THE DERIVE - NEWSLETTER #132

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+ CAS-TI

Contents:

December 2023

All links are from mathematical articles published in the 2023 German issues of the Scientific American. It can be that the direct links to arxiv.org don't work (on my PC, on the laptop they work!). Then copy the link and paste it into the entry line of your browser. Then they should work properly.

The local-global conjecture for Apollonian circle packings is false
https://arxiv.org/pdf/2307.02749

The Pro- étale topology for schemes
https://arxiv.org/pdf/1309.1198

Lectures on Condensed Mathematics
https://www.math.uni-bonn.de/people/scholze/Condensed.pdf

Lectures on Analytic Geometry
https://www.math.uni-bonn.de/people/scholze/Analytic.pdf

Looking at Euler flows through a contact mirror: Universality and undecidability
https://arxiv.org/pdf/2107.09471

Computability and Beltrami fields in Euclidean space (Simulating a Turing machine)
https://arxiv.org/pdf/2111.03559

Sums of square roots that are close to an integer
https://arxiv.org/pdf/2401.10152

The singular set in the Stefan problem
https://arxiv.org/pdf/2103.13379

Integers expressible as the sum of two rational cubes
https://arxiv.org/pdf/2210.10730

Autoformalization with Large Language Models
https://arxiv.org/pdf/2205.12615

Solving Quantitative Reasoning Problems with Language Models
https://arxiv.org/pdf/2206.14858

Dear DUG-Members,

This DNL should have been the last one of 2023. Unfortunately, it is very late due to some health problems. My wife and I caught the COVID-virus just at Christmas time. It was only a light version because we both were vaccinated. Then I lost all my energy for almost two weeks. I even didn't want to turn on the computer! Now I feel well again – more or less – and could finalize the overdue newsletter. It contains two presentations from ACA23:

Elena Varbanova's contribution might be a challenge for you. Try to give answers to 30 problems presented on pages 8 and 9. You can download more collections from the COMPMATH-website. I had an intense mail communication with Magda Skrzypiec and am very grateful for her patience answering my questions concerning the Isoptics. I was not aware that so many mathematicians have published papers on isoptics.

My contribution about the Apollonian Cubics might be of interest for students when they have learned about Apollonian circles.

I am working on some exciting projects for the next newsletters. Have a look on the last lines of the preview next page and you will see the long list of newly added contributions.

I collected the links presented on the first page in the 2023 issues of the German version of Scientific American. The papers are all in English and can be downloaded for free. Some of them are really "high" mathematics.

With best regards, Josef

Liebe DUG-Mitglieder,

Dieser DNL hätte der letzte in 2023 sein sollen. Leider ist er wegen einiger gesundheitlichen Probleme sehr verspätet. Meine Frau und ich haben gerade zur Weihnachtszeit noch den COVID-Virus eingefangen. Da wir beide geimpft waren, war es nur eine leichte Version. Ich habe aber trotzdem all meine Energie für fast zwei Wochen verloren. Nicht einmal den PC wollte ich aufdrehen! Jetzt fühle ich mich besser – mehr oder weniger – und kann den überfälligen DNL fertigstellen. Er enthält zwei Vorträge von der ACA 2023.

Elena Varbanova´s Beitrag könnte auch eine Herausforderung für Euch sein. Versucht die 30 Aufgaben auf Seiten 8 und 9 zu lösen. Mehr Aufgaben könnt Ihr von der COMPMATH-Website downloaden. Mit Magda Skrzypiec hatte ich eine intensive Kommunikation und ich bin sehr dankbar für ihre Geduld, meine Fragen zu den Isoptics zu beantworten. Ich habe nicht gewusst, dass so viele Mathematiker Arbeiten über Isoptics veröffentlicht haben.

Mein Beitrag über Apollonische Kubiken könnte für Studenten, die die Apollonischen Kreise kennen gelernt haben interessant sein.

Für die nächsten DNLs habe ich einige spannende Projekte in Arbeit. Ihr könnt sie in den letzten Zeilen des Previews finden.

Auf der ersten Seite habe ich Links zusammengestellt, die ich in den 2023 Ausgaben des „Spektrums der Wissenschaft" gefunden habe.

Mit besten Grüßen

Josef

The *DERIVE-NEWSLETTER* is the Bulletin of the *DERIVE* & CAS-*TI User Group*. It is published at least four times a year with a content of 40 pages minimum. The goals of the *DNL* are to enable the exchange of experiences made with *DERIVE*, *TI*-CAS and other CAS as well to create a group to discuss the possibilities of new methodical and didactical manners in teaching mathematics.

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Contributions:
Please send all contributions to the Editor. Non-English speakers are encouraged to write their contributions in English to reinforce the international touch of the *DNL*. It must be said, though, that non-English articles will be warmly welcomed nonetheless. Your contributions will be edited but not assessed. By submitting articles, the author gives his consent for reprinting it in the *DNL*. The more contributions you will send, the more lively and richer in contents the *DERIVE* & CAS-*TI Newsletter* will be.

Next issue: March 2024

Preview: Contributions waiting to be published

Some simulations of Random Experiments, J. Böhm, AUT, Lorenz Kopp, GER
Wonderful World of Pedal Curves, J. Böhm, AUT
Simulating a Graphing Calculator in *DERIVE*, J. Böhm, AUT
Logos of Companies as an Inspiration for Math Teaching
Exciting Surfaces in the FAZ, BooleanPlots.mth, P. Schofield, UK
Old traditional examples for a CAS – What´s new? J. Böhm, AUT
Tutorials for the NSpireCAS, G. Herweyers, BEL
Dirac Algebra, Clifford Algebra, Vector-Matrix-Extension, D. R. Lunsford, USA
Another Approach to Taylor Series, D. Oertel, GER
Charge in a Magnetic Field, H. Ludwig, GER
More Applications of TI-Innovator™ Hub and TI-Innovator™ Rover
A Collection of Special Problems, Puzzles, W. Alvermann, GER
DERIVE Bugs? D. Welz, GER
Tweening & Morphing with TI-NspireCX-II-T, J. Böhm. AUT
The Gap between Poor and Rich, J. Böhm, AUT
More functions from M. Myers and from Bhuvanesh's Mathtools-library
TaxiCab Conics, Two alternate Approaches to Conics, R. Haas, USA
QR-Code light, Random numbers following a given distribution
153 is another Special Number
Quartiles, Numeros Primos, F. de Jesús Martínez Vargas, Mexico
Stepwise First Order Partial Differential Equations Solver, J. L. Galan a. o., ESP
Stepwise Ordinary Differential Equations Solver, J. L. Galan and colleagues, ESP
Stepwise Ordinary Differential Equations Solver, J. L. Galan and colleagues, ESP
Math Lessons with/despite AI, H. Heugl, AUT
Integration Tables, Summary 2, M. Beaudin, CAN
Dual Surfaces, J. Böhm, AUT
Th. Dana-Picard and others

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Assessment of Students' Knowledge and Abilities in Technology enriched Learning Environment

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In memory of Eugenio Roanes-Lozano

HavingComputer Algebra Systems (CAS)at disposal the teaching and learning of mathematics undergoes somechanges. But care must be taken to changes: the things we think are changing aren't always what's changing.In the triad Teaching-Learning-Assessment (TLA) the components are to be considered in tandem.Theyare to be interrelated, because what gets assessed is what gets taught.Why is it necessary to focus also on purposeful assessment. Because:1) teaching and learning are carried out in technology (incl. CAS) supported environment; 2) hybrid and fully remote/online education exist. The students need to build habits and qualities of mind that are useful for the real life and, above all, for their work. New professions require not only knowledge and skills, but alsological, critical and creative thinking. The assessment of students' knowledge and abilities hasto assure the accomplishmentof these qualities.

In textbooks and in exam questions mostly the verbs find, determine, calculate and solve are used. The abilities to perform these activities show that the student has accomplished the goal " Remember, Understand and Apply"(in Blooms taxonomy), i.e.Lower Order Learning. Is that enough in order to test students'comprehension of a given material? In technology enriched teaching-learning environment there are opportunities to interpret this goal and change the way of itsaccomplishment. Higher Order Learning(HOL: development ofabilities to analyze, synthesize, create)is to be also achieved and assessed.

Here I would like to share some thoughts and experience in reconsideration of exam questions.

Using a CAS (I use Derive) any Taylor polynomial of interest can be obtained. Then instead of itsdeterminationfurther questions for testing the deepness of students' knowledge of this concept can be set up. For instance, consider the following task.

Task 1. Given the 5th degree Taylor polynomial $T_5(x)=\frac{x}{2}+\frac{x^2}{8}+\frac{x^3}{24}+\frac{x^4}{64}+\frac{x^5}{160}$ at the point $x_0 = 0$ of the function $f(x)=\ln\frac{2}{2-x}$.

(a) Show that the first term is correct.

(b) Calculate an approximation of $f(0.5)$ (or of $\ln\frac{4}{3}$) using the second degree Taylor polynomial $T_2(x)$ at the point $x_0 = 0$.

(c) Evaluate $f^{(5)}(0)$.

Solution. The students are supposed to write the formula

$$T_n(f;x;x_0) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \ldots \quad \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

and, in particular,

$$T_5(x) = T_5(f,x,0) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 +$$
$$\ldots \quad \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \frac{f^{(5)}(0)}{5!}x^5.$$

Then correct interpretation and extraction of information from the given Taylor polynomial allow to answer the questions in Task 1:

(a) $f(0) = \ln\left.\frac{2}{2-x}\right|_{x=0} = \ln 1 = 0;$ hence, the first term is correct,

(b) $T_2(x) = \frac{x}{2} + \frac{x^2}{8} \Rightarrow f(0.5) = \frac{0.5}{2} + \frac{0.5^2}{8} = 0.28125 \approx \ln\frac{2}{2-0.5} = \ln\frac{4}{3},$

(c) $\frac{f^{(5)}(0)}{5!}x^5 = \frac{x^5}{160} \Rightarrow \frac{f^{(5)}(0)}{5!} = \frac{1}{160} \Rightarrow f^{(5)}(0) = \frac{5!}{160} = \frac{120}{160} = \frac{3}{4}.$

In (b) the error estimation could be also found and interpreted using the rest term of the second degree Taylor polynomial.

The aim of such kind of questions is to help students consolidate key knowledge about Taylor polynomials: their construction, their applications: for calculating values of functions, for solving approximately integrals and differential equations. The solution also aims at HOL and development of the habit to solve problems not just somehow, to work (consequently, perform any activity) smarter, not harder.

In case of Fourier series similar questions can be formulated.

Task 2. Given the Fourier series

$$f(x) = \frac{\pi}{2} + \frac{4}{\pi}\cos x + \ldots \qquad \ldots$$

for the periodic function $f(x) = \begin{cases} \pi + x, & -\pi \le x \le 0 \\ \pi - x, & 0 \le x \le \pi \end{cases}, \quad f(x+2\pi) = f(x),\ \forall x \in \mathbb{R}$

Sketch the graph of f and justify the form of the series. Show that the second term is correct.

When the student is checking up the result, for instance, the correctness of a term, the form of the series -he/she develops the habit to control the results using different prototypes (analytical, numerical, graphical) including those obtained by application of CAS. "Reading" the given series, the student has first to check up that the function is even: this is the purpose of the first question in the above Task 2.

The sense organs of different people are developed to varying degrees. The strongest is the visual channel. It plays a crucial role in overall perception. CASs allow to formulate questions based on visual information(Task 3 and Task 4).

Task 3. The area Di s bounded by the curves $y = 9 + 3x$ and $y = 9 - x^2$ (Fig. 1). Describe D in terms of double inequalities in two ways. Calculate the integral $\iint_D dydx$ and interpret the result.

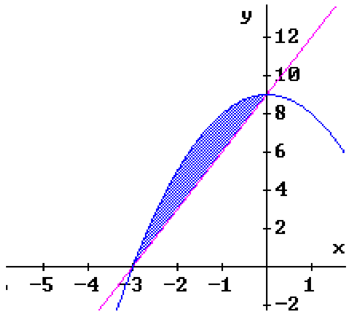

Figure 1. Task 3

Visualisation can support students in learning theoretical facts. The next question can be set up to check up ("Trust but check up") students' comprehension of the relationship between the signs of the derivatives and the shape of the function's graph.

Task 4. In Fig. 2 (a) the graphs of the first and the second derivative of a function $f(x)$ are given. Which is the graph of $f(x)$: in Fig 2 (b) or in Fig. 2 (c)?

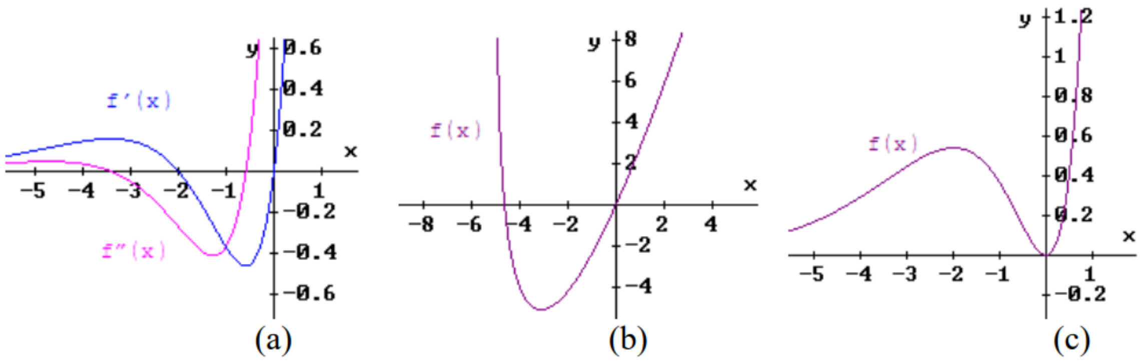

Figure 2. Task 4

One more question: How to solve problems/How to teach solving problems? Definitely, not just anyhow. "Work smarter, not harder" is related to effective solutions, i.e. to effective learning. In connection to effectiveness consider the following

Task 5. Evaluate the expression $\int_0^1 \sqrt{1-x^2}\,dx - \int_0^{\frac{1}{2}} \sqrt{\frac{1}{4}-x^2}\,dx$ and interpret the result.

Solution.

By the choice of an approach to the solution the basic question is whether to apply a method if the solution can be obtained by "conventional weapons": direct application of definitions, properties or/and graphical images (geometrical meaning). For this task CAS again comes to the rescue: plot each of the integrals on the same coordinate system and find the solution by observation, at a glance, taking into account the geometrical meaning of the concept of definite integral. In this particular case solution by another approach would not be adequate: it would be a "demonstration" of the proverb "To kill sparrow by cannon".
Such kind of problems and approaches to solutions could help students develop the good habit to look for effective and concise solutions of any other problem.

