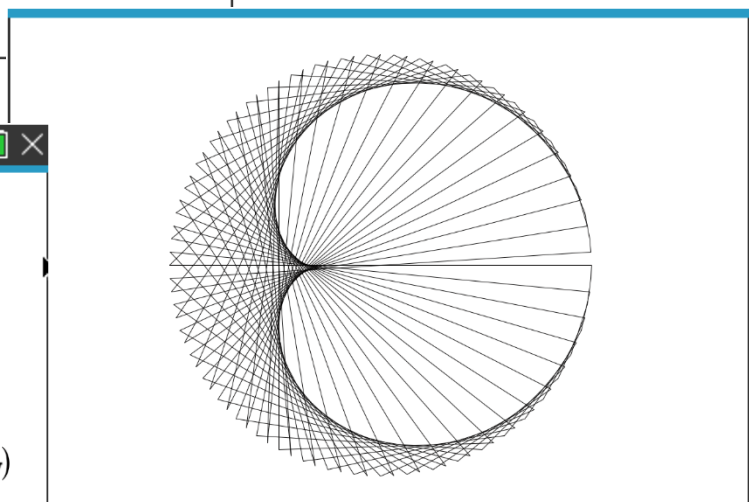
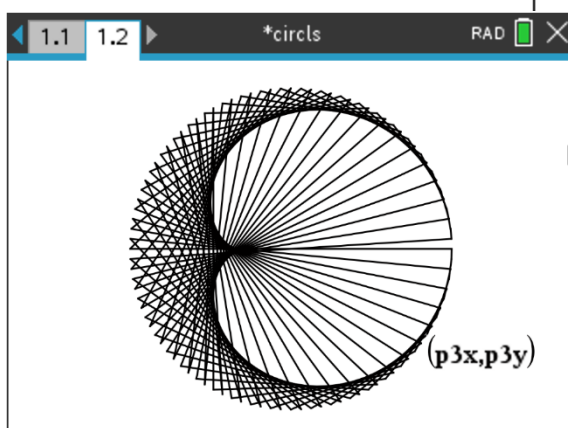
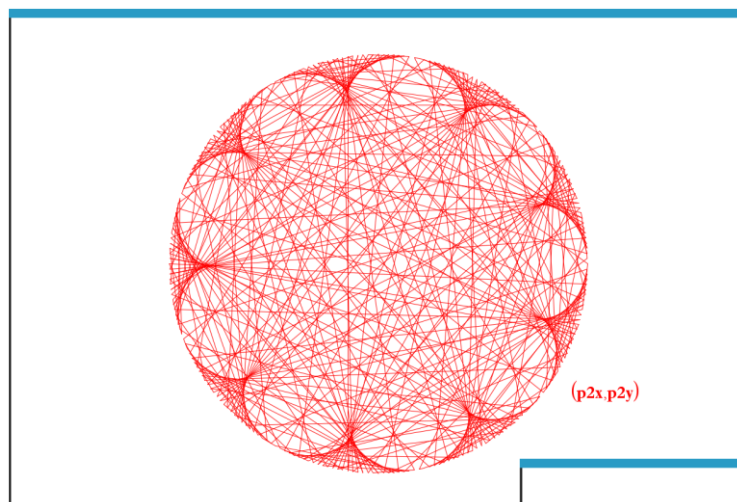
$p2x:=cx(10,257)$

$p2y:=cy(10,257)$

The Cardioid

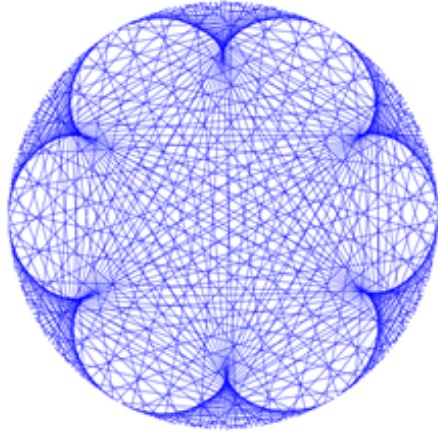
$p3x:=cx(2,101)$

$p3y:=cy(2,101)$

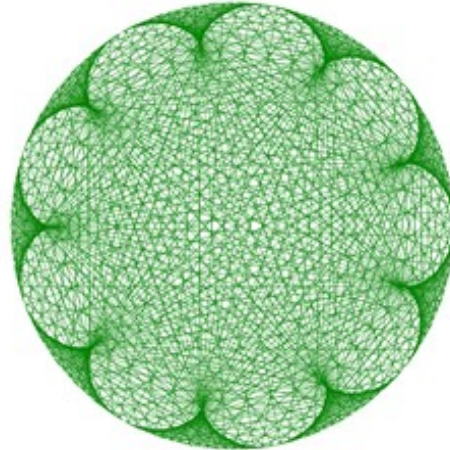





$$b=7, p=263$$

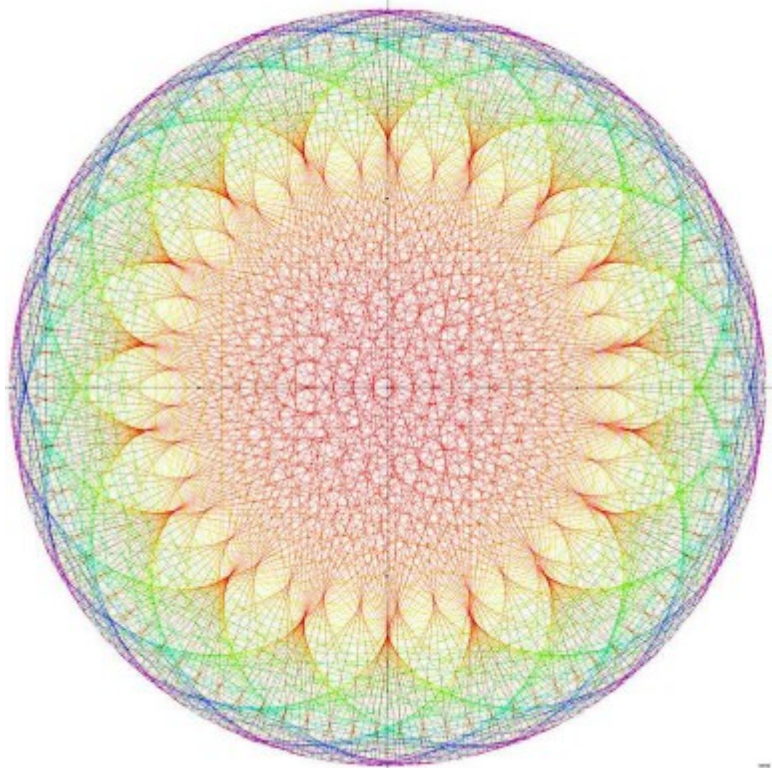


$$b=10, p=503$$



I am quite sure that Simon Plouffe will not mind presenting one of his wonderful pictures using colors depending on the length of the point connecting segments within the circle:

$$\text{base } b = 60 \text{ and prime number } p = 1103$$



Take it as a mouthwatering for downloading his papers:

<http://plouffe.fr/Inverseofprimes/La%20forme%20de%20bn%20mod%20p.pdf> (French)

<http://plouffe.fr/Inverseofprimes/The%20shape%20of%20b%5En%20mod%20p.pdf> (English)

You will find a lot of references in Plouffe's papers.

