THE DERIVE - NEWSLETTER #133

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THE BULLETIN OF THE

DERIVE

USER GROUP

+ CAS-TI

Contents:

March 2024

Richard J. Morris: Visualising Duals of Surfaces
https://www.singsurf.org/

US Government Sites

Bureau of Labor Statistics

This government site provides relevant economic data including unemployment rates and consumer price index.
http://www.bls.gov/

Centers for Disease Control and Prevention
This site has links to statistics and data concerning health statistics, health surveillance and laboratory information.

http://www.edc.gov/

United States Census Bureau
This government site contains data as well as estimates and projections on housing, income, poverty, genealogy, foreign trade, state profiles and other aspects of American government and society
http://www.census.gov/

United States Energy Information Administration
This site provides useful data on energy resources and energy consumption.
http://www.eia.d.e.gov

United States Environmental Protection Agency
This site provides access to environmental data

http://www.epa.gov

United States Geological Survey
This site provides access to United States Geological Survey including data on earthquakes, minerals, geospatial holdings
http://www.usgs.gov

United Nations
This is the home page of the U.N. which has many links to the organizations' departments and some of its free databases.
http://www.un.org/

Distance Learning References

United States Distance Learning Association
http://www.usdla.org/

University of Phoenix
http://www.phoenix.edu/

Dear DUG-Members,

Today it is Holy Saturday and I am happy that I can finalize this DNL adding my letter.

We have only few, but very extended contributions in this issue. I included the written version of my presentation at ACA 23 in Warsaw. I didn't want to split.

Some weeks ago, I received a paper from Helmut Heugl which fits excellent in the many discussions about the influence of AI in education, and then particularly in math education. The paper was in German.

I thought that it might be of interest for all of us and asked Helmut for permission to translate it for the DNL. He did not hesitate to give his allowance and added new results. Many thanks to Helmut. Please read my introduction to his paper on page 30.

I collected some links for websites which provide a lot of materials for statistic investigations. The User Forum (page 3) shows an example of an exam question of the very hard end examination for South Korean students who want to study at one of the country's universities.

In the preview you can see a screen shot of the TI-Nspire. Jose Luis Galan and his colleagues from the Malaga University produced great DERIVE tools for performing stepwise solutions for 1st order partial differential equations, for ordinary DEs and for multiple integration problems. I got permission to transfer these tools for TI-Nspire – as far as possible, of course.

Then we will have interesting paper on integration tables (Michel Beaudin) and a contribution showing that there are several ways to define quartiles in statistics.

With a picture from our spring time garden I wish you a wonderful spring time and remain with my best regards until next time
always Yours Josef

The *DERIVE-NEWSLETTER* is the Bulletin of the *DERIVE* & CAS-*TI User Group*. It is published at least four times a year with a content of 40 pages minimum. The goals of the *DNL* are to enable the exchange of experiences made with *DERIVE*, *TI*-CAS and other CAS as well to create a group to discuss the possibilities of new methodical and didactical manners in teaching mathematics.

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Contributions:
Please send all contributions to the Editor. Non-English speakers are encouraged to write their contributions in English to reinforce the international touch of the *DNL*. It must be said, though, that non-English articles will be warmly welcomed nonetheless. Your contributions will be edited but not assessed. By submitting articles, the author gives his consent for reprinting it in the *DNL*. The more contributions you will send, the more lively and richer in contents the *DERIVE* & CAS-*TI Newsletter* will be.

Next issue: June 2024

Preview: Contributions waiting to be published

Some simulations of Random Experiments, J. Böhm, AUT, Lorenz Kopp, GER
Wonderful World of Pedal Curves, J. Böhm, AUT
Simulating a Graphing Calculator in *DERIVE*, J. Böhm, AUT
Logos of Companies as an Inspiration for Math Teaching
Exciting Surfaces in the FAZ, BooleanPlots.mth, P. Schofield, UK
Old traditional examples for a CAS – What´s new? J. Böhm, AUT
Tutorials for the NSpireCAS, G. Herweyers, BEL
Dirac Algebra, Clifford Algebra, Vector-Matrix-Extension, D. R. Lunsford, USA
Another Approach to Taylor Series, D. Oertel, GER
Charge in a Magnetic Field, H. Ludwig, GER
More Applications of TI-Innovator™ Hub and TI-Innovator™ Rover
A Collection of Special Problems, Puzzles, W. Alvermann, GER
DERIVE Bugs? D. Welz, GER
Tweening & Morphing with TI-NspireCX-II-T, J. Böhm. AUT
The Gap between Poor and Rich, J. Böhm, AUT
More functions from M. Myers and from Bhuvanesh's Mathtools-library
TaxiCab Conics, Two alternate Approaches to Conics, R. Haas, USA
QR-Code light, Random numbers following a given distribution
153 is another Special Number
Quartiles, Numeros Primos, F. de Jesús Martínez Vargas, Mexico
A Stepwise First Order Partial Differential Equations Solver, J. L. Galan a. o., ESP
A Stepwise Ordinary Differential Equations Solver, J. L. Galan and colleagues, ESP
A Stepwise Multiple Integration Solver Using a CAS, J. L. Galan and colleagues, ESP
Math Lessons with/despite AI – Part 2, H. Heugl, AUT
Integration Tables, Summary 2, M. Beaudin, CAN
Th. Dana-Picard and others

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Do you pass the SUNEUNG?

In the 03/2024 (18. 1. 2014) issue of „Die Zeit" was an article about the end examination for South Korean students which is the entrance examination for the universities. The day of "SUNEUNG" is a very important day for the whole country.

This exam is very difficult and the students are preparing very hard for it. "Die Zeit" showed one task of this 9 hours examination:

Eine ganzrationale Funktion dritten Grades mit führendem Koeffizienten 1 erfüllt folgende Eigenschaften:
(1) Es gibt keine ganze Zahl k mit $f(k-1)\cdot f(k+1) < 0$
(2) $f'(-\tfrac{1}{4}) = -\tfrac{1}{4}$
(3) $f'(\tfrac{1}{4}) < 0$
Bestimmen Sie den Funktionswert $f(8)$.

An integer rational (polynomial) function of degree 3 with leading coefficient 1 fulfills the following properties:

(1) There is no integer k with $f(k-1)\cdot f(k+1) < 0$

(2) $f'(-\tfrac{1}{4}) = -\tfrac{1}{4}$

(3) $f'(-\tfrac{1}{4}) < 0$

Find the function value $f(8)$.

You are invited to solve the problem! Let us know!

DERIVE and Windows 11 once more.

Fred Tydeman had problems to install DERIVE 6 on his Laptop. Once more Günter Schödl, our expert gave advice. Many thanks.

*) Remove all DERIVE remains from your PC (folder of installation)
*) Newstart
*) Make sure that there is no antivirus or data protecting running (except Windows Defender). Don't install a second antivirus program, because Windows Defender is excellent.)
*) Switch off all Security Locks of win10/11 - Settings - Data protection and Security - Windows security (memorize the settings)
*) Switch off the User Account Control (Search for it or ask Google)
*) Start the installation as administrator after mouse right click - (absolutely necessary)
*) Then install the Help as described earlier (like with win10)

And further:

dmp files indicate a hardware crash - memory or graphics/driver problem - I'm guessing graphics.
How to evaluate this can be found here:
https://www.windowscentral.com/how-open-and-analyze-dump-error-files-windows-10

I would first check if the process (derive) was started at all, this can be done with the task manager :
https://www.whatismyip.com/task-manager/

Check if the Derive folder grants access for EVERYONE
https://www.isumsoft.com/windows-10/how-to-check-if-i-have-administrator-rights-windows-10.html
https://www.microcenter.com/tech_center/article/10900/how-to-check-or-change-file-permissions-in-windows-10
I hope this helps - I suspect a driver problem with the graphics - can be found out via the dmp file...

Surfaces and their Duals

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ACDCA (Austrian Center for Didactics of Computer Algebra)
DUG (International DERIVE and TI-CAS User Group)

This paper is dedicated to the memory of my close friend Eugenio Roanes-Lozano, a driving force of Applications of Computer Algebra.

The pictures were taken at the occasion of a private visit and stay of Eugenio in our home in Würmla.

Introduction

I have a large collection of papers and articles from various journals, conference proceedings, seminars and other publications.

From time to time I like to browse through my 10 or 11 thick folders. Then I remove one or the other paper which is out of time or find it interesting enough for treating it by means of Dynamic Geometry and/or Computer Algebra.

So, I came across two pages from a Maths&Stats Journal from 2002 where Dr Richard Morris presented Dual Surfaces.

I had the idea to transfer the ideas presented by a short text and some pictures of low quality to my preferred CAS tools: DERIVE and TI-NspireCAS. This was an exciting task, although remaining on the level of Upper Secondary school – or just a little bit above.

This was Dr Morris' article:

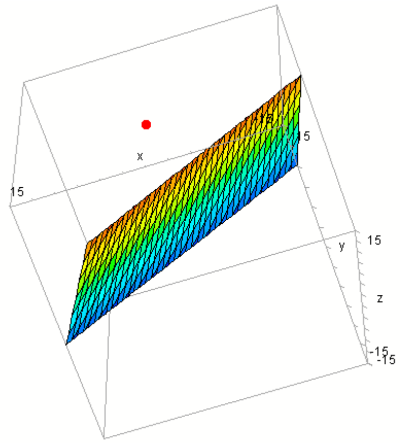
Dr Richard Morris, Liverpool University

The pictures in this issue of Maths&Stats are all of various surfaces and their duals. The dual of a surface is the set of planes tangent to the surface. Any plane, a x + b y + c z = d, can be thought of as a point (a,b,c,d) in RP^2. Hence the dual of a surface forms a surface in RP^2. Each point on the surface will have a tangent plane and hence give a point on the dual. What is actually shown here are projections of the duals into R^3 using the map (a,b,c,d)->(a/c, b/c, d/c).

The pictures here have been produced using the Liverpool Surface Package and Geomview. The Liverpool Surface Package is a set of programs which can generate a variety of mathematical curves and surface, defined either by a parametrization or by implicit algebraic equations. Other options such as performing mapping from R^3 to R^3 and constructing duals are also possible. Geomview is an interactive 3D geometry viewer, written at the Geometry Center, University of Minneapolis, Minnasota. Geomview is public domain software available via anonymous ftp from geom.edu.umn. Enquiries about the Liverpool Surface Package should be e-mailed to rmorris@liverpool.ac.uk. Both packages run on Silicon Graphics Irises.

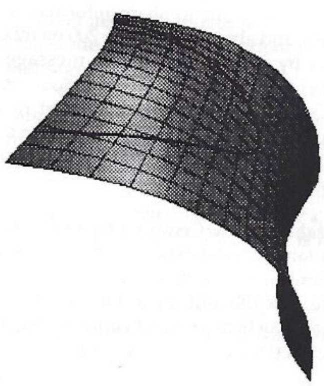
3·x + 7·y + z = 12

[3, 7, 12]



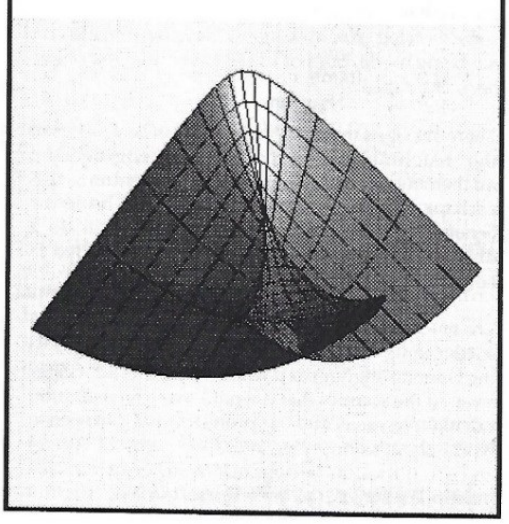
A plane and its dual point

$z = x^2 + y^3$

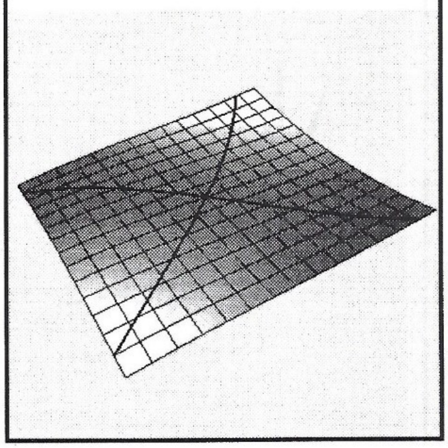


1. A surface with a parabolic line running along it - see page 6 for its dual.

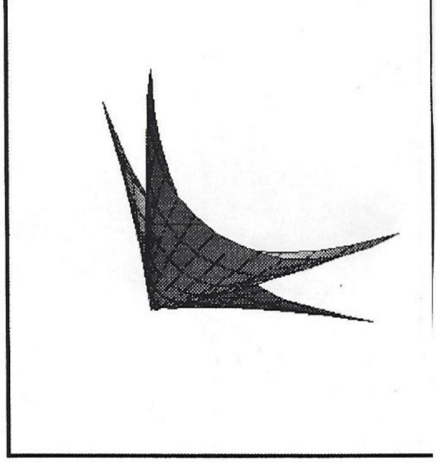
2. The dual of the surface on page 1



$z = x^3 + xy^2$ hyperbolic cubic form

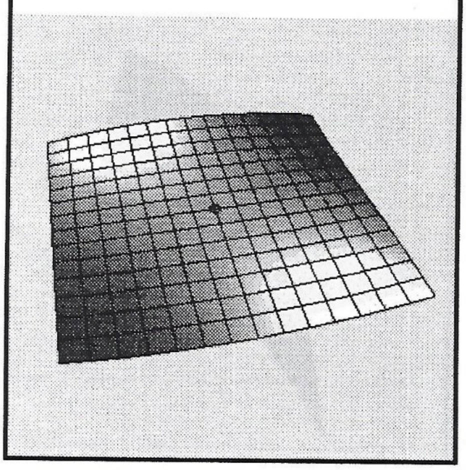


3. A flat hyperbolic umbilic surface - see over for its dual

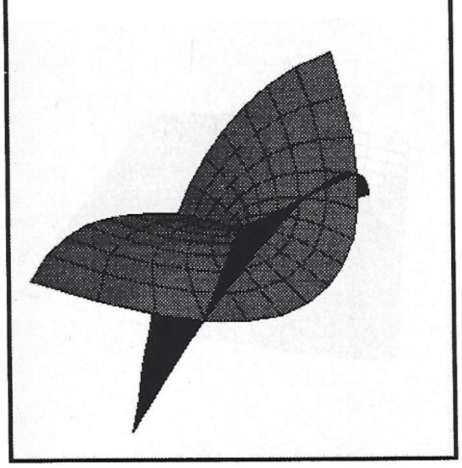


4. Dual of surface on previous page - see page 23

$z = x^3 - xy^2$ (elliptical cubic form)



7. A flat elliptical umbilic surface - see next page for dual



8. Dual of surface on previous page - see page 35

I was confused, because Dr Morris writes about RP^2 in his short article, but then about RP^4 in his extended paper – which seems to have more sense.

I will perform this mapping using my favourite Computer Algebra systems: DERIVE and TI-NspireCAS.

I must admit that I don't know about RP^2 (Real Projective Plane). I wanted to download the program mentioned Geomview. It is free but it runs only under UNIX.

https://journals.gre.ac.uk/index.php/msor/issue/archive

Searching for the original article from 2002 I found a much more extended paper on the same issue, also from Dr Morris and also from 2002 at

www.singsurf.org/papers/dual/dual.pdf

What do we need? The Tangent Plane of a Surface

Surface	Tangent Plane
explicit – implicit form $z = f(x,y)$ $F(x,y,z) = f(x,y) - z = 0$	$grad(F(u,v,w)) \cdot \begin{pmatrix} x-u \\ y-v \\ z-w \end{pmatrix} = 0$
parameter form $p(u,v) = \begin{pmatrix} px(u,v) \\ py(u,v) \\ pz(u,v) \end{pmatrix}$ $f(x,y) \to p(u,v) = \begin{pmatrix} u \\ v \\ f(u,v) \end{pmatrix}$	$\begin{pmatrix} x-px(u,v) \\ y-py(u,v) \\ z-pz(u,v) \end{pmatrix} \cdot \left(\frac{d(p(u,v))}{du} \times \frac{d(p(u,v))}{dv} \right) = 0$

Morris' first example stepwise:

```
#1:   tf_expl(f) := SUBST(GRAD(f - z), [x, y], [u, v])·([x, y, z] -
        SUBST([x, y, f], [x, y], [u, v])) = 0
```

$$\#2: \quad f1 := x^2 + y^3$$

$$\#3: \quad tf\_expl(f1) = (2 \cdot u \cdot x + 3 \cdot v^2 \cdot y - z - u^2 - 2 \cdot v^3 = 0)$$

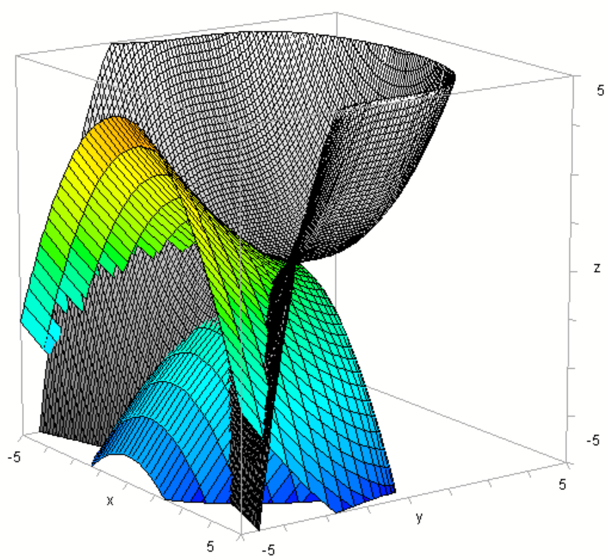
$$\#4: \quad 2 \cdot u \cdot x + 3 \cdot v^2 \cdot y - z - u^2 - 2 \cdot v^3 = 0$$

(a/c,b/c,d/c)

$$\#5: \quad \left[\frac{2 \cdot u}{-1}, \frac{3 \cdot v^2}{-1}, \frac{u^2 + 2 \cdot v^3}{-1} \right]$$

Parameter form of the Dual:

$$\#6: \quad \left[-2 \cdot u, -3 \cdot v^2, -u^2 - 2 \cdot v^3 \right]$$



Origin in Gray Scale, Dual in Rainbow

Later we will come back to the cuspidal edge of the dual.