Mathematics and computer science education is of great importance to society. Any activity to stimulate and provoke the interest of young people to these fields will be for the benefit of society. This would produce added value to this education. The second author of this paper proposed the idea of organizing national student contest in mathematics with application of CASs. He shared it with university teachers keen to implement CASs in the teaching-learning process. As a result of teachers' and students' enthusiasm the experimental competition in Computer Mathematics took place in 2011 in Bulgaria [1]. The 9th edition of CompMath was held in October 2022.

It has to be mentioned that in the past several years Geo-Gebra is intensively used at Bulgarian math high schools. Advanced students and students who prefer doing mathematics in technology-supported environment are the majority of participants in this competition. All they need to be stimulated by additional activities at the universities: establishment of initiatives such as informal education, visiting lecturers - distinguished professionals in computer mathematics and computer science, clubs for exchange of ideas and experience. Based on their achievements they need to be advised to acquire additional learning material, so that to develop their full potential.

CompMath is now a traditional annual forum for students at Bulgarian universities. It proved to be useful for stimulating students' interest in mathematics and the opportunities of CASs for solving both theoretical and applied problems. It helps to create new as well as best practices in mathematics education: they could serve as models for suitable purposeful problems and assessment criteria.

The participants in the CompMath are given 30 mathematical problems to be solved within four hours. Based on their bachelor degree program they are divided into two groups:

- Group A: Mathematics, Informatics, Computer Science;

- Group B: Engineering, Natural Sciences.

All topics from the mathematics courses are covered. The students solve the problems in different ways depending on the level of their mathematical knowledge and programming skills. In the presentation some interesting problems and solutions created by participants will be demonstrated [2]. Mathematica, Maple, Maxima, Derive and MATLAB are mostly used in CompMath.

Example:
https://www.compmath.eu/tasks/2015A_en.pdf

References

[1] COMPMATH, `www.compmath.eu`, Accessed20/05/2023
[2] S.KAPRALOV, S.BOUYUKLIEVA, P.KOPANOV, Compmath2018–TheSeventhStudent OlympiadinComputerMathematics.In *ProceedingsoftheEDULEARN19Conference*, 9964-9970, Palma,Mallorca,Spain,1st-3rdJuly2019,ISBN:978-84-09-12031-4.

SGGW Warsaw, the location of ACA 2023, July 2023

ТЕХНИЧЕСКИ УНИВЕРСИТЕТ – ГАБРОВО
КАТЕДРА „МАТЕМАТИКА“

СТУДЕНТСКА ОЛИМПИАДА ПО КОМПЮТЪРНА МАТЕМАТИКА
CompMath – 2011
20-22. X. 2011, Габрово

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1. Calculate the value of the expression $\sqrt{\dfrac{x}{x+y}+\dfrac{y}{x-y}}$ for $x=1{,}1$ and $y=3{,}14$.
2. Calculate $\left(\dfrac{\sqrt{3}+5i}{4+2\sqrt{3}i}\right)^{2011}$.
3. Find the standard form of the polynomial $(x-1)^7+(x-3)^3$.
4. Factor the polynomial x^8+1 into irreducible factors with real coefficients.
5. Find the partial fraction decomposition of the rational function
$$\frac{18x^5+19x^4+37x^3+166x^2-90x+270}{x^6+2x^5+2x^4+25x^3-36x^2+78x-72}.$$
6. Solve the inequality $x+2\ge\sqrt{x+4}$.
7. Solve in complex numbers the equation
$$\begin{vmatrix} 1 & 2 & 2 & 1 \\ -1 & x & 0 & 0 \\ 0 & -1 & x & 0 \\ 0 & 0 & -1 & x \end{vmatrix}=0.$$
8. Solve the equation $X\,A=B$ if $A=\begin{pmatrix} 1 & 3 & -5 \\ 8 & 6 & 9 \\ 3 & -1 & 4 \end{pmatrix}$ and $B=\begin{pmatrix} 3 & 2 & -9 \\ 1 & 6 & 4 \\ 7 & 4 & -7 \end{pmatrix}$.
9. Solve the equation $X^2=A$ if $A=\begin{pmatrix} 13 & 9 \\ 12 & 16 \end{pmatrix}$.
10. Solve the system
$$\left|\begin{array}{l} 3x_1-3x_2-x_3-4x_4=26 \\ 2x_1+7x_2+6x_3+15x_4=-5 \\ 3x_1-x_2+2x_3+6x_4=18 \end{array}\right.$$
11. Find the area of the triangle ABC if $A(45,28)$, $B(23,92)$ and $C(11,13)$.
12. Find the coordinates of the intersection point of the line $g:\ \dfrac{x-1}{1}=\dfrac{y+2}{6}=\dfrac{z-7}{8}$ and the plane $\alpha:\ x+2y+3z+4=0$.
13. Find the intersection points of the ellipse $\dfrac{x^2}{4}+\dfrac{y^2}{9}=1$ and the parabola $y=(x+1)^2$.

14. Find the volume of the triangle pyramid $ABCD$ with vertices $A(1,3,6)$, $B(4,7,2)$, $C(8,1,-2)$ and $D(3,2,7)$.

15. Plot the graph of the quadratic function $f(x)=ax^2+bx+c$, given it passes through the points $A(1,2)$, $B(2,3)$ and $C(5,4)$.

16. Calculate $\lim\limits_{n\to\infty}\dfrac{n^{2012}-(n-2011)^{2012}}{2012\,n^{2011}}$.

17. Calculate the third derivative $f(x)=x^{2012}\ln x$.

18. Find $f'(0)$ given $f(x)=\operatorname{arctg}\left(\sqrt{1+x}-\sqrt{1-x}\right)$.

19. Find the values of x, such that the function $f(x)=\sqrt[3]{(x-8)^2}+\sqrt[3]{(x+1)^2}$ has a local extreme.

20. Find a function $F(x)$, which is the antiderivative of the function $f(x)=\dfrac{1}{x\sqrt{x^2-1}}$ in the interval $(1,\infty)$ and $F\left(\sqrt{2}\right)=0$.

21. Calculate $\displaystyle\int_0^1\frac{x^4(1-x)^4}{1+x^2}\mathrm{d}x$.

22. Calculate $\displaystyle\int_0^1\sqrt{x+1}.\sin x^2\mathrm{d}x$.

23. Which are the values of $x\in\left(\sqrt{2},\infty\right)$ such that the equality $\displaystyle\int_{\sqrt{2}}^{x}\frac{\mathrm{d}t}{t\sqrt{t^2-1}}=\frac{\pi}{12}$ holds.

24. Calculate the area of the figure, defined by the graphs of the functions $y=\sin x$ and $y=\cos 2x$ for $x\in[0,\pi]$.

25. Solve the differential equation $y'+y=x\sqrt{y}$ with initial condition $y(1)=\mathrm{e}$.

26. Calculate the sum $\dfrac{1}{2}-\dfrac{2}{3}+\dfrac{3}{4}-\dfrac{4}{5}+\cdots+\dfrac{99}{100}$.

27. Calculate $\lim\limits_{n\to\infty}\dfrac{\dfrac{1}{n+2011}+\dfrac{1}{n+2012}+\cdots+\dfrac{1}{2n}-\ln 2}{\dfrac{1}{n+1}+\dfrac{1}{n+2}+\cdots+\dfrac{1}{2n}-\ln 2}$.

28. Solve the equation $2\operatorname{arctg}x+\arcsin\dfrac{2x}{1+x^2}=\pi$.

29. Find the real roots of the equation $x^3-10x^2+6=0$.

30. For which real values of the real parameter m the equation $x\ln^2 x=m$ has exactly two real roots?

Each problem is worth 2 points.
All numerical calculations must be performed with the expected computing mathematical accuracy for the corresponding computer algebra system.

https://www.compmath.eu/tasks/2011_en.pdf

Find more collection of problems:
https://www.compmath.eu/2019/#problems

Isoptic or Apollonian Cubics

Josef Böhm, Würmla, AUT

Gegeben sind zwei Strecken AB und CD. Die Apollonische Kurve ist dann die Menge aller Punkte P, von denen aus beide Strecken unter dem gleichen Winkel erscheinen.

Die Kurve ist damit die isoptische Kurve der zwei Strecken AB und CD und gleichzeitig eine Verallgemeinerung der Apollonischen Kreise.

1829 beschrieb van Rees die Kurve erstmals, 1852 gab Steiner die Konstruktionsmöglichkeit an. Erst 1915 konnte Gomes Texeira die Äquivalenz nachweisen. Weitere wichtige Untersuchungen stammen von Brocard, Chasles, Dandelin, Darboux und Salmon.

Die A. K. kann aus zwei Teilen bestehen.

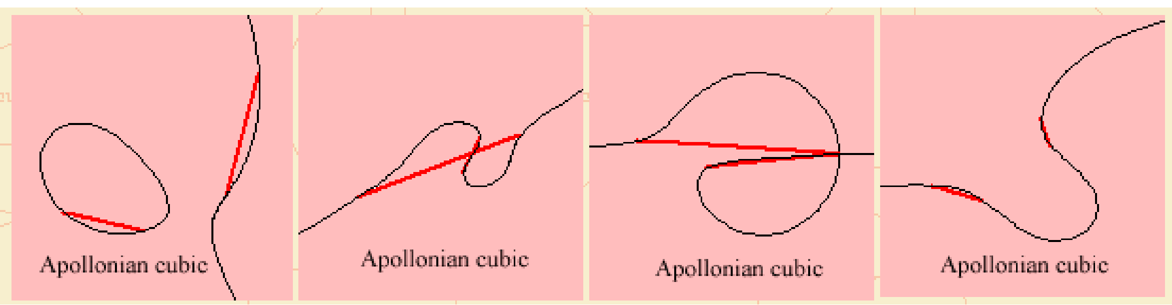
Ihre allgemeine Gleichung ist

$$(x - a)(x^2 + y^2) + bx + cy = 0$$

Two segments AB and CD are given. The Apollonian curve is then the set of all points P from which both segments appear at the same angle.
The curve is therefore the isoptic curve of the two segments AB and CD and at the same time a generalization of the Apollonian
circles.
In 1829, van Rees described the curve for the first time, and in 1852 Steiner gave the construction possibility. It was not until 1915 that Gomes Texeira was able to prove the equivalence. Other important studies were carried out by Brocard, Chasles, Dandelin, Darboux and Salmon.
The A.C. can consist of two parts.
Its general equation is

$$(x-a)(x^2+y^2)+bx+cy=0$$

Nowadays, at the University of Crete, a group around *Paris Pamfilos* is working on these cubics. They gave the curve the name **Apollonian cubic** or **isoptic cubic** and they constructed a tool, named [Isoptikon (796 kB)] to draw an Apollonian cubic, given the two segments.



https://www.2dcurves.com/cubic/cubica.html

I wanted to calculate the equation of this curve using means of upper secondary mathematics following the construction by intersecting two Apollonian circles with same angles and then finding the locus of these points when varying the angle:

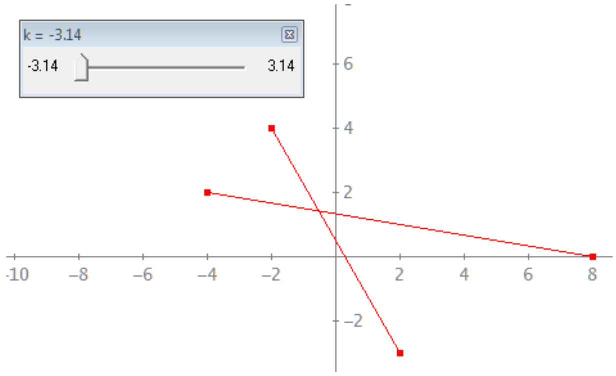
```
[CaseMode := Sensitive, InputMode := Word]

[A := [-4, 2], B := [8, 0], C := [-2, 4], D := [2, -3]]
```

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

$$\left[m1 := \frac{B_2 - A_2}{B_1 - A_1},\ m2 := \frac{D_2 - C_2}{D_1 - C_1}\right] = \left[m1 := -\frac{1}{6},\ m2 := -\frac{7}{4}\right]$$

$$\left[k1 := \mathrm{TAN}\left(\mathrm{ATAN}(m1) + \frac{\pi}{2} - k\right),\ k2 := \mathrm{TAN}\left(\mathrm{ATAN}(m2) + \frac{\pi}{2} - k\right)\right]$$



$$(l1 := y - B_2 = k1\cdot(x - B_1)) = l1 := y = \frac{(x - 8)\cdot(6\cdot\mathrm{COS}(k) - \mathrm{SIN}(k))}{\mathrm{COS}(k) + 6\cdot\mathrm{SIN}(k)}$$

$$(l2 := y - C_2 = k2\cdot(x - C_1)) = l2 := y - 4 = \frac{4\cdot(x + 2)\cdot\mathrm{COS}(k) - 7\cdot(x + 2)\cdot\mathrm{SIN}(k)}{7\cdot\mathrm{COS}(k) + 4\cdot\mathrm{SIN}(k)}$$

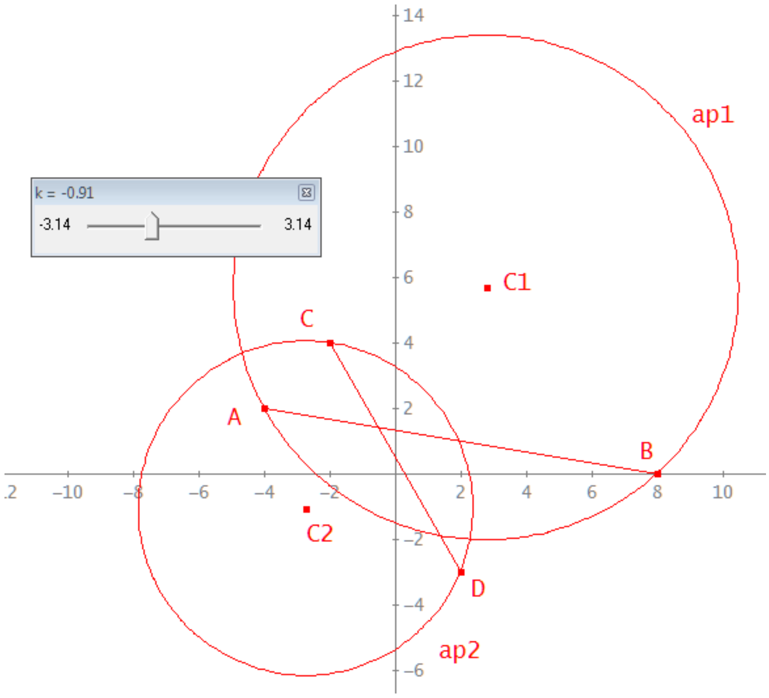
$$\left[M1 := \frac{A + B}{2},\ M2 := \frac{C + D}{2}\right] = \left[M1 := [2, 1],\ M2 := \left[0, \frac{1}{2}\right]\right]$$

$$\left(s1 := y - M1_2 = -\frac{1}{m1}\cdot(x - M1_1)\right) = s1 := y - 1 = 6\cdot(x - 2)$$

$$\left(s2 := y - M2_2 = -\frac{1}{m2}\cdot(x - M2_1)\right) = s2 := y - \frac{1}{2} = \frac{4\cdot x}{7}$$

$$(C1 := (\mathrm{SOLUTIONS}([l1, s1], [x, y]))_1) = C1 := \left[2 - \frac{\mathrm{COS}(k)}{\mathrm{SIN}(k)},\ 1 - \frac{6\cdot\mathrm{COS}(k)}{\mathrm{SIN}(k)}\right]$$

$$(C2 := (\mathrm{SOLUTIONS}([l2, s2], [x, y]))_1) = C2 := \left[\frac{7\cdot\mathrm{COS}(k)}{2\cdot\mathrm{SIN}(k)},\ \frac{2\cdot\mathrm{COS}(k)}{\mathrm{SIN}(k)} + \frac{1}{2}\right]$$