Reference: Chr. Pöppe in Spektrum der Wissenschaft 7/22

**Warsaw University of Life Sciences – SGGW**  
**Institute of Information Technology**  
**July 17 - 21, 2023**

Session Computer Algebra in Education

**Elena Varbanova**, Stoyan Kapralov, Stanislav Simeonov

Assessment of students' knowledge and abilities in undergraduate Mathematics

**Setsuo Takato**, Hideyo Makishita

Online drills created by extended CindyJS and scoring them with Maxima

**Eli Bagno**, Thierry Dana-Picard, Shulamit Reches

ChatGPT excels in medicine but falters in basic algebra

Johannes Middeke, **David J. Jeffrey**, Aishat Olagunju

Orthogonal matrices: third time around

**Michel Beaudin**

Using CAS in the classroom: personal thoughts (Part III)

**Josef Böhm**

Surfaces and their Duals

**David J. Jeffrey**, Albert D. Rich

Rubi gems

**Thierry Dana-Picard**, Tomas Recio

Automated computation of geometric Loci in Mathematics Education

Zoltán Kovács, Tomás Recio, **M. Pilar Vélez**

GeoGebra Automated Reasoning Tools: why and how (to use them in the classroom)

**Magdalena Skrzypiec**, W. Mozgawa, A. Naiman, P. Pikuta

Orthogonal trajectories to isoptics of ovals

**Hideyo Makishita**

Using CAS in mathematics education with the quadratic curve addition method



### Michel Beaudin's Exercise list for Summary 1 – Part 4

#### Problem 7 First-order linear differential equations

a) Solve the first order DE  $\frac{dy}{dx} - x y = 1, y(0) = 0$  and give your answer in terms of the error function erf.

b) Show that the solution of the differential equation  $\frac{dy}{dx} + 2x y = 5, y(0) = 3$  is given by

$$y = 3e^{-x^2} - \frac{5i\sqrt{\pi}}{2} e^{-x^2} \operatorname{erf}(ix).$$

c) Solve the DE  $\frac{dy}{dx} - \frac{y}{x} = \sin x, y(1) = 1$ . Answer in terms of function Si.

#### Problem 8 Systems of differential equations

Convert each of the following DE systems into a single DE (order to be determined) and then solve the DE. (e.g., by using the Laplace transform).

$$\text{a) } \begin{cases} \frac{dx}{dt} = y & x(0) = 1 \\ \frac{dy}{dt} = -x & y(0) = 0 \end{cases}$$

$$\text{b) } \begin{cases} \frac{d^2x}{dt^2} = y & x(0) = 1, x'(0) = 0 \\ \frac{d^2y}{dt^2} = x & y(0) = 2, y'(0) = 0 \end{cases}$$

And redo the previous exercises a) and b), leaving them as a system and applying the Laplace transform.

#### Problem 9 Undamped harmonic motion, variable frequency input

Let's consider the problem

$$\frac{d^2y}{dt^2} + y = \cos(\omega t), y(0) = 0, y'(0) = 0 \quad (\omega \neq 0).$$

a) For  $\omega^2 = 1$ , show that the solution is  $y = \frac{t \sin t}{2}$  (“resonance”).

b) For  $\omega^2 \neq 1$ , show that the solution to the problem can always be written as

$$y = \frac{\cos(t) - \cos(\omega t)}{\omega^2 - 1} = \frac{2}{1 - \omega^2} \sin\left(\frac{1}{2}(1 + \omega)t\right) \sin\left(\frac{1}{2}(1 - \omega)t\right) \quad [\omega^2 \neq 1]$$

c) Is the answer to b) a periodic function? If yes, what is the period?

**Problem 10** Solve the following differential equations, showing your calculations.

- a)  $y'' - 3y' + 2y = 10\sin x$ ,  $y(0) = 1$ ,  $y'(0) = -6$ . Review the method of indeterminate coefficients to find a particular solution.
- b)  $(D^3 - D^2 - D + 1)y = 0$ ,  $y(0) = 2$ ,  $y'(0) = 1$ ,  $y''(0) = 0$

**Solution of Problem 10 a)** Let's solve the following differential equation with initial condition:

- a)  $y'' - 3y' + 2y = 10\sin x$ ,  $y(0) = 1$ ,  $y'(0) = -6$ . We will apply the method of indeterminate coefficients. The roots of the characteristic equation  $r^2 - 3r + 2 = (r - 2)(r - 1)$  are 1 and 2. Hence the complementary (or "homogeneous") solution is:  $y_c = c_1 e^x + c_2 e^{2x}$ . The particular solution is necessarily of the form  $y_p = A\cos(x) + B\sin(x)$  since  $\sin(x)$  is not part of the complementary solution (this could not happen since the coefficient of in the DE was not zero). Remember: to have a pure sine or cosine in the solution of the DE  $ay'' + by' + cy = 0$ , it is necessary that  $b = 0$ . Find  $A$  and  $B$ . Substitute  $y_p$  in the left-hand side of the DE and obtain, grouping similar terms:

$$\begin{cases} y_p = A\cos(x) + B\sin(x) \\ y'_p = -A\sin(x) + B\cos(x) \\ y''_p = -A\cos(x) - B\sin(x) \end{cases} \Rightarrow y''_p - 3y'_p + 2y_p = (A - 3B)\cos(x) + (3A + B)\sin(x)$$

We solve the linear system  $\begin{cases} A - 3B = 0 \\ 3A + B = 10 \end{cases}$  and easily find  $A = 3$  et  $B = 1$ . So, the general solution of the DE  $y'' - 3y' + 2y = 10\sin x$  is  $y = c_1 e^x + c_2 e^{2x} + 3\cos(x) + \sin(x)$ . The initial conditions  $y(0) = 1$ ,  $y'(0) = -6$  substituted in the last equality and using the fact that  $y' = c_1 e^x + 2c_2 e^{2x} - 3\sin(x) + \cos(x)$  lead us to the following system of linear equations:

$$\begin{cases} c_1 + c_2 + 3 = 1 \\ c_1 + 2c_2 + 1 = -6 \end{cases} \Leftrightarrow c_1 = 3 \text{ et } c_2 = -5.$$

The only solution to our problem is thus  $y = 3e^x - 5e^{2x} + 3\cos(x) + \sin(x)$ . Everything we've just done by hand can be automated, and this will be very useful when the expressions become heavier. The fact that a symbolic system like Nspire always groups similar terms together will be of main importance.

**Problem 11** Laplace transforms and their properties ( $f(t) \leftrightarrow F(s)$ )

Let  $f(t)$  denote the Laplace transform by  $F(s)$ . Use ONLY factorization in  $\mathbb{R}$ , square completion and the following properties:

P1) If  $f(0) = 0$ , then  $f'(t) \leftrightarrow sF(s)$ .

P2) If,  $x(t) \leftrightarrow X(s)$ ,  $h(t) \leftrightarrow H(s)$ , then  $\int_0^t x(\tau)h(t-\tau) d\tau \leftrightarrow X(s)H(s)$

P3)  $\frac{\omega}{(s+a)^2 + \omega^2} \leftrightarrow e^{-at} \sin(\omega t)$

Prove the following relationships (in the following,  $a$  is a positive constant):

a)  $\frac{1}{s^4 + 4a^4} \leftrightarrow \frac{1}{4a^3} (\cosh(at) \sin(at) - \sinh(at) \cos(at))$

b)  $\frac{s}{s^4 + 4a^4} \leftrightarrow \frac{1}{2a^2} \sinh(at) \sin(at)$

c)  $\frac{s^2}{s^4 + 4a^4} \leftrightarrow \frac{1}{2a} (\cosh(at) \sin(at) + \sinh(at) \cos(at))$

d)  $\frac{s^3}{s^4 + 4a^4} \leftrightarrow \cosh(at) \cos(at)$

Reminder:  $\cosh x = \frac{e^x + e^{-x}}{2}$ ,  $\sinh x = \frac{e^x - e^{-x}}{2}$  ( $x \in \mathbb{R}$ ). These functions will be extended to the complex plane at a later date.