For Morris's second example we will work with the parameter form:

Instead of $z(x,y)=x^3+xy^2$ we take $p(u,v)=\begin{pmatrix} u \\ v \\ u^3+uv^2 \end{pmatrix}$.

#7: $f2 := x^3 + x \cdot y^2$

#8: $p2 := [u, v, u^3 + u \cdot v^2]$

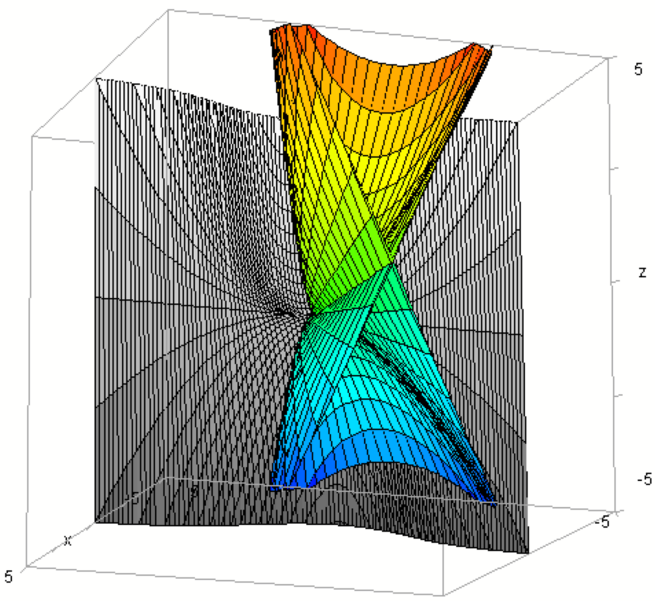
#9: $tp(p) := \text{EXPAND}\left([x - p_1, y - p_2, z - p_3] \cdot \left(\left(\frac{d}{du} p\right) \times \frac{d}{dv} p\right)\right) = 0$

#10: tp(p2)

#11: $2 \cdot u^3 - 3 \cdot u^2 \cdot x + 2 \cdot u \cdot v^2 - 2 \cdot u \cdot v \cdot y - v^2 \cdot x + z = 0$

#12: $- 3 \cdot u^2 \cdot x - 2 \cdot u \cdot v \cdot y - v^2 \cdot x + z = - 2 \cdot u^3 - 2 \cdot u \cdot v^2$

#13: $[- 3 \cdot u^2 - v^2, - 2 \cdot u \cdot v, - 2 \cdot u^3 - 2 \cdot u \cdot v^2]$



We collect the steps in two – very – short programs (functions) for parameter and explicit form:

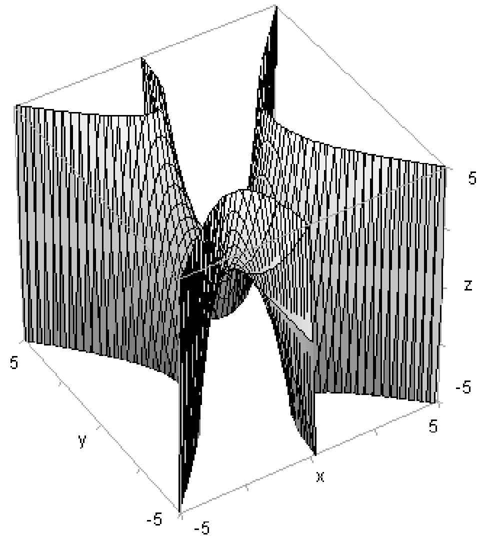
```
        dualp(p, dt, d) :=
          Prog
#14:        dt := [x - p↓1, y - p↓2, z - p↓3]·(∂(p, u) × ∂(p, v))
            d := - SUBST(dt, [x, y, z], [0, 0, 0])
            [(GRAD(dt))↓1, (GRAD(dt))↓2, d]/(GRAD(dt))↓3

        dualf(f, p, dt, d) :=
          Prog
#15:        p := [u, v, SUBST(f, [x, y], [u, v])]
            dualp(p)
```

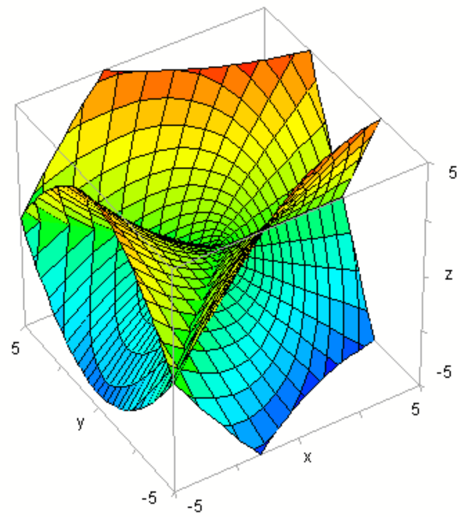
Let's test them with Morris' third example:

#16: $f3 := x^3 - x\cdot y^2$

#17: $\text{dualf}(f3) = [v^2 - 3\cdot u^2,\ 2\cdot u\cdot v,\ 2\cdot u\cdot (v^2 - u^2)]$

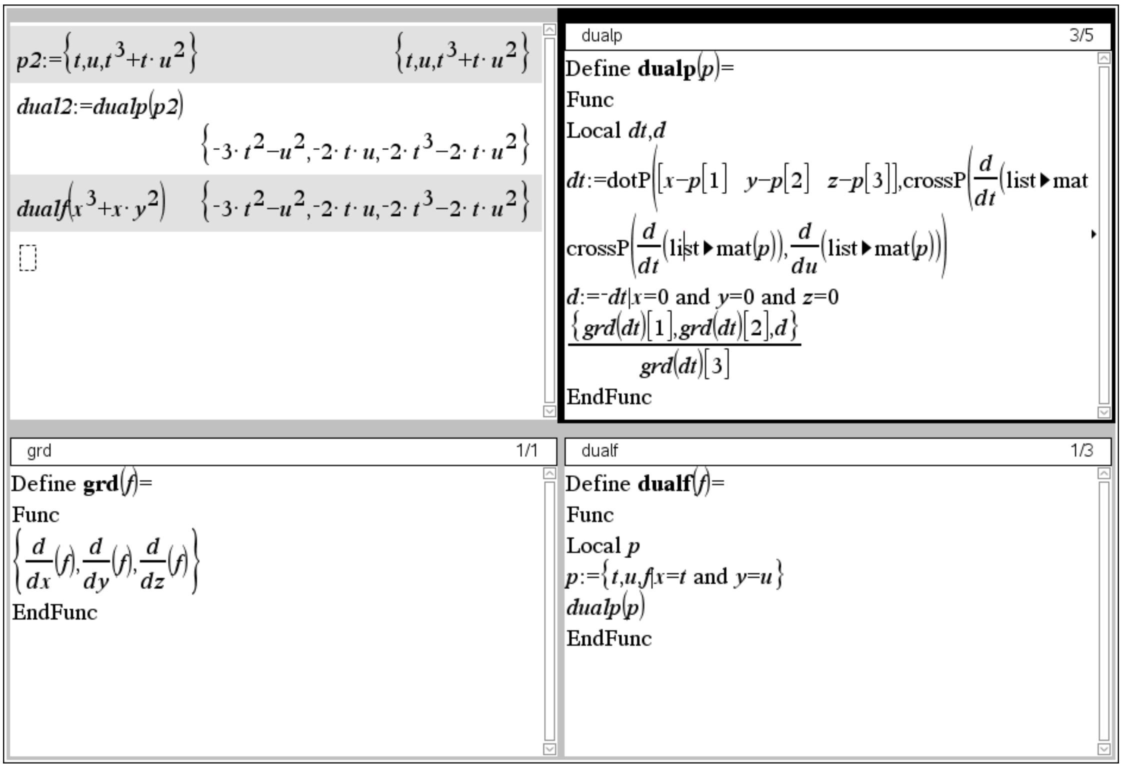
#18: $\text{dualp}([u,\ v,\ u^3 - u\cdot v^2]) = [v^2 - 3\cdot u^2,\ 2\cdot u\cdot v,\ 2\cdot u\cdot (v^2 - u^2)]$

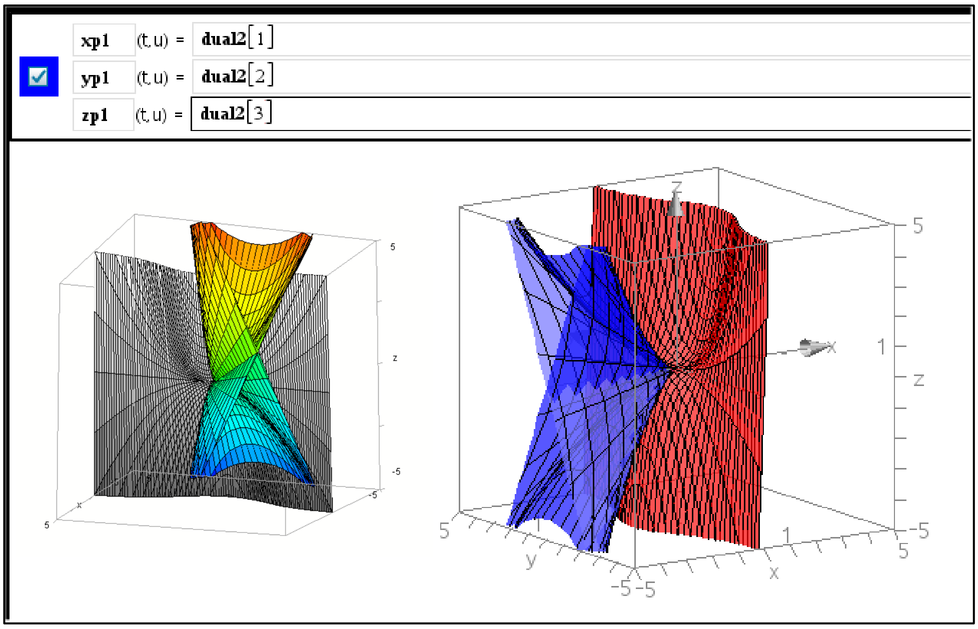




I will come back to Morris' examples at the end of the paper.

Have a look how TI-NspireCAS performs with the tasks:





Compare with the DERIVE plot (left)!

Until here this is the realization of the short article in the journal.

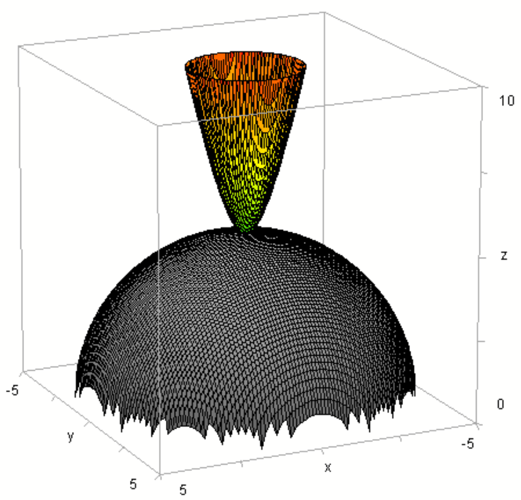
We can start experimenting with surfaces of our choice.

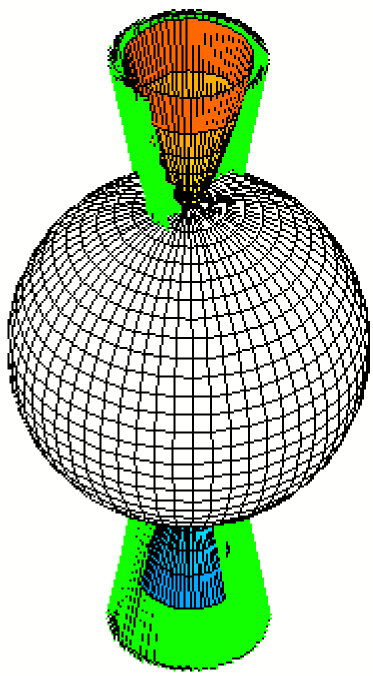
I begin with a simple sphere:

#20: $\text{dualf}\left(\sqrt{25 - x^2 - y^2}\right)$

#21: $$\left[\frac{u}{\sqrt{-u^2 - v^2 + 25}}, \frac{v}{\sqrt{-u^2 - v^2 + 25}}, \frac{25}{\sqrt{-u^2 - v^2 + 25}}\right]$$

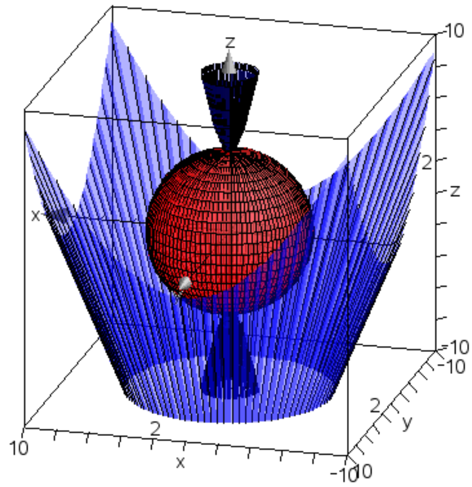
Below the dual surface of the function and right the dual surface of the complete sphere (parameter form).





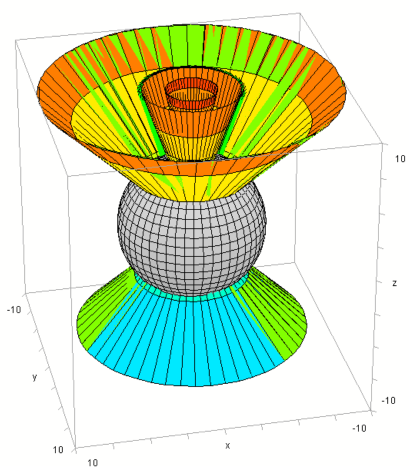
Let's try the parameter form on the Nspire:

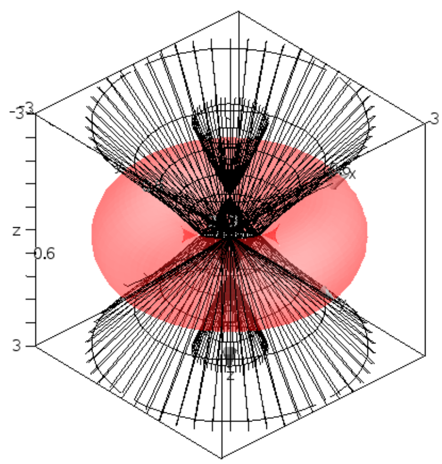
$$dualsph:=dualp(\{5\cdot\sin(t)\cdot\cos(u),5\cdot\sin(t)\cdot\sin(u),5\cdot\cos(t)\}) \blacktriangleright \left\{\tan(t)\cdot\cos(u),\tan(t)\cdot\sin(u),\frac{5}{\cos(t)}\right\}$$



Nice "crown with four spikes".

Torus





$$\mathbf{torus} \blacktriangleright \{\cos(t)\cdot(\cos(u)+2),\sin(t)\cdot(\cos(u)+2),\sin(u)\}$$

$$\mathbf{dualp(torus)} \blacktriangleright \left\{\frac{\cos(t)}{\tan(u)},\frac{\sin(t)}{\tan(u)},\frac{2\cdot\cos(u)+1}{\sin(u)}\right\}$$

I compare the 3rd component of the dual given by DERIVE (left) and by TI-NspireCAS (right):

$$\frac{\mathrm{COS}(u)^3\cdot\mathrm{COS}(v)^2+2\cdot\mathrm{COS}(u)^2+\mathrm{COS}(u)\cdot(\mathrm{SIN}(u)^2\cdot\mathrm{COS}(v)^2+\mathrm{SIN}(v)^2+4)+2}{\mathrm{SIN}(u)\cdot(\mathrm{COS}(u)+2)}-\frac{2\cdot\mathrm{COS}(u)+1}{\mathrm{SIN}(u)}$$

Simplifying the difference of both expressions gives 0 – both expressions are identical.

0

Trigonometry := Collect

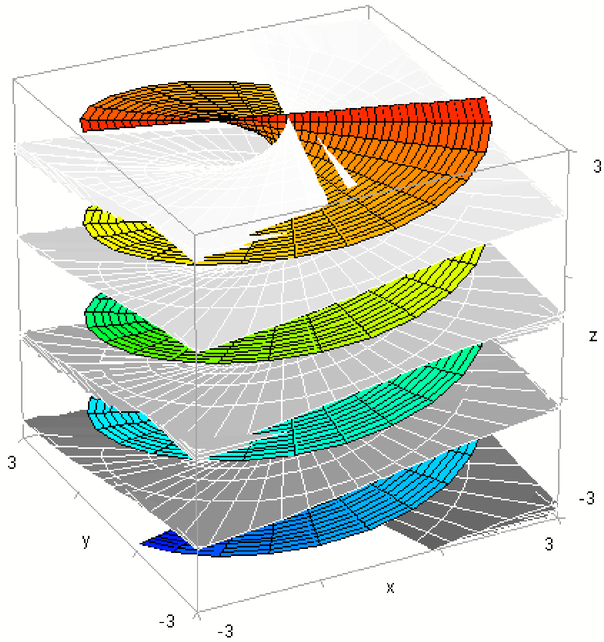
$$\frac{2\cdot(\cos(2\cdot u) + 5\cdot\cos(u) + 3)}{\sin(2\cdot u) + 4\cdot\sin(u)} - \frac{2\cdot\cos(u) + 1}{\sin(u)}$$

Trigonometry := Expand

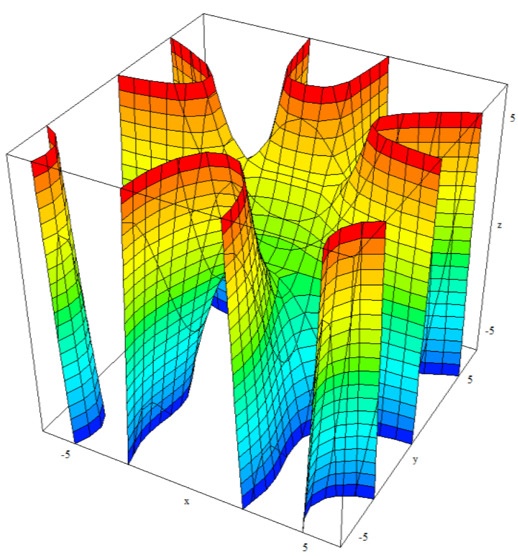
$$\left(\frac{2\cdot\cos(u)}{\sin(u)} + \frac{1}{\sin(u)}\right) - \frac{2\cdot\cos(u) + 1}{\sin(u)}$$

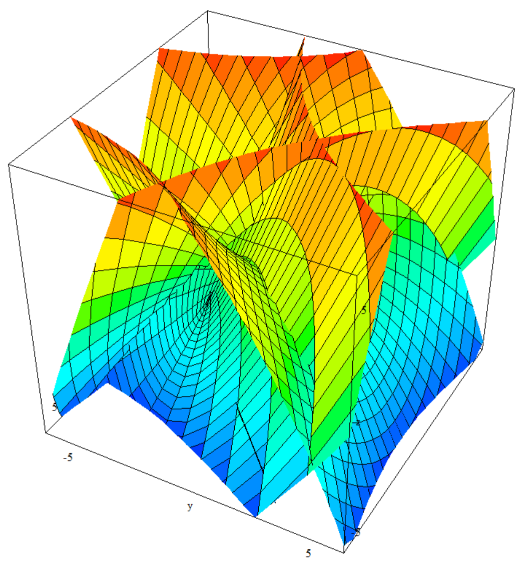
Short manipulation with trig expressions shows the identity, too. (Let's try manually ☺)

Helicoid: [3·v·COS(u), 3·v·SIN(u), 0.5·u]



Finally, I'd like to find the duals of surfaces of my phantasy: z=x^2·SIN(y) + y^2·COS(x)





Graphs are produced using David Parker's DPGraph.