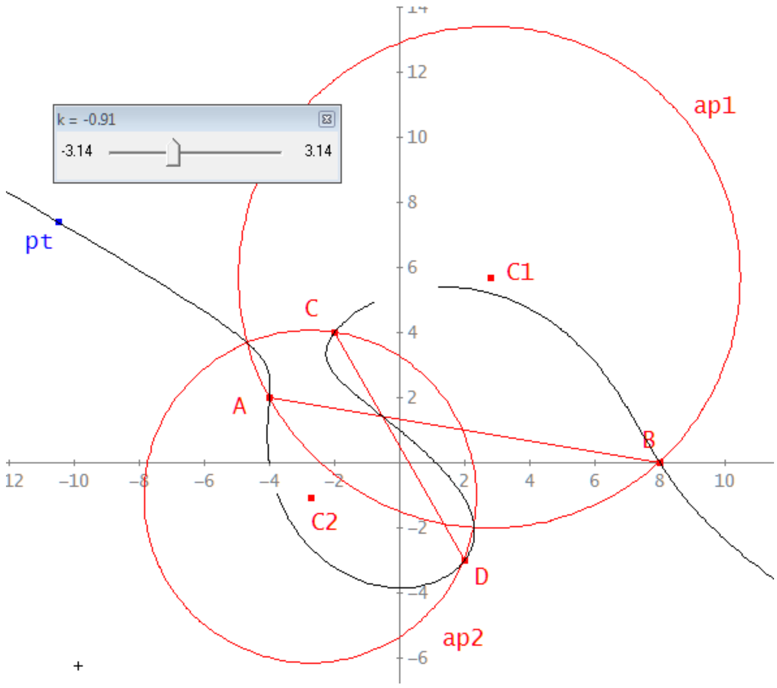



```
isopts := SOLUTIONS([ap1, ap2], [x, y])

isop(t) := SUBST(isopts  , k, t)
                       2

isop(t)
```

$$\left(pt := isop\left(-\frac{\pi}{10}\right)\right) = pt := [-10.46388209, 7.386003211]$$



$$(C3 := 2\cdot M1 - C1) = C3 := \left[\frac{\text{COS}(k)}{\text{SIN}(k)} + 2,\ \frac{6\cdot\text{COS}(k)}{\text{SIN}(k)} + 1\right]$$

$$(C4 := 2\cdot M2 - C2) = C4 := \left[-\frac{7\cdot\text{COS}(k)}{2\cdot\text{SIN}(k)},\ \frac{1}{2} - \frac{2\cdot\text{COS}(k)}{\text{SIN}(k)}\right]$$

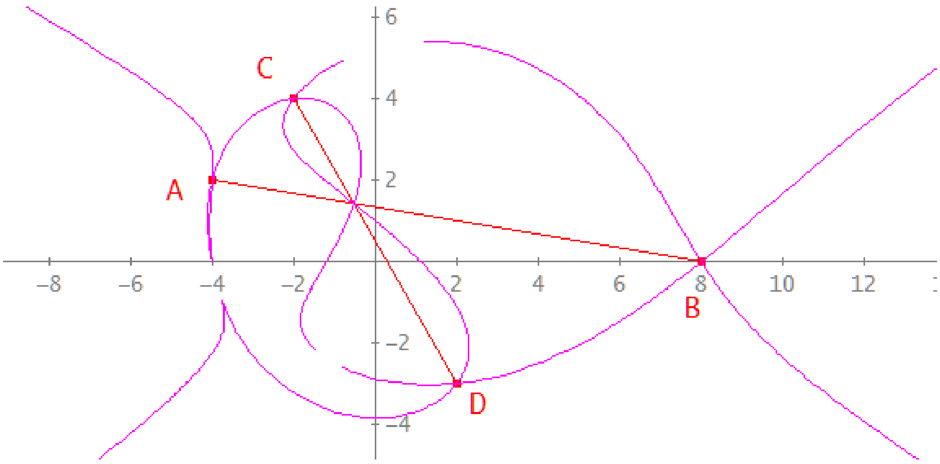
$$ap3 := (x - C3_1)^2 + (y - C3_2)^2 = |C3 - A|^2$$

$$ap4 := (x - C4_1)^2 + (y - C4_2)^2 = |C4 - C|^2$$

$$\text{SUBST}((\text{SOLUTIONS}([ap1,\ ap4],\ [x,\ y]))_1,\ k,\ t)$$

$$\text{SUBST}((\text{SOLUTIONS}([ap3,\ ap4],\ [x,\ y]))_1,\ k,\ t)$$

$$\text{SUBST}((\text{SOLUTIONS}([ap3,\ ap2],\ [x,\ y]))_1,\ k,\ t)$$

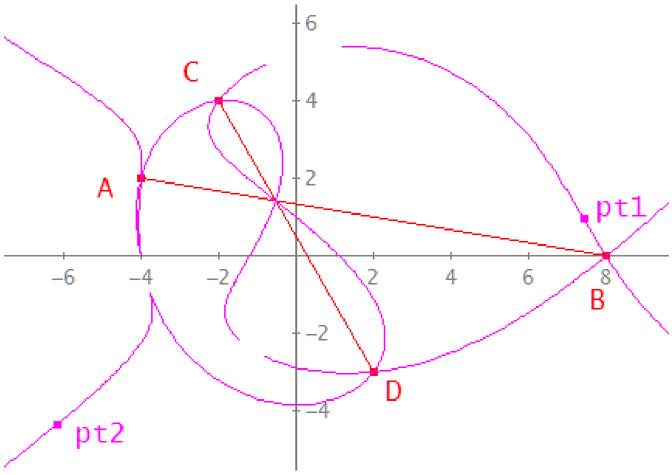


$$\text{isop}\left(\frac{3\cdot\pi}{10}\right) = [7.426140502,\ 0.9622562782]$$

$$pt1 := \text{SUBST}\left(\text{SUBST}((\text{SOLUTIONS}([ap1,\ ap2],\ [x,\ y]))_2,\ k,\ t),\ t,\ \frac{3\cdot\pi}{10}\right)$$

$$pt1 := [7.426140503,\ 0.9622562782]$$

$$pt2 := \text{SUBST}\left(\text{SUBST}((\text{SOLUTIONS}([ap1,\ ap4],\ [x,\ y]))_2,\ k,\ t),\ t,\ \frac{3\cdot\pi}{10}\right)$$



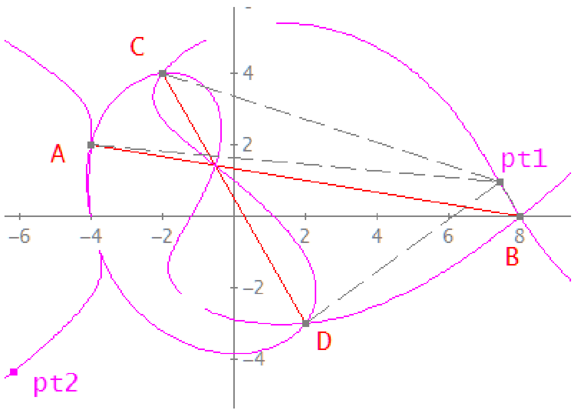
Check:

$k = \pi/5$ --> Angle $= \pi/2 - \pi/5 = 3\pi/10 = 54°$

$$\begin{bmatrix} A & pt1 & B \\ C & pt1 & D \end{bmatrix}$$

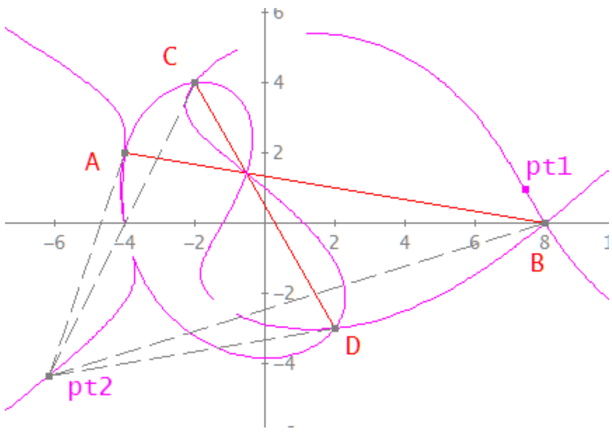
$$\text{angle}(P, Q, R) := \frac{\text{ACOS}\left(\dfrac{(P - Q)\cdot(R - Q)}{|P - Q|\cdot|R - Q|}\right)}{1°}$$

$$\begin{bmatrix} \text{angle}(A, pt1, B) \\ \text{angle}(C, pt1, D) \end{bmatrix} = \begin{bmatrix} 126 \\ 54 \end{bmatrix}$$



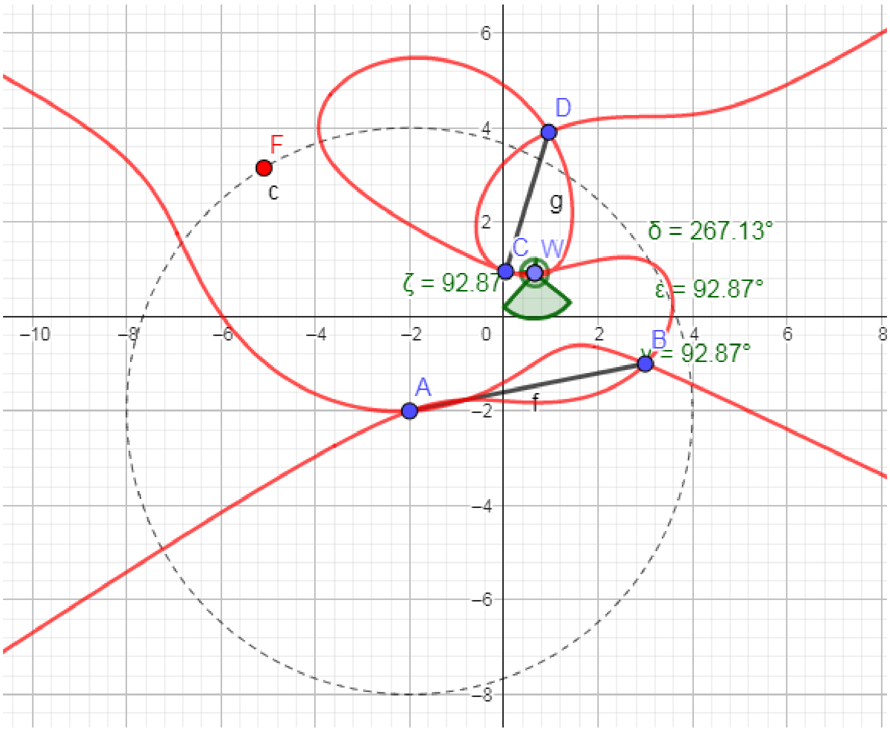
$$\begin{bmatrix} A & pt2 & B \\ C & pt2 & D \end{bmatrix}$$

$$\begin{bmatrix} \text{angle}(A, pt2, B) \\ \text{angle}(C, pt2, D) \end{bmatrix} = \begin{bmatrix} 54 \\ 54 \end{bmatrix}$$



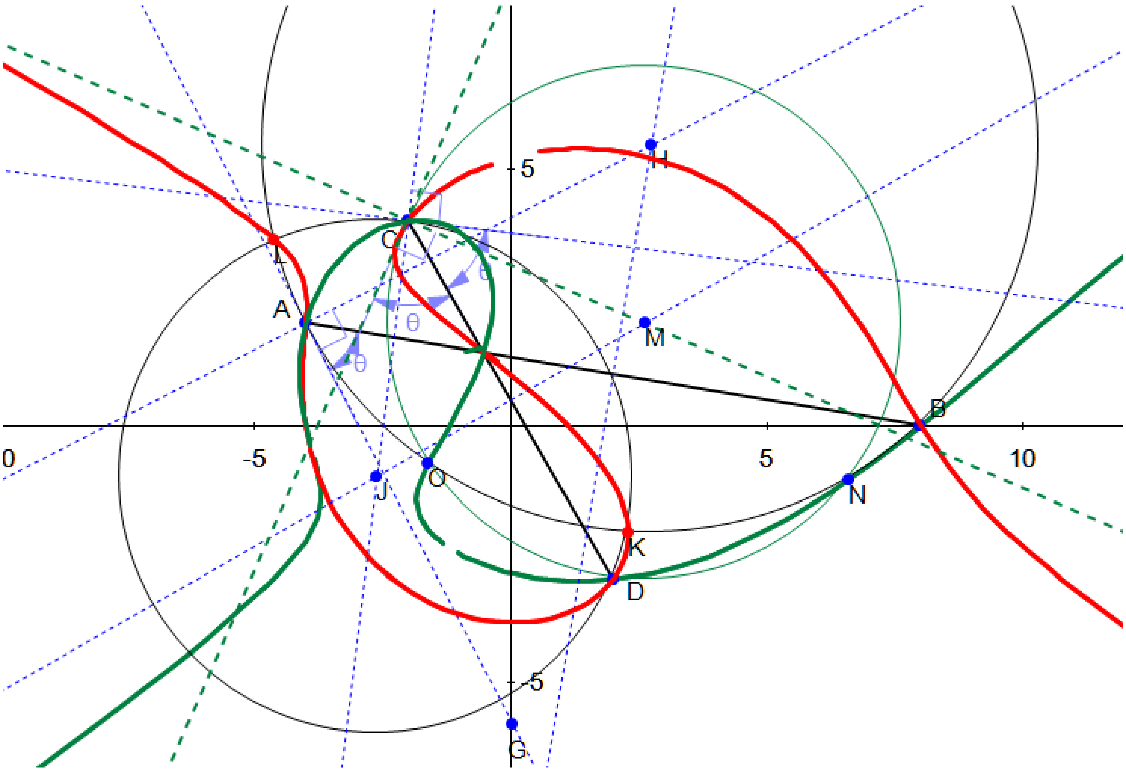
This is, what I wanted, but I could not derive the implicit form shown above from the bulky parameter form containing a lot of trigonometric expressions!