**Problem 12** Some applications of the Laplace transform

a) Let's consider the problem

$$y'' + y = \sum_{n=0}^{\infty} \delta(t - n\pi), \quad y(0) = 0, y'(0) = 0.$$

Here we have a periodic input ("pulse train") with period  $\pi$  (i.e. natural frequency of 2), while the system has a natural frequency  $\omega = 1$ , so no resonance is in sight.

Show that the solution  $s(t)$  of this problem is the following periodic function:

$$s(t) = \begin{cases} \sin t & \text{if } 0 < t < \pi \\ 0 & \text{if } \pi < t < 2\pi \end{cases} \quad s(t+2\pi) = s(t)$$

b) How can the situation described in a) be "realized" in concrete terms? Let

$$\text{CHI}(a, t, b) = \begin{cases} 1 & \text{if } a < t < b \\ 0 & \text{else} \end{cases}$$

Show that the problem

$$y'' + y = \sum_{n=0}^N \frac{1}{\varepsilon} \text{CHI}(n\pi, t, n\pi + \varepsilon), \quad y(0) = 0, y'(0) = 0$$

makes it possible to "realize" the situation described in a) if, after solving the DE,  $\varepsilon$  is allowed to go to 0 from the right and  $N$  to infinity.

c) Let's now consider the problem

$$y'' + y = \sum_{n=0}^{\infty} \delta(t - 2n\pi), \quad y(0) = 0, \quad y'(0) = 0.$$

Here we have a periodic input ("pulse train") with a period of  $2\pi$ . Show that there is resonance.

### Problem 12 Solutions of a) and c)

a) Note: it is clear that the input of this mass-spring system (the right-hand side of the DE) is

in fact a limit of the type  $\lim_{N \rightarrow \infty} \sum_{n=0}^N \delta(t - n\pi)$ . The linearity of the Laplace transform would

allow us to solve and then move on to the limit. We might as well keep the infinite sum and use the Laplace transform right away!

Let  $Y(s)$  be the Laplace transform of  $y(t)$ . Applying the Laplace transform on both sides and using the important property P22 of the table, we find the following:

$$s^2 Y(s) + Y(s) = \sum_{n=0}^{\infty} e^{-n\pi s} \Leftrightarrow Y(s) = \sum_{n=0}^{\infty} \frac{e^{-n\pi s}}{s^2 + 1} \Leftrightarrow y(t) = \sum_{n=0}^{\infty} \sin(t - n\pi) u(t - n\pi).$$

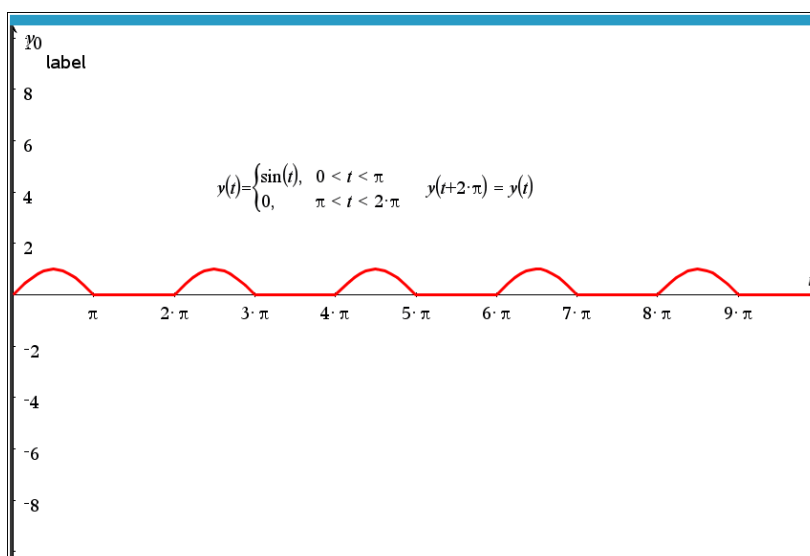
Expanding the summation, we find:

$$y(t) = \sin(t)u(t) + \sin(t - \pi)u(t - \pi) + \sin(t - 2\pi)u(t - 2\pi) + \sin(t - 3\pi)u(t - 3\pi) + \dots$$

$$y(t) = \sin(t)u(t) - \sin(t)u(t - \pi) + \sin(t)u(t - 2\pi) - \sin(t)u(t - 3\pi) + \dots$$

$$y(t) = \begin{cases} \sin(t), & 0 < t < \pi \\ 0, & \pi < t < 2\pi \end{cases} \quad P = 2\pi$$

The solution is continuous everywhere (so we could have included the ends of the subintervals). The graph of the solution is given in figure below and shown here over the interval  $0 < t < 10\pi$ .



c) Now let's consider  $y'' + y = \sum_{n=0}^{\infty} \delta(t - 2n\pi)$ ,  $y(0) = 0$ ,  $y'(0) = 0$ .

Here we have a periodic input ("pulse train") with period  $2\pi$ . Let's show that there is resonance. The difference with the previous problem is that we arrive at

$$s^2 Y(s) + Y(s) = \sum_{n=0}^{\infty} e^{-2n\pi s} \Leftrightarrow Y(s) = \sum_{n=0}^{\infty} \frac{e^{-2n\pi s}}{s^2 + 1} \Leftrightarrow y(t) = \sum_{n=0}^{\infty} \sin(t - 2n\pi) u(t - 2n\pi).$$

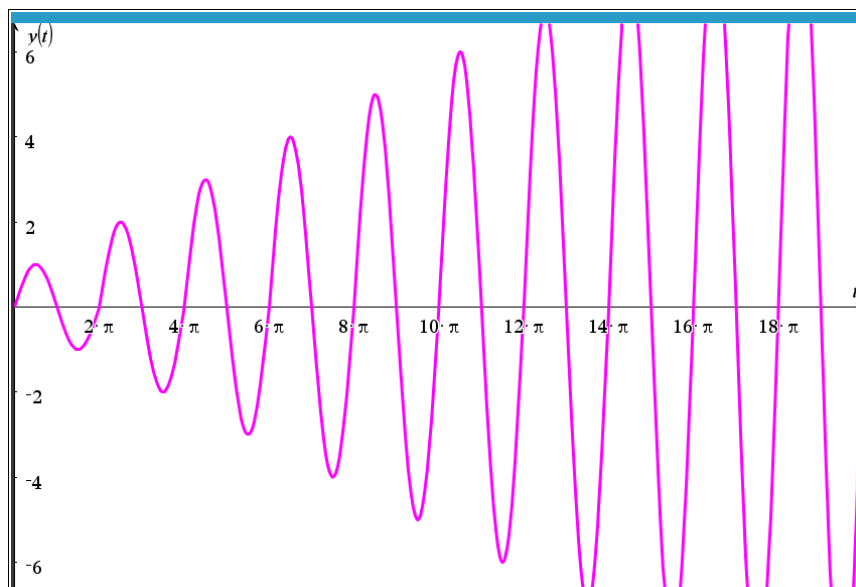
Expanding this last summation, we find

$$y(t) = \sin(t)u(t) + \sin(t - 2\pi)u(t - 2\pi) + \sin(t - 4\pi)u(t - 4\pi) + \sin(t - 6\pi)u(t - 6\pi) + \dots$$

$$y(t) = \sin(t)u(t) + \sin(t)u(t - 2\pi) + \sin(t)u(t - 4\pi) + \sin(t)u(t - 6\pi) + \dots$$

$$y(t) = \begin{cases} \sin(t), & 0 < t < 2\pi \\ 2\sin(t), & 2\pi < t < 4\pi \\ 3\sin(t), & 4\pi < t < 6\pi \\ 4\sin(t), & 6\pi < t < 8\pi \\ \vdots & \vdots \\ \vdots & \vdots \end{cases}$$