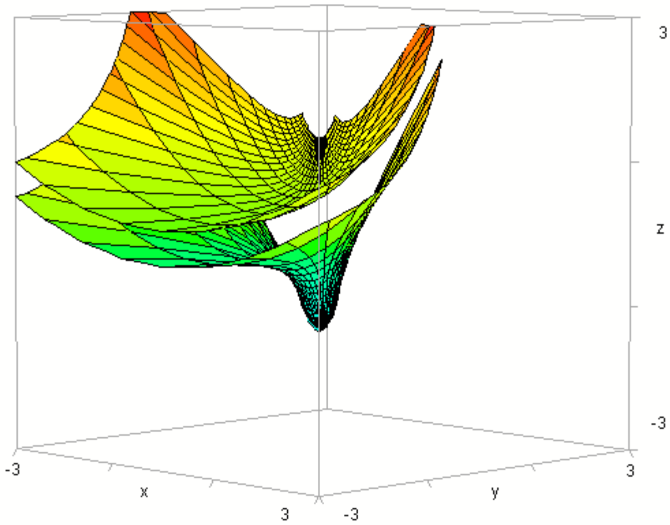
Another dual?

Dr. Morris: "… using the map (a,b,c,d) → (a/c,b/c,d/c)". Why not (a,b,c,d) → (a/b,c/b,d/b)?

We will apply this map on the first example:

$$2\cdot u\cdot x + 3\cdot v^2\cdot y - z - u^2 - 2\cdot v^3 = 0$$

$$\left[\frac{2\cdot u}{3\cdot v^2}, -\frac{1}{3\cdot v^2}, \frac{u^2 + 2\cdot v^3}{3\cdot v^2}\right]$$

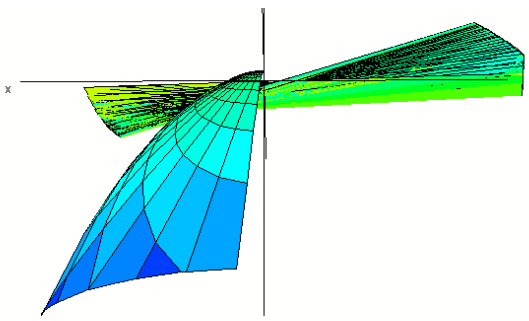


Looks quite different!

Here are the other duals:

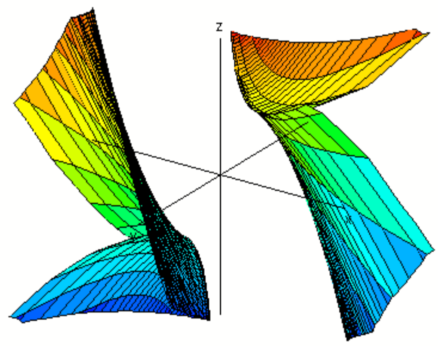
(a/d,b/d,c/d)

$$\left[\frac{2*u}{u^2 + 2*v^3}, \frac{3*v^2}{u^2 + 2*v^3}, -\frac{1}{u^2 + 2*v^3}\right]$$



(b/a,c/a,d/a)

$$\left[\frac{3\cdot v^2}{2\cdot u}, -\frac{1}{2\cdot u}, \frac{u^2 + 2\cdot v^3}{2\cdot u}\right]$$



I will remain at this first easy surface and follow the next idea:

I intend to find the mappings of parameter curves.

#63 and #64 are the v-lines on surface (#61) and its dual (#62) (blue), #65 and #66 are the u-lines (red).

#61: $[u, v, u^2 + v^3]$

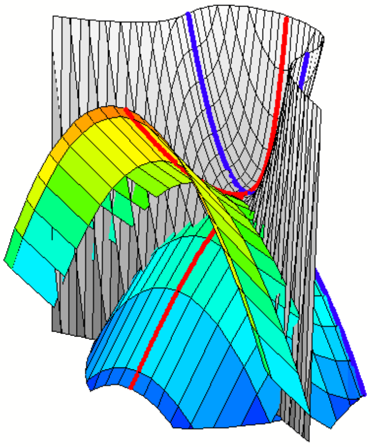
#62: $[-2\cdot u, -3\cdot v^2, -u^2 - 2\cdot v^3]$

#63: VECTOR($[[0, v, v^3]]$, v, -5, 5, 0.01)

#64: VECTOR($[[0, -3\cdot v^2, 0 - 2\cdot v^3]]$, v, -5, 5, 0.01)

#65: VECTOR($[[u, 0, u^2]]$, u, -5, 5, 0.01)

#66: VECTOR($[[-2\cdot u, 0, -u^2]]$, u, -5, 5, 0.01)

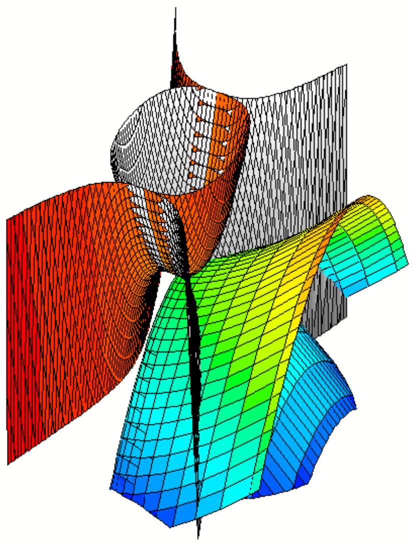


How does the **dual of the dual** of a surface look like?

#74: $f1 = x^2 + y^3$

#75: $\mathrm{dualf}(f1) = \left[-2\cdot u,\ -3\cdot v^2,\ -u^2 - 2\cdot v^3 \right]$

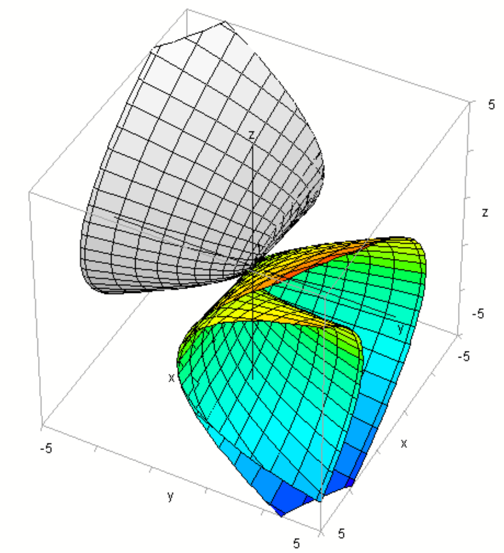
#76: $\mathrm{dualp}(\mathrm{dualf}(f1)) = \left[-u,\ -v,\ u^2 + v^3 \right]$

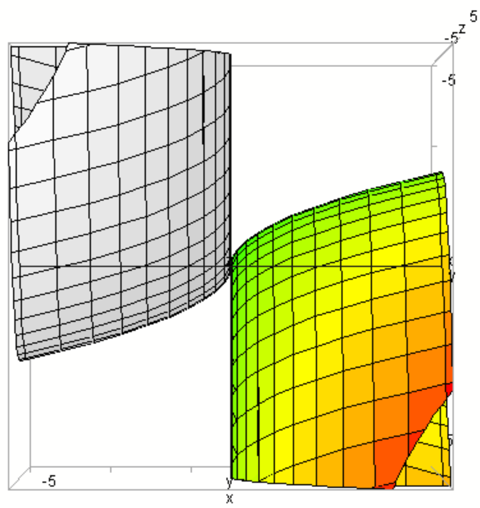


#80: $\left[u^2 - v,\ v^2,\ u\cdot v \right]$

#81: $\mathrm{dualp}(\mathrm{dualp}(\left[u^2 - v,\ v^2,\ u\cdot v \right]))$

#82: $\left[v - u^2,\ -v^2,\ u\cdot v \right]$





It seems to be that the dual of the dual results in a surface symmetric wrt the z-axis to the origin figure? Can we prove this?

Barack Obama: "Yes, we can!"

[f(u, v) :=, h(u, v) :=, g(u, v) :=]

$$\mathrm{dualp}([g(u,v),\ h(u,v),\ f(u,v)]) = \left[\frac{\left(\frac{d}{dv}h(u,v)\right)\cdot\frac{d}{du}f(u,v) - \left(\frac{d}{du}h(u,v)\right)\cdot\frac{d}{dv}f(u,v)}{\left(\frac{d}{du}h(u,v)\right)\cdot\frac{d}{dv}g(u,v) - \left(\frac{d}{dv}h(u,v)\right)\cdot\frac{d}{du}g(u,v)},\ \frac{\left(\frac{d}{dv}g(u,v)\right)\cdot\frac{d}{du}f(u,v) - \left(\frac{d}{du}g(u,v)\right)\cdot\frac{d}{dv}f(u,v)}{\left(\frac{d}{dv}h(u,v)\right)\cdot\frac{d}{du}g(u,v) - \left(\frac{d}{du}h(u,v)\right)\cdot\frac{d}{dv}g(u,v)},\right.$$

$$\left.\frac{\left(\frac{d}{du}f(u,v)\right)\cdot\left(h(u,v)\cdot\frac{d}{dv}g(u,v) - g(u,v)\cdot\frac{d}{dv}h(u,v)\right) + \left(\frac{d}{dv}f(u,v)\right)\cdot\left(g(u,v)\cdot\frac{d}{du}h(u,v) - h(u,v)\cdot\frac{d}{du}g(u,v)\right) + f(u,v)\cdot\left(\left(\frac{d}{dv}h(u,v)\right)\cdot\frac{d}{du}g(u,v) - \left(\frac{d}{du}h(u,v)\right)\cdot\frac{d}{dv}g(u,v)\right)}{\left(\frac{d}{dv}h(u,v)\right)\cdot\frac{d}{du}g(u,v) - \left(\frac{d}{du}h(u,v)\right)\cdot\frac{d}{dv}g(u,v)}\right]$$

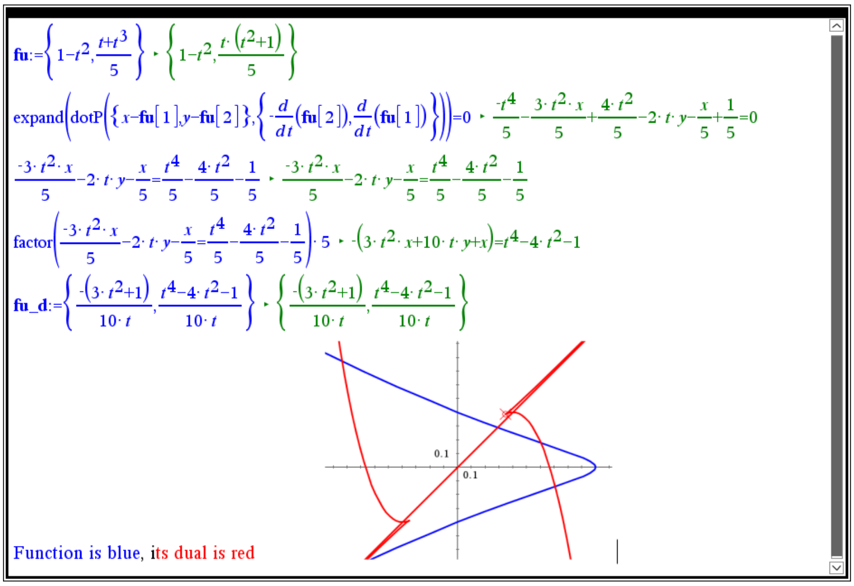
Pretty bulky expression, isn't it? Give it a try:

dualp(dualp([g(u, v), h(u, v), f(u, v)])) = [– g(u, v), – h(u, v), f(u, v)]

Usually we raise a problem from the second dimension up to the third one – from plane to space. This time I do it the other way. I make a step down:

I map all tangents of a curve in the plane on points: $a\,x + b\,y = c \rightarrow \left(\frac{a}{b}, \frac{c}{b}\right)$.

I assume that the function to be mapped is given in parameter form $f(u) = (x(u), y(u))$.



The straight red line seems to be an asymptote?

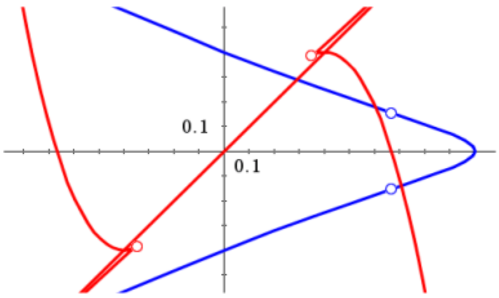
Inspecting the red graph, we cannot avoid the next question: There are two cusps. Where are they? What are their origins on the given blue curve?

At first, we try to find the singularities and then we track them back to the blue curve:

$\text{solve}\left(\left\{\frac{d}{dt}(\text{fu\_d}[1])=0,\frac{d}{dt}(\text{fu\_d}[2])=0\right\},t\right) \blacktriangleright t=\frac{-\sqrt{3}}{3} \text{ or } t=\frac{\sqrt{3}}{3}$ ⚠

$\text{fu\_d}|t=\left\{\frac{-\sqrt{3}}{3},\frac{\sqrt{3}}{3}\right\} \blacktriangleright \begin{bmatrix}\frac{\sqrt{3}}{5} & \frac{-\sqrt{3}}{5}\\ \frac{2\cdot\sqrt{3}}{9} & \frac{-2\cdot\sqrt{3}}{9}\end{bmatrix}$ **the cusps on the dual curve …**

$\text{fu}|t=\left\{\frac{-\sqrt{3}}{3},\frac{\sqrt{3}}{3}\right\} \blacktriangleright \begin{bmatrix}\frac{2}{3} & \frac{2}{3}\\ \frac{-4\cdot\sqrt{3}}{45} & \frac{4\cdot\sqrt{3}}{45}\end{bmatrix}$ **… and where they come from.**



Can it be, that the cusps are the mappings of the inflection points?

The points of inflection are where the curvature changes its sign.
So we calculate the zeros of the curvature of a curve given in parameter form -
- we need the zeros of the numerator:

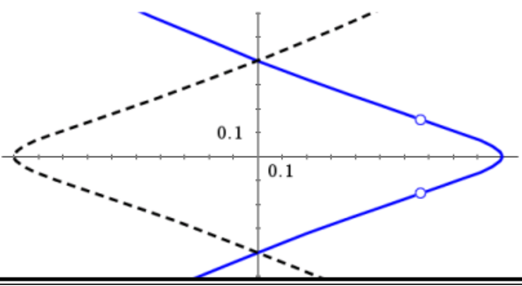
$\text{zeros}\left(\frac{d}{dt}(\text{fu}[1])\cdot\frac{d^2}{dt^2}(\text{fu}[2])-\frac{d^2}{dt^2}(\text{fu}[1])\cdot\frac{d}{dt}(\text{fu}[2]),t\right) \blacktriangleright \left\{\frac{-\sqrt{3}}{3},\frac{\sqrt{3}}{3}\right\}$

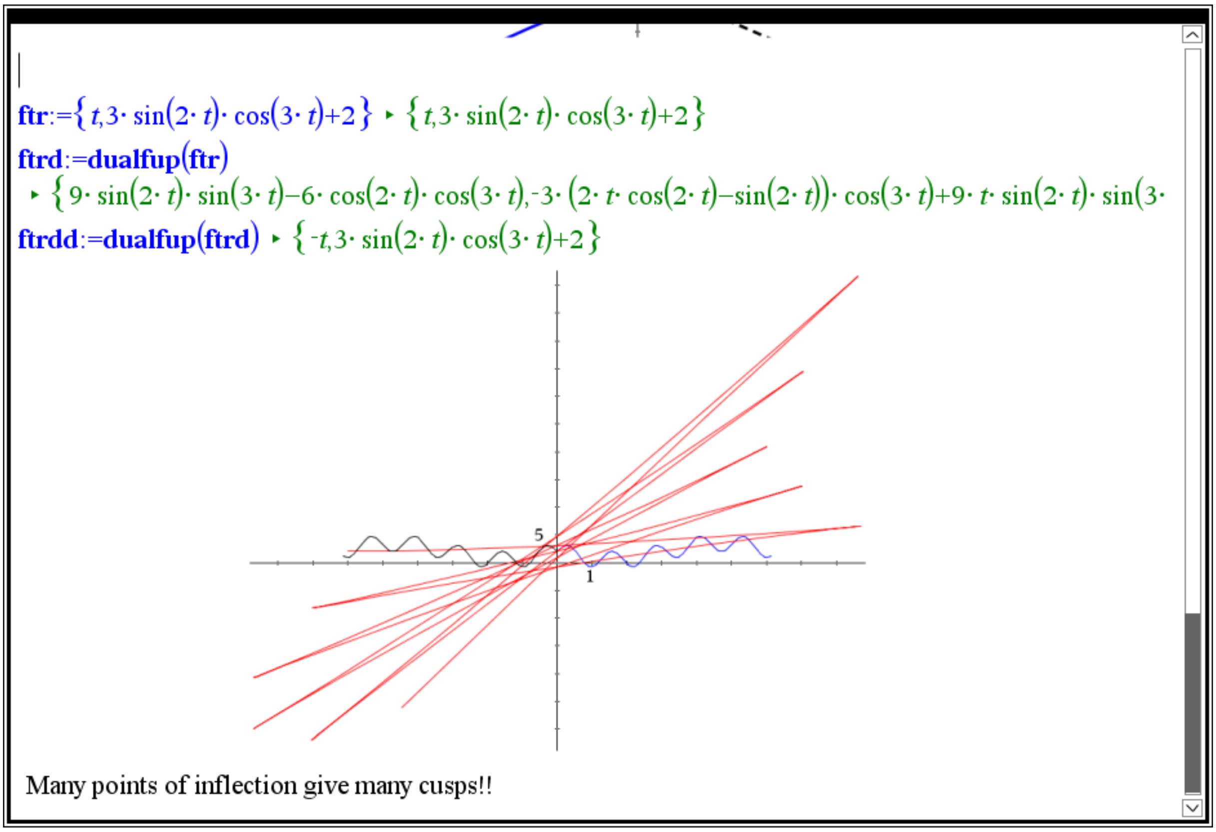
The dual points of the inflection points are the cusps.