A GeoGebra plot of the Apollonian Cubic:



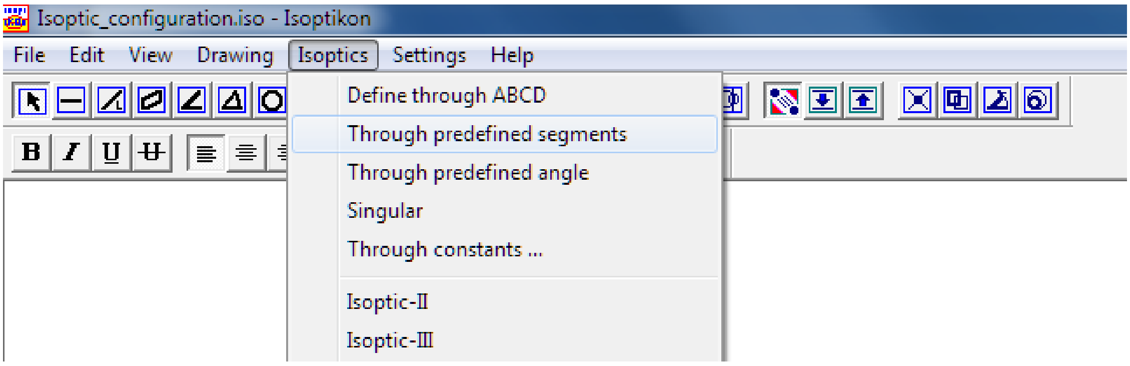
isoptik.ggb

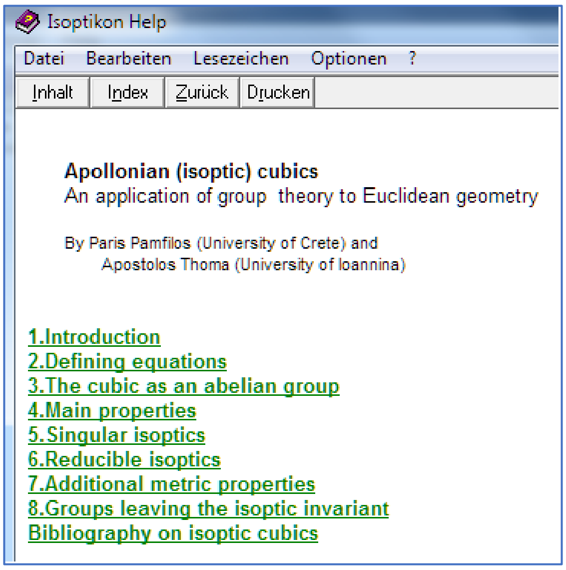
Screen shot from a Geometric Expression construction:



isoptic.gx

Screen shot from Isoptikon.exe (http://users.math.uoc.gr/~pamfilos/#iso):





The locus of points viewing the segment AB under the angle f, consists of two circles, described by the equations:
<X,X> - <X, (A+B)> + <A, B> = ±cotf(<X, J(B-A)> - <A, JB>). (*)
J denotes the +90-rotation of which in coordinates is given by:
(x,y) |--> (-y, x).
The sign ± corresponds to the two sides of the segment AB (segments and circles are supposed to be oriented). Analogous equations will describe the locus of points viewing CD under the angle f :

<X,X> - <X, (C+D)> + <C, D> = ±cotf(<X, J(D-C)> - <C, JD>).

2.3 Eliminating cotf among these equations we obtain the equations of the cubic curves in the form of determinants (intersections of pairs of circles: such cubics are called cyclic).

(<X,X>-<X,A+B>+<A,B>)(<X,J(D-C)>-<C,JD>) - (<X,J(B-A)>-<A,JB>)(<X,X>-<X,C+D>+<C,D>)=0, (*)

(<X,X>-<X,A+B>+<A,B>)(<X,J(D-C)>-<C,JD>)+(<X,J(B-A)>-<A,JB>)(<X,X>-<X,C+D>+<C,D>)=0. (**)

The two equations correspond to intersections of equally or inversely oriented circles viewing AB and CD under equal (or supplementary) angles.

Elliptic Curves of Conic Sections
https://www.geometrie.tuwien.ac.at/odehnal/tr200.pdf

An Integration Problem

Dear Michel,

I am working on Jose Luis´ library SMIS (Stepwise Multiple Integration Solver) library and came across a strange behavior of TI-Nspire?

$\int \frac{-x\cdot(x^2+y^2-1)\cdot y}{2}\,dy$	$\frac{-(y^2+x^2-1)^2\cdot x}{8}$
$\text{expand}\left(\frac{-x\cdot(x^2+y^2-1)\cdot y}{2}\right)$	$\frac{-x^3\cdot y}{2}-\frac{x\cdot y^3}{2}+\frac{x\cdot y}{2}$
$\int \left(\frac{-x^3\cdot y}{2}-\frac{x\cdot y^3}{2}+\frac{x\cdot y}{2}\right)dy$	$\frac{-y^4\cdot x}{8}-\frac{y^2\cdot x\cdot(x^2-1)}{4}$

Regards as ever,
Josef

Michel's answer:

Dear Josef, instead of "x" in your expression, take "a" or if you want, take a number, say x = 3. Integrate and observe the result. Expand, integrate and take a look at the result. You will see that the difference of the 2 answers is a constant.

I agree with you: it can confuse the user. But one thing that should be more documented in our textbooks is the fact that 2 antiderivatives of a continuous function over an interval differ by a constant (and this constant can surprise us !). Here are some examples.

Example 1 is about "first1" and "second1" where you integrate and then Expand before integrate. Observe the difference between the 2 answers.

Example 2 concerns "first2" and "second2" : to compute "first2", you need to expand and then integrate. But Derive (and now Rubi) knows a more compact antiderivative ("second2").

$$\text{first1}:=\int x^3\cdot(5+7\cdot x^4)^2\,dx \blacktriangleright \frac{(7\cdot x^4+5)^3}{84}$$

$$\text{second1}:=\int \text{expand}\left(x^3\cdot(5+7\cdot x^4)^2,x\right)\,dx \blacktriangleright \frac{49\cdot x^{12}}{12}+\frac{35\cdot x^8}{4}+\frac{25\cdot x^4}{4}$$

$$\text{first1}-\text{second1} \blacktriangleright \frac{125}{84}$$

$$\text{first2}:=\int \frac{x^3}{(x+1)^5}\,dx \blacktriangleright \frac{-(4\cdot x^3+6\cdot x^2+4\cdot x+1)}{4\cdot(x+1)^4}$$

$$\text{second2}:=\frac{x^4}{4\cdot(x+1)^4} \blacktriangleright \frac{x^4}{4\cdot(x+1)^4}$$ (this comes from goodl old Derive and Rubi)

$$\text{first2}-\text{second2} \blacktriangleright \frac{-1}{4}$$

Michel

Hi Michel,

I understand that this is correct, but it might confuse the user?

Thanks for removing my confusion.

I came across this behavior when transferred another function from Jose Luis' SIMS.dfw.by applying the "trick" by integrating the intermediate expanded expression.

Regards

Josef

I'll include this in our next Newsletter's User Forum.

Another interesting website (including *Isoptikon*):

http://users.math.uoc.gr/~pamfilos/#abo

Orthogonal trajectories to isoptics of ovals

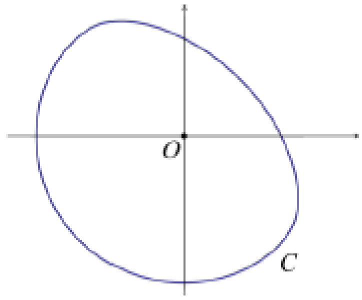
Magdalena Skrzypiec[1], Witold Mozgawa[2],
Aharon Naiman[3], Piotr Pikuta[4]

[1] Institute of Mathematics, Maria Curie-Sklodowska University, Lublin, Poland
[2] Institute of Social and Economic Sciences, Academy of Zamo´s´c, Zamo´s´c, Poland
[3] Department of Applied Mathematics, Jerusalem College of Technology-Machon Lev, Jerusalem, Israel
[4] Department of Theoretical Chemistry, Maria Curie-Sklodowska University, Lublin, Poland

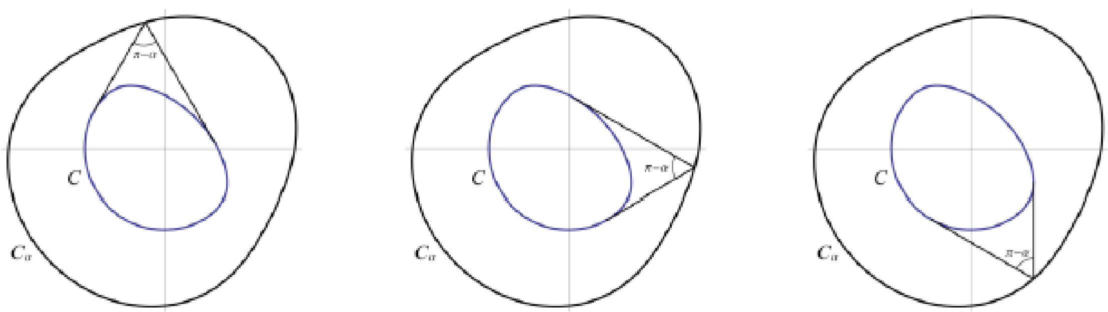
(Important note: Text in Calibri are my - Josef's - comments, Text in Times New Roman is Magda's original presentation))

Isoptics of Ovals

An oval – a simple closed convex plane curve of class C^3 with positive curvature.



An **α-isoptic** – the set of points at which two support lines of C intersect at angle $\pi - \alpha$ [Philippe de La Hire]



Support function parametrization

Comment: The support function is defined to be ***h*=*h*(*u*,*v*)=⟨*X*,*N*⟩ where *N* is a unit normal**. (This function measures the oriented distance from a tangent plane to the origin.) For curves in the plane there is a nifty result along these lines.

Parametrization of C with a support function $p(t)$:

$z(t) = p(t)e^{it} + p'(t)ie^{it}$ for $t \in [0, 2\pi)$.

For ovals $p(t) \in C^2$ the radius of curvature $R(t) = p(t) + p''(t)$ is positive for all $t \in [0, 2\pi)$.

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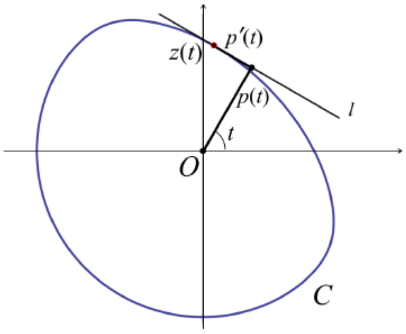


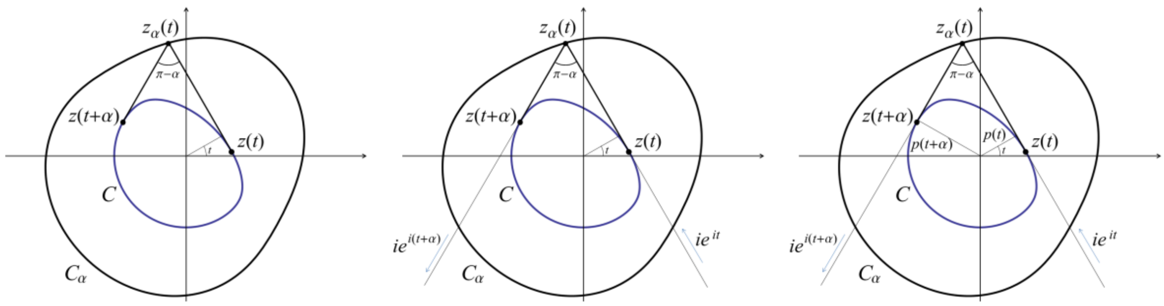
L. Santalo, Integral geometry and geometric probability, Encyclopedia of Mathematics and its Applications, Vol. 1. Addison-Wesley Publishing Co., Reading, Mass.-London-Amsterdam, 1976.

For *C* given by support function parametrization

$$C : z(t) = p(t)e^{it} + p'(t)ie^{it}$$

an α-isoptic C_α is given by

$$z_\alpha(t) = p(t)e^{it} + \left\{ -p(t)\cot(\alpha) + \frac{1}{\sin\alpha} p(t+\alpha) \right\} ie^{it},\ t \in [0, 2\pi).$$



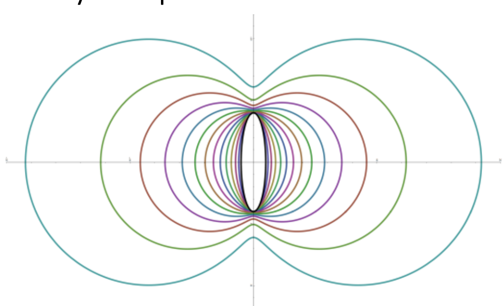
W. Cieślak, A. Miernowski, W. Mozgawa, Isoptics of a closed strictly convex curve, Global differential geometry and global analysis (Berlin, 1990), 28-35, Lecture Notes in Math., 1481, Springer, Berlin, 1991.

Comment: A method to obtain the equation of the isoptic of the curve f (x, y) = 0: eliminate x1, y1, x2, y2 between the 5 equations:

(1) f(x1,y1) = 0,
(2) f(x2,y2) = 0,
(3) f_x(x1,y1) * (x–x1) + f_y(x1,y1) * (y–y1) = 0,
(4) f_x(x2,y2) * (x–x2) + f_y (x2,y2) * (y–y2) = 0, and
(5) tan(alpha) = (f_x(x1,y1) – f_y(x2,y2))/(1 + f_x(x1,y1) * fy(x2,y2)).

https://mathcurve.com/courbes2d.gb/isoptic/isoptic.shtml

A family of isoptics of an oval



We will consider a family of curves

$$\{C_\alpha, \alpha \in (0, \pi)\}$$

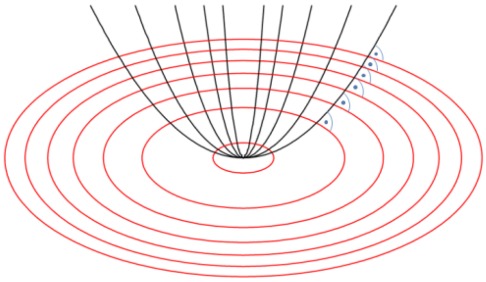
and consider a Cauchy problem, the solution of which can be used to obtain the parametric form of their orthogonal trajectories (curves, which intersect any curve of a given family orthogonally).

Definition:

An orthogonal trajectory is a curve, which intersects any curve of a given pencil of (planar) curves orthogonally.

Classical approach in cartesian coordinates:

- implicitly given pencil of curves $F(x,y,c) = 0$,
- a first order differential equation for those curves $F_x(x,y,c) + F_y(x,y,c)\,y' = 0$,
- simplified by elimination of the parameter c: $y' = f(x,y)$,
- differential equation for orthogonal trajectories of those curves: $y' = -\dfrac{1}{f(x,y)}$,
- solving the above equation.