The solution is unbounded but cancels out at all multiples of  $\pi$  as shown in the graph in the figure given below, plotted here in a window which clearly shows that the amplitudes are getting out of control:



**Problem 13** A bit of complex numbers

Consider the formulas in the following table:

<b>Euler's formula</b>	$e^{i\theta} = \cos \theta + i \sin \theta$
Sum of a geometric sequence	$\sum_{k=0}^n z^k = 1 + z + z^2 + \dots \quad \frac{z^{n+1} - 1}{z - 1} \quad (z \neq 1)$ <p>ou encore écrite sous la forme</p> $1 + z + z^2 + \dots \quad \frac{z^{n+1}}{1 - z} = \frac{1}{1 - z} \quad (z \neq 1)$
Newton's binomial formula	$(a + b)^n = \sum_{k=0}^n \binom{n}{k} a^k b^{n-k} \quad \text{où} \quad \binom{n}{k} \equiv \frac{n!}{k!(n-k)!}$
De Moivre's formula	$(r(\cos \theta + i \sin \theta))^n = r^n (\cos(n\theta) + i \sin(n\theta)) \quad (n \in \mathbb{N})$

Figure for problem 13

Using them, demonstrate the following results:

a) If  $\alpha$  and  $\theta$  are real, if  $\alpha$  is not a multiple of  $2\pi$ , then

$$\text{i) } \sum_{k=0}^n \cos(\theta + k\alpha) = \frac{\sin((n+1)\alpha/2)}{\sin(\alpha/2)} \cos(\theta + n\alpha/2)$$

$$\text{ii) } \sum_{k=0}^n \sin(\theta + k\alpha) = \frac{\sin((n+1)\alpha/2)}{\sin(\alpha/2)} \sin(\theta + n\alpha/2).$$

b) "trigonometrical identities":

$$\text{i) } \cos(n\theta) = \cos^n \theta - \binom{n}{2} \cos^{n-2} \theta \sin^2 \theta + \binom{n}{4} \cos^{n-4} \theta \sin^4 \theta - \dots$$

$$\text{ii) } \sin(n\theta) = \binom{n}{1} \cos^{n-1} \theta \sin \theta - \binom{n}{3} \cos^{n-3} \theta \sin^3 \theta + \dots$$

iii) Conversely, how do you replace  $\cos^n \theta$  or  $\sin^n \theta$  by a linear combination of cosines (sines)?



**Problem 14** Complex numbers, functions of a complex variable and solving an equation of the form  $z^a = w$  where  $a$  is real and not necessarily an integer. This exercise prepares us for the chapter on complex analysis. First, we need to give some definitions.

Definition: Let  $z = x + iy$  a complex number. The exponential function of  $z$  is defined by

$$e^z = e^x e^{iy} = e^x \cos y + i e^x \sin y.$$

This follows from the Euler formula. We can therefore see that the function is  $P$ -periodic with complex period  $P = 2\pi i$ .

Definition: Let  $z = x + iy$  a non-zero complex number. The logarithm function in the base  $e$  of  $z$  is defined by

$$\text{Ln } z = \ln(|z|) + i \text{Arg}(z).$$

Here "ln" stands for the usual logarithm in base  $e$ ,  $|z|$  is the modulus of  $z$  and  $\text{Arg}(z)$  is the argument of  $z$  (always in the interval  $]-\pi, \pi]$ ).

Consequence: if  $w$  is a complex number, then the solutions of the equation  $e^z = w$  are given by  $z = \text{Ln}(w) + 2k\pi i$ ,  $k \in \mathbb{Z}$ . And those for the equation  $\text{Ln}(z) = w$  are given by  $z = e^w$  with the condition that  $-\pi < \text{Im}(w) \leq \pi$ .

Definition: Let  $z$  and  $c$  be two non-zero complex numbers. General power  $z^c$  defined as

$$z^c = e^{c \text{Ln} z}$$

Consequence: we will therefore solve an equation of the form  $z^c = w$  by applying the previous definitions.

Now here are the exercises:

- a) The equation  $z^2 = 4$  admits only 2 and -2 as solutions. Show that this is also confirmed if we write the equation as  $e^{2 \text{Ln} z} = 4$ .
- b) Find all solutions (real and complex) to the equation  $z^{\sqrt{2}} = 10$ .
- c) Find all solutions (real and complex) to the equation  $z^\pi = 10$ .
- d) Both equations  $e^x = x$  and  $\text{Ln}(x) = x$  have no real solutions. Show that the first one has an infinite number of complex solutions, but that the second one has only two. These solutions must be expressed in terms of the LambertW function.

## A Mixture of Problems with Polynomials

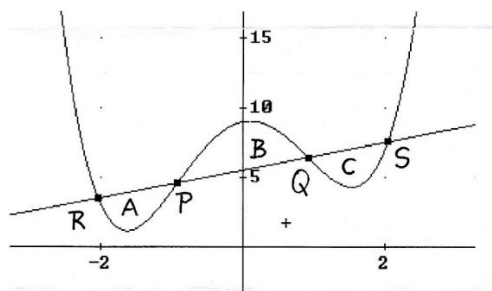
R. Baumann, Lin McMullin, Tania Koller, Susanne Waach

### Problem 1 (sent by Rüdiger Baumann, 2002)

Hello Josef,

Here is a task which I posed in class. Unfortunately, DERIVE let down a student. Can any DERIVE specialist in the know help out?

Let  $f$  a polynomial function of order 4 and  $P$  and  $Q$  its inflection points. Line  $PQ$  intersects the quartic in points  $R$  and  $S$ . So, we obtain areas  $A$ ,  $B$  and  $C$ . Show, that  $A = C$  and  $B = 2A$ .



*Solution given by Gertrude de Morçeau (Leibnizschool Hannover)*

“I start with the general investigation:

```
#1: f(x) := a4·x^4 + a3·x^3 + a2·x^2 + a1·x + a0
#2 : SOLUTIONS(DIF(f(x), x, 2) = 0, x)
#3: [x2 := -√3·(√(3·a3^2 - 8·a2·a4) + √3·a3)/(12·a4),
     x3 := √3·(√(3·a3^2 - 8·a2·a4) - √3·a3)/(12·a4)]
#4: POLY_INTERPOLATE([x2, f(x2); x3, f(x3)], x)
#5: g(x) := (9·x·(8·a1·a4^2 - a3·(4·a2·a4 - a3^2))
           + 72·a0·a4^2 - 10·a2^2·a4 + 3·a2·a3^2)/(72·a4^2)
#6: SOLUTIONS(f(x) = g(x), x)
#7: [x1 := -√3·(√5·√(3·a3^2 - 8·a2·a4) + √3·a3)/(12·a4),
     x4 := √3·(√5·√(3·a3^2 - 8·a2·a4) - √3·a3)/(12·a4)]
#8: Compare(Obj1, Obj2) := IF(Obj1 = Obj2, "equal", "not equal")
#9: A := INT(g(x) - f(x), x, x1, x2)
#10: C := INT(g(x) - f(x), x, x3, x4)
#11: B := INT(f(x) - g(x), x, x2, x3)
#12: Compare(A, C) = equal
```

Wonderful, DERIVE confirms that the first statement is correct. I continue with the second one:

```
#13: Compare(2·A, B) =
```

DERIVE keeps silent – or returns an incomprehensible output. Did I make any mistake? As a control I substitute numbers for the coefficients (which results in the figure given above):

```
#20: Speci(Term) = SUBST(Term, [a4,a3,a2,a1,a0], [1,0,-5,1,9])
#21: Figure := Speci([f(x), g(x), [x1, g(x1)], [x2, g(x2)],
                    [x3, g(x3)], [x4, g(x4)]])
#22: [Speci(A), Speci(B), Speci(C)]
#23: [10*sqrt(30/27), 10*sqrt(30/27), 10*sqrt(30/27)]
```

Numerically, it works, but symbolically it does not. I tried to manipulate with the complicated root-expressions, but without any success.”