Last question:

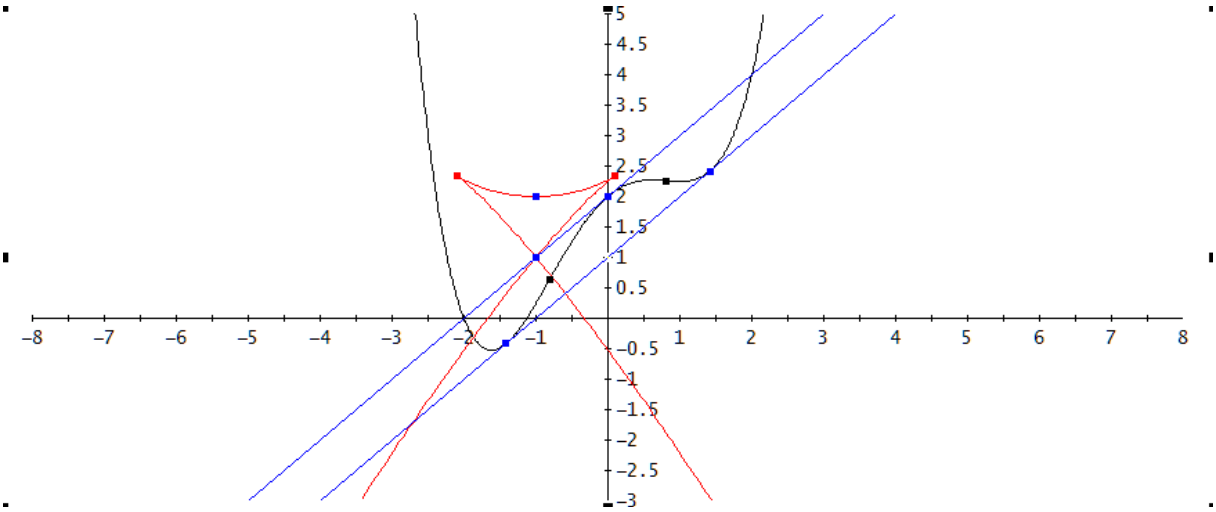
Do you guess how the dual of the dual will look like?
You are right:

$\text{fudd}:=\text{dualfup}(\text{dualfup}(\text{fu})) \blacktriangleright \left\{t^2-1,\frac{t\cdot(t^2+1)}{5}\right\}$





A quartic together with its dual:



Other investigations are pressing:

Function – Dual function

Inflection point – Cusp

Double tangent – Self intersection

I investigate a polynomial of order five together with its “dual”:

Double tangents on a quintic …

$$z(x) := \frac{x^5 + 3\cdot x^4 - 11\cdot x^3 - 27\cdot x^2 + 10\cdot x + 64}{20}$$

$$z'(a) - z'(b) = 0$$

$$\text{FACTOR}(z(b) - z(a) - z'(a)\cdot(b - a) = 0)$$

$$eq1 := 5\cdot a^3 + a^2\cdot(5\cdot b + 12) + a\cdot(5\cdot b^2 + 12\cdot b - 33) + 5\cdot b^3 + 12\cdot b^2 - 33\cdot b - 54$$

$$eq2 := 4\cdot a^3 + a^2\cdot(3\cdot b + 9) + a\cdot(2\cdot b^2 + 6\cdot b - 22) + b^3 + 3\cdot b^2 - 11\cdot b - 27$$

$$\text{GROEBNER\_BASIS}([eq1,\ eq2],\ [a,\ b])$$

$$\text{NSOLUTIONS}(500\cdot b^9 + 2700\cdot b^8 - 3740\cdot b^7 - 35004\cdot b^6 - 1837\cdot b^5 + 149937\cdot b^4 + 53017\cdot b^3 - 217071\cdot b^2 - 69909\cdot b + 38367,\ b)$$

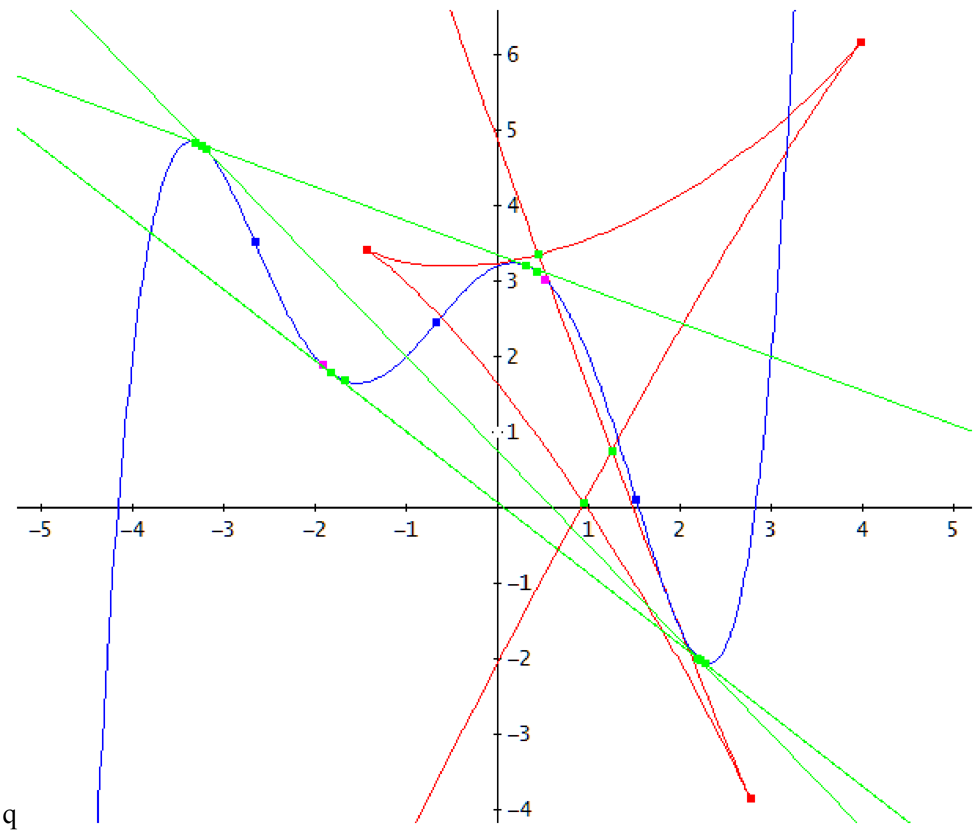
[2.224845673, -3.302775637, 2.192582403, -3.192582403, -1.824845673, 1.524977841, 0.3027756377, -0.6658185230, -2.659159318]

The Cusps

$$\begin{bmatrix} -1.438273175 & 3.411523680 \\ 3.974701467 & 6.157388302 \\ 2.769371708 & -3.845895982 \end{bmatrix}$$

The Self Intersection points

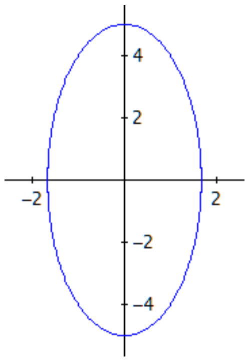
$$\begin{bmatrix} 0.9413000011 & 0.06811600289 \\ 0.45 & 3.35 \\ 1.25 & 0.75 \end{bmatrix}$$

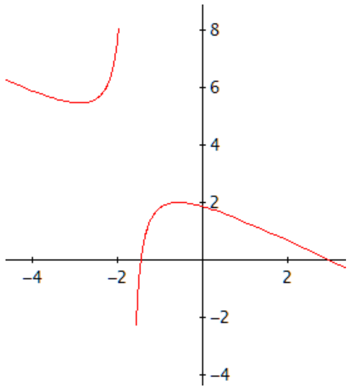


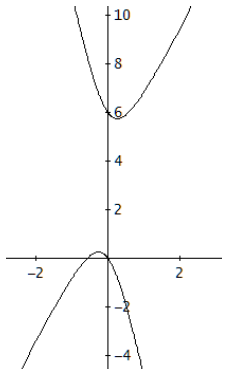
q

Note: It can happen that you have interested and curious students. So, they might ask: "What are the duals of conics?"

Present the following graphs and invite them to guess from which conic they are the dual?







If necessary, provide a hint: points with vertical tangents are mapped to points of infinity!
(I must admit that I am not able to find the equation of the asymptotes of the duals. Any idea?)

(Solution: hyperbola (3 sinh(u), ±5 cosh(u)), parabola $((u^2 + 2\sqrt{3}u)/4,\ (u^2\sqrt{3} + 2u+8)/4$,
ellipse (5 cos(u) – 2, 3 sin(u) + 3)).

This is a nice calculation:

$$\text{hyperbola1} := [3\cdot \text{SINH}(u),\ 5\cdot \text{COSH}(u)]$$

$$\text{dualfup(hyperbola1)} = \left[\frac{5\cdot(1 - e^{2\cdot u})}{3\cdot(e^{2\cdot u} + 1)},\ \frac{10\cdot e^{u}}{e^{2\cdot u} + 1}\right]$$

Is this really an ellipse?

$$\left[\frac{5\cdot(e^{2\cdot u} - 1)}{3\cdot(e^{2\cdot u} + 1)},\ -\frac{10\cdot e^{u}}{e^{2\cdot u} + 1}\right]$$

$$\left[\frac{5\cdot(e^{2\cdot \text{LOG}(t)} - 1)}{3\cdot(e^{2\cdot \text{LOG}(t)} + 1)},\ -\frac{10\cdot e^{\text{LOG}(t)}}{e^{2\cdot \text{LOG}(t)} + 1}\right] = \left[\frac{5\cdot(t^2 - 1)}{3\cdot(t^2 + 1)},\ -\frac{10\cdot t}{t^2 + 1}\right]$$

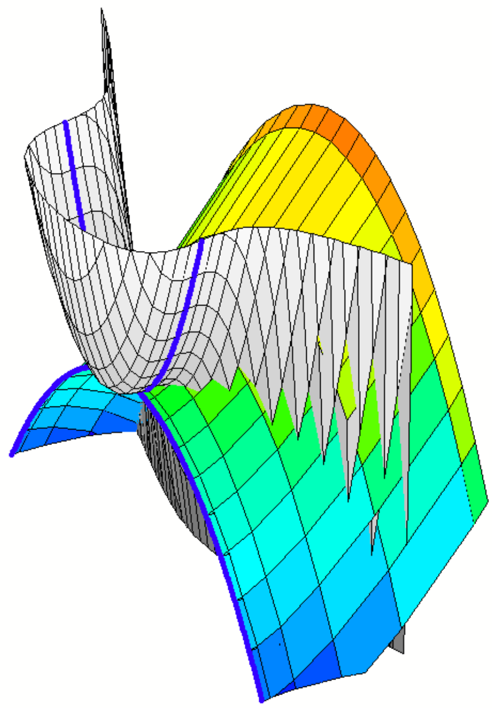
$$\text{SOLVE}\left(x = \frac{5\cdot(t^2 - 1)}{3\cdot(t^2 + 1)},\ t\right) = \left(t = -\frac{\sqrt{(-3\cdot x - 5)}}{\sqrt{(3\cdot x - 5)}} \vee t = \frac{\sqrt{(-3\cdot x - 5)}}{\sqrt{(3\cdot x - 5)}}\right)$$

$$\text{SUBST}\left(y = -\frac{10\cdot t}{t^2 + 1},\ t,\ -\frac{\sqrt{(-3\cdot x - 5)}}{\sqrt{(3\cdot x - 5)}}\right)$$

$$y = -\sqrt{(3\cdot x - 5)}\cdot\sqrt{(-3\cdot x - 5)}$$

$$\left(y = -\sqrt{(3\cdot x - 5)}\cdot\sqrt{(-3\cdot x - 5)}\right)^2 = \left(y^2 = 25 - 9\cdot x^2\right)$$

Back to the Surfaces!



The inflection points and the cusps reminded me on the parameter curves of the first surface.

The mapping of the parameter curve on the given surface is the sharp – cuspidal – edge of its dual.

I get the idea that the origin curve is a locus of something like inflection points. It looks like a curve on a "terrace in the landscape" …

It is not long ago that I found a more extended Morris paper on the same subject Visualising Duals of Surfaces:

www.singsurf.org/papers/dual/dual.pdf.

(Unfortunately, this website seems to be not available any longer.)

"If x lies on a parabolic line on S then the corresponding point on the dual, x*, lies on a cuspidal edge on S*."

So, what is a parabolic line?

Points with Gauss Curvature = 0 are parabolic points and the set of parabolic points form a parabolic line. The respective maps are cuspidal points and cuspidal edges.

Next function gives the Gauss curvature of function p(u,v):

I found the Gaussian Curvature and how to calculate it in Alfred Gray's "Modern Differential Geometry …"

This is beyond Upper Secondary level, but it might be a good exercise providing for the students A. Gray's text and ask them to produce a working function (DERIVE, TI-NspireCAS, wxMaxima, …)

```
gauss_curv(p, u, v, e_, f_, g_, e, f, g, h) :=
  Prog
    e_ := ∂(p, u)·∂(p, u)
    f_ := ∂(p, u)·∂(p, v)
    g_ := ∂(p, v)·∂(p, v)
    h := √(e_·g_ - f_^2)
    e := DET([∂(p, u, 2), ∂(p, u), ∂(p, v)])/h
    f := DET([∂(∂(p, u), v), ∂(p, u), ∂(p, v)])/h
    g := DET([∂(p, v, 2), ∂(p, u), ∂(p, v)])/h
    (e·g - f^2)/h^2
```

$$\mathrm{gauss\_curv}\left(\left[u,\ v,\ u^2 + v^3\right]\right) = \frac{12\cdot v}{(4\cdot u^2 + 9\cdot v^4 + 1)^2}$$

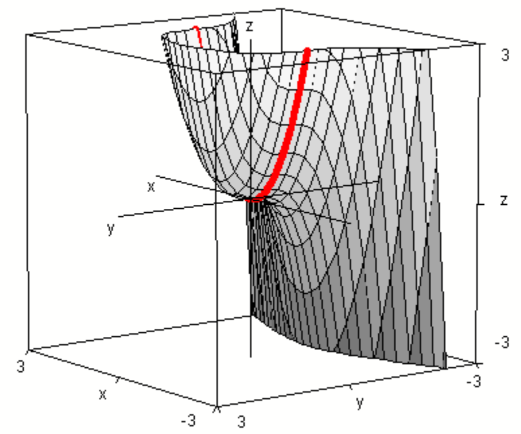
v = 0 gives the parabolic line on the given surface and the cuspidal edge on its dual.

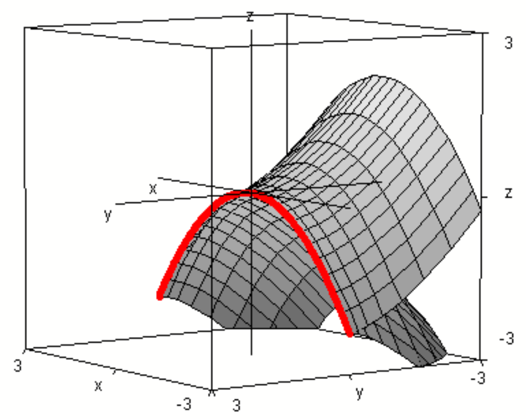
$$\mathrm{VECTOR}\left(\left[\left[u,\ 0,\ u^2\right]\right],\ u,\ -5,\ 5,\ 0.01\right)$$

The dual is:

$$\left[-2\cdot u,\ -3\cdot v^2,\ -u^2-2\cdot v^3\right]$$

```
VECTOR([[-2·u, 0, -u^2]], u, -5, 5, 0.01)
```





I come back to function f2 = $x^3 + x\,y^2$ in order to confirm my findings:

$$\text{surf} := \left[u,\ v,\ u^3 + u\cdot v^2\right]$$

$$\text{gauss\_curv(surf)} = \frac{4\cdot(3\cdot u^2 - v^2)}{(9\cdot u^4 + 10\cdot u^2\cdot v^2 + v^4 + 1)^2}$$

$$3u^2 - v^2 = 0$$

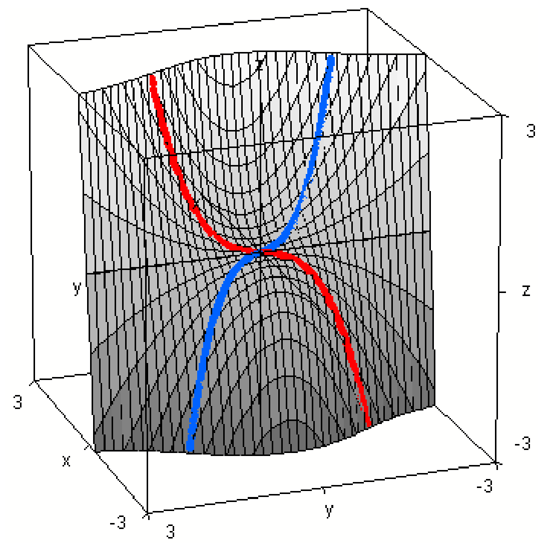
$$v = \pm u\sqrt{3}$$

$$\text{SUBST}\left(\left[u,\ v,\ u^3 + u\cdot v^2\right],\ v,\ u\cdot\sqrt{3}\right) = \left[u,\ \sqrt{3}\cdot u,\ 4\cdot u^3\right]$$

$$\text{SUBST}\left(\left[u,\ v,\ u^3 + u\cdot v^2\right],\ v,\ -u\cdot\sqrt{3}\right) = \left[u,\ -\sqrt{3}\cdot u,\ 4\cdot u^3\right]$$

```
VECTOR([[u, √3·u, 4·u^3]], u, -1, 1, 0.01)
VECTOR([[u, -√3·u, 4·u^3]], u, -1, 1, 0.01)
```