Example:

For a circle C: $x^2 + y^2 = r^2$ α-isoptics form a family of concentric circles given by the equation $C_\alpha : x^2 + y^2 - \dfrac{r^2}{\cos^2\dfrac{\alpha}{2}} = 0.$

Comment: Verification using a CAS (*DERIVE*) applying the formula given above:

$$C(\alpha) := r\cdot e^{i\cdot t} + \left(-r\cdot \mathrm{COT}(\alpha) + \frac{1}{\mathrm{SIN}(\alpha)}\cdot r\right)\cdot i\cdot e^{i\cdot t}$$

[RE(C(α)), IM(C(α))]

$$\left[r\cdot \mathrm{COS}(t) - \frac{r\cdot \mathrm{SIN}(t)\cdot \mathrm{SIN}(\alpha)}{\mathrm{COS}(\alpha) + 1},\ \frac{r\cdot \mathrm{COS}(t)\cdot \mathrm{SIN}(\alpha)}{\mathrm{COS}(\alpha) + 1} + r\cdot \mathrm{SIN}(t)\right]$$

$$\left[x := r\cdot \mathrm{COS}(t) - \frac{r\cdot \mathrm{SIN}(t)\cdot \mathrm{SIN}(\alpha)}{\mathrm{COS}(\alpha) + 1},\ y := \frac{r\cdot \mathrm{COS}(t)\cdot \mathrm{SIN}(\alpha)}{\mathrm{COS}(\alpha) + 1} + r\cdot \mathrm{SIN}(t)\right]$$

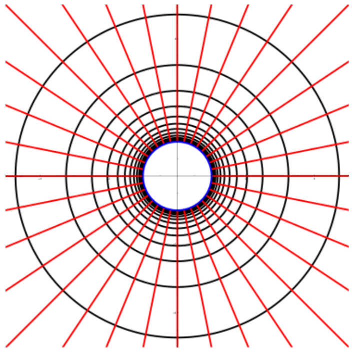
$$x^2 + y^2 = \frac{2\cdot r^2}{\mathrm{COS}(\alpha) + 1}$$

$$x^2 + y^2 = \frac{2\cdot r^2}{\mathrm{COS}(2\cdot \alpha\_) + 1}$$

$$x^2 + y^2 = \frac{r^2}{\mathrm{COS}(\alpha\_)^2}$$

$$x^2 + y^2 = \frac{r^2}{\mathrm{COS}\left(\frac{\alpha}{2}\right)^2}$$

- Then the first order ordinary differential equation for those curves is $2x + 2y\,y' = 0$,
- Which simplifies to the form $y' = -\dfrac{x}{y}$.
- The differential equation for orthogonal trajectories of those curves is $y' = \dfrac{y}{x}$,
- $\rightarrow$ y = c x, where $c \in \mathbb{R}$



For an ellipse C: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ α-isoptics are given by the equation:

$$C_\alpha : \frac{(x^2 + y^2 - a^2 - b^2)^2}{a^2 b^2} - 4\cot^2\alpha \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 \right) = 0.$$

Dana-Picard, T., Zehavi, N., Mann, G., From conic intersections to toric intersections: The case of the isoptic curves of an ellipse, The Mathematics Enthusiast 9 (2012), no. 1, Article 4, https://scholar-works.umt.edu/tme/vol9/iss1/4.

For the proof of the above formula see also: https://en.wikipedia.org/wiki/Orthoptic_(geome-try)#Isoptic_of_a_parabola,_an_ellipse_and_a_hyperbola

Comment: It is easy to obtain the formula. We need neither the procedure using a support function nor the system of equations both given above. Knowledge from upper secondary school is sufficient: equation of the ellipse, tangent in one point $P(x_0, y_0)$ the condition of tangency for a line $y = k\,x + d$ and how to calculate the angle between two lines with slopes k_1, k_2.

$$\left[\mathrm{ell} := b^2 \cdot x^2 + a^2 \cdot y^2 = a^2 \cdot b^2,\ \mathrm{tg} := y = k \cdot x + d\right]$$

$$\mathrm{SOLUTIONS}(a^2 \cdot k^2 + b^2 = d^2,\ d) = \left[\sqrt{(a^2 \cdot k^2 + b^2)},\ -\sqrt{(a^2 \cdot k^2 + b^2)}\right]$$

$$\mathrm{SUBST}(\mathrm{tg},\ d,\ \sqrt{(a^2 \cdot k^2 + b^2)}) = (y = k \cdot x + \sqrt{(a^2 \cdot k^2 + b^2)})$$

$$\mathrm{SUBST}(\mathrm{tg},\ d,\ -\sqrt{(a^2 \cdot k^2 + b^2)}) = (y = k \cdot x - \sqrt{(a^2 \cdot k^2 + b^2)})$$

$$\mathrm{SOLUTIONS}(y = \sqrt{(a^2 \cdot k^2 + b^2)} + k \cdot x,\ k) = \left[\frac{\sqrt{(b^2 \cdot x^2 + a^2 \cdot (y^2 - b^2))} + x \cdot y}{x^2 - a^2},\ \frac{\sqrt{(b^2 \cdot x^2 + a^2 \cdot (y^2 - b^2))} - x \cdot y}{a^2 - x^2}\right]$$

$$\left[k1 := \frac{\sqrt{(b^2 \cdot x^2 + a^2 \cdot (y^2 - b^2))} + x \cdot y}{x^2 - a^2},\ k2 := \frac{\sqrt{(b^2 \cdot x^2 + a^2 \cdot (y^2 - b^2))} - x \cdot y}{a^2 - x^2}\right]$$

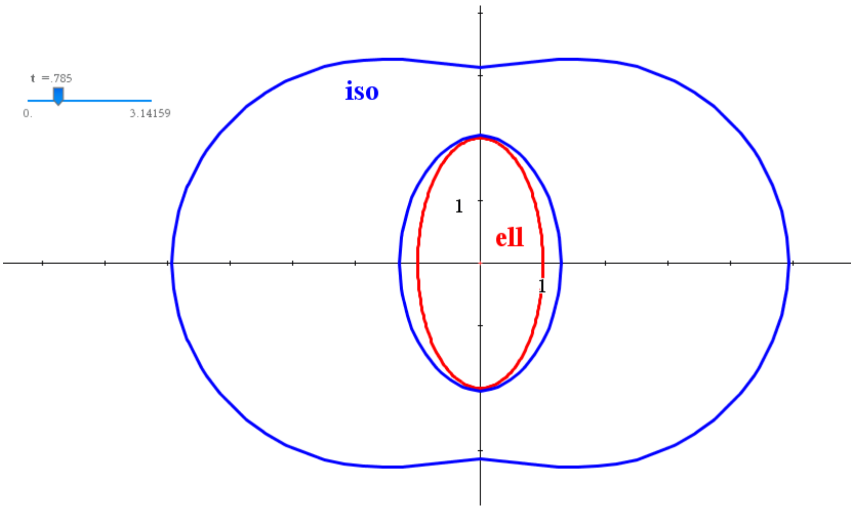
$$\mathrm{TAN}(\pi - \alpha) = \frac{k1 - k2}{1 + k1\cdot k2}$$

$$-\mathrm{TAN}(\alpha) = \frac{2\cdot\sqrt{(b^2\cdot x^2 + a^2\cdot(y^2 - b^2))}}{x^2 + y^2 - a^2 - b^2}$$

$$\left(-\mathrm{TAN}(\alpha) = \frac{2\cdot\sqrt{(b^2\cdot x^2 + a^2\cdot(y^2 - b^2))}}{x^2 + y^2 - a^2 - b^2}\right)^2 = \left(\mathrm{TAN}(\alpha)^2 = \frac{4\cdot(b^2\cdot x^2 + a^2\cdot(y^2 - b^2))}{(x^2 + y^2 - a^2 - b^2)^2}\right)$$

$$\mathrm{isop\_ell} := 4\cdot(b^2\cdot x^2 + a^2\cdot(y^2 - b^2)) - (x^2 + y^2 - a^2 - b^2)^2\cdot\mathrm{TAN}(\alpha)^2 = 0$$

I substitute for $a = 1$ and $b = 2$, install a slider for angle π (renamed as t in the plot) and receive the isoptic curve. To my surprise I see two curves (compare with Magda's pictures above!).



Let me try calculating the isoptic once more using the formula given above (now with TI-NspireCAS):

parametrization of the ellipse using support function p(t):

$\mathrm{p}(t):=\mathrm{norm}([a\cdot\cos(t)\;\; b\cdot\sin(t)])$ ▸ *Done* $\quad \mathrm{p}(t)$ ▸ $\sqrt{(a^2-b^2)\cdot(\cos(t))^2+b^2}$

$\mathrm{ell}(t):=\mathrm{p}(t)\cdot \mathbf{e}^{i\cdot t}+\frac{d}{dt}(\mathrm{p}(t))\cdot i\cdot \mathbf{e}^{i\cdot t}$

$\mathrm{xe}(t):=\mathrm{real}(\mathrm{ell}(t))|a=1 \text{ and } b=2$ ▸ *Done*

$\mathrm{ye}(t):=\mathrm{imag}(\mathrm{ell}(t))|a=1 \text{ and } b=2$ ▸ *Done*

$[\mathrm{xe}(t)\;\; \mathrm{ye}(t)]$ ▸ $\left[\frac{\cos(t)}{\sqrt{3\cdot(\sin(t))^2+1}}\;\; \frac{4\cdot\sin(t)}{\sqrt{3\cdot(\sin(t))^2+1}}\right]$

Parameter form of the π/4− and π−π/4 − isoptic of the ellipse:

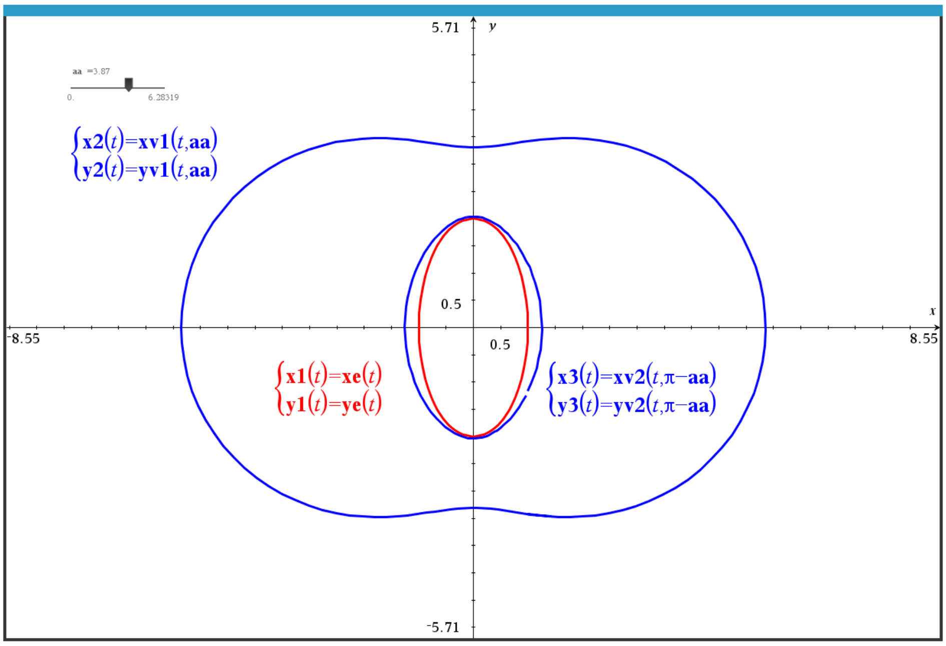
$\mathrm{isop}(t,\alpha):=\mathrm{p}(t)\cdot \mathbf{e}^{i\cdot t}+\left(-\mathrm{p}(t)\cdot\cot(\alpha)+\frac{1}{\sin(\alpha)}\cdot\mathrm{p}(t+\alpha)\right)\cdot i\cdot \mathbf{e}^{i\cdot t}$ ▸ *Done*

$\mathrm{xv1}(t,\mathrm{aa}):=\mathrm{real}(\mathrm{isop}(t,\mathrm{aa}))|a=1 \text{ and } b=2$ ▸ *Done*

$\mathrm{yv1}(t,\mathrm{aa}):=\mathrm{imag}(\mathrm{isop}(t,\mathrm{aa}))|a=1 \text{ and } b=2$ ▸ *Done*

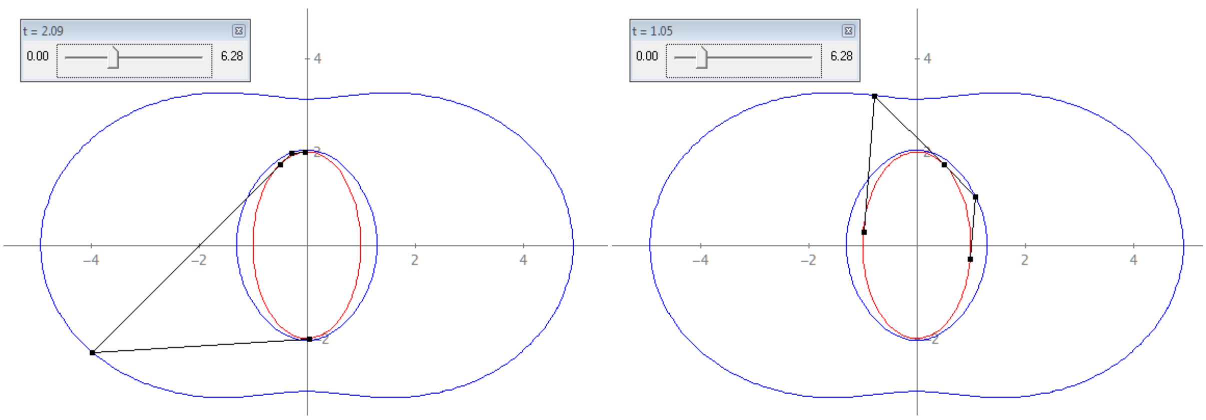
$\mathrm{xv2}(t,\mathrm{aa}):=\mathrm{real}(\mathrm{isop}(t,\mathrm{aa}))|a=1 \text{ and } b=2$ ▸ *Done*

$\mathrm{yv2}(t,\mathrm{aa}):=\mathrm{imag}(\mathrm{isop}(t,\mathrm{aa}))|a=1 \text{ and } b=2$ ▸ *Done*



The isoptic of an ellipse with a slider for the angle between the tangents)

Finally, I wanted to plot the ellipse, the isoptic and the pairs of tangents interesting on the isoptic. I fixed the angle as π/4. The slider moves the point on the ellipse (red) and we can observe two pairs of tangents with their intersection point on the blue curve. They form angles π/4 and π −π/4.