So far Gertrude. One could translate the curve such that the inflection points are lying on the x-axis. Then one might be successful. But I didn't want to demand such an idea from the students. They should use the CAS-power of DERIVE. Any idea?

Regards Rüdiger

### DUG:

I don't know which DERIVE Version was used by the student. I could not reproduce the IF-construct to obtain #11. I worked with DERIVE 5 and 6, and this does not work, too.

Instead of applying the IF-construction, I did the following to confirm both statements:

$$\text{FACTOR}(A - C) = \frac{0}{4608 \cdot a^4}$$

$$\text{FACTOR}(A - C) = 0$$

$$\text{FACTOR}(2 \cdot A - B) = \frac{0 \cdot \sqrt{3}}{69120 \cdot a^4}$$

$$\text{FACTOR}(2 \cdot A - B) = 0$$

### Problem 2 (sent by Lin McMullin, 2004)

Show that any cubic polynomial is symmetric about its point of inflection.

$$\#1: f(x) := a \cdot x^3 + b \cdot x^2 + c \cdot x + d$$

$$\#2: \text{SOLUTIONS}\left(\left(\frac{d}{dx}\right)^2 f(x) = 0, x\right) = \left[-\frac{b}{3 \cdot a}\right]$$

$$\#3: \frac{f\left(-\frac{b}{3 \cdot a} - u\right) + f\left(-\frac{b}{3 \cdot a} + u\right)}{2} = \frac{27 \cdot a^2 \cdot d - 9 \cdot a \cdot b \cdot c + 2 \cdot b^3}{27 \cdot a^2}$$

$$\#4: f\left(-\frac{b}{3 \cdot a}\right) = \frac{27 \cdot a^2 \cdot d - 9 \cdot a \cdot b \cdot c + 2 \cdot b^3}{27 \cdot a^2}$$

Another way could be:

Shift  $f(x)$  such, that the point of inflection lies at the origin

$$g(x) := f\left(x - \frac{b}{3 \cdot a}\right) - \frac{27 \cdot a^2 \cdot d - 9 \cdot a \cdot b \cdot c + 2 \cdot b^3}{27 \cdot a^2}$$

$$g(x) := \frac{x \cdot (3 \cdot a^2 \cdot x^2 + 3 \cdot a \cdot c - b^2)}{3 \cdot a}$$

$g(x)$  is an odd function, which is symmetric about the origin (= point of inflection).

### Problem 3 (sent by Lin McMullin, 2004)

This one is, I believe a heretofore unnoticed result that I found recently working a different problem on a CAS:

Let  $q(x)$  be any fourth-degree polynomial with points of inflection at  $x = a$  and  $x = b$ . Show that the line joining the points of inflection intersects  $q(x)$  at two other points with  $x$ -coordinates of

$$x_1 = 1/2 ((a(1+\sqrt{5}) + b(1-\sqrt{5}))) \text{ and } x_2 = 1/2 ((b(1+\sqrt{5}) + a(1-\sqrt{5}))).$$

Notice that the coefficients are the Golden Ratio and its conjugate (Phi and little phi).

Solving the problem with TI-NspireCAS:

$$\begin{aligned} f_2(x) &:= k \cdot (x-a) \cdot (x-b) \quad \blacktriangleright \text{ Done} \\ f_1(x) &:= \int f_2(x) \, dx + c \quad \blacktriangleright \text{ Done} \\ f(x) &:= \int f_1(x) \, dx + d \quad \blacktriangleright \text{ Done} \\ f(x) &\blacktriangleright \frac{k \cdot x^4}{12} - \frac{(a+b) \cdot k \cdot x^3}{6} + \frac{a \cdot b \cdot k \cdot x^2}{2} + c \cdot x + d \\ f_0(x) &:= \frac{k \cdot x^4}{12} - \frac{(a+b) \cdot k \cdot x^3}{6} + \frac{a \cdot b \cdot k \cdot x^2}{2} + c \cdot x + d \quad \blacktriangleright \text{ Done} \\ g(x) &:= \frac{f_0(a) - f_0(b)}{a-b} \cdot (x-a) + f_0(a) \quad \blacktriangleright \text{ Done} \\ g(x) &\blacktriangleright \frac{-(a^3 \cdot k - 3 \cdot a^2 \cdot b \cdot k - 3 \cdot a \cdot b^2 \cdot k + b^3 \cdot k - 12 \cdot c) \cdot x}{12} + \frac{a^3 \cdot b \cdot k}{12} - \frac{a^2 \cdot b^2 \cdot k}{4} + \frac{a \cdot b^3 \cdot k}{12} + d \quad \triangle \\ \frac{f(x) - g(x)}{(x-a) \cdot (x-b)} &\blacktriangleright \frac{k \cdot (x^2 - (a+b) \cdot x - a^2 + 3 \cdot a \cdot b - b^2)}{12} \quad \triangle \\ \text{solve}\left(\frac{f(x) - g(x)}{(x-a) \cdot (x-b)} = 0, x\right) &\blacktriangleright x = \frac{a \cdot (\sqrt{5} + 1) - b \cdot (\sqrt{5} - 1)}{2} \text{ or } x = \frac{-(a \cdot (\sqrt{5} - 1) - b \cdot (\sqrt{5} + 1))}{2} \text{ or } k=0 \quad \triangle \end{aligned}$$

I found out that it is important to define  $f_0(x)$  and not proceed with  $f(x)$ !

Another access to solve the problem (DERIVE):

InputMode := Word

$$f(x) := a_4 \cdot x^4 + a_3 \cdot x^3 + a_2 \cdot x^2 + a_1 \cdot x + a_0$$

$$\text{SOLUTIONS} \left( \left( \frac{d}{dx} \right)^2 f(x) = 0, x \right)$$

$$\left[ - \frac{\sqrt{3} \cdot (\sqrt{(3 \cdot a_3^2 - 8 \cdot a_2 \cdot a_4)} + \sqrt{3} \cdot a_3)}{12 \cdot a_4}, \frac{\sqrt{3} \cdot (\sqrt{(3 \cdot a_3^2 - 8 \cdot a_2 \cdot a_4)} - \sqrt{3} \cdot a_3)}{12 \cdot a_4} \right]$$

$$\left[ a := - \frac{\sqrt{3} \cdot (\sqrt{(3 \cdot a_3^2 - 8 \cdot a_2 \cdot a_4)} + \sqrt{3} \cdot a_3)}{12 \cdot a_4}, b := \frac{\sqrt{3} \cdot (\sqrt{(3 \cdot a_3^2 - 8 \cdot a_2 \cdot a_4)} - \sqrt{3} \cdot a_3)}{12 \cdot a_4} \right]$$

$$g(x) := \text{POLY\_INTERPOLATE} \left( \begin{bmatrix} a & f(a) \\ b & f(b) \end{bmatrix}, x \right)$$