We see two parabolic lines on the given surface and plot their respective maps on its dual (next page):



dualp(surf)

$$[-3\cdot u^2 - v^2,\ -2\cdot u\cdot v,\ -2\cdot u\cdot (u^2 + v^2)]$$

$$[-3\cdot u^2 - (u\cdot\sqrt{3})^2,\ -2\cdot u\cdot (u\cdot\sqrt{3}),\ -2\cdot u\cdot (u^2 + (u\cdot\sqrt{3})^2)]$$

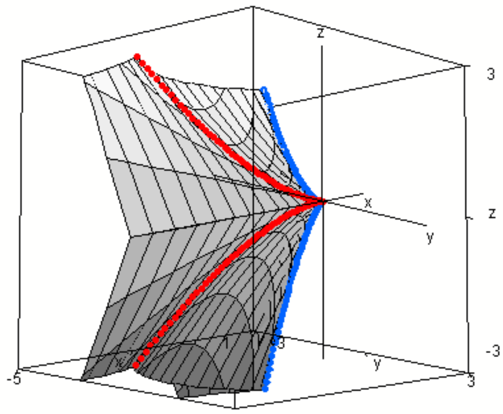
$$[-6\cdot u^2,\ -2\cdot\sqrt{3}\cdot u^2,\ -8\cdot u^3]$$

$$[-3\cdot u^2 - (-u\cdot\sqrt{3})^2,\ -2\cdot u\cdot (-u\cdot\sqrt{3}),\ -2\cdot u\cdot (u^2 + (-u\cdot\sqrt{3})^2)]$$

$$[-6\cdot u^2,\ 2\cdot\sqrt{3}\cdot u^2,\ -8\cdot u^3]$$

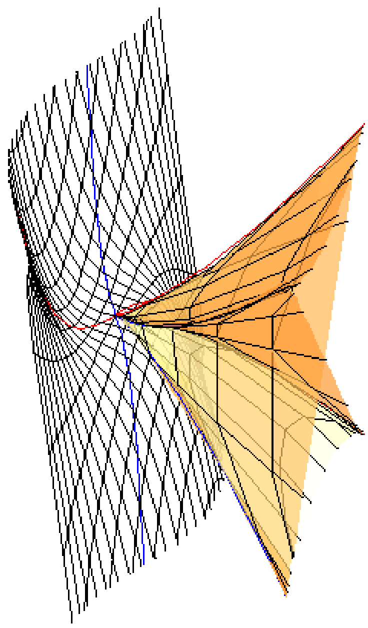
$$\text{VECTOR}([[-6\cdot u^2,\ -2\cdot\sqrt{3}\cdot u^2,\ -8\cdot u^3]],\ u,\ -1,\ 1,\ 0.01)$$

$$\text{VECTOR}([[-6\cdot u^2,\ 2\cdot\sqrt{3}\cdot u^2,\ -8\cdot u^3]],\ u,\ -1,\ 1,\ 0.01)$$



Same procedure with TI-NspireCAS:

$surf:=\{t,u,t^3+t\cdot u^2\}$	$\{t,u,t^3+t\cdot u^2\}$
$gauss\_curv(surf)$	$\dfrac{4\cdot(3\cdot t^2-u^2)}{(9\cdot t^4+10\cdot t^2\cdot u^2+u^4+1)^2}$
$surf\vert u=t\cdot\sqrt{3}$	$\{t,\sqrt{3}\cdot t,4\cdot t^3\}$
$surf\vert u=-t\cdot\sqrt{3}$	$\{t,-\sqrt{3}\cdot t,4\cdot t^3\}$
$dualsurf:=dualp(surf)$	$\{-3\cdot t^2-u^2,-2\cdot t\cdot u,-2\cdot t^3-2\cdot t\cdot u^2\}$
$dualsurf\vert u=t\cdot\sqrt{3}$	$\{-6\cdot t^2,-2\cdot\sqrt{3}\cdot t^2,-8\cdot t^3\}$
$dualsurf\vert u=-t\cdot\sqrt{3}$	$\{-6\cdot t^2,2\cdot\sqrt{3}\cdot t^2,-8\cdot t^3\}$



As expected, we receive two cuspidal edges. Unfortunately, we cannot plot the parabolic lines and their respective maps, the cuspidal edges as thick lines as we can do with DERIVE. I worked with parameters t and u, because these two parameters are requested for 3D-plots with TI-Npsire.

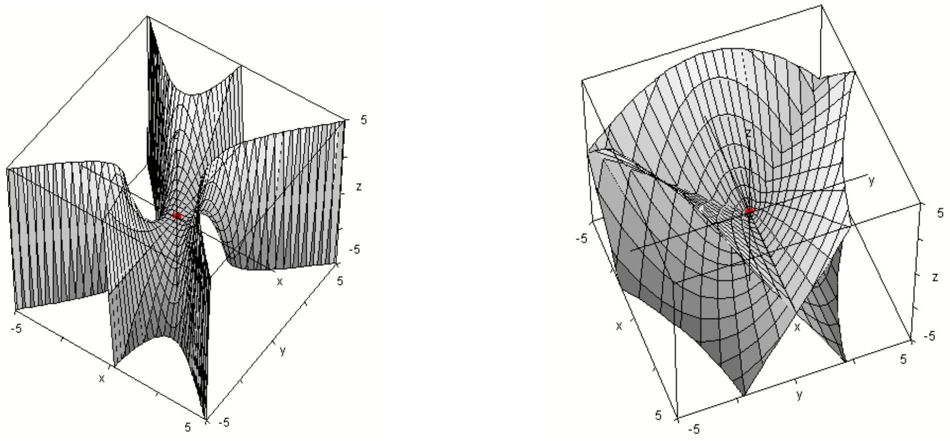
Morris writes:

If S has a cusp of Gauss at x, then the dual has a swallowtail point at x*.

Next surface has only one – isolated – parabolic point which results a swallow tail point on the dual:

$$f3 = x^3 - x\cdot y^2$$

$$\text{gauss\_curv}([u,\ v,\ u^3 - u\cdot v^2]) = -\frac{4\cdot(3\cdot u^2 + v^2)}{(9\cdot u^4 - 2\cdot u^2\cdot v^2 + v^4 + 1)^2}$$



The swallow tail point (right)

A real swallow tail butterfly in our garden.

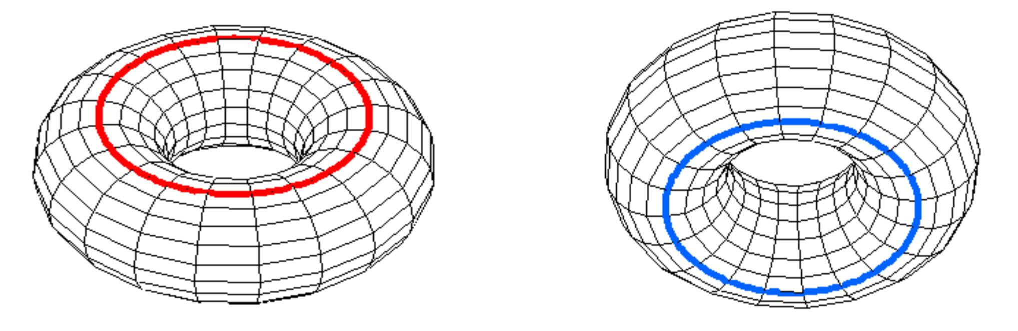
Once more the Torus – and its parabolic lines.

$$\text{gauss\_curv(torus)} = \frac{\text{COS(u)}}{\text{COS(u)} + 2}$$

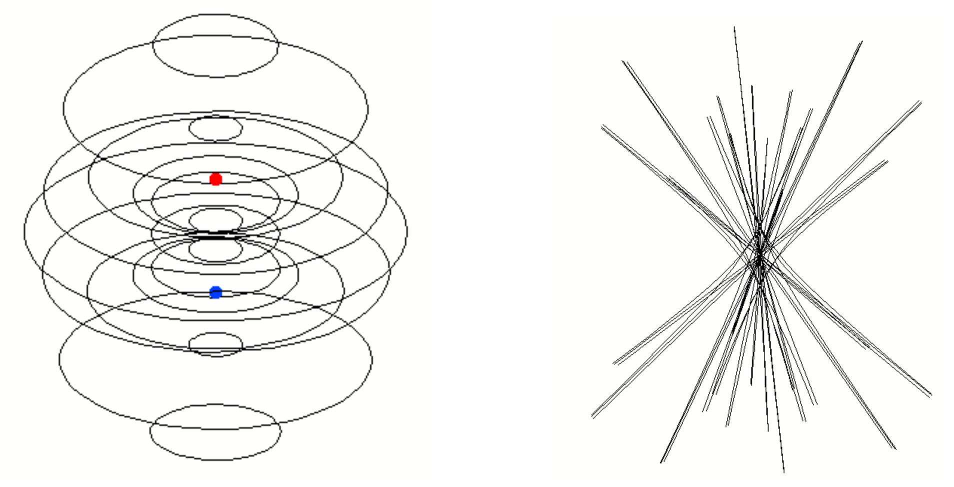
$$\text{SOLVE(COS(u) = 0, u)} = \left(u = \frac{3\cdot\pi}{2} \vee u = -\frac{\pi}{2} \vee u = \frac{\pi}{2}\right)$$

$$\text{SUBST}\left(\text{torus, u, } \frac{\pi}{2}\right) = [2\cdot\text{COS(v)}, 2\cdot\text{SIN(v)}, 1]$$

$$\text{SUBST}\left(\text{torus, u, } \frac{3\cdot\pi}{2}\right) = [2\cdot\text{COS(v)}, 2\cdot\text{SIN(v)}, -1]$$

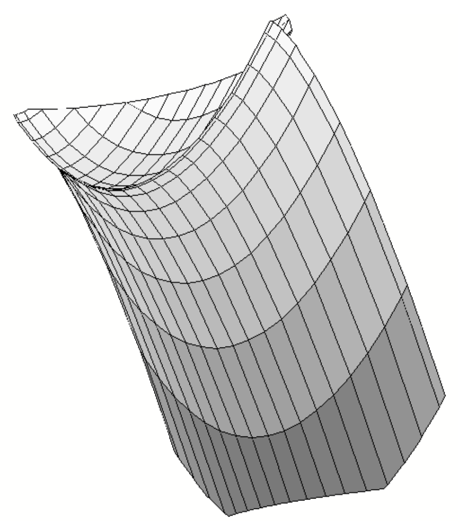


The parameter curves on the dual of the torus. The parabolic lines degenerate to one point each.



Finally, I had one question (for me, for students, for colleagues, …): How to find directly the cuspidal edge of a surface if there is one? Take $f(u,v)=\left(3u,2v^2,-u^2+2v^3+2\right)$.

The answer is easy:
Cusps are there where the Gauss curvature is not defined. So set the denominator of the GC = 0.

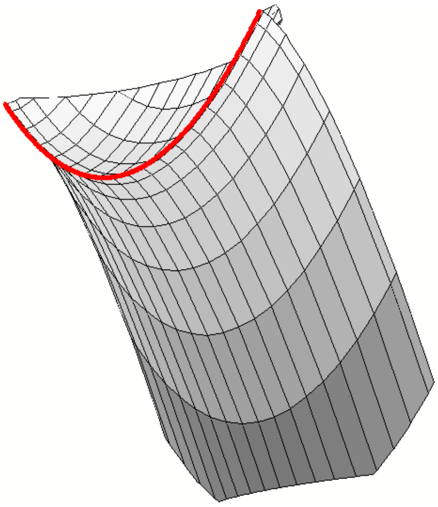


$$\text{gauss\_curv}([3\cdot u,\ -2\cdot v^2,\ -u^2+2\cdot v^3+2]) = -\frac{108}{v\cdot(16\cdot u^2+9\cdot(9\cdot v^2+4))^2}$$

$$\text{SOLUTIONS}(v\cdot(16\cdot u^2+9\cdot(9\cdot v^2+4))^2,\ v) = \left[0,\ \frac{2\cdot i\cdot\sqrt{(4\cdot u^2+9)}}{9},\ -\frac{2\cdot i\cdot\sqrt{(4\cdot u^2+9)}}{9}\right]$$

$$\text{VECTOR}([[3\cdot u,\ 0,\ -u^2+2]],\ u,\ -5,\ 5,\ 0.01)$$

We have only one real solution v = 0.



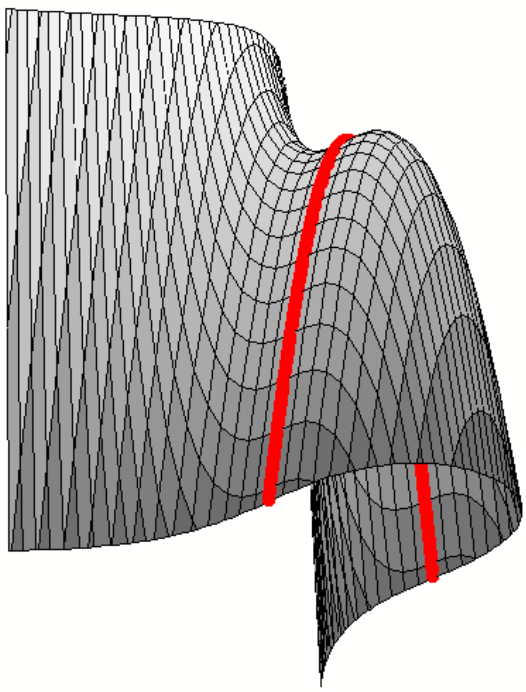
$$\text{dualp}([3\cdot u,\ -2\cdot v^2,\ u^2+2\cdot v^3+2])$$

$$\left[-\frac{2\cdot u}{3},\ \frac{3\cdot v}{2},\ -u^2-v^3+2\right]$$

$$\text{gauss\_curv}\left[-\frac{2\cdot u}{3},\ \frac{3\cdot v}{2},\ -u^2-v^3+2\right]$$

$$\frac{12\cdot v}{(9\cdot u^2+4\cdot v^4+1)^2}$$

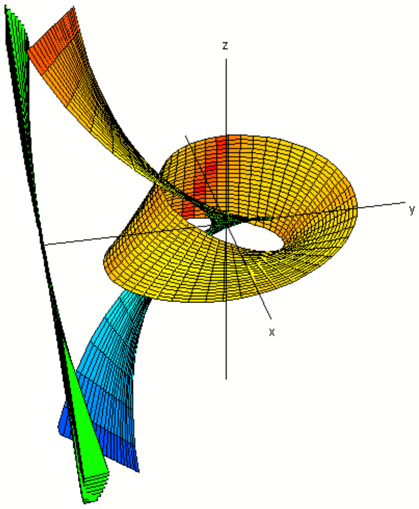
The dual has – as expected – for v = 0 a parabolic line.



I will finish with Dr Morris‘ favorite surface: the double twisted Moebius band and one possible form of its dual:

$$mb := \left[COS(u)\cdot\left(1 + \frac{v}{2}\cdot COS\left(\frac{u}{2}\right)\right), SIN(u)\cdot\left(1 + \frac{v}{2}\cdot COS\left(\frac{u}{2}\right)\right), \frac{v}{2}\cdot SIN\left(\frac{u}{2}\right)\right]$$

dualp(mb)



My Conclusion:

The two short paragraphs in a journal from more than 20 years ago gave reason for an extended investigation. Every question answered offered new questions. I am quite sure that all of them can be treated in depth. I remained only on the surface. To cite David Halprin (Australia): “I opened a can of worms”.

I must admit that Morris’ extended paper – also from 2002 – opened further interesting insights for me. I started admiring fascinating pictures but then including the Gauss curvature it got more mathematical contents.

In my opinion, many parts of this talk could be treated in Upper Secondary classes. They might inspire students and Teachers for further investigations. Morris’ paper offers more ways to visualise surfaces and their duals which are well worth to be tried to realize with our tools.

It was funny for me that I stepwise remembered many facts which I have heard long ago. Computer Algebra and graphing tools helped immensely.

References:

www.singsurf.org/papers/dual/dual.pdf (can be obtained from the author of this paper)

Alfred Gray, Modern Differential Geometry

R. Haas & J. Böhm, Cello Tangents, DERIVE Newsletter 125

DP Graph, http://www.dpgraph.com/

https://github.com/RichardMorris/SingSurf

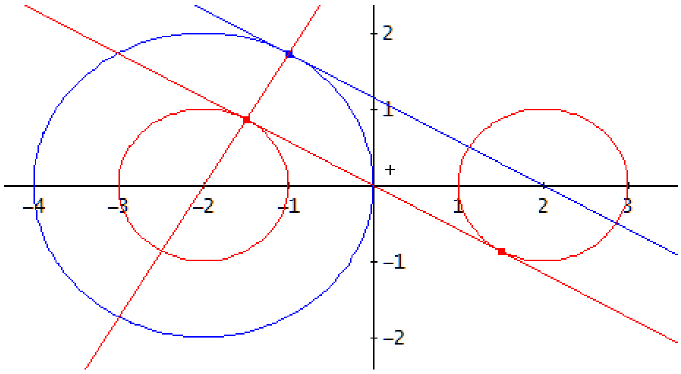
Roman Hašek, Exploration of dual curves using a dynamic geometry, and computer algebra system, ACA 2018

Appendix

Remember the double points of the dual curves of the quartic and quantic above. What about self-intersection of dual surfaces?

Dr. Morris writes in his extended paper: "If there is a plane tangent to S at x and y then S* has a self-intersection at x* = y*."

As it is not so easy to calculate bi-tangent planes, I took again the torus for demonstrating this property of dual surfaces. It gives another opportunity for a students' project.



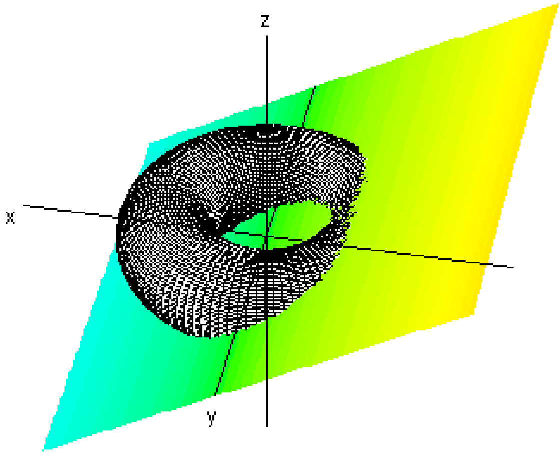
We find the double tangent of the cross section of the torus following the well-known construction of the common tangent on two circles.

$$\left[-\frac{3}{2}, \frac{\sqrt{3}}{2}\right]$$

$$y - \frac{\sqrt{3}}{2} = -\frac{1}{\sqrt{3}}\cdot\left(x + \frac{3}{2}\right)$$

$$pl0 := \left[-\frac{3}{2}, 0, \frac{\sqrt{3}}{2}\right] + u\cdot[0, 1, 0] + v\cdot\left[1, 0, -\frac{1}{\sqrt{3}}\right]$$

Transferring the result in 3D space gives one bi-tangent plane pl10.



Back in the 3D-world, we have to find out the respective u and v values for the point - in order to apply my functions -, which are $u = 2\pi/3$ and $v = \pi$.

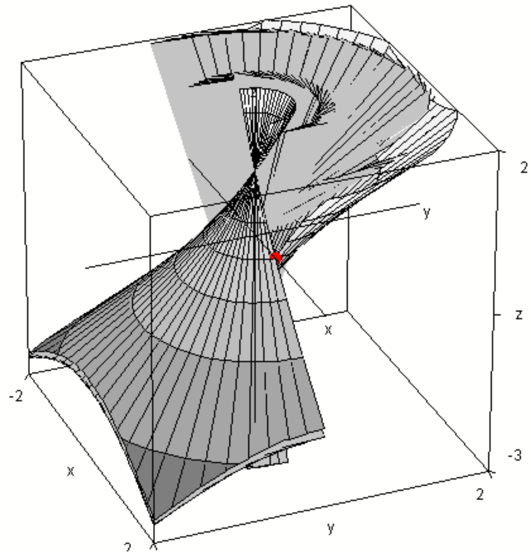
Then we calculate the tangent plane in this point and its corresponding point on the dual surface, which should turn out to be a point of the self-intersecting curve!

$$-\frac{3\cdot x}{4}-\frac{3\cdot\sqrt{3}\cdot z}{4}=0$$

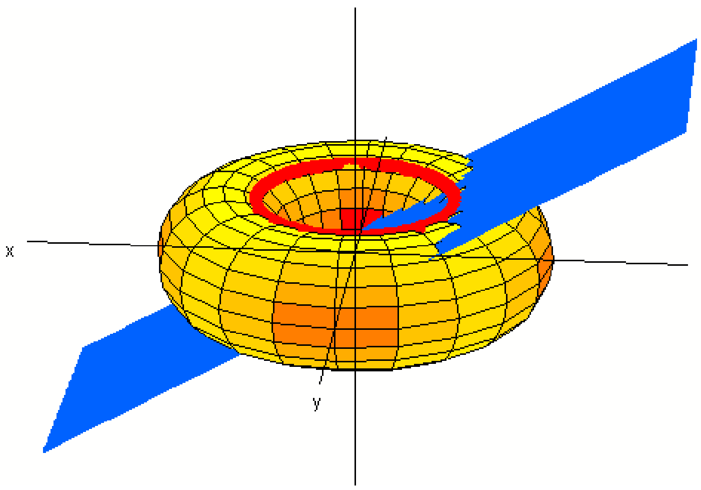
$$-\frac{1}{\sqrt{3}}\cdot x$$

$$\mathrm{dualf}\left(-\frac{1}{\sqrt{3}}\cdot x\right)$$

$$\left[\frac{\sqrt{3}}{3},\ 0,\ 0\right]$$



The plot shows only a part of the dual surface to make this point (red) better visible.



The torus together with the bi-tangent plane and the set of all points on the torus with points giving double tangential planes. The mapping of the red circle should result in the self-intersection of the dual surface.

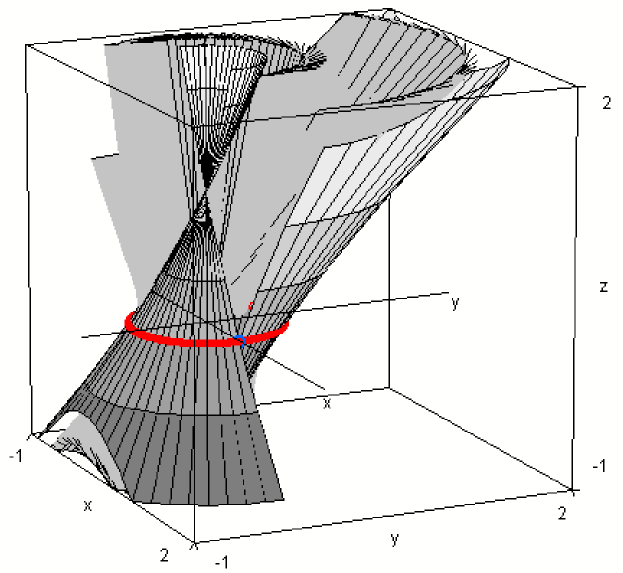
$$\left[\frac{\mathrm{COS}(u)\cdot\mathrm{COS}(v)}{\mathrm{SIN}(u)},\ \frac{\mathrm{COS}(u)\cdot\mathrm{SIN}(v)}{\mathrm{SIN}(u)},\ \frac{\mathrm{COS}(u)^3+2\cdot\mathrm{COS}(u)^2+\mathrm{COS}(u)\cdot(\mathrm{SIN}(u)^2+4)+2}{\mathrm{SIN}(u)\cdot(\mathrm{COS}(u)+2)}\right]$$

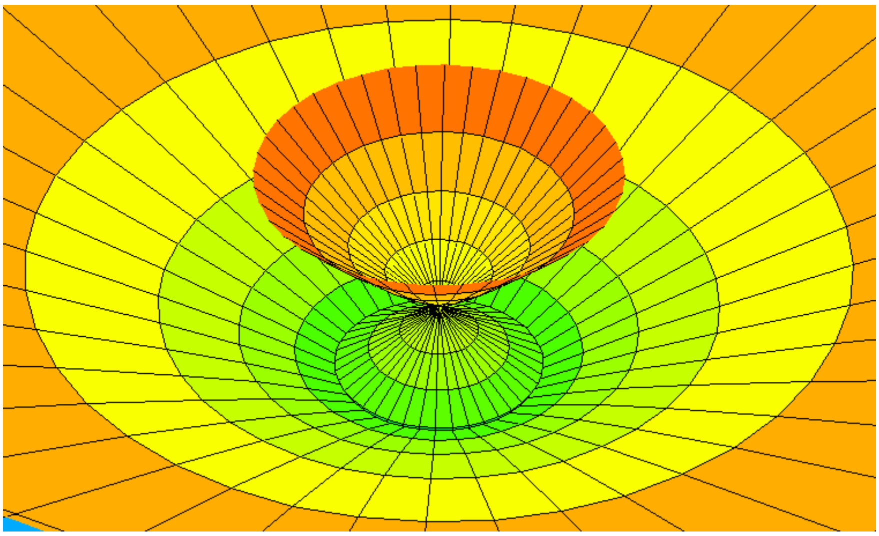
$$\left[\frac{\mathrm{COS}\left(\frac{2\cdot\pi}{3}\right)\cdot\mathrm{COS}(v)}{\mathrm{SIN}\left(\frac{2\cdot\pi}{3}\right)},\ \frac{\mathrm{COS}\left(\frac{2\cdot\pi}{3}\right)\cdot\mathrm{SIN}(v)}{\mathrm{SIN}\left(\frac{2\cdot\pi}{3}\right)},\ \frac{\mathrm{COS}\left(\frac{2\cdot\pi}{3}\right)^3+2\cdot\mathrm{COS}\left(\frac{2\cdot\pi}{3}\right)^2+\mathrm{COS}\left(\frac{2\cdot\pi}{3}\right)\cdot\left(\mathrm{SIN}\left(\frac{2\cdot\pi}{3}\right)^2+4\right)+2}{\mathrm{SIN}\left(\frac{2\cdot\pi}{3}\right)\cdot\left(\mathrm{COS}\left(\frac{2\cdot\pi}{3}\right)+2\right)}\right]$$

$$\left[-\frac{\sqrt{3}\cdot\mathrm{COS}(v)}{3},\ -\frac{\sqrt{3}\cdot\mathrm{SIN}(v)}{3},\ 0\right]$$

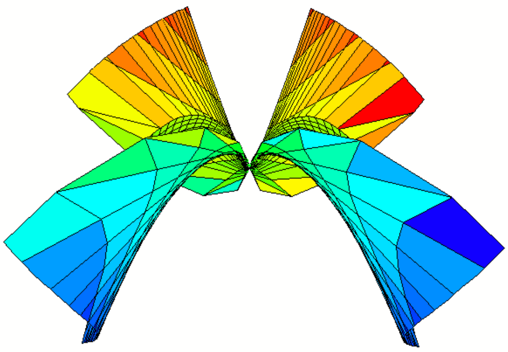
$$\mathrm{VECTOR}\left(\left[\left[-\frac{\sqrt{3}\cdot\mathrm{COS}(v)}{3},\ -\frac{\sqrt{3}\cdot\mathrm{SIN}(v)}{3},\ 0\right]\right],\ v,\ 0,\ 2\cdot\pi,\ 0.01\right)$$

Parameter curve with u = 2π/3:





DPGraph-view from above into the interior of the dual. We can see the self-intersecting curve (horizontal circle).



Thanks for your attention

Some weeks ago, I received a paper from my friend and colleague from our very first days as young teachers. Helmut has been an advocate and propagator for technology supported math teaching from the very beginning. He was the driving force that Austria was the first country with a general license for DERIVE for all Secondary schools. He co-organized conferences starting with the Summer school in Krems 1991. Other conferences followed (Vienna, Krems, ...).

He gave numerous talks in Europe and USA demonstrating new didactics in math education – with DERIVE and later with TI-Nspire-CAS.

Two weeks ago, Helmut celebrated his 80th birthday. It is an honor for me that at this occasion he gave permission to translate his paper showing how AI treats tasks from secondary school mathematics.

The DERIVE- and TI-Nspire community wishes the best for his birthday and we hope to receive new papers and lectures in the future.

Josef

Math Lessons with/despite AI? (Part 1)

Helmut Heugl, Austria

It is undeniable that generative AI technologies are exceeding our expectations and have the potential to change the educational landscape sustainably. In view of these rapid technological advances, both students and teachers as well are facing new challenges and questions regarding the division of labor between humans and machines, learning objectives and forms of assessment of academic performance. [Helfrich-Schkarbanenko, 2023]

There have always been tools that have influenced and changed mathematics. Now there are the powerful tools of AI systems, which are not just computing or graphics tools, but problem-solving media and communication media. And many see the opportunities that arise from this as a threat to mathematics as a subject. However, we should also see the opportunities that arise from this.

The discussion about the role and influence of AI does not surprise me. I experienced exactly the same discussion in the early 1990s when we started using CAS in the classroom.

There are two poles [Buchberger, 2002]:

Populist view:

- by recent advances, (math) teaching, can be automated;
- thus, let's automate it and dismiss the teachers!

Purist view:

- (math) teaching is an art;
- thus, let’s protect it against automation!

Bruno Buchberger was my most important teacher back then and still is today. He is not only a great mathematician, his theory of Gröbner bases is probably the most internationally cited Austrian research result in the field of computer mathematics. Through his research in the field of computer-mathematics, he has also developed a high level of didactic competence by linking the question "how does mathematics develop in the computer?" with the question "how does mathematics develop in the human brain?".

In my article, I would first like to discuss two questions: "Why ChatGPT?" and, above all, the question "Why math?" Only then can we think about possible consequences for the teaching and learning of mathematics.

1. What can ChatGPT do in the field of mathematics?

Of course, the paper does not deal in detail with the question: "How does ChatGPT perform?", but it is important for me that "artificial intelligence" is also a product of "human intelligence", or more precisely "mathematical intelligence".

"*I personally therefore like to expand the abbreviation "AI" to "algorithmic intelligence", i.e. human intelligence for finding programs at ever higher levels!*" [Buchberger, 2023_3].

ChatGPT is a text-based tool with impressive performance in various linguistic tasks such as text generation, as it has been pre-trained on extensive data. It is important, that ChatGPT is not optimized for correctness, but for probability [Spannagel, 2023]. Therefore, ChatGPT also makes mistakes and still has difficulties with some logical tasks that require logical thinking and multiple steps to solve. This must be taken into account when using it in mathematics. However, the new version 4.0 has expert systems working in the background that do not make mistakes, such as the "Wolfram Alpha Plugin".

Some examples of the performance, but also of the weaknesses of ChatGPT:

a) **A project entitled „Is ChatGPT Smarter than Master’s Applicants? “**
[Buchberger, 2023_1]

As part of the selection process for a special computer science scholarship program from Upper Austrian companies, I ask the applicants a few simple technical questions about programming etc. in a Zoom meeting.

I ask the same questions the ChatGPT dialog system, [ChatGPT].

The result

I was surprised by the result: almost all of ChatGPT's answers were completely correct and well explained. ChatGPT's explanations in the dialogs to clarify some of the critical points in the responses were also surprisingly clear and to the point.

In comparison: I tried out the same questions in the personal Zoom interviews with around 30 applicants from five countries. Only the top three candidates (those who had a GPA of 1.0, i.e. the highest possible GPA, in their Bachelor's degree program) performed about equally well in the interview. All others performed (far) worse than ChatGPT. And of course, all of ChatGPT's answers came within 1 to 10 seconds, while most of the human applicants' answers required quite lengthy and difficult dialogs.

I am particularly impressed with ChatGPT's ability to extract meaningful and well-structured programs from problem specifications in natural language. The performance of ChatGPT was quite impressive concerning formalization and propositional logic. In examples where predicate logic is required, ChatGPT's responses are not yet perfect.

b) The transcript of an interview of Bruno Buchberger with ChatGPT

BB

Suppose we have 10 red cards with the numbers 1, ..., 10; 5 blue cards with the numbers 1, ..., 5; 7 green cards with the numbers 1, ..., 7. What is the probability of finding a blue card with the number 2 or 3?

ChatGPT:

To calculate the probability of drawing a blue card with the number 2 or 3, we need the total number of possible outcomes (i.e. the total number of cards). There are 10 + 5 + 7 = 22 cards and two of them (cards with numbers 2 or 3 are successful results. The probability to draw a blue card with number 2 or 3 is therefore 2/22 = 1/11.

I hope this is helpful! Let me know if you have any further questions.

BB

Do you know Bayes' theorem? How were you able to apply Bayes' theorem in the previous task?

ChatGPT:

Yes, Bayes' theorem is a mathematical formula for calculating the probability of an event based on prior knowledge of conditions that might be related to the event. It is often used in machine learning and statistics to make predictions or derive the probability of an event based on data.

$P(A|B) = P(B|A) * P(A) / P(B)$

Whereby:

- *P(A|B) is the probability that event A will occur if event B has occurred,*
- *P(B|A) is the probability that event B will occur if event A has occurred,*
- *P(A) is the probability of the occurrence of event A,*
- *P(B) is the probability of the occurrence of event A.*

To apply Bayes' theorem for calculating the probability of drawing a blue card with the number 2 or 3, we need to define the events A and B and calculate the probabilities for each event. For example, we could define event A as "drawing a blue card with the number 2 or 3" and event B as "drawing a blue card". We can then use the information in the question to calculate the probabilities for each event as follows:

- *P(A) = 2/22 = 1/11 (as there are 2 successful outcomes from a total of 22 cards)*
- *P(B) = 5/22 = 5/22 (as there are 5 blue cards out of a total of 22 cards)*
- *P(B|A) = 1 (because all successful outcomes are blue cards)*

If you insert these values into Bayes' theorem, you can calculate the probability of drawing a blue card with the number 2 or 3, assuming we have drawn a blue card:

*P(A|B) = P(B|A) * P(A) / P(B) = (1) * (1/11) / (5/22) = 2/5*

Therefore, the probability of drawing a blue card with the number 2 or 3 having drawn a blue card is 2/5.

I hope this is helpful! Let me know if you have any further questions.

Comment: To show that ChatGPT knows the Bayes' theorem the KI tool formulates a new task with the condition *"having drawn a blue card"* and solves this new task correctly

c) **Problem 01: Proof of Pythagoras' theorem with discussion**
[ChatGPT 4, 2023] (Original German version)

You
Beweise den Lehrsatz von Pythagoras

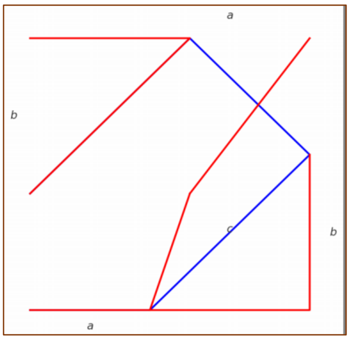
ChatGPT
Um den Lehrsatz von Pythagoras zu beweisen, gibt es viele verschiedene Methoden. Eine der bekanntesten ist der geometrische Beweis durch Umordnung. Hier wird die Gültigkeit der Formel $a^2 + b^2 = c^2$ für ein rechtwinkliges Dreieck gezeigt, wobei a und b die Längen der Katheten und c die Länge der Hypotenuse darstellen.

Der Beweis geht folgendermaßen:

1. Zeichne ein Quadrat mit der Seitenlänge $a + b$. Dieses Quadrat hat eine Gesamtfläche von $(a + b)^2$.
2. Zeichne innerhalb dieses großen Quadrats vier rechtwinklige Dreiecke, deren Katheten die Längen a und b haben. Platziere die Dreiecke so, dass ihre Hypotenusen ein kleineres Quadrat in der Mitte des großen Quadrats bilden.
3. Die Fläche des kleineren Quadrats in der Mitte entspricht der Fläche einer Hypotenuse im Quadrat, also c^2.
4. Die verbleibende Fläche, die von den vier Dreiecken umgeben wird, entspricht viermal der Fläche eines der rechtwinkligen Dreiecke. Die Gesamtfläche der vier Dreiecke ist $4 \times \frac{1}{2}ab = 2ab$.
5. Somit kann die Gesamtfläche des großen Quadrats auf zwei Arten berechnet werden:
   - Als Fläche des gesamten großen Quadrats: $(a + b)^2$
   - Als Summe der Flächen des kleineren Quadrats und der vier Dreiecke: $c^2 + 2ab$
6. Da beide Berechnungen die Fläche desselben Quadrats beschreiben, müssen sie gleich sein. Das führt zu $(a + b)^2 = c^2 + 2ab$.
7. Nach der Ausmultiplizierung der linken Seite erhält man $a^2 + 2ab + b^2 = c^2 + 2ab$.
8. Subtrahiere $2ab$ von beiden Seiten der Gleichung: $a^2 + b^2 = c^2$.