I proceed with Magda's lecture:

Then a first order ordinary differential equation for those curves is

$$2(x^2+y^2-a^2-b^2)\frac{2x}{a^2b^2}-4\cot^2\alpha\frac{2x}{a^2}+\left(2(x^2+y^2-a^2-b^2)\frac{2y}{a^2b^2}-4\cot^2\alpha\frac{2y}{b^2}\right)y'=0.$$

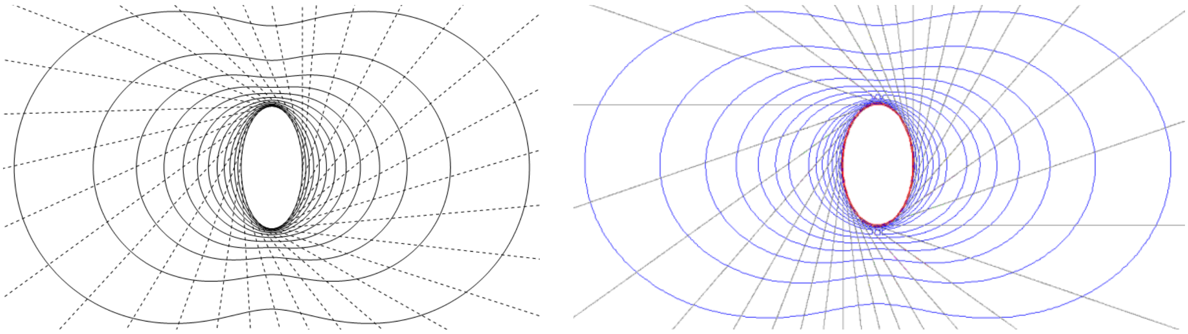
Let

$F(\alpha,t)=z_\alpha(t),$

where $\alpha\in(0,\pi), t\in(0,2\pi)$. The mapping *F* is a diffeomorphism between rectangle $(0,\pi)\times(0,2\pi)$ and the exterior of *C* without one half line.

Comment: In mathematics, a diffeomorphism is an isomorphism of smooth manifolds. It is an invertible function that maps one differentiable manifold to another such that both the function and its inverse are differentiable (Wikipedia & Wolfram Mathworld).

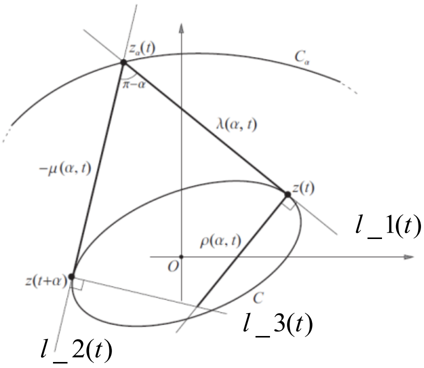
The set $F(\alpha, t_0)$ for $\alpha \in (0,\pi)$, t_0 = const. is a curve formed of points $z(t_0)$ and $z_\alpha(t_0)$. It is a half line from $z(t_0)$ in direction ie^{it_0}.



The DERIVE plot

Let us consider other curves $\gamma(\alpha) = F(\alpha, t(\alpha))$, which starts at the point on the oval C and have one common point with each isoptic. There $t(\alpha)$ is a function of variable α. The function $t(\alpha)$ allows us to move along the isoptic C_α.

Standard notation for isoptics



$$\lambda(\alpha,t)=\frac{1}{\sin\alpha}(p(t+\alpha)-p(t)\cos\alpha-p'(t)\sin\alpha),$$
$$\mu(\alpha,t)=-\frac{1}{\sin\alpha}(p(t)-p(t+\alpha)\cos\alpha+p'(t+\alpha)\sin\alpha),$$
$$\rho(\alpha,t)=\frac{1}{\sin\alpha}(p(t)\sin\alpha-p'(t)\cos\alpha+p'(t+\alpha)),$$
where $\alpha\in(0,\pi),\ t\in(0,2\pi)$.

Comment (many thanks to Magda for explaining the sketch and the respective formulas):

These notations were introduced in

https://link.springer.com/content/pdf/10.1007/BFb0083621.pdf#page=35 and http://www.numdam.org/article/RSMUP_1996__96__37_0.pdf

The values of those functions are lengths of the segments indicated in the sketch. I attached this slide complemented by parametric equations of considered tangents. In those formulas t and α are fixed and $s \in \mathbb{R}$ is the parameter. So, the directional vectors of these lines are i e^{it}, I e^{$i(t+\alpha)$} and e^{it}, respectively.

The case of ρ is more complicated. This function can have negative values. Then lines perpendicular to tangents intersect on the opposite side of the line l_1 and the length if the segment between $z(t)$ and the intersection point is equal to minus ρ.

$$l\_1(t)=z(t)+ie^{it}\cdot s$$
$$l\_2(t)=z(t+\alpha)+ie^{i(t+\alpha)}\cdot s$$
$$l\_3(t)=z(t)+e^{it}\cdot s$$

Comment: This could be a good occasion for an exercise using CAS and practicing working with complex direction vectors. We accept the challenge as follows:

$$\mathrm{ell}(t) := [4\cdot \mathrm{COS}(t),\ 2\cdot \mathrm{SIN}(t)]$$

$$p(t) := |\mathrm{ell}(t)|$$

$$p(t) := 2\cdot\sqrt{(3\cdot \mathrm{COS}(t)^2 + 1)}$$

$$z(t) := p(t)\cdot e^{i\cdot t} + p'(t)\cdot i\cdot e^{i\cdot t}$$

$$\mathrm{ell}(t) := [\mathrm{RE}(z(t)),\ \mathrm{IM}(z(t))]$$

$$\mathrm{ell}(t) := \left[\frac{8\cdot \mathrm{COS}(t)}{\sqrt{(3\cdot \mathrm{COS}(t)^2 + 1)}},\ \frac{2\cdot \mathrm{SIN}(t)}{\sqrt{(3\cdot \mathrm{COS}(t)^2 + 1)}}\right]$$

$$\mathrm{isop}(t,\ \alpha) := p(t)\cdot e^{i\cdot t} + \left(-\ p(t)\cdot \mathrm{COT}(\alpha) + \frac{1}{\mathrm{SIN}(\alpha)}\cdot p(t + \alpha)\right)\cdot i\cdot e^{i\cdot t}$$

$$\mathrm{isop\_}(t,\ \alpha) := [\mathrm{RE}(\mathrm{isop}(t,\ \pi - \alpha)),\ \mathrm{IM}(\mathrm{isop}(t,\ \pi - \alpha))]$$

I choose t = π/4 and α = π/3. I plot the ellipse ell(t) and the isoptic isop_(t, π/3). followed by the points on the ellipse and on the isoptic together with the connecting segment. The length of this segment should be given by function λ(α,t) presented above.

$$\mathrm{ell}\left(\frac{\pi}{4}\right) = \left[\frac{8\cdot\sqrt{5}}{5},\ \frac{2\cdot\sqrt{5}}{5}\right]$$

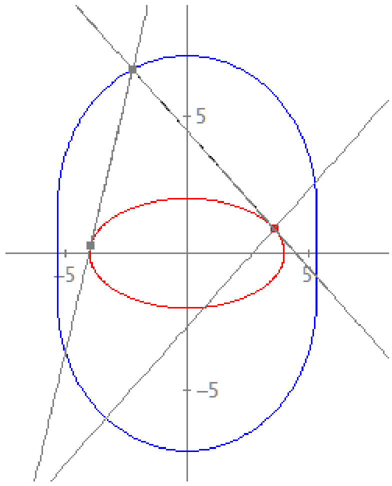
$$\mathrm{isop\_}\left(\frac{\pi}{4},\ \frac{\pi}{3}\right)$$

$$\left[\mathrm{ell\_p}\left(\frac{\pi}{4}\right),\ \mathrm{isop\_p}\left(\frac{\pi}{4},\ \frac{\pi}{3}\right)\right]$$

$$\left|\mathrm{ell\_p}\left(\frac{\pi}{4}\right) - \mathrm{isop\_p}\left(\frac{\pi}{4},\ \frac{\pi}{3}\right)\right| = \frac{2\cdot\sqrt{(9\cdot\sqrt{3} + 30)}}{3} + \frac{\sqrt{30}}{3} + \frac{3\cdot\sqrt{10}}{5}$$

$$\lambda(\alpha,\ t) := \frac{1}{\mathrm{SIN}(\alpha)}\cdot(p(t + \alpha) - p(t)\cdot \mathrm{COS}(\alpha) - p'(t)\cdot \mathrm{SIN}(\alpha))$$

$$\lambda\left(\pi - \frac{\pi}{3},\ \frac{\pi}{4}\right) = \frac{2\cdot\sqrt{(9\cdot\sqrt{3} + 30)}}{3} + \frac{\sqrt{30}}{3} + \frac{3\cdot\sqrt{10}}{5}$$



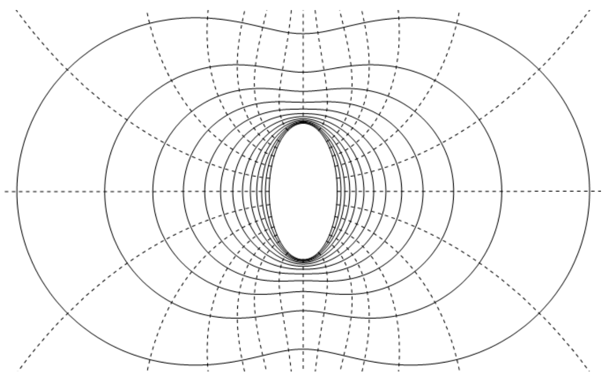
It is no problem to obtain the second point on the ellipse, the respective segment and finally, the three straight lines l_1(t), l_2(t) and l_3(t).

Left you can see the curves together with the points and lines.

To get a curve γ, which in each its point is perpendicular to appropriate isoptic we have to guarantee, that the tangent vector of this curve is perpendicular to the tangent vector of this isoptic in intersection point of considered curves.

This condition can be written in form

$$\langle \gamma'(\alpha), z'_\alpha(t(\alpha)) \rangle = 0$$



Orthogonal trajectories for parametric curves

Since the condition for orthogonal trajectories for isoptics can be written in form

$$\langle \gamma'(\alpha), z'_\alpha(t(\alpha)) \rangle = 0$$

and

$$\gamma'(\alpha) = \frac{\partial F}{\partial \alpha}(\alpha, t(\alpha)) + \frac{\partial F}{\partial t}(\alpha, t(\alpha)) \cdot t'(\alpha) =$$

$$= -\lambda(\alpha, t(\alpha)) t'(\alpha) e^{it(\alpha)} + \left(-\frac{\mu(\alpha, t(\alpha))}{\sin \alpha} + \rho(\alpha, t(\alpha)) t'(\alpha) \right) i e^{it(\alpha)},$$

$$z'_\alpha(t(\alpha)) = \frac{\partial F}{\partial t}(\alpha, t(\alpha)) = -\lambda(\alpha, t) e^{it(\alpha)} + \rho(\alpha, t(\alpha)) i\, e^{it(\alpha)}$$

It leads to the differential equation for the following function of $t(\alpha)$

$$t'(\alpha) = \frac{\mu(\alpha, t(\alpha))\, \rho(\alpha, t(\alpha))}{(\lambda^2(\alpha, t(a)) + \rho^2(\alpha, t(\alpha))) \sin \alpha}$$

This procedure could be applied for other evolutes of a given curve.

Extended definitions of known functions

For $(\alpha,t) \in [0,\pi) \times \mathbb{R}$ let us define

$$\lambda(\alpha,t)=\begin{cases}\frac{1}{\sin\alpha}\left(p(t+\alpha)-p(t)\cos\alpha-p'(t)\sin\alpha\right), & (\alpha,t)\in(0,\pi)\times\mathbb{R},\\ 0, & (\alpha,t)\in\{0\}\times\mathbb{R},\end{cases}$$

$$\mu(\alpha,t)=\begin{cases}-\frac{1}{\sin\alpha}\left(p(t)-p(t+\alpha)\cos\alpha+p'(t+\alpha)\sin\alpha\right), & (\alpha,t)\in(0,\pi)\times\mathbb{R},\\ 0, & (\alpha,t)\in\{0\}\times\mathbb{R},\end{cases}$$

$$\nu(\alpha,t)=\begin{cases}\frac{\mu(\alpha,t)}{\sin\alpha}, & (\alpha,t)\in(0,\pi)\times\mathbb{R},\\ -\frac{1}{2}R(t), & (\alpha,t)\in\{0\}\times\mathbb{R},\end{cases}$$

$$\rho(\alpha,t)=\begin{cases}\frac{1}{\sin\alpha}\left(p(t)\sin\alpha-p'(t)\cos\alpha+p'(t+\alpha)\right), & (\alpha,t)\in(0,\pi)\times\mathbb{R},\\ R(t), & (\alpha,t)\in\{0\}\times\mathbb{R},\end{cases}$$

$$F(\alpha,t)=\begin{cases}z_\alpha(t), & (\alpha,t)\in(0,\pi)\times\mathbb{R},\\ z(t), & (\alpha,t)\in\{0\}\times\mathbb{R}.\end{cases}$$

The Cauchy problem

Let us define the function

$$H(\alpha,t)=\begin{cases}\dfrac{\nu(\alpha,t)\rho(\alpha,t)}{\lambda^2(\alpha,t)+\rho^2(\alpha,t)}, & (\alpha,t)\in(0,\pi)\times\mathbb{R},\\ -\dfrac{1}{2}, & (\alpha,t)\in\{0\}\times\mathbb{R}.\end{cases}$$

For technical reasons, we define $H(\alpha,t)=H(-\alpha,t)\ for\ (\alpha,t)\in(-\pi,0)\times\mathbb{R}$

We will consider the following Cauchy problem:

$$\begin{cases}t'(\alpha)=H(\alpha,t(\alpha)),\\ t(0)=t_0.\end{cases}$$

later in this talk.

Continuity of *H*

Lemma

If p is a C^2 function, then H is continuous in $(-\pi,\pi)\times\mathbb{R}$.

Proof. Since the functions λ, ν and ρ are continuous in $(0,\pi)\times\mathbb{R}$ we only need to prove that H is continuous at $(0,t)$ for all $t\in\mathbb{R}$ which we obtain by showing that

$$\lim_{(a,s)\to(0^+,t)}\lambda(a,s)=0, \qquad \lim_{(a,s)\to(0^+,t)}\nu(a,s)=-\frac{1}{2}R(t)$$

and

$$\lim_{(a,s)\to(0^+,t)}\rho(a,s)=R(t),$$

where the limits are taken as $(a,s)\to(0,t)$ with $a>0$.

To this purpose we use some versions of l'Hôpital's rule for multivariable functions.