We know that a and b must be solutions of  $f(x) = g(x)$  - to obtain the remaining two intersection points:

$$\text{SOLUTIONS} \left( \frac{f(x) - g(x)}{(x - a) \cdot (x - b)} = 0, x \right)$$

$$\left[ - \frac{\sqrt{3} \cdot (\sqrt{5} \cdot \sqrt{(3 \cdot a_3^2 - 8 \cdot a_2 \cdot a_4)} + \sqrt{3} \cdot a_3)}{12 \cdot a_4}, \frac{\sqrt{3} \cdot (\sqrt{5} \cdot \sqrt{(3 \cdot a_3^2 - 8 \cdot a_2 \cdot a_4)} - \sqrt{3} \cdot a_3)}{12 \cdot a_4} \right]$$

$$\left[ \frac{1}{2} \cdot (1 + \sqrt{5}) \cdot a + \frac{1}{2} \cdot (1 - \sqrt{5}) \cdot b, \frac{1}{2} \cdot (1 + \sqrt{5}) \cdot b + \frac{1}{2} \cdot (1 - \sqrt{5}) \cdot a \right]$$

$$\left[ - \frac{\sqrt{3} \cdot (\sqrt{5} \cdot \sqrt{(3 \cdot a_3^2 - 8 \cdot a_2 \cdot a_4)} + \sqrt{3} \cdot a_3)}{12 \cdot a_4}, \frac{\sqrt{3} \cdot (\sqrt{5} \cdot \sqrt{(3 \cdot a_3^2 - 8 \cdot a_2 \cdot a_4)} - \sqrt{3} \cdot a_3)}{12 \cdot a_4} \right]$$

Problem 4 (sent by Tania Koller and Susanne Waach, my colleagues at HAK St. Pölten, )

Tania and Susi found a task to find the equation of a cubic in a textbook:

Which 3<sup>th</sup> degree polynomial has in  $P(-3/2)$  an extremal point, passes point  $Q(3/7)$  and has for  $x=-1$  its inflection point.

Tania and Susi taught mathematics supported by DERIVE in the computer lab.

See, what happened in class.

We set the four equations and tried to solve for the coefficients  $a, b, c, d$ :

$$f(x) := a \cdot x^3 + b \cdot x^2 + c \cdot x + d$$

$$f(-3) = 2$$

$$-27 \cdot a + 9 \cdot b - 3 \cdot c + d = 2$$

$$f'(-3) = 0$$

$$27 \cdot a - 6 \cdot b + c = 0$$

$$f(3) = 7$$

$$27 \cdot a + 9 \cdot b + 3 \cdot c + d = 7$$

$$f''(-1) = 0$$

$$2 \cdot b - 6 \cdot a = 0$$

$$\text{SOLVE}([f(-3) = 2, f'(-3) = 0, f(3) = 7, f''(-1) = 0], [a, b, c, d])$$



The system has no solution! How can this happen? We tried to solve the system with paper and pen:

$$\begin{array}{rcl}
 -27a + 9b - 3c + d & = & 2 \quad (1) \\
 27a - 6b + c & = & 0 \quad (2) \\
 27a + 9b + 3c + d & = & 7 \quad (3) \\
 -6a + 2b & = & 0 \quad (4) \\
 \hline
 (1)+(2) & 3b - 2c + d & = 2 \quad (5) \\
 (1)+(3) & 18b + 2d & = 9 \quad (6) \\
 2 \cdot (2) + 9 \cdot (4) & 6b + 2c & = 0 \quad (7) \\
 \hline
 (5)+(7) & 9b + d & = 2 \quad (8) \\
 (6) & 18b + 2d & = 9 \quad (9) \\
 \hline
 (9) - 2 \cdot (8) & 0 & = 5 \\
 & & \text{Widerspruch !!}
 \end{array}$$

After discussion in the class, the majority blames the inflection point for the bad result and checks its conjecture by taking a generalized inflection point for  $x = t$ .

$$\left[ -\frac{5}{108 \cdot (t+1)}, \frac{5 \cdot t}{36 \cdot (t+1)}, \frac{5}{12 \cdot (t+1)} + \frac{5}{6}, \frac{13 \cdot t + 18}{4 \cdot (t+1)} \right]$$

$$ff(x) := -\frac{5 \cdot x^3}{108 \cdot (t+1)} + \frac{5 \cdot t \cdot x^2}{36 \cdot (t+1)} + x \cdot \left( \frac{5}{12 \cdot (t+1)} + \frac{5}{6} \right) + \frac{13 \cdot t + 18}{4 \cdot (t+1)}$$

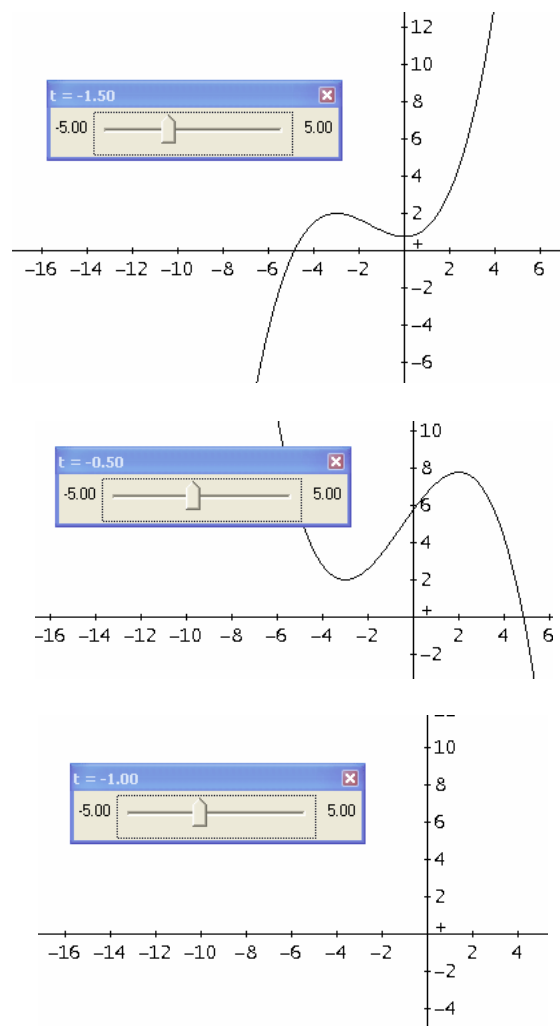
If we don't recognize the cause for failing at this point, we look for the inflection point to be sure:

$$\left( \frac{d}{dx} \right)^2 ff(x) = \frac{5 \cdot t}{18 \cdot (t+1)} - \frac{5 \cdot x}{18 \cdot (t+1)}$$

Question: Which values for  $t$  don't give a result for the inflection point?

**It is  $t = -1$ , of course. That's too bad!**

The students introduced a slider for  $t$ :



They observed the graph for  $t < -1$ ,  $t > -1$  and  $t = -1$ .

This is what Tania and Susi sent.

When I received the mail + dfw-files from my former colleagues (I was already retired) I said to myself: But that can't be all ...

**Why is the turning point the culprit?**

**Maybe it is the interaction of all specification parameters?**

$$f(x) := a \cdot x^3 + b \cdot x^2 + c \cdot x + d$$

$$[x_e :=, x_w :=, x_p :=, y_e :=]$$

$$\begin{bmatrix} f(x_e) = y_e \\ f'(x_e) = 0 \\ f(x_p) = y_p \\ f''(x_w) = 0 \end{bmatrix} = \begin{bmatrix} a \cdot x_e^3 + b \cdot x_e^2 + c \cdot x_e + d = y_e \\ 3 \cdot a \cdot x_e^2 + 2 \cdot b \cdot x_e + c = 0 \\ a \cdot x_p^3 + b \cdot x_p^2 + c \cdot x_p + d = y_p \\ 6 \cdot a \cdot x_w + 2 \cdot b = 0 \end{bmatrix}$$

(Note:  $x_w$  is the  $x$ -coordinate for the inflection point which is in German “Wendepunkt”).