You

Kannst du den Beweis mit Hilfe von TikZ graphisch darstellen?



Im obigen Diagramm wird der geometrische Beweis des Satzes von Pythagoras dargestellt:

- Das große Quadrat, mit der Seitenlänge $a + b$, ist in blau gezeichnet.
- Vier rote rechtwinklige Dreiecke sind innerhalb des großen Quadrats platziert. Jedes Dreieck hat die Katheten a und b und die Hypotenuse c.
- Das kleinere Quadrat, gebildet durch die Hypotenusen der roten Dreiecke, repräsentiert c^2.

Dieses Diagramm visualisiert, wie die Gesamtfläche des großen Quadrats sowohl durch $(a + b)^2$ als auch durch die Kombination der Fläche des inneren Quadrats c^2 und der Flächen der vier Dreiecke (zusammen $2ab$) ausgedrückt werden kann, was den Lehrsatz von Pythagoras beweist.

Kommentar: siehe in der englischen Übersetzung weiter unten.

Now in English (translated by Josef by means of AI – DeepL)

You

Prove Pythagoras' theorem

ChatGPT

There are many different methods for proving Pythagoras' theorem. One of the best known is the geometric proof by rearrangement. Here is the validity of the formula $a^2 + b^2 = c^2$ for a right-angled triangle shown, where a and b are the lengths of the legs and c is the length of the hypotenuse.

The proof is as follows

1. Draw a square with side length $a + b$. This square has a total area of $(a + b)2$.
2. Within this large square, draw four right triangles whose legs have the lengths a and b. Place these triangles so that their hypotenuses form a smaller square in the middle of the large square.
3. The area of the smaller square in the center corresponds to the area of a hypotenuse squared, i.e.c^2.
4. The remaining area surrounded by the four triangles is four times the area of one of the right triangles. The total area of the four triangles is $4 \times ½ ab = 2ab$.

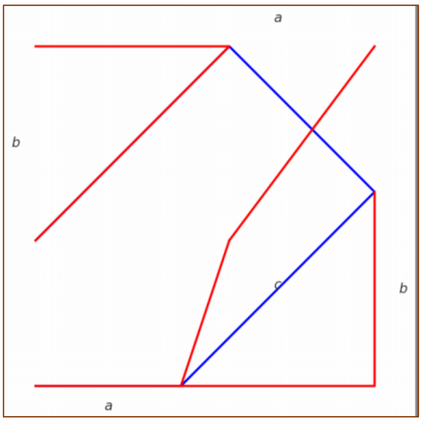
5. This means that the total area of the large square can be calculated in two ways:
   - as the area of the entire large square $(a + b)^2$
   - as the sum of the areas of the small square and the four triangles $c^2 + 2ab$
6. Since both calculations describe the area of the same square, they must be the same. This leads to $(a + b)^2 = c^2 + 2ab$.
7. After expanding the left-hand side, you get $a^2 + 2ab + b^2 = c^2 + 2ab$.
8. Subtract $2ab$ from both sides of the equation: $a^2 + b^2 = c^2$.

   This is Pythagoras' theorem. This elegant geometric method shows, that in every right triangle the sum of the squares of the legs equals the square of the hypotenuse.

You

Can you present the proof graphically using TikZ?

ChatGPT



The diagram above shows the geometric proof of Pythagoras' theorem.

- The large square with side length $a + b$ is drawn in blue.
- Four red right-triangles are placed within the large square. Each triangle has the legs a and b and the hypotenuse c.
- The smaller square formed by the hypotenuses of the red triangles represents c^2.

This diagram visualizes how the total area of the large square can be expressed both by $(a + b)^2$ and by combining the area of the inner square c^2 and the areas of the four triangles (together $2ab$), which proves Pythagoras' theorem.

Comment:

Thanks to plugins, ChatGPT version 4.0 has become mathematically more powerful compared to version 3.5. The Wolfram plugin makes ChatGPT smarter by leveraging powerful computation, precise mathematics, curated knowledge, real-time data and visualization from Wolfram|Alpha and Wolfram Language.

However, the graphical representation shows that there are still weaknesses in graphical representations. It is not only the quality of the plot, the assignment of the colors to the squares is incorrect.

Josef: Helmut carried out an interesting experiment. He wanted to find out, if ChatGPT acts different if the problems are given in different languages. So, he posed problems 01 and 03 once more, but in English. I attach both protocols to the appropriate tasks.

The protocol of Helmut's ChatGPT session in English follows:

Task example: Proof of the Pythagorean theorem

You
I am looking for a proof of the Pythagorean theorem

ChatGPT
The Pythagorean theorem is a fundamental principle in geometry, discovered by the ancient Greek mathematician Pythagoras. It relates to right-angled triangles and can be stated as follows:

In a right-angled triangle, the square of the length of the hypotenuse (the side opposite the right angle) is equal to the sum of the squares of the lengths of the other two sides.

This can be expressed algebraically as:

$$c^2 = a^2 + b^2$$

where c represents the length of the hypotenuse, and a and b represent the lengths of the triangle's other two sides.

There are many proofs of the Pythagorean theorem, each using different approaches.

One of the most famous and simplest is by rearrangement, often attributed to Bhaskara, an Indian mathematician. This proof involves creating a large square and rearranging the components to show that the area of the square formed by the hypotenuse is equal to the sum of the areas of the squares formed by the other two sides.

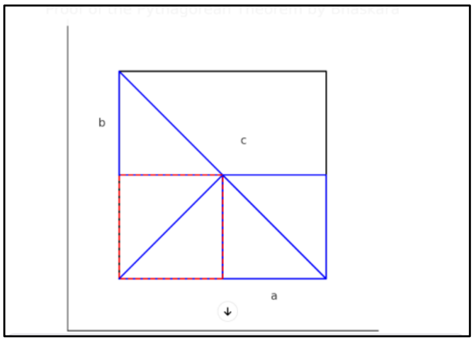
Would you like a detailed explanation or visual representation of one of these proofs?

You

yes please both, the detailed explanation and the visual representation

ChatGPT

Proof of the Pythagorean Theorem by Bhaskara



Here's a visual representation of one of the proofs of the Pythagorean theorem, specifically Bhaskara's proof:

- In the figure, we have a large square with a side length of $a + b$.
- Inside this large square, four right-angled triangles are placed at the corners. Each of these triangles has sides of length a and b, and a hypotenuse of length c.
- The remaining central area forms a smaller square with sides of length c, the hypotenuse of the triangles.

The proof works by comparing the areas of the large square and the sum of the areas of the four triangles and the smaller square.

1. **Area of the large square**: The area of the outer square is $(a + b)^2$.
2. **Area of the four triangles**: Each triangle has an area of $\frac{1}{2}ab$, so four triangles have a total area of $4 \times \frac{1}{2}ab = 2ab$.
3. **Area of the smaller square**: The area of the inner square is c^2.

Putting these together:

- The total area of the large square can also be calculated as the sum of the areas of the four triangles and the smaller square, which gives us $2ab + c^2$.
- Therefore, we have $(a + b)^2 = 2ab + c^2$.

Expanding $(a + b)^2$, we get $a^2 + 2ab + b^2$. This simplifies to:

$$a^2 + b^2 = c^2$$

This is the statement of the Pythagorean theorem, proving that in a right-angled triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides. [>-]

Comment: The proof is correct, the drawing is useless

2. Why math? How does math develop in science and in the learner?

Here are some answers:

(1) *My favorite definition of mathematics: "Mathematics is the technique of problem solving by reasoning, developed over centuries". (Buchberger)*

(2) *"Mathematics as a possible mode of encountering the world, as a specific way of seeing or modeling the world around us"*. (*Austrian end examination: Excerpt from the educational theory orientation*) [IDM, 2009].

(3) *The most important yield of the subject of mathematics is the thinking technology that is necessary to do mathematics (Buchberger). This thinking technology is an essential contribution to the general ability to solve problems*.

(4) *"Teaching without teachers?"* [Buchberger, 2003]
(*Lecture Buchberger at VISITME Conference, Vienna 2002*)

Mathematics and Automation

- *The goal of mathematics is automation.*
- *The goal of mathematics is to trivialize mathematics.*
- *The goal of mathematics is explanation (= making things of high dimension plane = making complicated things simple).*
- *Mathematics is didactics.*
- *The process of trivialization is completely non-trivial.*
- *Think nontrivially once and act trivially infinitely often.*

Global Math Teaching

- *Teaching will never be obsolete in this global magma.*
- *Rather, teaching will be more and more important and challenging.*
- *Everyone should be a teacher and nobody should be a teacher only.*
- *Math teaching is a paradigm: If we manage math teaching, we master teaching in (all, some, many?) other areas.*

(5) *"The Future of Mathematics – a personal view and comments on math education"*
(*Lecture Buchberger at TIME Conference, Krems 2014*)

Conclusions 2014

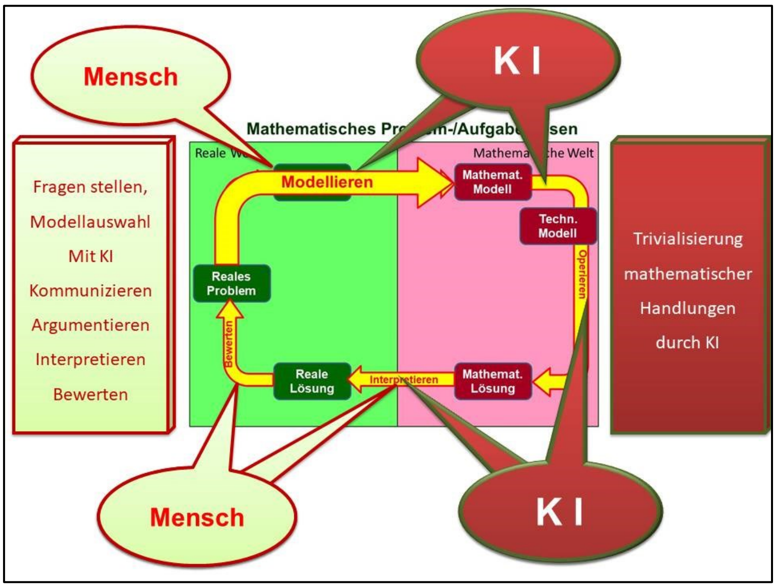
- *Using technology in math education? Math = technology!*
- *Algorithms (on higher and higher levels) are goal and means of math and math education*
- *Teaching math using "technology": Give students a chance to repeat evolution! (The White-Box / Black-Box Principle, BB 1989).*
- *The higher the "technology", the more important the personal teacher!*

If one identifies with the 5 answers, there are consequences regarding the role of AI in mathematics teaching.

3. What contribution can AI make to the mathematic teaching as characterized in paragraph 2?

3.1 Conclusions on response (1)

The problem-solving cycle:



Despite AI, there is still plenty for learners to do:

The model decision is not subject to the two-valued logic of true/false. Based on the mathematical tools available in the learning process, it is necessary to weigh up the possible mathematical models for the real problem.

Left Box:
Pose questions, Model selection Communicate with AI Argue Interpreting Evaluate

Right Box:
Trivialization of mathematical actions by AI

[Heugl 2014]

In an investigation by Buchberger, a problem was posed five times, each time ChatGPT suggested a different algorithm, but always correctly.
[Buchberger, 2023_1]

It is fascinating how ChatGPT develops a mathematical model from a voice input of a problem:

Problem 02: Home loan and savings credit (Original German version)

You
Ich benötige für einen Hausbau einen Kredit von € 140000 und möchte in 30 Jahresraten tilgen. Der Zinssatz ist derzeit 3,5%. Wie hoch ist die Jahresrate?

ChatGPT
Die Jahresrate für einen Kredit von 140.000 € mit einem Zinssatz von 3,5% und einer Laufzeit von 30 Jahren beträgt etwa 7.612 €. [>-]

You

I need a loan of € 140000€ to build a house and would like to pay it off in 30 annual installments. The interest rate is currently 3.5%. How high is the annual installment?

ChatGPT

The annual installment for a loan of 140000 € with an interest rate of 3.5% and a term of 30 years is €7.612 €.

An interesting task for trivialization using the AI tool. But not only the result is important for the learning process, the path to trivialization is more important. Now you can ask ChatGPT further questions to explain the path.

You

Wie kommt man zu diesem Ergebnis?

ChatGPT

Um die Jahresrate eines Kredits zu berechnen, der in regelmäßigen Raten über eine feste Laufzeit zurückgezahlt wird, verwenden wir die Formel für ein Annuitätendarlehen. Die Formel lautet:

$$R = P \times \frac{i \times (1+i)^n}{(1+i)^n - 1}$$

Dabei steht:

- R für die Jahresrate (die regelmäßige Zahlung, die Sie leisten müssen).
- P für den Kreditbetrag (in Ihrem Fall 140.000 €).
- i für den jährlichen Zinssatz, ausgedrückt als Dezimalzahl (also 3,5% oder 0,035).
- n für die Anzahl der Jahre, über die der Kredit zurückgezahlt wird (hier 30 Jahre).

Die Formel berechnet, wie viel Sie jedes Jahr zahlen müssen, um sowohl den Kreditbetrag als auch die darauf anfallenden Zinsen über die Laufzeit des Kredits vollständig zu begleichen.

In Ihrem Fall setzen wir die Werte in die Formel ein:

$$R = 140.000 \times \frac{0{,}035 \times (1+0{,}035)^{30}}{(1+0{,}035)^{30} - 1}$$

Durch diese Berechnung ermitteln wir, dass die jährliche Zahlung, die Sie leisten müssen, etwa 7.612 € beträgt.

The English translation follows on the next page

You

How do you arrive at this result?

ChatGPT

To calculate the annual installment of a loan that is repaid in regular installments over a fixed term, we use the formula for an annuity loan. The formula is as follows:

$$R = P \times \frac{i \times (1+i)^n}{(1+i)^n - 1}$$

Where:

- R for the annual installment (the regular payment you have to make).
- P for the loan amount (in your case 140.000 €).
- i for the annual interest rate, expressed as decimal number (hence 5.5% or 0.0035).
- n for the number of years over which the loan amount is repaid (here 30 years).

The formula calculates how much you have to pay each year to settle both the loan amount and the interest in full over the term of the loan.

In your case we insert the values into the formula:

$$R = 140.000 \times \frac{0{,}035 \times (1+0.035)^{30}}{(1+0.035)^{30} - 1}$$

Using this calculation, we obtain that the annual payment you will have to make is approximately €7.612 €.

However, you cannot always rely on the model selected by ChatGPT. For this check, however, you need your own mathematical skills on the respective topic.

Problem 03: Flu epidemic

(Original German version)

HH **You**

Aufgabe: „Grippeepidemie"

Betrachtet man den Verlauf einer Grippewelle in einer Stadt mit 5 000 Einwohnern, so lässt sich die Anzahl an Erkrankten E in Abhängigkeit von der Zeit t (in Tagen) annähernd durch eine Polynomfunktion 3. Grades beschreiben.

Folgende Informationen liegen vor:

(1) Zu Beginn der Beobachtungen sind 10 Personen mit dem Grippevirus infiziert.

(2) Nach einem Tag sind bereits 100 Personen an Grippe erkrankt.

(3) Am 3. Tag nimmt die Anzahl an Erkrankten am stärksten zu.

(4) Am 10. Tag erreicht die Grippewelle (d. h. die Anzahl an Erkrankten) ihr Maximum.

Aufgabenstellung:

Ermitteln sie die Funktionsgleichung der Funktion E aus den verfügbaren Informationen und zeichnen Sie den Graphen.

ChatGPT

Die Funktionsgleichung der Funktion $E(t)$, die den Verlauf der Grippeepidemie in einer Stadt mit 5.000 Einwohnern beschreibt, lautet:

$$E(t) = \frac{180}{143}t^3 - \frac{270}{11}t^2 + \frac{16200}{143}t + 10$$

Dieses Polynom 3. Grades wurde basierend auf den gegebenen Bedingungen ermittelt:

1. Zu Beginn (t = 0) sind 10 Personen infiziert.
2. Nach einem Tag (t = 1) sind bereits 100 Personen erkrankt.
3. Am 3. Tag (t = 3) nimmt die Anzahl der Erkrankten am stärksten zu (dies entspricht dem Punkt, an dem die erste Ableitung des Polynoms maximal ist).
4. Am 10. Tag (t = 10) erreicht die Anzahl der Erkrankten ihr Maximum (dies entspricht dem Punkt, an dem die erste Ableitung des Polynoms null wird).

Der beigefügte Graph illustriert den Verlauf der Grippewelle. Hierbei zeigt die rote gestrichelte Linie den 3. Tag an, an dem die Zunahme der Erkrankten am stärksten ist, und die grüne gestrichelte Linie den 10. Tag, an dem die Anzahl der Erkrankten ihr Maximum erreicht.[>-]

Translated (Josef):

You

Problem: Flu Epidemic

If we look at the course of a flu epidemic in a town with 5000 inhabitants, the number of infected persons E can be approximately described by a 3rd degree polynomial function as a function of time t (in days).

The following information is available:

(1) At the beginning of the observation, 10 people are infected with the flu virus.
(2) After one day, 100 people have already fallen ill with the flu.
(3) On the third day, the number of sick people increases the most.
(4) On day 10, the flu epidemic (i.e. the number of people infected) reaches its maximum.

Task:
Determine the functional equation E from the available information and draw the graph.

ChatGPT

The function equation of the function *E*(*t*), which describes the course of the flu epidemic in a town with 5000 inhabitants, is as follows:

$$E(t) = \frac{180}{143}t^3 - \frac{270}{11}t^2 + \frac{16200}{143}t + 10$$

This polynomial of 3rd degree was determined based on the given conditions.

1. At the beginning (t = 0), 10 people are infected.
2. After one day (t = 1), 100 people are already infected.
3. On the third day (t=3), the number of infected people increases the most (this corresponds to the point at which the first derivative of the polynomial becomes its maximum).
4. On the 10th day (t=10), the number of sick people reaches its maximum (this corresponds to the point at which the first derivative of the polynomial becomes zero).

The attached graph illustrates the course of the flu epidemic. The red dashed line shows the 3rd day, on which the increase in the number of sick people is highest, and the green dashed line shows the 10th day, on which the number of sick people reaches its maximum.

In order to trust the result, you should look at the model, i.e. the associated system of equations. And you will find a logical error:

You

Wie lautet das Gleichungssystem?

ChatGPT

Das Gleichungssystem zur Bestimmung der Funktionsgleichung der Grippeepidemie besteht aus vier Gleichungen, die sich aus den gegebenen Informationen ableiten lassen. Die unbekannten Koeffizienten der Polynomfunktion 3. Grades $E(t) = at^3 + bt^2 + ct + d$ werden durch die folgenden Bedingungen bestimmt:

1. Zu Beginn der Beobachtungen (t = 0) sind 10 Personen mit dem Grippevirus infiziert:
   $E(0) = 10$
   $d = 10$
2. Nach einem Tag (t = 1) sind bereits 100 Personen an Grippe erkrankt:
   $E(1) = 100$
   $a + b + c + 10 = 100$

logischer Fehler

3. Am 3. Tag (t = 3) nimmt die Anzahl an Erkrankten am stärksten zu. Dies bedeutet, dass die erste Ableitung von E(t) an diesem Tag einen Ext[…]mwert (hier ein Maximum) hat:
   $E'(3) = 0$
4. Am 10. Tag (t = 10) erreicht die Grippewelle ihr Maximum. Das bedeutet, dass die erste Ableitung von E(t) an diesem Tag null ist:
   $E'(10) = 0$
   $3a(10)^2 + 2b(10) + c = 0$
   $300a + 20b + c = 0$

Diese vier Gleichungen bilden das Gleichungssystem, das gelöst werden muss, um die Koeffizienten a, b, c und d der Funktion $E(t)$ zu bestimmen.

Zum “logischen Fehler”: ChatGPT formuliert den Sachverhalt richtig (Suche nach dem Extremwert der ersten Ableitung. Das wäre $(E')' = 0$, übersetzt aber falsch in die mathematische Formulierung!

English translation follows:

You

What is the system of equations?

ChatGPT

The system of equations for determining the function equation of the flu epidemic consists of four equations, which can be deduced from the given information. The unknown coefficients of the order three polynomial $E(t) = at^3 + bt^2 + ct + d$ are determined by the following conditions.

1. At the beginning of the observation (t = 0), 10 people are infected.
   $E(0) = 10$
   $d = 10$
2. After just one day (t = 1), 100 people have already come down with the flu.
   $E(1) = 100$
   $a + b + c + d = 100$

logical error

3. On the 3rd day (t = 3), the number of patients increases the most. This means that the first derivative of E(t) has an extreme value (in this case a maximum) on this day:
   $E'(3) = 0$
4. The flu epidemic reaches its maximum on the 10th day (t = 10). This means that the first derivative of E(t) is zero on this day.
   $E'(10) = 0$
   $3a(10)^2 + 2b(10) + c = 0$
   $300a + 20b + c = 0$

These four equations for the system of equations, which is to be solved, in order to determine the coefficients a, b, c, and d of the function $E(t)$.

Comment to the "logical error": ChatGPT gives a correct answer: we know the extreme value of the 1st derivative at t =3, but translates it incorrect into a mathematical expression.

But ChatGPT is capable of learning. The learners become teachers by using their math skills to make the system see reason.

You

Nein wenn die Anzahl am stärksten zunimmt, hat die Geschwindigkeit ein Maximum, also ist die zweite Ableitung gleich 0

ChatGPT

Sie haben recht, ich entschuldige mich für das Missverständnis. Wenn die Anzahl der Erkrankten am stärksten zunimmt, erreicht die Geschwindigkeit, mit der sich die Anzahl der Erkrankten ändert (dargestellt durch die erste Ableitung der Funktion $E(t)$), ihr Maximum. Dies bedeutet, dass die zweite Ableitung der Funktion $E(t)$ zu diesem Zeitpunkt gleich null sein muss.

Das korrekte Gleichungssystem, das wir aufstellen müssen, sieht dann wie folgt aus:

1. Zu Beginn der Beobachtungen (t = 0) sind 10 Personen mit dem Grippevirus infiziert:
   $E(0) = 10$
   $d = 10$
2. Nach einem Tag (t = 1) sind bereits 100 Personen an Grippe erkrankt:
   $E(1) = 100$
   $a + b + c + 10 = 100$
3. Am 3. Tag (t = 3) erreicht die Geschwindigkeit der Zunahme der Anzahl der Erkrankten ihr Maximum, also ist die zweite Ableitung vo ↓ $'(t)$ zu diesem Zeitpunkt gleich null:
   $E''(3) = 0$
   $6a(3) + 2b = 0$
   $18a + 2b = 0$
4. Am 10. Tag (t = 10) erreicht die Anzahl der Erkrankten ihr Maximum, was bedeutet, dass die erste Ableitung von $E(t)$ an diesem Tag null ist:
   $E'(10) = 0$
   $3a(10)^2 + 2b(10) + c = 0$
   $300a + 20b + c = 0$

Dieses korrigierte Gleichungssystem werden wir nun lösen, um die richtigen Koeffizienten zu ermitteln.

Die endgültige korrigierte Funktionsgleichung für die Grippeepidemie, basierend auf der korrekten Annahme, dass am 3. Tag die Geschwindigkeit der Zunahme der Anzahl der Erkrankten (erste Ableitung von $E(t)$) ihr Maximum erreicht und somit die zweite Ableitung $E''(t)$ zu diesem Zeitpunkt null ist, lautet:

$$E(t) = -\frac{45}{64}t^3 + \frac{405}{64}t^2 + \frac{675}{8}t + 10$$

Der Graph zeigt den Verlauf der Anzahl der Infizierten über die Zeit. Die rote gestrichelte Linie markiert den 3. Tag, an dem die Zunahmerate der Infektionen (Geschwindigkeit) ihr Maximum erreicht, und die grüne gestrichelte Linie den 10. Tag, an dem die Gesamtzahl der Infektionen ihren Höchststand erreicht.[>_]

Translation:

You

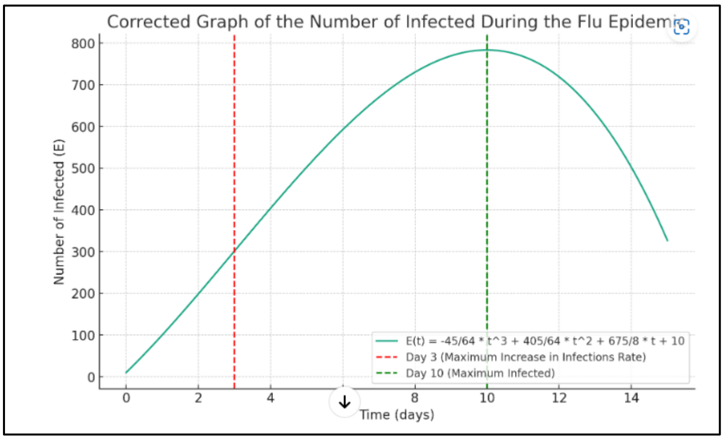
No, when the number increases the most, the speed has a maximum, so the second derivative is zero.

ChatGPT

You are right, I apologize for the misunderstanding. When the number of sick people increases the most, the rate at which the number of sick people changes (represented by the first derivative of the function E(t)) reaches its maximum. This means that the second derivative of the function E(t) must be zero at this time.

It is no need to translate the rest of the message. One can see the change in point 3. The correct function $E(t)$ is given.

The correct graph closes the message:



Even the graph of the polynomial function is drawn in the defined interval.

Helmut's conversation with ChatGPT in English:

You

If we look at the course of a flu epidemic in a city with 5000 inhabitants, the number of infected persons E can be approximately described by a polynomial function as a function of time t (in days).

The following information is available:

1) At the beginning of the observations, 10 people are infected with the flu virus.

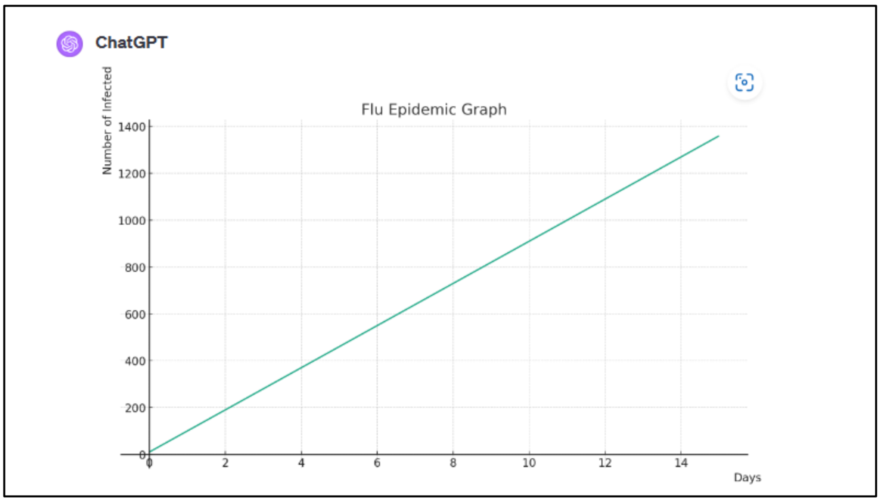
2) After one day, 100 people have already contracted the flu.

3) On the 3rd day, the number of people infected increases the most.

4) On the 10th day, the flu epidemic (i.e. the number of people infected) reaches its maximum.

Task:

Determine the functional equation of the function E from the available information and draw the graph.



The functional equation of the flu epidemic can be described as $E(t) = 90t + 10$, where E is the number of infected persons and t is the time in days.

Here's the graph of this function, illustrating the trend of the flu epidemic over time:

From the graph, we can observe a linear increase in the number of infected individuals over time, starting with 10 people and increasing by 90 each day. This aligns with the given information, including the rapid rise in cases from the initial observations. [>-]

You
How did you find this solution?

ChatGPT
To find the solution for the flu epidemic's functional equation, I followed these steps:

1. **Assumed a Polynomial Function**: Based on the information given, I assumed the function describing the number of infected persons, $E(t)$, would be a polynomial. Since the epidemic has a point of maximum increase and a peak (maximum number of infections), a cubic polynomial (degree 3) was a suitable choice. A cubic polynomial has the general form $E(t) = at^3 + bt^2 + ct + d$.
2. **Used Given Data Points**:
   - At the start (day 0), 10 people were infected: $E(0) = 10$.
   - On the first day (day 1), 100 people were infected: $E(1) = 100$.

   These provided two equations to solve for the coefficients a, b, c, and d.
3. **Incorporated Additional Information**:
   - On the 3rd day, the rate of infection increased the most. This implies that the first derivative of $E(t)$, which represents the rate of change of infections, had a peak at day 3. Mathematically, this translates to the second derivative (rate of change of the rate) being zero at day 3.
   - On the 10th day, the number of infections reached its maximum. This means the first derivative of $E(t)$ was zero on day 10, indicating a peak in the original function.
4. **Formulated and Solved Equations:** I set up four equations based on these points and solved them for the coefficients a, b, c, and d.
5. **Result and Graph**: The solution turned out to be a linear function $E(t) = 90t + 10$, which was unexpected for a cubic polynomial. This implies that the constraints led to a simplification where the higher-degree terms (cubic and quadratic) were zero. I then plotted this function to visualize the epidemic's progression over time.

In real-world scenarios, the progression of an epidemic is often more complex and may not adhere strictly to simple polynomial models. However, for the purpose of this exercise, the given data and constraints directed us towards this particular solution.

You
Can we see the system of equations?

ChatGPT

The system of equations derived from the given information about the flu epidemic is as follows:

1. $E(0) = 10$ → This equation comes from the information that 10 people are infected at the beginning of the observations. It translates to $d = 10$.
2. $E(1) = 100$ → This represents that after one day, 100 people are infected. It gives the equation $a + b + c + d = 100$.
3. The derivative of $E(t)$ is at its maximum on the 3rd day. This translates to the second derivative being zero on the 3rd day: $18a + 2b = 0$.
4. The function $E(t)$ reaches its maximum on the 10th day, which means the first derivative is zero on the 10th day: $60a + 2b = 0$.

These equations form the basis to solve for the coefficients a, b, c, and d in the cubic polynomial $E(t) = at^3 + bt^2 + ct + d$. Solving these equations led to the surprising result of a linear function, indicating that the higher degree terms (cubic and quadratic) effectively became zero.

You

The fourth equation is not correct. E has a maximum at the 10th day

ChatGPT

You're correct, and I apologize for the oversight. To accurately reflect the condition that the function $E(t)$ has a maximum at the 10th day, we should set the first derivative of $E(t)$ to zero at $t = 10$, not the second derivative. Let's correct the fourth equation and recompute the solution.

The correct fourth equation should be:
$E'(10) = 0$

Let's solve the system again with this correction.

With the corrected fourth equation, the solutions for the coefficients a, b, c, and d in the cubic polynomial $E(t) = at^3 + bt^2 + ct + d$ are:

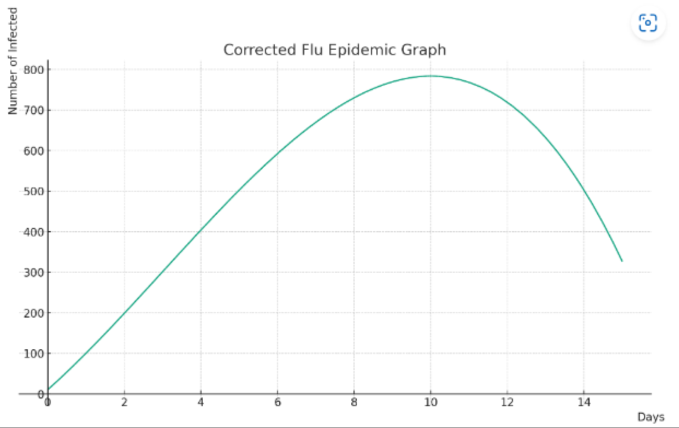
- $a = -\frac{45}{64}$
- $b = \frac{405}{64}$
- $c = \frac{675}{8}$
- $d = 10$

This gives us a more complex cubic polynomial equation to describe the flu epidemic, taking into account the maximum number of infected individuals on the 10th day.

You

Please show the graph

ChatGPT



Here is the graph of the corrected function for the flu epidemic, $E(t)$, based on the cubic polynomial with the updated coefficients. This graph represents the trend of the flu epidemic over time, taking into account the maximum number of infected individuals on the 10th day. [>-]

Comment (H. Heugl):

I have made two attempts: Two errors can be observed.

In the first one: Although it was described verbally by ChatGPT that the second derivative must be equal to 0, the equation that means the translation into the symbolic language, was wrong.

In the second attempt: The mistake is similar. Verbally ChatGPT recognized the necessary condition that the function E has a maximum on the 10^{th} day. But again, the translation into the symbolic language was wrong.

With such incorrect results, the most important outcome of the learning process is the dialog with ChatGPT, the competence to ask appropriate questions. Without the learner's own mathematical competence, such a dialog cannot lead to a comprehensible result.

This task is also an example of "distributed cognition". It makes sense to use a second tool, for example CAS, to check and correct the solution.

Will be continued in next DNL.

https://onramps.utexas.edu/what-we-offer/catalog/

https://highschool.utexas.edu/remote-learning-resources

http://shodor.org/interactivate/sitemap/

J. L. Galan & colleagues from Malaga University: A Stepwise Multiple Integration Solver

Done

$triplecyl\left(x \cdot y \cdot z, z, 0, \sqrt{1-\rho^2}, \rho, 0, 1, \theta, 0, \frac{\pi}{2}, 1, 1\right)$

Cylindrical coordinates are useful when the expression x^2+y^2 appreas in the function to be integrated
or in the region of integration and limits of z are easy to esrablish.
A triple integral in cylindrical coordinates is computed by means of three definite integrals ina given order.
Previously, the change of variables to cylindrical coordinates has to be done.
Let us consider the cylindrical coordinates change $x = \rho \cos(\theta)$, $x = \rho \sin(\theta)$, $z = z$
The 1st step is the substitution of this variable change in function $x \cdot y \cdot z$ and multiply this result by the Jacobian ρ.
In this case, the substitutions lead to integrate function $\sin(\theta) \cdot \cos(\theta) \cdot \rho^3 \cdot z$

Integrating the function $\sin(\theta) \cdot \cos(\theta) \cdot \rho^3 \cdot z$ with respect to variable z we get $\frac{\sin(\theta) \cdot \cos(\theta) \cdot \rho^3 \cdot z^2}{2}$.

Considering the limits of integration for this variable, we get $\frac{-\sin(\theta) \cdot \cos(\theta) \cdot \rho^3 \cdot (\rho^2-1)}{2}$.

Integrating this result wrt variable ρ the result is $\frac{-\sin(\theta) \cdot \cos(\theta) \cdot \rho^4 \cdot (2 \cdot \rho^2-3)}{24}$.

Considering the limits of integration, we get $\frac{\sin(\theta) \cdot \cos(\theta)}{24}$.

Finally, integrating this result wrt variable θ the result is $\frac{-(\cos(\theta))^2}{48}$.

Considering the limits of integration, the final result is:

$\frac{1}{48}$

Reflections on Derivative and Integral Tables

Michel Beaudin, DEG-mathematiques

Version of October 9, 2023

1 Motivation

We think the time has come to give fewer formulas in our tables and more justification for those that will remain. As the years have gone by and the courses have become full of contents, we have run out of time to justify, here and there, the origin of certain formulas.
As a first step, therefore, we suggest that there should be fewer formulas in our tables. To take account of the ubiquity of symbolic calculus systems, and for the sake of symmetry, our table of integrals will be quite different from the usual ones. And for those that could be called basic formulas, it seems to us that a minimum of memorization on the part of the students should be required.

de Jesús Martínez Vargas: Do you know that there are several ways to define quartiles?

`cuartile.dfw` (and quartiles.tns)

⋘ Hinge and Quartile Methods ⋙
1 ⇨ Tukey (Inclusive Hinge)
2 ⇨ Moore & McCabe (Exclusive Hinge)
3 ⇨ CDF (Empirical Distribution Function)
4 ⇨ N+1 Basis Interpolation
5 ⇨ N Basis Interpolations
6 ⇨ N-1 Basis Interpolations