G. R. Lawlor, A l'Hôpital's rule for multivariable functions, 2012. https://arxiv.org/pdf/1209.0363.pdf.

G. R. Lawlor, A l'Hôpital's rule for multivariable functions, The American Mathematical Monthly, 127:8:717725, 2020. DOI: 10.1080/00029890.2020.1793635.

For $(a,s)\in[0,\frac{\pi}{2})\times\mathbb{R}$ we define

$$f(a,s)=p(s+a)-p(s)\cos a-p'(s)\sin a \text{ and } g(a,s)=\sin a$$

So that we have

$$\lambda(a,s)=\frac{f(a,s)}{g(a,s)},\quad (a,s)\in\left(0,\tfrac{\pi}{2}\right)\times\mathbb{R}$$

Since

$$\lim_{(a,s)\to(0^+,t)} f(a,s)=\lim_{(a,s)\to(0^+,t)} f(a,s)=0.$$

Following the arguments from the papers mentioned above, let us fix an arbitrary point $(0,t), t\in\mathbb{R}$ and take any sequence $(a_n,t_n)\to(0,t)$ such that $a_n\in(0,\frac{\pi}{2})$ and $t_n\in\mathbb{R}$ Since the functions $a\mapsto$ $\mapsto$ are differentiable in $(0,\frac{\pi}{2})$ and continuous in $[0,\frac{\pi}{2}]$, and $\frac{\partial g}{\partial a}(a,s)>0$ for $(a,s)\in(0,\frac{\pi}{2})\times\mathbb{R}$ we can apply the Cauchy Mean Value Theorem and obtain

$$\frac{f(a_n,t_n)}{g(a_n,t_n)}=\frac{f(a_n,t_n)-f(0,t_n)}{g(a_n,t_n)-g(0,t_n)}=\frac{\frac{\partial f}{\partial a}(c_n,t_n)}{\frac{\partial g}{\partial a}(c_n,t_n)},$$

where $c_n\in(0,a_n)$ *for all* $n\in\mathbb{N}$

Since $(c_n,t_n)\to(0^+,t)$, we have

$$\lim_{(a,s)\to(0+,t)}\frac{f(a,s)}{g(a,s)}=\lim_{(a,s)\to(0+,t)}\frac{\frac{\partial f}{\partial \alpha}(a,s)}{\frac{\partial g}{\partial \alpha}(a,s)}$$
$$=\lim_{(a,s)\to(0+,t)}\frac{p'(s+a)+p(s)\sin a-p'(s)\cos a}{\cos a}=0,$$

and therefore $\lim_{(a,s)\to(0^+,t)}\lambda(a,s)=0$ *for all* $t\in\mathbb{R}$

The limits of $\nu(a,s)$ and $\rho(a,s)$ as $(a,s)\to(0^+,t)$ can be calculated in the same manner, defining

$$f(a,s)=\mu(a,s)\sin a=-p(t)+p(t+a)\cos a-p'(t+a)\sin a,$$

$$g(a,s)=\sin^2 a,$$

and

$$f(a,s)=p(s)\sin a-p'(t+a)\cos a+p'(s+a),$$

$$g(a,s)=\sin a,$$

respectively.

Uniqueness of the solution of the Cauchy problem

Lemma

If p is a C^3 function, then for each $(\alpha_0, t_0) \in (-\pi, \pi) \times \mathbb{R}$ the Cauchy problem

$$\begin{cases} t'(\alpha) = H(\alpha, t(\alpha)), & \alpha \in (-\pi, \pi), \\ t(\alpha_0) = t_0, \end{cases} \tag{1}$$

has a unique solution.

Proof. Since the continuity of $\frac{\partial H}{\partial t}$ in $(-\pi,\pi)\times\mathbb{R}$ implies that in every compact subset $(-\pi,\pi)\times\mathbb{R}$ the derivative $\frac{\partial H}{\partial t}$ is bounded and, consequently, H is locally Lipschitz continuous with respect to t, we only need to prove that $\frac{\partial H}{\partial t}$ is continuous in $(-\pi,\pi)\times\mathbb{R}$.

For $(\alpha,t)\in(0,\pi)\times\mathbb{R}$ we have

$$\frac{\partial H}{\partial t} = \frac{(\frac{\partial \nu}{\partial t}\rho + \nu\frac{\partial \rho}{\partial t})(\lambda^2+\rho^2) - 2\nu\rho(\lambda\frac{\partial \lambda}{\partial t} + \rho\frac{\partial \rho}{\partial t})}{(\lambda^2+\rho^2)^2}$$

Moreover, $\frac{\partial H}{\partial t}(0,t) = 0$ for all $t \in \mathbb{R}$ $\mathbb{R}$

Straightforward calculations yield

$$\frac{\partial \nu}{\partial t}(\alpha, t) = \begin{cases} -\dfrac{p'(t) - p'(t+\alpha)\cos\alpha + p''(t+\alpha)\sin\alpha}{\sin^2\alpha}, & (\alpha,t) \in (0,\pi)\times\mathbb{R}, \\ -\frac{1}{2}R'(t), & (\alpha,t)\in\{0\}\times\mathbb{R}, \end{cases}$$

$$\frac{\partial \rho}{\partial t}(\alpha, t) = \begin{cases} \dfrac{p'(t)\sin\alpha - p''(t)\cos\alpha + p''(t+\alpha)}{\sin\alpha}, & (\alpha,t) \in (0,\pi)\times\mathbb{R}, \\ R'(t), & (\alpha,t)\in\{0\}\times\mathbb{R}, \end{cases}$$

$$\frac{\partial \lambda}{\partial t}(\alpha, t) = \begin{cases} \dfrac{p'(t+\alpha) - p'(t)\cos\alpha - p''(t)\sin\alpha}{\sin\alpha}, & (\alpha,t) \in (0,\pi)\times\mathbb{R}, \\ 0, & (\alpha,t)\in\{0\}\times\mathbb{R}. \end{cases}$$

Continuity of $\frac{\partial \nu}{\partial t}, \frac{\partial \rho}{\partial t}$ and $\frac{\partial \lambda}{\partial t}$ in $[0,\pi)\times\mathbb{R}$ can easily be established by using l'Hôpital's rule to calculate the following limits

$$\lim_{(a,s)\to(0^+,t)} \frac{\partial \nu}{\partial t}(a,s) = -\frac{1}{2}R'(t), \qquad \lim_{(a,s)\to(0^+,t)} \frac{\partial \rho}{\partial t}(a,s) = -\frac{1}{2}R'(t)$$

and $$\lim_{(a,s)\to(0^+,t)} \frac{\partial H}{\partial t}(a,s) = 0, \quad t\in\mathbb{R}$$ Now we can calculate

$$\lim_{(a,s)\to(0^+,t)} \frac{\partial H}{\partial t}(a,s) = \frac{\partial H}{\partial t}(0,t),$$

and the continuity of $\frac{\partial H}{\partial t}$ in $(-\pi,\pi)\times\mathbb{R}$ follows easily.

Orthogonal trajectories ot isoptics of an oval

Theorem

Orthogonal trajectories to isoptics of an oval C, where C is parametrized in terms of the support function p of class C^3, are the curves parameterized by functions $\gamma_{\tau_0}\colon [0,\pi) \to \mathbb{R}^2$ for $\tau_0 \in [0, 2\pi)$, defined by

$$\gamma_{\tau_0}(\alpha) = F(\alpha, t(\alpha)),$$

where $t\colon [0,\pi) \to \mathbb{R}$ is the solution to the Cauchy problem

$$\begin{cases} t'(\alpha) = H(\alpha, t(\alpha)), & \alpha \in (0,\pi), \\ t(0) = \tau_0. \end{cases} \tag{2}$$

Comment: A Cauchy problem in mathematics **asks for the solution of a partial differential equation that satisfies certain conditions that are given on a hypersurface in the domain**. A Cauchy problem can be an initial value problem or a boundary value problem.
https://encyclopediaofmath.org/wiki/Cauchy_problem
https://homepage.univie.ac.at/piotr.chrusciel/teaching/Cauchy/1-Cauchy-Problem-Ehlers-et-al-p1-p19.pdf

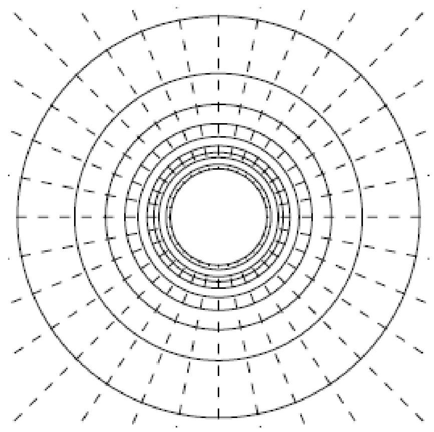
For C being the circle of radius $r > 0$ centered at (0,0) we have $p(t) = r$ for $t \in \mathbb{R}$
$H(\alpha,t) = -\frac{1}{2}$ for $(\alpha,t) \in [0,\pi) \times \mathbb{R}$ The solution to our Cauchy problem is

$$t(\alpha) = -\frac{1}{2}\alpha + \tau_0, \quad \alpha \in [0,\pi),$$

and orthogonal trajectories to the isoptics to the circle C are half lines

$$\gamma(\alpha) = F\left(\alpha, \tau_0 - \frac{1}{2}\alpha\right) = \frac{r}{\cos\frac{\alpha}{2}} e^{i\tau_0}, \quad \alpha \in [0,\pi),$$

starting from $z(\tau_0) = re^{i\tau_0}$, *where* $\tau_0 \in [0, 2\pi)$.



Comment: Calculation of $H(\alpha,t)$ is easy work. It can be done by paper and pencil following the “recipe” from above.

$$p(t) = r$$

$$\lambda = \frac{r - r\cos\alpha - 0}{\sin\alpha} = \frac{r(1-\cos\alpha)}{\sin\alpha}$$

$$\rho = \frac{r\sin\alpha - 0 + 0}{\sin\alpha} = r$$

$$\nu = \frac{\mu}{\sin\alpha} = -\frac{1}{\sin^2\alpha}\left(r - r\cos\alpha + 0\right) = -\frac{r(1-\cos\alpha)}{\sin^2\alpha}$$

$$H = \frac{\nu \cdot \rho}{\lambda^2 + \rho^2} = \frac{-\dfrac{r^2(1-\cos\alpha)}{\sin^2\alpha}}{\dfrac{r^2(1-\cos\alpha)^2}{\sin^2\alpha} + r^2} = \frac{-(1-\cos\alpha)}{(1-\cos\alpha)^2 + \sin^2\alpha} =$$

$$= \frac{-(1-\cos\alpha)}{1 - 2\cos\alpha + \cos^2\alpha + \sin^2\alpha} = \frac{-1(1-\cos\alpha)}{2 - 2\cos\alpha} = -\frac{1}{2}$$

This is the DERIVE procedure, followed by the DERIVE plot on the next page:

#1: $p(t) :=$

#2: $\lambda(\alpha,\ t) := \dfrac{1}{\mathrm{SIN}(\alpha)}\cdot(p(t + \alpha) - p(t)\cdot \mathrm{COS}(\alpha) - p'(t))$

#3: $\mu(\alpha,\ t) := -\dfrac{1}{\mathrm{SIN}(\alpha)}\cdot(p(t) - p(t + \alpha)\cdot \mathrm{COS}(\alpha) + p'(t + \alpha)\cdot \mathrm{SIN}(\alpha))$

#4: $\nu(\alpha,\ t) := \dfrac{\mu(\alpha,\ t)}{\mathrm{SIN}(\alpha)}$

#5: $\rho(\alpha,\ t) := \dfrac{1}{\mathrm{SIN}(\alpha)}\cdot(p(t)\cdot \mathrm{SIN}(\alpha) - p'(t)\cdot \mathrm{COS}(\alpha) + p'(t + \alpha))$

#6: $h(\alpha,\ t) := \dfrac{\nu(\alpha,\ t)\cdot \rho(\alpha,\ t)}{\lambda(\alpha,\ t)^2 + \rho(\alpha,\ t)^2}$

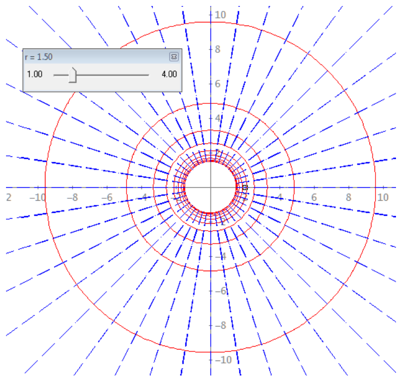
Circle: p(t) = r

#7: $p(t) := r$

#8: $[\lambda(\alpha,\ t),\ \mu(\alpha,\ t),\ \nu(\alpha,\ t),\ \rho(\alpha,\ t)]$

#9: $\left[r\cdot \mathrm{TAN}\left(\dfrac{\alpha}{2}\right),\ -r\cdot \mathrm{TAN}\left(\dfrac{\alpha}{2}\right),\ -\dfrac{r}{\mathrm{COS}(\alpha) + 1},\ r\right]$

#10: $h(\alpha,\ t) = -\dfrac{1}{2}$



Now I wanted to perform the same procedure for the ellipse with the respective support function for an ellipse with a = 1, b= 2:

```
#11:  p(t) := √(COS(t)^2 + (2·SIN(t))^2)

#12:  p(t) := √(3·SIN(t)^2 + 1)
```

The result for *h* was a very bulky expression:

```
h(α, t) :=

√(3·SIN(t)^2 + 1)·(9·COS(α)·SIN(4·t + 4·α)·√(3·SIN(t)^2 + 1) + 36·SIN(α)·SIN(t + α)^2·COS(2·t + 2·α)·√(3·SIN(t)^2 + 1) - 18·COS(α)·SIN(2·t~
4·SIN(α)·(9·COS(2·t + 2·α)·(3·SIN(t)^2 + 1) + 12·SIN(t + α)·COS(t + α)·√(3·SIN(t)^2 + 1)·(3·SIN(t)·C~
 + 2·α)·√(3·SIN(t)^2 + 1) - 24·SIN(t + α)·COS(t + α)·((3·SIN(t)·COS(t)·SIN(α)·COS(α) - 3·SIN(t)^2·(SIN(α)^2 + 1) - SIN(α)^2 - 1)·√(3·SIN(t ~
OS(t)·COS(α) - SIN(α)·(3·SIN(t)^2 + 1))·√(3·SIN(t + α)^2 + 1) + 4·√(3·SIN(t)^2 + 1)·(3·SIN(t)·COS(t) + COS(α)·(3·SIN(t)^2 + 1))·(3·SIN(t + ~
+ α)^2 + 1) + COS(α)·√(3·SIN(t)^2 + 1)) + 8·COS(α)·(3·SIN(t)·COS(t)·COS(α) - SIN(α)·(3·SIN(t)^2 + 1))·(3·SIN(t + α)^2 + 1)^3/2 + 12·SIN(t + ~
α)^2 + 1)^3/2 + 18·SIN(t + α)^2·(2·SIN(t)·COS(t)·COS(α)·(3·SIN(t)^2 + 1)·(SIN(α) - 1) + 3·SIN(t)^4·COS(α)^2 - SIN(t)^2·(3·COS(α)^2 + 7) - 1) + ~
α)^2·√(3·SIN(t)^2 + 1)·(SIN(α)·(6·SIN(t)^2 + 5) - 6·SIN(t)·COS(t)·COS(α)) - 8·√(3·SIN(t)^2 + 1)·(3·SIN(t)·COS(t)·COS(α) - SIN(α)·(3·SIN(t)^2~
12·SIN(t)·COS(t)·COS(α)·(3·SIN(t)^2 + 1)·(SIN(α) - 1) + 18·SIN(t)^4·COS(α)^2 - 9·SIN(t)^2·(2·COS(α)^2 + 7) - 13)
```