The equations with the parameters look very wild.

Who wants to solve this system manually for  $a, b, c$  and  $d$ ?

Fortunately, we have a CAS as our servant!

$$\text{SOLUTIONS}(f(x_e) = y_e \wedge f'(x_e) = 0 \wedge f(x_p) = y_p \wedge f''(x_w) = 0, [a, b, c, d])'$$

$$\begin{bmatrix} \frac{y_p - y_e}{2 \cdot x_e^3 - 3 \cdot x_e^2 \cdot (x_p + x_w) + 6 \cdot x_e \cdot x_p \cdot x_w + x_p^2 \cdot (x_p - 3 \cdot x_w)} \\ \frac{3 \cdot x_w \cdot (y_e - y_p)}{2 \cdot x_e^3 - 3 \cdot x_e^2 \cdot (x_p + x_w) + 6 \cdot x_e \cdot x_p \cdot x_w + x_p^2 \cdot (x_p - 3 \cdot x_w)} \\ \frac{3 \cdot x_e \cdot (x_e - 2 \cdot x_w) \cdot (y_e - y_p)}{2 \cdot x_e^3 - 3 \cdot x_e^2 \cdot (x_p + x_w) + 6 \cdot x_e \cdot x_p \cdot x_w + x_p^2 \cdot (x_p - 3 \cdot x_w)} \\ \frac{2 \cdot x_e^3 \cdot y_p - 3 \cdot x_e^2 \cdot (x_p \cdot y_e + x_w \cdot y_p) + 6 \cdot x_e \cdot x_p \cdot x_w \cdot y_e + x_p^2 \cdot y_e \cdot (x_p - 3 \cdot x_w)}{2 \cdot x_e^3 - 3 \cdot x_e^2 \cdot (x_p + x_w) + 6 \cdot x_e \cdot x_p \cdot x_w + x_p^2 \cdot (x_p - 3 \cdot x_w)} \end{bmatrix}$$

Question: What can you observe?

All denominators are the same!

Give reasons!

In case that the students are familiar with Cramer's rule they might know that the denominator is the determinant of the system matrix.



Remark: The teacher could take the occasion to repeat Cramer's rule and/or DERIVE's (or TI-Nspire's) `row_reduce()` (`ref()` with TI-Nspire) command and/or Gauß' elimination method?

$$\text{mat} := \begin{bmatrix} x^3 & x^2 & x & 1 & y \\ x^3 & x^2 & x & 1 & 0 \\ 3 \cdot x^2 & 2 \cdot x & 1 & 0 & y \\ 6 \cdot x & 2 & 0 & 0 & 0 \end{bmatrix}$$

DET(mat<sub>↓↓</sub>[1, ..., 4])

$$2 \cdot (2 \cdot x^3 - 3 \cdot x^2 \cdot (x + x) + 6 \cdot x \cdot x \cdot x + x^3 - 3 \cdot x^2 \cdot x)$$

$$\text{FACTOR}(2 \cdot x^3 - 3 \cdot x^2 \cdot (x + x) + 6 \cdot x \cdot x \cdot x + x^3 - 3 \cdot x^2 \cdot x)$$

$$(x - x)^2 \cdot (2 \cdot x + x - 3 \cdot x)$$

So, what does the (factorized) denominator tell us?

Interpretation of the result:

$x$  must be  $\neq x$  (obvious)

$2 \cdot x + x - 3 \cdot x$  must not become 0!! or  $x = 3 \cdot x - 2 \cdot x$  provides a contradiction!

I try with another example:  $x = 5, x = -2, x = -16$ . ( $3 \cdot (-2) - 2 \cdot 5 = -16$ )

**Express the problem in your own words:**

**A cubic has one turning point in (5,3), for  $x = -2$  an inflection point and passes P(-16,4).**

$$\text{SOLVE}(f(-16) = 4 \wedge f'(5) = 0 \wedge f(5) = -3 \wedge f''(-2) = 0, [a, b, c, d])$$

false

Give at least one more example! Take P(16,4) instead!

What can we learn from the nominators?

$$\begin{bmatrix} a & y - y \\ b & 3 \cdot x \cdot (y - y) \\ c & 3 \cdot x \cdot (x - 2 \cdot x) \cdot (y - y) \\ d & 2 \cdot x^3 \cdot y - 3 \cdot x^2 \cdot (x \cdot y + x \cdot y) + 6 \cdot x \cdot x \cdot x \cdot y + x^3 \cdot y \cdot (x - 3 \cdot x) \end{bmatrix}$$

- (1)  $y = y$  (express in your own words?)  
 $a, b$  and  $c$  disappear and a horizontal line is remaining

Confirm with a self-chosen example:

$$f(-3) = 7, f'(-3) = 0, f(3) = 7, f''(-2) = 0$$

$$\text{SOLVE}(f(-3) = 7 \wedge f'(-3) = 0 \wedge f(3) = 7 \wedge f''(-2) = 0, [a, b, c, d])$$

$$a = 0 \wedge b = 0 \wedge c = 0 \wedge d = 7$$

- (2)  $yw = 0$  (express in your own words?)

Coefficient  $b$  disappears and we don't have a quadratic term in the polynomial.

Confirm with a self-chosen example:

$$f(-3) = 2, f'(-3) = 0, f(3) = 7, f''(0) = 0$$

$$\text{SOLVE}(f(-3) = 2 \wedge f'(-3) = 0 \wedge f(3) = 7 \wedge f''(0) = 0, [a, b, c, d])$$

$$a = -\frac{5}{108} \wedge b = 0 \wedge c = \frac{5}{4} \wedge d = \frac{9}{2}$$

- (3)  $xe = 0$  or  $xe = 2 \cdot xw$  (express in your own words?)

$c$  disappears, and we don't have a linear term.

Confirm with a self-chosen example for both cases:

$$f(0) = 2, f'(0) = 0, f(3) = 7, f''(-1) = 0$$

or

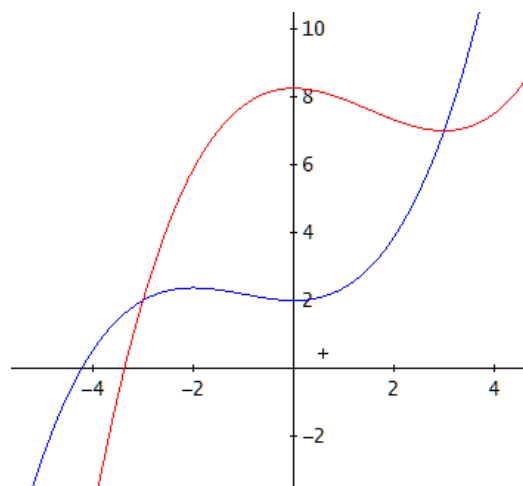
$$f(-3) = 2, f'(3) = 0, f(3) = 7, f''(3/2) = 0$$

$$\text{SOLVE}(f(0) = 2 \wedge f'(0) = 0 \wedge f(3) = 7 \wedge f''(-1) = 0, [a, b, c, d])$$

$$a = \frac{5}{54} \wedge b = \frac{5}{18} \wedge c = 0 \wedge d = 2$$

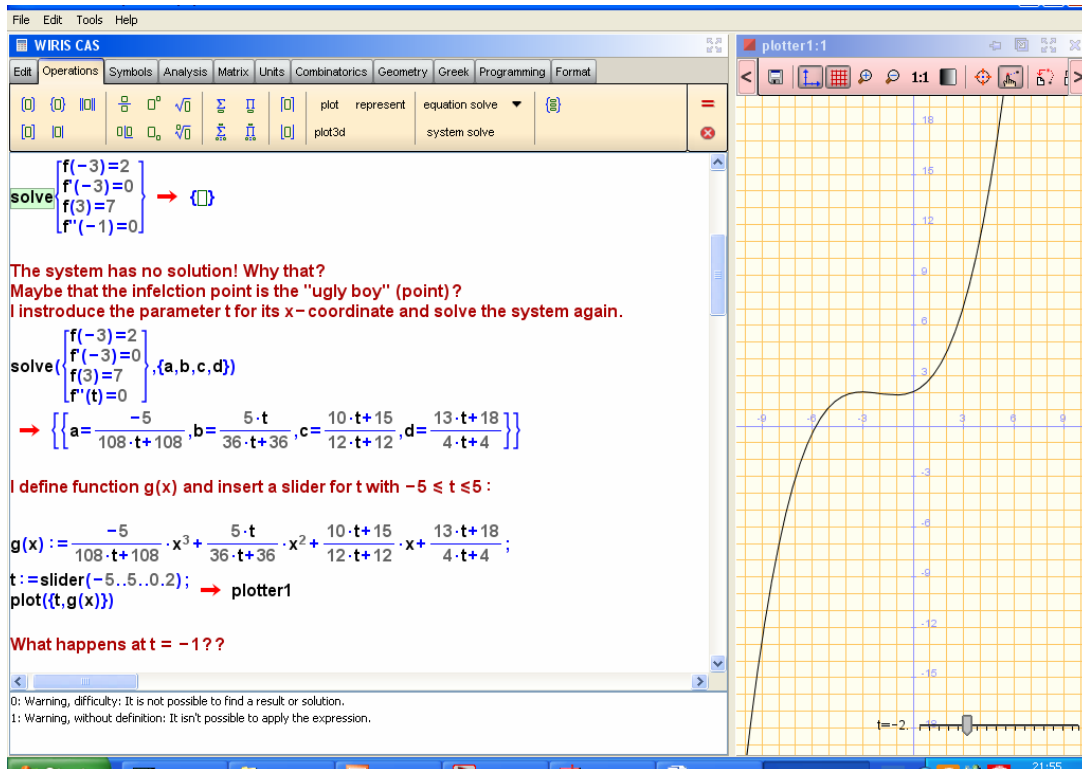
$$\text{SOLVE}\left(f(-3) = 2 \wedge f'(3) = 0 \wedge f(3) = 7 \wedge f''\left(\frac{3}{2}\right) = 0, [a, b, c, d]\right)$$

$$a = \frac{5}{54} \wedge b = -\frac{5}{12} \wedge c = 0 \wedge d = \frac{33}{4}$$

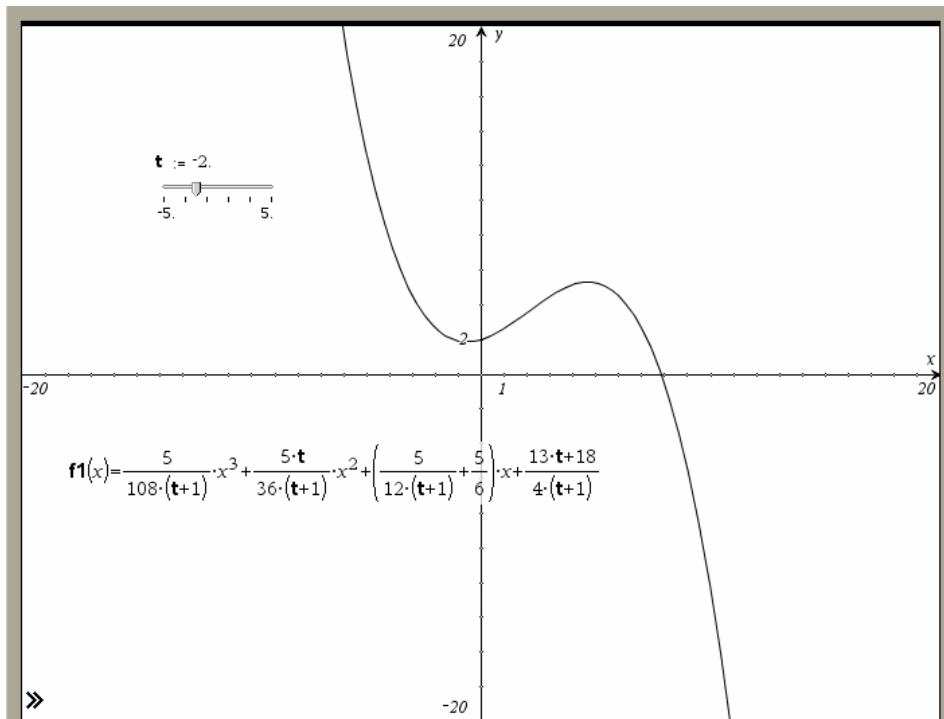


These tasks can be carried out with any CAS.

I show as an example a screen shot with WIRIS and then two pictures of a treatment with TI-NspireCAS:



$g12 := \frac{d}{dx}(f(x))=0 x=xe$	$3 \cdot a \cdot xe^2 + 2 \cdot b \cdot xe + c = 0$
$g13 := f(xp) = xp$	$a \cdot xp^3 + b \cdot xp^2 + c \cdot xp + d = xp$
$g14 := \frac{d^2}{dx^2}(f(x))=0 x=xw$	$6 \cdot a \cdot xw + 2 \cdot b = 0$
$\text{solve}(g11 \text{ and } g12 \text{ and } g13 \text{ and } g14, \{a, b, c, d\})$	
$a = \frac{xp - ye}{2 \cdot xe^3 - 3 \cdot xe^2 \cdot (xp + xw) + 6 \cdot xe \cdot xp \cdot xw + xp^2 \cdot (xp - 3 \cdot xw)}$	and $b = \frac{-3 \cdot (xp - ye) \cdot xw}{2 \cdot xe^3 - 3 \cdot xe^2 \cdot (xp + xw) + 6 \cdot xe \cdot xp \cdot xw + xp^2 \cdot (xp - 3 \cdot xw)}$
$g1 := f(-3) = 2$	$-27 \cdot a + 9 \cdot b - 3 \cdot c + d = 2$
$g2 := \frac{d}{dx}(f(x))=0 x=-3$	$27 \cdot a - 6 \cdot b + c = 0$
$g3 := f(3) = 7$	$27 \cdot a + 9 \cdot b + 3 \cdot c + d = 7$
$g4 := \frac{d^2}{dx^2}(f(x))=0 x=t_$	$6 \cdot a \cdot t_ + 2 \cdot b = 0$
$\text{solve}(g1 \text{ and } g2 \text{ and } g3 \text{ and } g4, \{a, b, c, d\})$	
$a = \frac{-5}{108 \cdot (t_ + 1)}$	and $b = \frac{5 \cdot t_}{36 \cdot (t_ + 1)}$ and $c = \frac{5 \cdot (2 \cdot t_ + 3)}{12 \cdot (t_ + 1)}$ and $d = \frac{13 \cdot t_ + 18}{4 \cdot (t_ + 1)}$



Of course, higher order polynomial functions or rational functions or, or, or ... can also be used for similar tasks.

I suggest that students invent tasks by themselves and investigate the conditions under which the tasks become "meaningful."

## What is the "value added"?

- Questioning a "non-result"
- Generalization of a task
- Conscious work with parameters
- Solution of a complex linear system of equations  
(BlackBox, Gauss, Cramer's rule)
- Interpretation of the solutions & verbalization of the answers
- Verification by graphical representation (possibly using sliders)
- Inventing own tasks
- New interpretation of trad. Textbook tasks