I was able to open Magda's MATHEMATICA file and find her expression for h, which is named L corresponds to *t´* and T corresponds to *t* in the differential equation.

```
ρ[p_][t_, α_] :=
 p[tt] - D[p[tt], tt] Cot[αα] +
   D[p[tt + αα], tt] / Sin[αα] /. {tt → t, αα → α};
μ[p_][t_, α_] :=
 -p[tt] / Sin[αα] - D[p[tt + αα], tt] + p[tt + αα] Cot[αα] /.
  {tt → t, αα → α};
L = Simplify[μ[p[r, a]][T[αα], αα]
    ρ[p[r, a]][T[αα], αα] /
     (Sin[αα] ((λ[p[r, a]][T[αα], αα])^2 +
          (ρ[p[r, a]][T[αα], αα])^2))];
Simplify[Expand[FunctionExpand[L]]]
```

```
(Csc[αα] (-295 Sin[αα] + 34 √(5 - 3 Cos[2 T[αα]])
     √(5 - 3 Cos[2 (αα + T[αα])]) Sin[2 αα] -
    3 (15 Sin[3 αα] - 25 Sin[αα - 2 T[αα]] +
       20 √(5 - 3 Cos[2 T[αα]]) √(5 - 3 Cos[2 (αα + T[αα])])
        Sin[2 (αα + T[αα])] - 75 Sin[αα + 2 T[αα]] -
       6 √(5 - 3 Cos[2 T[αα]]) √(5 - 3 Cos[2 (αα + T[αα])])
        Sin[2 (αα + 2 T[αα])] - 9 Sin[3 (αα + 2 T[αα])] -
       59 Sin[3 αα + 2 T[αα]] + 15 Sin[αα + 4 T[αα]] +
       45 Sin[3 αα + 4 T[αα]]))) /
 (4 (170 + 45 Cos[2 αα] - 126 Cos[2 T[αα]] -
    34 Cos[αα] √(5 - 3 Cos[2 T[αα]])
     √(5 - 3 Cos[2 (αα + T[αα])]) - 126 Cos[2 (αα + T[αα])] +
    30 √(5 - 3 Cos[2 T[αα]]) √(5 - 3 Cos[2 (αα + T[αα])])
     Cos[αα + 2 T[αα]] + 45 Cos[2 (αα + 2 T[αα])]))
```

This is the result for H how it is shown in Magda's presentation:

Orthogonal trajectories for isoptics of an ellipse

The support function

$$p(t) = \sqrt{\cos^2 t + 4\sin^2 t}, \quad t \in \mathbb{R},$$

defines an ellipse. For $(\alpha, t) \in (0, \pi) \times \mathbb{R}$ we have

$$H(\alpha, t) = \frac{\mathcal{N}(\alpha, t)}{\mathcal{D}(\alpha, t)},$$

where

$$\begin{aligned}\mathcal{N}(\alpha, t) = {} & -75\sin(2t-\alpha) - 295\sin\alpha - 45\sin 3\alpha + 225\sin(2t+\alpha) \\ & + 34\sin 2\alpha\sqrt{(5-3\cos 2t)(5-3\cos(2t+2\alpha))} \\ & - 60\sin(2t+2\alpha)\sqrt{(5-3\cos 2t)(5-3\cos(2t+2\alpha))} \\ & + 18\sin(4t+2\alpha)\sqrt{(5-3\cos 2t)(5-3\cos(2t+2\alpha))} \\ & + 27\sin(6t+3\alpha) - 45\sin(4t+\alpha) + 177\sin(2t+3\alpha) \\ & - 135\sin(4t+3\alpha)\end{aligned}$$

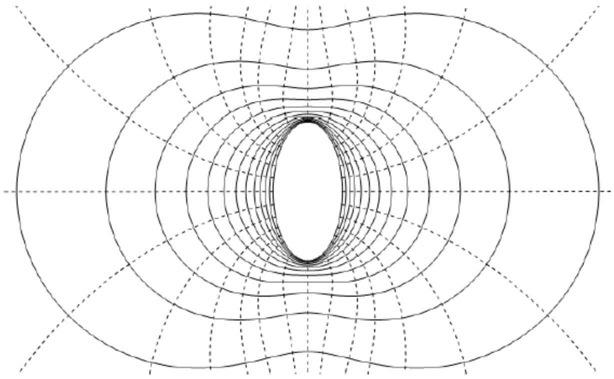
and

$$\begin{aligned}\mathcal{D}(\alpha, t) = 4\sin\alpha\big(170 & - 126\cos 2t + 45\cos 2\alpha - 126\cos(2t+2\alpha) \\ & - 34\cos\alpha\sqrt{(5-3\cos 2t)(5-3\cos(2t+2\alpha))} \\ & + 30\cos(2t+\alpha)\sqrt{(5-3\cos 2t)(5-3\cos(2t+2\alpha))} \\ & + 45\cos(4t+2\alpha)\big).\end{aligned}$$

Since we do not have an analytic solution to

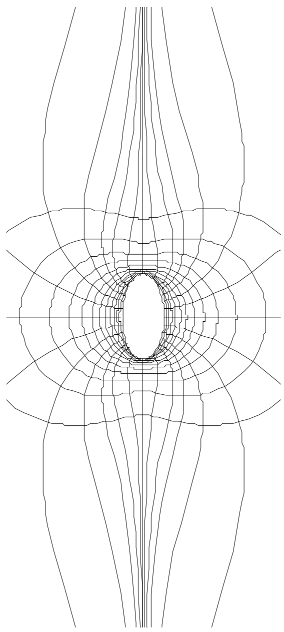
$$\begin{cases} t'(\alpha) = H(\alpha, t(\alpha)), & \alpha \in (0,\pi), \\ t(0) = \tau_0 \end{cases}$$

for $H(\alpha,t) = \frac{\mathcal{N}(\alpha,t)}{\mathcal{D}(\alpha,t)}$, orthogonal trajectories to the isoptics are obtained numerically.



I must admit that I was not able to perform the calculation using DERIVE. I would be happy if one of our readers could do it. The respective MATHEMATICA file is contained in MTH132.zip.

Magda's file gives the graph as shown below:

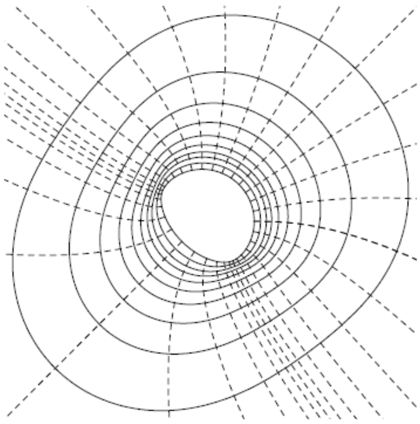


Orthogonal trajectories for isoptics of an oval with $p \in C^3$

Graphs of orthogonal trajectories for isoptics of the curve with the support function

$$p(t) = \begin{cases} r, & t \in (0, \frac{\pi}{2}) \\ r - 17a + 20a\cos t + 20a\sin t - 16a\sin 2t - 4\cos 3t + \\ +4a\sin 3t + a\cos 4t, & t \in \left[\frac{\pi}{2}, 2\pi\right] \end{cases}$$

where $r = 150$, $a = 1$.



I try to plot the oval together with some Isoptics:

$$p(t) := 150 - 17 + 20 \cdot COS(t) + 20 \cdot SIN(t) - 16 \cdot SIN(2 \cdot t) - 4 \cdot COS(3 \cdot t) + 4 \cdot SIN(3 \cdot t) + COS(4 \cdot t)$$

$$z(t) := p(t) \cdot e^{i \cdot t} + \left(\frac{d}{dt}\, p(t)\right) \cdot i \cdot e^{i \cdot t}$$

The Oval:

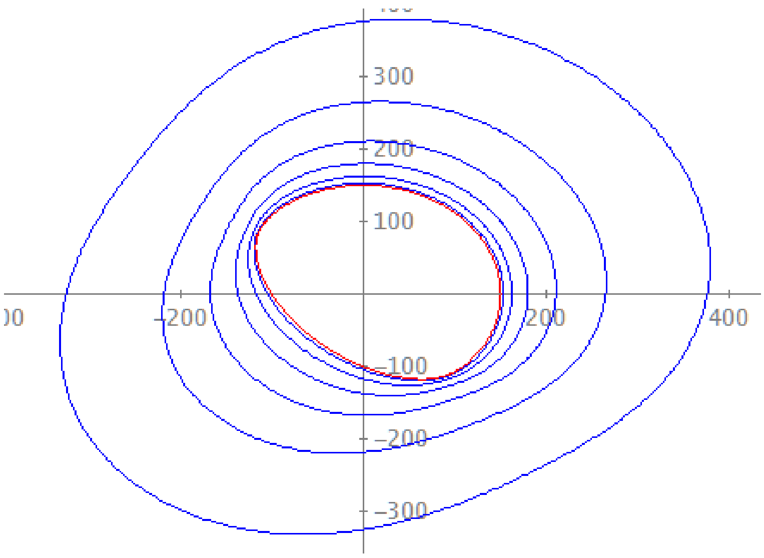
[RE(z(t)), IM(z(t))]

$$isop(t,\ \alpha) := p(t) \cdot e^{i \cdot t} + \left(- p(t) \cdot COT(\alpha) + \frac{1}{SIN(\alpha)} \cdot p(t + \alpha)\right) \cdot i \cdot e^{i \cdot t}$$

[RE(isop(t, α)), IM(isop(t, α))]

Some isoptics:

$$VECTOR\left([RE(isop(t,\ \alpha)),\ IM(isop(t,\ \alpha))],\ \alpha,\ \frac{\pi}{8},\ \pi,\ \frac{\pi}{8}\right)$$

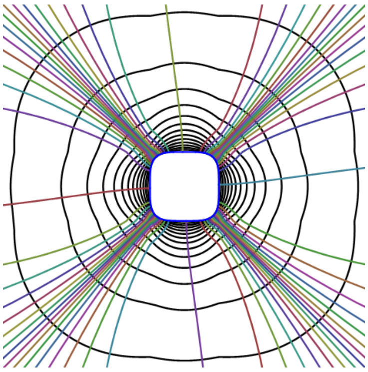


Orthogonal trajectories for isoptics of $x^4 + y^4 = 1$

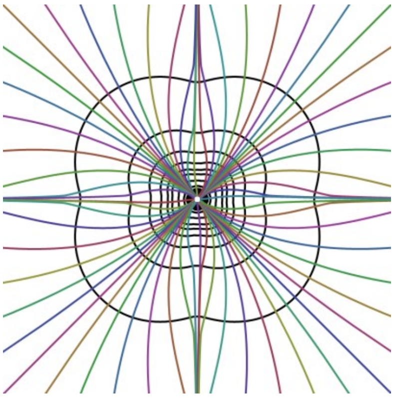
The Fermat curve is not an oval, but numerical solution of

$$t'(\alpha) = \frac{\mu(\alpha,t(\alpha))\,\rho(\alpha,t(\alpha))}{(\lambda^2(\alpha,t(\alpha)) + \rho^2(\alpha,t(\alpha))\sin\alpha}.$$

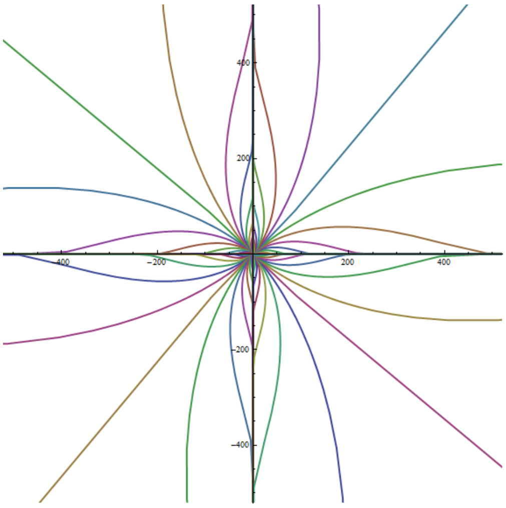
led us to such curves …



… and when we look at a greater distance …



… and a greater distance …



Thank you for your attention

A collection of links as references:

https://www.geogebra.org/search/isoptics
https://www.2dcurves.com/
https://mathcurve.com/courbes2d.gb/isoptic/isoptic.shtml
https://www.researchgate.net/publication/337743100_Exploring_the_Isoptics_of_Fermat_Curves_in_the_Affine_Plane_Using_DGS_and_CAS/figures?lo=1
https://cicm-conference.org/2018/slides/T2.pdf

Parametric forms of inner isoptics of ellipses

https://www.sciencedirect.com/science/article/abs/pii/S0393044022001620

About the support function:
https://scialert.net/fulltext/fulltextpdf.php?pdf=ansinet/jas/2008/383-386.pdf
https://en.wikipedia.org/wiki/Support_function

In the Proceedings of an ICTCM (International Conference on Technology in Collegiate Mathematics) in the early nineties I found this nice graphic.

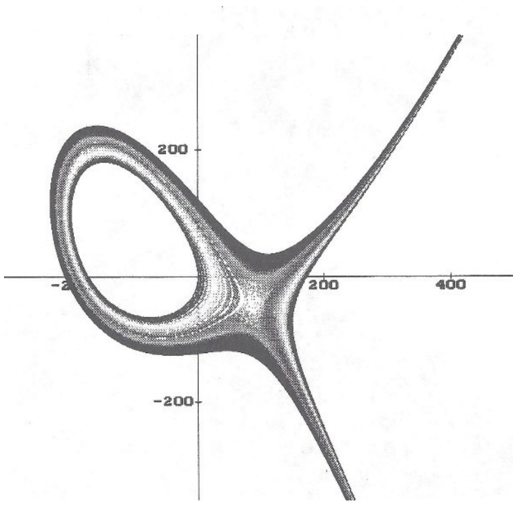


Figure 2. A family of elliptic curves given by
$X^3 - 30000x + u = 200y^2 + 200xy$
where $0 \le u \le 3000000$

See the DERIVE plot of this figure: