# THE DERIVE - NEWSLETTER #57

## THE BULLETIN OF THE

## USER GROUP

## + CAS-TI

## Contents:

# Ein großzügiges Angebot / A Generous Offer

Lieber Herr Böhm,

meine *DERIVE*-Dateien (Version 5.06) zum Unterricht in der Sekundarstufe II (Fach- und Berufsoberschule Bayern) habe ich zusammen mit einer kurzen Inhaltsangabe auf meiner neulich erstellten Internetseite in komprimierter Form zum Download bereitgestellt

http://lzkopp.gmxhome.de/index.htm

Zum Inhalt der Homepage:

- ➢ 35 Dateien in analysis.zip (Analysis für die 11. und 12. Klasse);
- ➢ 32 Dateien in geometrie.zip (Vektorgeometrie 12. Klasse, Vektoren in Spaltenschreibweise, 3D-Grafik);
- ➢ 22 Dateien in stochastik.zip (zB. grafische Simulation von Zufallsexperimenten);
- ➢ 13 Dateien in physik.zip (zB. Auswertung von Messreihen);
- ➢ 38 Dateien in diverses1.zip und diverses2.zip (Was einem so einfällt, nicht nur für den Unterricht);
- ➢ 3 Dateien in ergaenzung.zip (Radioaktiver Zerfall, harm. Schwingungen, Beugung, Interferenz von Wellen).

Sicher kann mancher *DERIVER* unter den Fachkolleginnen und –kollegen mit der einen oder anderen Datei etwas anfangen. Ich bitte Sie daher, meine Internetadresse im *DNL* zu veröffentlichen. Vielen Dank!

Schöne Osterfeiertage wünscht Ihnen und Ihrer Familie
Lorenz Kopp

DUG-member Lorenz Kopp offers a huge bundle of DERIVE files for download from his website

http://lzkopp.gmxhome.de/index.htm

You can find more than 140 files from Calculus, Geometry, Probability theory, Physics and others.
I browsed in Lorenz´ collection and I was overwhelmed by the plenty of ideas. Many thanks for your generous offer.

# GeoGebra - Dynamic Geometry, Algebra and Calculus



**GeoGebra** is an open source software that joins geometry, algebra and calculus in a new way. It is developed for education in secondary schools and already received several educational software awards.

On the one hand, **GeoGebra** is a dynamic geometry system: you can do constructions with the mouse and change them dynamically afterwards. On the other hand, you may also work with equations and coordinates or calculate derivatives and integrals of functions. These two views are characteristic for GeoGebra: an expression in the algebra window corresponds to an object in the geometry window and vice versa.

Markus Hohenwarter, University of Salzburg
www.geogebra.at

*(GeoGebra is written in many languages. One of its outstanding features is that you can produce interactive websites. It is really worth more than only a try. You can download GeoGebra for free from the above website but you can also work with GeoGebra online. Josef)*

Dear DUG members,

first of all I´d like to excuse for the delay in delivering DNL#57. This is due to two reasons: (1) Easter holidays and some courses and seminars which needed a lot of preparations and (2) the huge contents of this DNL. Collecting material for DNL#57 I intended to add Rüdeger Baumann´s great paper on *ENGEL-Sequences and Shift registers* but when inspecting the many mails which I received during the last three months I found them so interesting that I had to postpone this paper for DNL#58. I hope that you will find the questions and answers as useful as I do. In my opinion the DNL should be a plattform to exchange such findings.

Let me especially point to the exciting discussion on "Stepwise Simplification" (pp 36) and on Albert Rich´s proposal (p 38). You all are strongly invited to participate in improving *DERIVE*.

Before adding some notes on this DNL I´d like to inform you that you also can download revised DNL#5 which is another reason for the delay. It is great to reinvestigate the great ideas from 1992 using means of 2005. Among others there was a 3D-problem with a poor graphic representation – because it was not possible to have a better one in 1992 – and now we can use implicit and 3D-plotting. Some files must be rewritten to be compatible with *DERIVE* 5 and 6. You can download all the stuff.

What concerns DNL#57 you will find some contributions removed from the "Preview"-section on page 2 – they are published now – but you will also find some new contributions announced. I am very happy to have a pure TI-article in this *DNL* – contributed by a then 11 year old girl, Mandala von Westenholz. At this occasion I´d like to stimulate our TI-users to submit articles.

Donald McPhillips, busy as ever, provides a paper on *Actuarial Math* (*Versicherungsmathematik*). It reminded me that actuarial mathematics was part of the curriculum when I was student. One the endexamination tasks in secondary school was from this interesting application of probability theory. It might be that one or the other would be inspired to inform about this applied mathematics chapter.

Duncan McDougall from Canada joined the Usergroup and sent immediately after transmitting the application form an extended paper on *Diophantine Polynomials*. Many thanks.

Finally I´d like to emphasize three highlights of this DNL which I strongly recommend to use. On the information page you can find two great offers:

- Lorenz Kopp opens his *DERIVE* Box and invites us all to help ourselves,
- Markus Hohenwarter offers a great dynamic geometry program for free for you and your students,

and on page 49 is the third link

- Walter Wegscheider worked and is still working on Online-HTML-courses for Voyage200 and *DERIVE*.

Feedback is greatly appreciated.

I wish you a wonderful spring time (in the northern hemisphere) and a wonderful fall (south of the equator) and hope that there will be no delay in June for publishing DNL#58 (and revised DNL#6).

Best regards as ever

## Download all *DNL-DERIVE*- and TI-files from

The *DERIVE-NEWSLETTER* is the Bulletin of the *DERIVE* & CAS-*TI User Group*. It is published at least four times a year with a contents of 44 pages minimum. The goals of the *DNL* are to enable the exchange of experiences made with *DERIVE* and the *TI*-89/92/*Titanium*/*Voyage* 200 as well as to create a group to discuss the possibilities of new methodical and didactical manners in teaching mathematics.

As many of the *DERIVE* Users are also using the CAS-*TIs* the *DNL* tries to combine the applications of these modern technologies.

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**Contributions:**
Please send all contributions to the Editor. Non-English speakers are encouraged to write their contributions in English to reinforce the international touch of the *DNL*. It must be said, though, that non-English articles will be warmly welcomed nonetheless. Your contributions will be edited but not assessed. By submitting articles the author gives his consent for reprinting it in the *DNL*. The more contributions you will send, the more lively and richer in contents the *DERIVE* & CAS-*TI Newsletter* will be.

Next issue:       June 2005
Deadline         15 May 2005

**Preview:    Contributions waiting to be published (selection)**

Pringles, B. Grabinger, GER
Two Stage Least Squares, M. R. Phillips, USA
Some simulations of Random Experiments, J. Böhm, AUT & L. Kopp, GER
Wonderful World of Pedal Curves, J. Böhm
Another Task for End Examination, J. Lechner, AUT
Tools for 3D-Problems, P. Lüke-Rosendahl, GER
ANOVA with *DERIVE* & *TI*, M. R. Phillips, USA
Hill-Encription, J. Böhm
CAD-Design with *DERIVE* and the *TI*, J. Böhm
Farey Sequences on the *TI,* M. Lesmes-Acosta, COL
Simulating a Graphing Calculator in *DERIVE*, J. Böhm, AUT
Henon & Co, J. Böhm
Challenges from Fermat, Bj. Felsager, DEN
Are all Bodies falling equally fast, J. Lechner, AUT
Modelling Traffic Density, Th. Himmelbauer, AUT
Do you know this? Cabri & CAS on PC and Handheld, W. Wegscheider, AUT
An Interesting Problem with a Triangle, P. Lüke-Rosendahl, GER
Diophantine Polynomials, Duncan McDougall, CAN
Rosettes, J. Lechner, AUT

and Setif, FRA; Vermeylen, BEL; Leinbach, USA; Koller, AUT; Baumann, GER; Keunecke, GER, .........and others

**Bill Wilburn**                WDWNORWALK@aol.com

ROTATION MATRIX_ANGLES

HELLO JOSEPH:

I'M HOPING THAT YOU CAN HELP ME WITH THIS 3x3 ROTATION MATRIX PROBLEM:

| 0.3535533905   -0.5732233047   0.7391989197 |
| 0.6123724356   0.7391989197   0.2803300858 |
|-0.7071067811   0.3535533905   0.6123724356 |.

IT'S A CONCATENATED ROTATION MATRIX, REPRESENTING A ROTATION OF 30 DEGREES CCW [1] ABOUT THE X AXIS; 45 DEGREES CCW ABOUT THE Y AXIS, AND 60 DEGREES CCW ABOUT THE Z AXIS.

IF I DIDN'T KNOW THE ROTATION ANGLES, AND I HAD ONLY THE 3x3 ROTATION MATRIX GIVEN ABOVE, HOW WOULD I CALCULATE THE ROTATION ANGLES ABOUT THE X,Y AND Z AXES?

NOTE: THE AXIS ROTATION SETS ARE IN ROWS, NOT COLUMNS.

I HOPE YOU CAN HELP OUT WITH THIS ONE.

THANKS,
BILL WILBURN

[1] CCW = counter clock wise

Hi Bill,

I believe that I can help you.
Please inspect the attached file. It is clear that working with inverse trig functions might cause ambiguous solutions, but I think that my process will work.
Best regards
Josef

This was an interesting problem and I am not quite sure if this is the best way to tackle it. But it is a nice opportunity to demonstrate the power of a CAS and it might be a challenge for students, too.

```
This is to have the concetenated matrix in your form:
```

```
(ROTATE_X(30·1°)'·ROTATE_Y(45·1°)'·ROTATE_Z(60·1°)')'
```

$$\begin{bmatrix} \dfrac{\sqrt{2}}{4} & \dfrac{\sqrt{2}}{8} - \dfrac{3}{4} & \dfrac{\sqrt{6}}{8} + \dfrac{\sqrt{3}}{4} \\[2mm] \dfrac{\sqrt{6}}{4} & \dfrac{\sqrt{6}}{8} + \dfrac{\sqrt{3}}{4} & \dfrac{3\cdot\sqrt{2}}{8} - \dfrac{1}{4} \\[2mm] -\dfrac{\sqrt{2}}{2} & \dfrac{\sqrt{2}}{4} & \dfrac{\sqrt{6}}{4} \end{bmatrix}$$

$$\begin{bmatrix} 0.3535533905 & -0.5732233047 & 0.7391989197 \\ 0.6123724356 & 0.7391989197 & 0.2803300858 \\ -0.7071067811 & 0.3535533905 & 0.6123724356 \end{bmatrix}$$

I compare with the generalized form:

(ROTATE_X(x)'·ROTATE_Y(y)'·ROTATE_Z(z)')'

$$\begin{bmatrix} \cos(y)\cdot\cos(z) & \sin(x)\cdot\sin(y)\cdot\cos(z) - \cos(x)\cdot\sin(z) & \cos(x)\cdot\sin(y)\cdot\cos(z) + \sin(x)\cdot\sin(z) \\ \cos(y)\cdot\sin(z) & \cos(x)\cdot\cos(z) + \sin(x)\cdot\sin(y)\cdot\sin(z) & \cos(x)\cdot\sin(y)\cdot\sin(z) - \sin(x)\cdot\cos(z) \\ -\sin(y) & \sin(x)\cdot\cos(y) & \cos(x)\cdot\cos(y) \end{bmatrix}$$

I select three elements containing x, y and z and try to solve the respective system of trig equations:

[COS(y)·COS(z) = 0.3535533905, COS(x)·COS(z) + SIN(x)·SIN(y)·SIN(z) = 0.7391989197, COS(x)·COS(y) =

  0.6123724356]

$$\left[\cos(y)\cdot\cos(z) = \frac{\sqrt{2}}{4}, \ \cos(x)\cdot\cos(z) + \sin(x)\cdot\sin(y)\cdot\sin(z) = \frac{\sqrt{6}}{8} + \frac{\sqrt{3}}{4}, \ \cos(x)\cdot\cos(y) = \frac{\sqrt{6}}{4}\right]$$

Set cos(x) = x, cos(y) = y and cos(z) = z and go on solving the system.

$$\left[y\cdot z = \frac{\sqrt{2}}{4}, \ x\cdot z + \sqrt{(1 - x^2)}\cdot\sqrt{(1 - y^2)}\cdot\sqrt{(1 - z^2)} = \frac{\sqrt{6}}{8} + \frac{\sqrt{3}}{4}, \ x\cdot y = \frac{\sqrt{6}}{4}\right]$$

$$\text{SOLVE}\left(y\cdot z = \frac{\sqrt{2}}{4} \wedge x\cdot y = \frac{\sqrt{6}}{4}, \ [x, y]\right) = \left(x = \sqrt{3}\cdot z \wedge y = \frac{\sqrt{2}}{4\cdot z}\right)$$

Substitute for x and y, Simplify and solve for z:

$$(\sqrt{3}\cdot z)\cdot z + \sqrt{(1 - (\sqrt{3}\cdot z)^2)}\cdot\sqrt{\left(1 - \left(\frac{\sqrt{2}}{4\cdot z}\right)^2\right)}\cdot\sqrt{(1 - z^2)} = \frac{\sqrt{6}}{8} + \frac{\sqrt{3}}{4}$$

$$\frac{\sqrt{2}\cdot\sqrt{(1 - z^2)}\cdot\sqrt{(1 - 3\cdot z^2)}\cdot\sqrt{(8\cdot z^2 - 1)}}{4\cdot|z|} + \sqrt{3}\cdot z^2 = \frac{\sqrt{6}}{8} + \frac{\sqrt{3}}{4}$$

$$\text{SOLVE}\left(\frac{\sqrt{2}\cdot\sqrt{(1 - z^2)}\cdot\sqrt{(1 - 3\cdot z^2)}\cdot\sqrt{(8\cdot z^2 - 1)}}{4\cdot|z|} + \sqrt{3}\cdot z^2 = \frac{\sqrt{6}}{8} + \frac{\sqrt{3}}{4}, \ z, \ \text{Real}\right)$$

$$z = -\frac{2\cdot\sqrt{(2742\cdot\sqrt{2} + 10511)}}{457} \ \vee \ z = \frac{2\cdot\sqrt{(2742\cdot\sqrt{2} + 10511)}}{457} \ \vee \ z = -\frac{1}{2} \ \vee \ z = \frac{1}{2}$$

**We obtain four solutions for z and proceed with z = 0.5 by substituting in the expressions for x and y above:**

$$x = \sqrt{3}\cdot z \wedge y = \frac{\sqrt{2}}{4\cdot z}$$

<span style="color:red">This is a solution of our problem.</span>

$$x = \sqrt{3}\cdot\frac{1}{2} \wedge y = \frac{\sqrt{2}}{4\cdot\frac{1}{2}}$$

<span style="color:red">Accomplishing all the other solutions and forming the rotation matrix shows that these matrices differ in signs!!</span>

$$x = \frac{\sqrt{3}}{2} \wedge y = \frac{\sqrt{2}}{2}$$

<span style="color:red">You can follow the process inspecting wilburn.dfw (or wilburn5.dfw). Josef</span>

$$\left[\text{ARCCOS}\left(\frac{\sqrt{3}}{2}\right), \ \text{ARCCOS}\left(\frac{\sqrt{2}}{2}\right), \ \text{ARCCOS}\left(\frac{1}{2}\right)\right] = [30, \ 45, \ 60]$$

$$\left[\frac{\text{ACOS}\left(\frac{\sqrt{3}}{2}\right)}{1°}, \ \frac{\text{ACOS}\left(\frac{\sqrt{2}}{2}\right)}{1°}, \ \frac{\text{ACOS}\left(\frac{1}{2}\right)}{1°}\right] = [30, \ 45, \ 60]$$

**Leopold Watzinger**

I wrote a program to investigate fractals. All single functions are doing fine, but if nesting IF-, ITERTAE and VECTOR commands I am failing. You can find my problems in the attached file.

Expression #14 shall decide the question of convergence for points from [0,0] to [0.2,0.2] after 20 iterations $z_{n+1} = z_n^2 + i$ and return the point in case of convergence and [-2,-2] otherwise.

$$\text{VECTOR(IF(ITERATE(IF(|[x, y]| < 2, } \begin{bmatrix} x^2 - y^2, & 2 \cdot x \cdot y + 1 \end{bmatrix}, [-2, -2]), [x, y], [a,$$

$$\text{b], 20) = [-2, -2], [-2, -2], [a, b]), [a, b], [0, 0], [0.2, 0.2], [0.1, 0.1])}$$

*__DNL:__ I hope that I can follow your ideas.*

*At first I collect all points of the respective region in a list using* `region(..)`

#1:    `region(xw, yw, sw) := APPEND(VECTOR([a, b], [a, b], xw, yw, [sw, sw]))`

#2:    `region([0, 0], [0.2, 0.2], 0.05)`

$$\begin{bmatrix} 0 & 0 \\ 0.05 & 0 \\ 0.1 & 0 \end{bmatrix}$$

*Then I include this function into function* `fract( ):`

#4:    `fract(xw, yw, sw, depth) := VECTOR(IF(ITERATE(IF(|[x, y]| < 2, ` $\begin{bmatrix} x^2 - y^2, & 2 \cdot x \cdot y + \end{bmatrix}$

      `1` $\Big]$ `, [-2, -2]), [x, y], v, depth) = [-2, -2], [-2, -2], v), v, region(xw, yw, sw))`

#5:    `fract([0, 0], [0.2, 0.2], 0.01, 20)`

$$\begin{bmatrix} 0 & 0 \\ 0.01 & 0 \\ -2 & -2 \\ 0.03 & 0 \end{bmatrix}$$

*This seems to work properly. It could be useful to select all points which satisfy the convergence criterium, i.e., which remain with abs(z) < 2 after* `depth` *iterations.*

#7:    `fract2(xw, yw, sw, depth) := SELECT(u ≠ [-2, -2], u, VECTOR(IF(ITERATE(IF(|[x, y]|`

      `< 2, ` $\begin{bmatrix} x^2 - y^2, & 2 \cdot x \cdot y + 1 \end{bmatrix}$ `, [-2, -2]), [x, y], v, depth) = [-2, -2], [-2, -2],`

      `v), v, region(xw, yw, sw)))`

#8:    `fract2([0, 0], [0.2, 0.2], 0.01, 20) = ` $\begin{bmatrix} 0 & 0 \\ 0.01 & 0 \\ 0.03 & 0 \\ 0.02 & 0.01 \\ 0.03 & 0.06 \\ 0.13 & 0.11 \\ 0.08 & 0.13 \\ 0.01 & 0.16 \end{bmatrix}$

```
fract2([0, 0], [0.2, 0.2], 0.01, 10)

DIM(fract2([0, 0], [0.2, 0.2], 0.01, 10))

161

fract2([0, 0], [0.2, 0.2], 0.01, 15)

DIM(fract2([0, 0], [0.2, 0.2], 0.01, 15))

35
```

*After 10 iterations remain 161 points, after 15 iterations 35 convergent points are remaining and after 20 there are 8 points left (expr #8).*
*We plot the points leaving this filter after 5, 10, 15 and 20 iterations in different colors (red, blue, yellow and black), which leads to a nice fractal picture:*
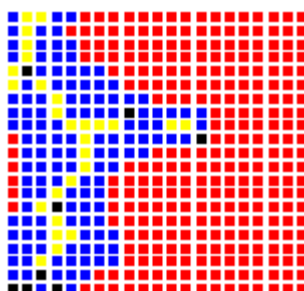


*Here again we can apply the powerful VECTOR command to produce all the point in one single step.*
*The lists can be selected and plotted in different colors, giving nice fractal pictures.*

```
big1 := VECTOR(fract2([0, 0], [0.2, 0.2], 0.005, k), k, 5, 20, 5)

big2 := VECTOR(fract2([-0.2, -0.2], [0, 0], 0.005, k), k, 5, 20, 5)

big3 := VECTOR(fract2([-0.2, 0], [0, 0.2], 0.005, k), k, 5, 20, 5)

big4 := VECTOR(fract2([0, -0.2], [0.2, 0], 0.005, k), k, 5, 20, 5)
```

**Plot red**

$$\begin{bmatrix} big1 & , & big2 & , & big3 & , & big4 \\ 1 & & 1 & & 1 & & 1 \end{bmatrix}$$

**Plot blue**

$$\begin{bmatrix} big1 & , & big2 & , & big3 & , & big4 \\ 2 & & 2 & & 2 & & 2 \end{bmatrix}$$

**Plot yellow**

$$\begin{bmatrix} big1 & , & big2 & , & big3 & , & big4 \\ 3 & & 3 & & 3 & & 3 \end{bmatrix}$$

**Plot black**

$$\begin{bmatrix} big1 & , & big2 & , & big3 & , & big4 \\ 4 & & 4 & & 4 & & 4 \end{bmatrix}$$
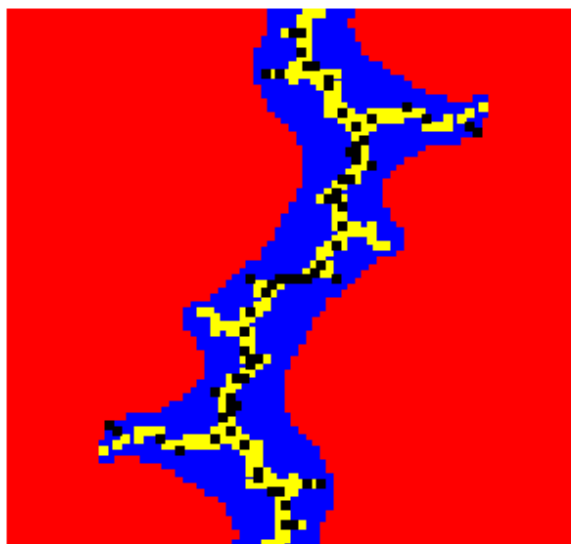
**Yves De Racker** $\hfill$ rc00028433@versateladsl.be

Mr Boehm,

As you know the EXPAND function can be used on an expression to expand an expression when it is simplified. Recently I tried it with,

$$\frac{3 \cdot x^4 - 11 \cdot x^3 + 9 \cdot x^2 + 5 \cdot x - 3}{x^2 - 4 \cdot x - 2}$$

Factorizing the denominator gives : $\qquad (x + \sqrt{6} - 2) \cdot (x - \sqrt{6} - 2)$

But with the EXPAND function you obtain:
$$\frac{83 \cdot x}{x^2 - 4 \cdot x - 2} + \frac{35}{x^2 - 4 \cdot x - 2} + 3 \cdot x^2 + x + 19$$

Instead of two fractions with a linear factor in the denominator! How is that possible? I'm using Derive 5.06. I send you also a small Derive file where you can see what I have discovered! Thank you very much for your help and I wish you a happy New Year for you and your family!

*DNL:*

$$\text{EXPAND}\left(\frac{3 \cdot x^4 - 11 \cdot x^3 + 9 \cdot x^2 + 5 \cdot x - 3}{x^2 - 4 \cdot x - 2}, \text{ Radical}, x\right)$$

$$\frac{67 \cdot \sqrt{6} + 166}{4 \cdot (x - \sqrt{6} - 2)} + \frac{166 - 67 \cdot \sqrt{6}}{4 \cdot (x + \sqrt{6} - 2)} + 3 \cdot x^2 + x + 19$$

**Ernest Carpenter (South Africa)**

Hi Josef,

I recently looked at some of the latest DNL's posted on the DUG website and must compliment all of you on excellent newsletters full of interesting and useful contributions!
I'm not a teacher or affiliated with any educational institution (actually to be honest, I'm not even good with maths, although I try my best to make sense of some topics☺), but I have a great admiration/ love for geometry, technology and of course my TI-92+! Thus I enjoy any program, article that involves geometry (especially 3D) and the TI-92+. (goes hand-in-hand with my work as CADD Co-ordinator and 3D modeler)

(I also must admit that I'm not a member…yet!)

A few years back I read a small paper you wrote on representing 3D objects on the TI-92 and you have also sent me the program which I used (and is still using) to visualize some basic concepts in 3D!

Anyway, the reason for my email is actually to ask you if it would be possible to get my hands on the papers mentioned in the "contributions waiting to be published" section of the last few DNL's:

- Tools for 3D-Problems, P. Lüke-Rosendahl, GER
- CAD-Design with *DERIVE* and the *TI*, J. Böhm

They sound intriguing!
If not possible, would they be in the March DNL?
Any other references in connection with TI & 3D geometry that might be useful?

Thank you for your time, and I hope you have a prosperous 2005!
Regards

# Type Checking, Finite Continued Fractions, and the Solution of the Linear Diophantine Equation $ax+by=c$

Stefan Welke, Bonn, Germany , Spwelke@aol.com

**Abstract**

A continued fraction approximation of the quotient $\frac{a}{b}$ is used to compute a solution of the diophantine equation $ax + by = c$.

## 1 Introduction

Let me first explain the bulky title. The main purpose of this paper is twofold: (1) To introduce the function CON_FRAC_RATIONAL(q) which computes the continued fraction expansion of a rational number. In this case the expansion is finite. The algorithm is recursive. Thus an irrational argument like $\sqrt{2}$ or $\pi$ would yield an infinite recursion and a resulting memory overflow. To avoid this dilemma the type checking function rational_Q(a) is introduced. (2) To present a canonical solution of the linear diophantine equation $ax + by = c$ as an application of the continued fraction algorithm.

## 2 Type checking with *DERIVE*

Some algorithms only work with numerical arguments of a special type, e.g.: natural, integer, rational, real or complex numbers. The following *DERIVE* functions return true, if the argument evaluates to the type which is specified in the function name. The "Q" at the end stands for query!

```
natural_Q(a) :=
  If a = FLOOR(a) ∧ a ≥ 0
     true
     false
     false

natural_Q(1.5) = false

integer_Q(a) :=
  If a = FLOOR(a)
     true
     false
     false

[integer_Q(-10), integer_Q(√2)] = [true, false]

[INTEGER?(-10), INTEGER?(√2)] = [true, false]

rational_Q(a) :=
  If integer_Q(NUMERATOR(a)) ∧ integer_Q(DENOMINATOR(a))
     true
     false
     false
```

$$\left[\text{rational\_Q}(0.28999), \text{rational\_Q}\left(\frac{\sqrt{2}}{2}\right)\right] = [\text{true, false}]$$

$$\left[\text{RATIONAL?}(0.28999), \text{RATIONAL?}\left(\frac{\sqrt{2}}{2}\right)\right] = [\text{true, false}]$$

There is no Expression Type function in *DERIVE* to recognize natural numbers.

INTEGER?(a) is implemented since *DERIVE* 5.

*DERIVE* distinguishes between INTEGER?(a) and INTEGER_TYPE?(a). Please inform yourself in the Online-Help!

```
number_Q(a) :=
  If DIMENSION(VARIABLES(a)) = 0
      true
      false
      false
```

[number_Q(2 + k), NUMBER?(2 + k)] = [false, false]

k := π

[number_Q(2 + k), NUMBER?(2 + k)] = [true, false]

NUMBER_TYPE?(2 + k) = true

```
real_Q(z) :=
  If number_Q(z) ∧ CONJ(z) = z
      true
      false
      false
```

There is a difference between Stefan´s `number_Q(a)` and *DERIVE*´s `NUMBER?(a)` treating expressions which do not appear as numbers but really are numbers.

`NUMBER_TYPE?(a)` is *Derivish*.

Note the behaviour of `number_Q` in the examples!

$$\left[ \text{real\_Q}(2.5), \ \text{real\_Q}\left( \frac{2.5}{1 + 2 \cdot i} \right) \right] = [\text{true}, \text{false}]$$

[real_Q(x + y), REAL_TYPE?(x + y)] = [false, true]

$$\left[ \text{real\_Q}(2.5), \ \text{real\_Q}\left( \frac{2.5}{1 + 2 \cdot i} \right) \right] = [\text{true}, \text{false}]$$

[real_Q(x + y), REAL_TYPE?(x + y)] = [false, true]

```
symbolic_Q(a) :=
  If DIM(VARIABLES(a)) > 0
      true
      false
      false
```

$$\left[ \text{number\_Q}(2 + m^2), \ \text{symbolic\_Q}(2 + m^2) \right] = [\text{false}, \text{true}]$$

The functions `number_Q(a)` and `symbolic_Q` are related by the equation

$$\text{number\_Q(expr)} = \text{not(symbolic\_Q(expr))}$$

so one of both is redundant.

## 3      Continued fraction expansion of rational numbers

Consider the following chain of equations,

$$\frac{127}{46} = 2 + \frac{35}{46} = 2 + \frac{1}{\frac{46}{35}} = 2 + \frac{1}{1 + \frac{11}{35}} = \ldots = 2 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{5 + \frac{1}{2}}}}$$

which defines the continued fraction expansion of the rational number $\frac{127}{46}$. The sequence of integers [2, 1, 3, 5, 2] contains all information and is commonly used to describe this expansion. We obtain this sequence by an interative algorithm:

$$a_0 := \text{FLOOR}\left( \frac{127}{46} \right) \quad \text{and} \quad a_{n+1} := \frac{1}{\text{MOD}(a_n, 1)}.$$

The function `CONTINUED_FRACTION(a,n)` in the `NUMBER.MTH` file computes $n$ iterations of this algorithm, eg.:

#21:   $\text{CONTINUED\_FRACTION}\left(\dfrac{127}{46}, 7\right) = [2, 1, 3, 5, 2, \pm\infty, ?, ?]$

This result shows us that we should have terminated the algorithm after four iterations because $a_4 = 0$ mod 1. The continued fraction algorithm applies to every real number and we have the following

**Theorem 1**   *The continued fraction algorithm of a real number q terminates iff q is rational.*

For a proof see ([1], p.135-136). We now replace the iteration in the definition of the `CONTINUED_FRACTION` function by recursion.

```
cf_vector(v) :=
  If v↓1 = ∞
     v↓2
     cf_vector([1/MOD(v↓1), APPEND(v↓2, [FLOOR(v↓1)])])
     ?
```

$\text{cf\_vector}\left[\dfrac{127}{46}, [\,]\right] = [2, 1, 3, 5, 2]$

As Stefan Welke´s contribution is from pre-*DERIVE*-programming times we can now use a short program to make the input more comfortable. This is an example how to improve your old *DERIVE*-functions.

#24:
```
cf_vector2(x, v) :=
  Prog
    v := [x, []]
    If v↓1 = ∞
       RETURN v↓2
       cf_vector([1/MOD(v↓1), APPEND(v↓2, [FLOOR(v↓1)])])
       ?
```

#25:   $\text{cf\_vector2}\left(\dfrac{127}{46}\right) = [2, 1, 3, 5, 2]$

This function terminates iff $a_n = 0$ mod 1 and we need not specify the number of iterations. But we must observe that an irrational number would yield infinite recursion and memory overflow. A combination with a type check prevents us from falling into this trap:

#26:
```
con_frac_rational(a) :=
  If rational_Q(a)
     cf_vector2(a)
     "not rational"
     "not rational"
```

#27:   $\text{con\_frac\_rational}(\pi) = \text{not rational}$

#28:   $\text{con\_frac\_rational}\left(\dfrac{17}{14}\right) = [1, 4, 1, 2]$

## 4      The diophantine equation $ax + by = c$

The main purpose of continued fraction expansions is to give good rational approximations of real numbers. Let us look at the following example:

**#29:**   `CONTINUED_FRACTION(`$\pi$`, 7) = [3, 7, 15, 1, 292, 1, 1, 1]`

The rational numbers

$$[3,7] = 3 + \frac{1}{7} = \frac{22}{7} \approx 3.1429 \quad \text{and} \quad [3,7,15] = 3 + \frac{1}{7 + \frac{1}{15}} = \frac{333}{106} \approx 3.1415$$

serve as very good approximations of $\pi$ that were already known by Archimedes. Indeed: $\frac{22}{7}$ is the best rational approximation with denominator $m \leq 7$ and $\frac{333}{106}$ is the best rational approximation with denominator $m \leq 106$. For general details see ([1], p. 151).

Now let us try to find an integer solution of the equation

$$127x - 46y = 1.$$

This equation is equivalent to

$$\frac{127}{46} - \frac{y}{x} = \frac{1}{46y}$$

and we can see that the quotient $\frac{y}{x}$ must be a very good rational approximation to $\frac{127}{46}$. In the theory of continued fractions the best possible rational approximation is $[2,1,3,5] = \frac{58}{21}$, i.e. we drop the last place of the continued fraction expansion.

$$2 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{5}}} = \frac{58}{21}$$

And surprisingly: $127 \cdot 21 - 46 \cdot 58 = -1$.

Thus $x = -21$ and $y = -58$. This fact is a consequence of the following general

**Theorem 2**    *Let $\xi = \frac{a}{b}$ a rational number and gcd(a,b) = 1. Let $[a_0, a_1, \ldots, a_n]$, $n \geq 1$, be the continued fraction expansion of $\xi$ and let $[a_0, a_1, \ldots, a_n] = \frac{p_{n-1}}{q_{n-1}}$ be the (n − 1)th convergent. Then $p_{n-1}$ and $q_{n-1}$ satisfy*

$$a \cdot q_{n-1} - b \cdot p_{n-1} = (-1)^n.$$

For a proof see ([1], p.131, Theorem 150). Thus continued fractions serve to compute the absolute values of solutions of diophantine equations of the form $ax + by = c$ with gcd($a,b$) = 1. The solvability of the general case is covered by the next

**Theorem 3**    *The linear equation $ax + by = c$ with integer coefficients has infinitely many integer solutions iff c is a multiple of gcd(a,b) and no solutions otherwise. Every integer solution has the form*

$$(x,y) = \frac{c}{\gcd(a,b)}(x_0, y_0) + \frac{n}{\gcd(a,b)}(-b, a), \quad n \in \mathbb{Z},$$

*where* $(x_0, y_0)$ *is an integer solution of* $\frac{a}{\gcd(a,b)} x + \frac{b}{\gcd(a,b)} y = 1$.

The proof follows from ([1], p.21, Theorem 24).

## 5     Solving $ax + by = c$ with *DERIVE*

The implemetation of the solution via continued fractions requires some technical functions which shall be briefly explained. It follows from Theorem 2 that the continued fraction solution only works if $|a| \neq |b|$.

We start with two functions borrowed from the file `NUMBER.MTH`:

(This is from NUMBER.MTH connected with Derive for Windows 4!!)

```
#31:    CONVERGENT_AUX(v, k_, a) :=
          If k_ = 0
            a
            CONVERGENT_AUX(v, k_ - 1, v↓k_ + 1/a)

#32:  CONVERGENT(v) := CONVERGENT_AUX(v, DIMENSION(v), ∞)
```

`cf_approx([a,b])` computes the rational approximation $\frac{p_{n-1}}{q_{n-1}}$ of $\frac{a}{b}$ given as a vector $[-a,b]$:

```
#33:  cf_approx(v) := CONVERGENT(DELETE_ELEMENT(v, DIMENSION(v)))
```

The next two functions convert fractions $\frac{a}{b}$ to vectors $[-a,b]$ and vectors $[x,y]$ to fractions $\frac{x}{y}$ :

```
#34:  frac_to_vec(a) := [- DENOMINATOR(a), NUMERATOR(a)]
```

```
#35:  frac_to_vec( 127/46 ) = [-46, 127]
```

```
#36:  vec_to_frac(v) := v↓1 / v↓2
```

```
#37:  vec_to_frac([127, 46]) = 127/46
```

`aux_sol_1([a,b])` computes a solution vector of $|ax + by| = 1$ for the case $\gcd(a,b) = 1$.

```
#38:  aux_sol_1(v) := frac_to_vec(cf_approx(con_frac_rational(vec_to_frac(v))))
```

```
#39:  aux_sol_1([7, 18]) = [-5, 2]
```

`co(eq,vars)` extracts the coefficient vector $[a,b]$ from the rationa equation *eq* := $ax + by = c$:

```
co(eq, vars) := [∂(LHS(eq), vars↓1), ∂(LHS(eq), vars↓2)]
```

```
co(27·x + 5·y = 7, [x, y]) = [27, 5]
```

`aux_sol_2([a,b])` computes a solution of $ax + by = \gcd(a,b)$ as a vector. Note that the factor `v*aux_sol_1(v)` corrects the sign of the solution `aux_sol_1(v)`:

#43: $\quad$ aux_sol_2(v) := $\dfrac{\dfrac{v \cdot \text{aux\_sol\_1(v)}}{2}}{GCD(v)} \cdot \text{aux\_sol\_1(v)}$

#44: $\quad$ aux_sol_2([127, 46]) = [-21, 58]

#45: $\quad$ aux_sol_2([6, 18]) = $\left[\dfrac{1}{6}, \ 0\right]$

aux_sol_3(eq) detects the variables in *eq* and computes a solution of *ax* + *by* = *c* if *c* is a multiple of gcd(*a*,*b*):

#46: $\quad$ aux_sol_3(eq) := VARIABLES(eq) = aux_sol_2(co(eq, VARIABLES(eq)))·RHS(eq)

#47: $\quad$ aux_sol_3(10·x + 12·y = 6) = (x = -3 ∧ y = 3)

#48: $\quad$ aux_sol_3(10·x - 17·y = 21) = (x = -105 ∧ y = -63)

#49: $\quad$ 10·(-105) - 17·(-63) = 21

Finally the equation is tested for the condition of the previous theorem and a solution is computed if it exists:

#50: $\quad$ 
```
lin_diophant_solve(eq) :=
   If MOD(RHS(eq)/GCD(co(eq, VARIABLES(eq)))) = 0
      aux_sol_3(eq)
      ?
      ?
```

The function detects the variables but it is crucial to use the correct order *ax* + *by* = *c*, i.e. variables on the left and numbers on the right hand side, and one must observe the conditiot |*a*| ≠ |*b*|:

#51: $\quad$ lin_diophant_solve(- 127·s + 46·t = 5) = (s = 105 ∧ t = 290)

#52: $\quad$ lin_diophant_solve(- 12·x + 18·y = 15) = ?

#53: $\quad$ lin_diophant_solve(1023797·u + 4601337·v = -15078)

#54: $\quad$ u = -29117563062 ∧ v = 6478654728

#55: $\quad$ lin_diophant_solve(- 127·s = - 46·t + 5) = ?

#56: $\quad$ lin_diophant_solve(x + y = 3) = (x = -∞ ∧ y = ∞)

Though –127*s* = –46*t* + 5 is equivalent to –127*s* + 46*t* = 5 this function in its present form cannot solve the first equation.

Let me finally explain the reason that makes the continued fraction solution so particular. *ax* + *by* = *c* is the equation of a straight line in $\mathbb{R}^2$, which is perpendicular to the vector (*a*, *b*). All integer solutions are points on this line with integer coordinate values. The following theorem explains the geometric significnace of our solution:

**Theorem 4** $\quad$ *Let the continued fraction solution* (*x*, *y*) *of ax* + *by* = *c exist. Then* (*x*, *y*) *has the least distance to the origin among all integer solutions.*

I have no reference for a proof of this theorem but I cannot imagine that no proof in the literature on number theory exists.

We already mentioned the impossibility to construct a continued fraction solution in the case |*a*| = |*b*|. See expression #56 above.

The arithmetic reason for this failure is the following fact: The continued fraction expansion of $\left|\dfrac{a}{b}\right|$ is [1], i.e. we cannot approximate by a shorter continued fraction. In geometric terms this difference is:

- If $|a| \neq |b|$ and an integer solution exists then exactly one integer solution minimizes the distance to the origin.

- If $|a| = |b|$ and an integer solution exists then exactly two integer solutions minimize the distance to the origin.

A piece of additional code that chooses one of the two minimizing solutions closes this gap. We resemble in this way the geometric flavour of the continued fraction solution.

$$\#57: \quad \text{aux\_sol\_4}(v, c) := \left[ x = \text{FLOOR}\left(\frac{c}{2 \cdot v_1}\right), \ y = \frac{c - v_1 \cdot \text{FLOOR}\left(\frac{c}{2 \cdot v_1}\right)}{v_2} \right]$$

$\#58:$  aux_sol_4([1, 1], 3) = [x = 1, y = 2]

$\#59:$  aux_sol_4([3, -3], 9) = [x = 1, y = -2]

The next function tests for the condition $|a| = |b|$.

```
eq_test(v) :=
  If ABS(v↓1) = ABS(v↓2)
     true
     false
     false
```
$\#60:$

We now enhance the solving function:

```
lin_diophant_solve(eq) :=
  If MOD(RHS(eq)/GCD(co(eq, VARIABLES(eq))), 1) = 0
     If eq_test(co(eq, VARIABLES(eq))) = true
        aux_sol_4(co(eq, VARIABLES(eq)), RHS(eq))
        aux_sol_3(eq)
     ?
     ?
```
$\#61:$

$\#62:$  lin_diophant_solve(3·x + 3·y = 18) = [x = 3, y = 3]

$\#63:$  lin_diophant_solve(3·x + 3·y = 178) = ?

$\#64:$  lin_diophant_solve(3·x − 3·y = 9) = [x = 1, y = −2]

$\#65:$  lin_diophant_solve(3·x + 5·y = 178) = (x = 356 ∧ y = −178)

$\#66:$  lin_diophant_solve(6·x + 18·y = 18) = (x = 3 ∧ y = 0)

$\#67:$  lin_diophant_solve(6·x + 6·y = 18) = [x = 1, y = 2]

Now the solving function covers all cases.

**Remarks:** (a) the method of solving linear equations with continued fractions is briefly described in ([2], p.195). The continued fractions solution is restricted by its nature to equations with two variables. (b) Historically the theory of continued fractions was further developed by Lagrange to solve Pell´s equation $x^2 - dy^2 = 1$ with $d \in \mathbb{N}$ and $\sqrt{d} \notin \mathbb{N}$. This method and its treatment with *DERIVE* wil be presented in a forthcoming paper.

**References**

[1]  Hardy, G. & Wright, E.M.: *An Introduction to the Theory of Numbers*, 5th edition, Oxford, At the Clarendon Press 1979

[2]  Scheid, H.: *Zahlentheorie*, 2. Auflage, BI-WIssenschaftsverlag, Mannheim 1994

# Actuarial Math with Derive 6

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This worksheet provides a set of basic functions, such as D, N, S, C, M, and R, for computing the actuarial present values of life insurance and life annuities, as well as premiums and contributions.  It assumes a basic working knowledge of actuarial mathematics and Derive.

Each actuarial function allows for calculations based on three different life tables: a primary life (X), a secondary life (Y), and a joint life (J).  Mortality for the primary and secondary lives is specified by selecting appropriate mortality tables.  A good source of hundreds of mortality tables is The Society of Actuaries Table Manager 3.0 sofware that can be found at www.soa.org.  Simply enter Table Manager in the search box on the home page and follow the instructions for downloading and using the software.  The individual mortality tables can be exported from the Table Manager to a text file.  If you cut out all the extraneous text and save only the age and mortality numbers in a text file, the file extenstion can be renamed to .dat and loaded into Derive as a data file.  The resulting 2-column matrix should be given an appropriate name and then saved as a .mth file for later use with the functions in this worksheet.

I used two sources in developing the actuarial functions.  The first is *Actuarial Mathematics*, by Bowers, et. al., The Society of Actuaries, 1986.  The second is *QLIB/48: Users Guide*, by Steve Lindblad, 1997; this is an actuarial library for the HP-48GX calculator.  Any errors in the concepts or code developed here are, of course, my own!  If, by chance, you find any errors, please let me know at one of the email addresses given above.

On the other hand, if you find this worksheet useful, I would also appreciate hearing from you.  Any comments, questions, or suggestions for improvement are, of course, most welcome.

The functions and their arguments included in this worksheet are (where x means a person age x):

1. CommutationFunctions(tabs,i,axmy:=0,tω,r:=10^9), compute commutation tables
2. lx(age,tab:=x), number living at age x
3. Qx(age,tab:=x), probability of x dying in one year
4. Px(age,tab:=x), probability of x living one year
5. nPx(age,n,tab:=x), probability of x living n years
6. nQx(age,n,tab:=x), probability of x dying within n years
7. tnQx(age,t,n,tab:=x), probability of x living t years and then dying within n years
8. D(age,tab:=x), D commutation function

```
9.  DM(age,m,tab:=x), DM commutation function
10. N(age,tab:=x), N commutation function
11. NM(age,m,tab:=x), NM commutation function
12. S(age,tab:=x), S commutation function
13. SM(age,m,tab:=x), SM commutation function
14. C(age,tab:=x), C commutation function
15. CM(age,m,tab:=x), CM commutation function
16. M(age,tab:=x), M commutation function
17. MM(age,m,tab:=x), MM commutation functions
18. R(age,tab:=x), R commutation function
19. RM(age,m,tab:=x), RM commutation function
20. CaM(n,m), n-year certain annuity payable mthly
21. CLM(age,n,m,tab:=x), n-year certain & life annuity payable mthly
22. CL12(age,n,tab:=x), n-year certain & life annuity payable monthly
22. JSM(age,p,m), joint & survivor life annuity payable mthly
23. JS12(age,p), join & survivor life annuity payable monthly
```

## 1. Setting the Assumptions

The commutation functions are determined by the mortality table(s) you select, any age set-backs or set forwards, the interest rate, age difference between the primary and secondary lives, terminal age of the table(s), and radix. All this information goes into the `Commutation-Functions(tabs,i,axmy:=0,tω,r:=10^9)` program.

The arguments are:

`tabs` -- a vector or 2-row matrix specifying a primary or a primary/seconday mortality table and any age setbacks or set forwards. Setbacks or forwards allow you to shift the mortality rates up to later ages (setback) or down to earlier ages (set forward). For example, a 2-year setback shifts the mortality rates up two years so that a person 50 years old, for example, will now have the same mortality rate as a person 48 years under the original, unshifted table. Setbacks therefore decrease the level of mortality for any given age (if the mortality rates are increasing, as is usually the case). Set forwards, on the other hand, increase the level of mortality for any given age. Setbacks are entered as positive numbers and set forwards as negative numbers. If there is no setback/forward simply enter a 0.

`i` -- the assumed interest rate expressed as a percentage, i.e., an interest rate of .06 is entered as 6.

`axmy` -- the age difference between x and y. For example, if x is 62 years old and y is 60, `axmy` = 2. On the other hand, if x is 60 and y is 62, `axmy` = -2. The default for `axmy` is 0 (zero).

`tω` -- the terminal age truncation options allows you to set an age beyond which the table are not permitted to extend. If the terminal age of the mortality table is greater than the `tω` value, mortality rates at the end of the table are discarded to force the desired terminal age. This can be useful when trying to match commutation functions produced with software based on

truncated tables.  For example, if tω is set to 100, then mortality rates above 99 are discarded and the mortality rate for age 99 will be set to 1, so that the terminal age of the table is 100.  If no value is set for tω or it is set to an age greater than the truncation age of the table, no truncation will be done.

r -- the radix of the table.  The radix is the initial number of lives assumed to exist at the initial age of the mortality table.  The default value is 10^9.

The mortality tables used in the examples are the Male UP-94 and Female UP-94 tables. "UP" refers to uninsured pensioner.  They are tables #832 and #833 in the Society of Actuaries Table Manager 3.0 software.  The tables are named mup1994 and fup1994 in this worksheet.  They may be viewed by going to Author/Variable Value... and selecting the name of the table.

These two tables are used with no age setbacks or forwards, an interest rate of 6 percent, an age difference of 2, no terminal age truncation, and the default radix.

$$
\#1: \quad \text{CommutationFunctions}\left(\begin{bmatrix} \text{mup1994} & 0 \\ \text{fup1994} & 0 \end{bmatrix}, \ 6, \ 2\right)
$$

Notice that a 2-row matrix was used to specify the mortality tables and age setbacks/forwards.  In this instance, there are no setbacks/forwards, they are set to zero (0).  In specifying two mortality tables, a primary and secondary table of commutation functions are computed, as well as a joint-life table calculated from the primary and secondard tables and based on the age difference you specify.  If you want to use only one table based on the primary life, simply enter a 2-element vector speficifying the mortality table and age setback/forward.  Any prior secondary and joint tables will be deleted.

Now, select the function and click on the = (Simplify) button in the toolbar.

#2:    Tables Done!

Since the tables are very large, they are not displayed.  They have, however, the global names of x, y, and j.  The first four rows of the x (primary) table are displayed below.  The first column is the age, the second is the number living at the beginning of that age, the third is the D commutation function, the fourth is the N commutation functions, the fifth is the C commutation function, and the last is the M commutation function.  In general, the D and N commutation functions are used for life annuities and the C and M commutation functions for life insurance.  All calculations done with the commutation functions use a ratio of at least two of the functions.

Two other global variables created are irate -- the interest rate, and qxmy -- the age difference between X and Y.

#3:
$$[1, ..., 4]^x$$

#4:
| Age | l | D | N | C | M |
|-----|---|---|---|---|---|
| 1 | 1000000000 | 943396226.4 | 16344980654.4 | 566927.7322 | 18208642.2 |
| 2 | 999363000 | 889429512.2 | 15401584427.9 | 360806.3115 | 17641714.47 |
| 3 | 998933273.9 | 838723639.2 | 14512154915.7 | 282475.7917 | 17280908.15 |

#5: irate

#6: 6

#7: qxmy

#8: 2

## 2. Using the Commutation Functions

Example 1: Compute the actuarial present value (APV) of a life annuity of 1 payable at the beginning of each year to X age 50 based on the assumptions in the commutation functions computed above.  (By stipulating "X age 50" I mean a person age 50 using the primary table of commutation functions.)

The APV of a life annuity of 1 payable at the beginning of the year is given by Nx/Dx, i.e., the N commutation function for age x divided by the D commutation function for age x.

#9:
$$\frac{N(50)}{D(50)}$$

#10:  13.99577103

The APV is 13.99577...  Notice that in the N and D commutation functions I did not specify a primary or secondary life; this is because the commutation functions default to the primary life (X).  If I wanted to compute the same APV for the secondary life (Y), I would enter:

#11:
$$\frac{N(50, y)}{D(50, y)}$$

#12:  14.78052895

Or, for an annuity payable on the joint life status of X and Y, i.e., as long as both X and Y are alive.

#13:
$$\frac{N(50, j)}{D(50, j)}$$

#14:  13.30680737

Example 2: Compute the same APV but with the annual annuity of 1 payable monthly.  That is, the annuitant receives 1/12 of the annuity every month.  For this caculation use the NM(age,m,tab:=x) function instead of the N function.  In this case, m would equal 12 since the annuity is paid monthly.  m can take on any value from 1 to 8; 8 is used for annuities payable continuously.  For X age 50 we have:

Example 2: Compute the same APV but with the annual annuity of 1 payable monthly. That is, the annuitant receives 1/12 of the annuity every month. For this caculation use the `NM(age,m,tab:=x)` function instead of the `N` function. In this case, m would equal 12 since the annuity is paid monthly. m can take on any value from 1 to 8; 8 is used for annuities payable continuously. For X age 50 we have:

$$\#15: \quad \frac{NM(50,\ 12)}{D(50)}$$

$$\#16: \quad 13.53158441$$

For Y age 50 we have:

$$\#17: \quad \frac{NM(50,\ 12,\ y)}{D(50,\ y)}$$

$$\#18: \quad 14.31656285$$

Example 3: Compute the same APVs but with the annuities payable at the end instead of the beginning of the period. For the annuity payable annually simply increase the age of the `N` commutation function by 1. For X age 50:

$$\#19: \quad \frac{N(51)}{D(50)}$$

$$\#20: \quad 12.99577103$$

To calculate the APV of annuities payable mthly at the end of the period, enter m as a negative value.

$$\#21 \quad \frac{NM(50,\ -12)}{D(50)}$$

$$\#22: \quad 13.44825108$$

(The fact that a negative payment frequency gives you annuities payable at the end of each period is a consequence of the mathematical definition of the actuarial functions.) The result given by `N(51)/D(50)` can be gotten, of course, from setting m to -1 in the `NM` function.

$$\#23: \quad \frac{NM(50,\ -1)}{D(50)}$$

$$\#24: \quad 12.99577103$$

Example 4: Calculating a 75% Joint-and-Survivor Annuity

The `j` table, along with the `x` and `y` tables can be used to calculate joint-and-survivor (or last-to-die) annuities. Assume X, age 65, receives an annuity of $30,000 a year as long as he lives and his wife Y receives 75% of $30,000 (or $22,500) for as long as she lives after X dies. Assume also that the annuity is payable monthly at the beginning of the month and

that Y is 2 years younger than X.  Use the commutation functions already computed with the mortality tables, setbacks/forwards, and interest rate as stipulated above.  Compute the APV of this annuity.

This APV can be computed in 3 steps.  First, X will receive an annuity as long as he lives of $30,000.  The APV is:

$$\#25: \quad \frac{30000 \cdot NM(65,\ 12)}{D(65)}$$

$$\#26: \quad 303285.7073$$

Second, Y will receive an annuity of $22,500 as long as she lives.  Since she is 2 years younger than X, enter her age as 63.  The APV is:

$$\#27: \quad \frac{22500 \cdot NM(63,\ 12,\ y)}{D(63,\ y)}$$

$$\#28: \quad 265181.3315$$

But Y will not recieve her annuity as long as both X and Y are alive.  From the addition of their annuities' APV, we need to subtract the APV of $22,500 for as long as both are alive.  The joint mortalilty is indexed to X's life with Y's age equal to X's minus their age difference.  So, when using the joint commutation functions enter only X's age.  The APV is:

$$\#29: \quad \frac{22500 \cdot NM(65,\ 12,\ j)}{D(65,\ j)}$$

$$\#30: \quad 202485.1758$$

The APV of this 75% joint-and-survivor annuity is therefore $365,982.

$$\#31: \quad 303285.7073 + 265181.3315 - 202485.1758 = 365981.863$$

If X is alive, he receives his annuity of $30,000; Y's annuity and the joint annuity of $22,500 cancel each other out.  If X dies and Y remains alive, X's annuity of $30,000 is no longer paid and the joint annuity of $22,500 is no longer viable either; this leaves Y with an annuity of $22,500 for as long as she lives.  The same APV can be computed with the JS12(age,p) function, where p is the percentage of X's annuity received by Y.

$$\#32: \quad 30000 \cdot JS12(65,\ 75)$$

$$\#33: \quad 365981.863$$

Example 5:  Suppose X, age 25, wants to provide for a retirement annuity of $2,500 per month when he retires at age 65.  What is the APV of this deferred annuity?

$$\#34: \quad \frac{2500 \cdot 12 \cdot NM(65,\ 12)}{D(25)}$$

$$\#35: \quad 25784.22814$$

If X wanted to make annual payments to fund his deferred annuity, what would they be?

$$\#36: \quad \frac{2500 \cdot 12 \cdot NM(65, \ 12)}{N(25) \ - \ N(65)}$$

$$\#37: \quad 1641.925657$$

X's contribution would be $1,641,93 payable at the beginning of each year until he was age 64.  Or X could make an annual contribution of $1,687.48 with 1/12th ($140.62) payable at the beginning of each month until he was age 64 and 11 months.

$$\#38: \quad \frac{2500 \cdot 12 \cdot NM(65, \ 12)}{NM(25, \ 12) \ - \ NM(65, \ 12)}$$

$$\#39: \quad 1687.477924$$


**Example 6:**  Y wants to receive a 10-year certain and life annuity of $3,000 per month payable at the beginning of the month when she retires at age 65.  What is the APV of the annuity at age 65?

This is a combination of a 10-year certain annuity and a 10-year deferred life annuity.  The function `CaM(n,m)` computes the present value of an annual annuity payable mthly for a certain number of years; if m is positive, it is an annuity due.  If m is negative, it is an annuity immediate.  And, if n is negative, `CaM(n,m)` computes the future value of the anuity certain. The APV of the certain and deferred life annuity, payable monthly, is $421,531.

$$\#40: \quad 3000 \cdot 12 \cdot \left( CaM(10, \ 12) + \frac{NM(75, \ 12, \ y)}{D(65, \ y)} \right)$$

$$\#41: \quad 421530.7227$$

If instead, the annuity was to be paid at the end of the month, simply enter -12 for m.  The APV is $418,763.

$$\#42: \quad 3000 \cdot 12 \cdot \left( CaM(10, \ -12) + \frac{NM(75, \ -12, \ y)}{D(65, \ y)} \right)$$

$$\#43: \quad 418762.6292$$


Two other functions included in the worksheet will compute a n-year certain and deferred life annuity.  They are `CLM(age,n,m,tab:=x)` and `CL12(age,n,tab:=x)`.

$$\#44: \quad 3000 \cdot 12 \cdot CLM(65, \ 10, \ 12, \ y)$$

$$\#45: \quad 421530.7227$$

$$\#46: \quad 3000 \cdot 12 \cdot CL12(65, \ 10, \ y)$$

$$\#47: \quad 421530.7227$$

**Example 7:** A firm's pension plan provides at retirement at age 65 $50 per month per year of service for its employees. It uses the Entry Age Normal (EAN) method of funding it's pension plan, an interest rate of 6% and the UP-1994 mortality tables. X, age 30, is just hired. What is the firm's normal cost, i.e., annual cost, for funding X's expected pension in the year he is hired and for all subsequent years assuming no changes in the retirement plan?

In this case, the normal cost is based on X's expected penion at age 65. This is $50 * 12 months * 35 years of service = $21,000. The normal cost is:

$$\#48: \quad \frac{nc \cdot (N(35) - N(65))}{D(65)} = \frac{21000 \cdot NM(65, 12)}{D(65)}$$

$$\#49: \quad \text{SOLVE}\left(\frac{nc \cdot (N(35) - N(65))}{D(65)} = \frac{21000 \cdot NM(65, 12)}{D(65)}, \ nc, \ \text{Real}\right)$$

$$\#50: \quad nc = 2276.983661$$

In this example, the APV of the pension at age 65 (right hand side of #48) is equated to the actuarial future value at age 65 (left hand side of #48) of the annuity used to fund the pension.

**Example 8:** What is the APV or net single premium of a whole life insurance policy of $100,000 payable at the end of the year of death for X age 25? The APV of a life insurance policy is given by Mx/Dx.

$$\#51: \quad \frac{100000 \cdot M(25)}{D(25)}$$

$$\#52: \quad 6022.617956$$

If X wants to pay annual premiums on this policy at the beginning of each year for as long as he lives, what would it be?

$$\#53: \quad \frac{100000 \cdot M(25)}{N(25)}$$

$$\#54: \quad 362.7499466$$

Monthly premiums would be:

$$\#55: \quad \frac{100000 \cdot M(25)}{\dfrac{NM(25, 12)}{12}}$$

$$\#56: \quad 31.09722343$$

**Example 9:** Y wants to insure her life for $1,000,000 during her working years from age 25 to 65, payable at the moment of death. But she only wants to pay annual premiums for 20 years. What is the amount of the premium?

For this example use the `MM(age,m,tab:=x)` function with m set to 8. For life insurance, m determines when the benefit is paid. A positive number means it is paid at the end of the period of death. A negative number means it is paid at the beginning of the period of death. For example, if m = 12, the death benefit is paid at the end of the month of death. If m = 1, it is paid at the end of the year of death. If m = 8, it is, of course, assumed to be paid at the moment of death. If you noticed above that the sign of m means the reverse for life annuities, you are correct. Again, this is a consequense of the mathematical definition of the actuarial functions.

$$#57: \quad \frac{1000000 \cdot (MM(25, \infty, y) - MM(65, \infty, y))}{N(25, y) - N(45, y)}$$

#58:   1230.800747

**Example 10:** The commutation functions `S(age,tab:=x)` and `R(age,tab:=x)` are used for calculations involving increasing annuities and increasing life insurance, respectively. `SM(age,m,tab:=x)` and `RM(age,m,tab:=x)` refer, of course to increasing life annuities and life insurance payable mthly. For instance, what is the APV of a life annuity of $1,000 that increases by $1,000 every year to X who is age 65?

$$#59: \quad \frac{1000 \cdot S(65)}{D(65)}$$

#60:   90139.53494

Or, what is the APV of a life insurance that pays $10,000 the first year if death occurs and increases by $10,000 each year after to Y who is age 35?

$$#61: \quad \frac{10000 \cdot R(35, y)}{D(35, y)}$$

#62:   28970.10445

## 3. Non-integral Ages and the Uniform Distribution of Deaths

All the commuation functions accept non-integral ages, e.g, 50.5 or 32.3. Their values are calculated on the assumption that deaths during a year follow a uniform distribution. However, if non-integral ages are used with functions such as NM or MM where abs(m) > 1, there are small relative errors in the calculations. Towards the end of the mortality table the errors can become large. If non-integral ages are used only when abs(m) = 1; or if only integral ages are used when abs(m) > 1, there is no error, given the uniform distribution of deaths assumption.

## 4. Q's and P's

The Q's and P's associated with actuarial math are the probabilities of dying and living. (In actuarial textbooks they are always lower case.  Since DERIVE does not do sub- or super-scripts very well, I have capitalized Q and P.)  For instance, based on the mortality table, Qx is the probability that a person age x will live one year.  Px is the probability that a person age x will live one year.  nQx is the probability that a person age x will die within n years; nPx is the probability that a person age x with live n years.  And finally, tnQx is the probability that a person age x will live t years and then die within n years.  (The function lx(age,tab:=x) gives the number living at any given age.  It is used to calculate the Q's and P's.)

The values of the q's and p's are available for X, Y, and J, that is, the primary life, secondary life, and joint life status of X:Y.


Example 11:  What is the probability that X age 50 will die in one year?  Ten years?  Will live 10 years and then die in the next 5 years?

        #63:  Qx(50)

        #64:  0.002773

        #65:  nQx(50, 10)

        #66:  0.04691685322

        #67:  tnQx(50, 10, 5)

        #68:  0.0516509792


Example 12:  What is the probability that the joint & survivor status of X:Y will last 10 years if X is 65 and Y 2 years younger?  Since qxmy = 2, we have:

      #69:  nPx(65, 10) + nPx(65 − qxmy, 10, y) − nPx(65, 10, j)

      #70:  0.97409016

The probability that either X or Y or both will be alive in 10 years is .974.

This is simply the probability that X and Y survive 10 years minus the joint probability that they both live 10 years.  Subtracting the joint probability basically eliminates double counting. See any textbook on elementary probability theory.


## 5.  Conclusion

This is just a taste of what can be done with actuarial mathematics.  The commutation func-tions can be used for calculating life insurance premiums, reserves and cash values, or the normal cost and actuarial value of pensions, or contributions to tax deferred annuities, etc., etc.

# AVOIDING CONVOLUTION AND TRANSFORM METHODS

By: Milton Lesmes Acosta. Bogotá, Colombia.

## 1. Intersection of areas and Convolution

When you need to express the amount of overlap of one object over another you will find some common aspects with the mathematical concept of convolution.

The following picture illustrates the area of intersection of a static triangle and a square which moves in the direction of V.
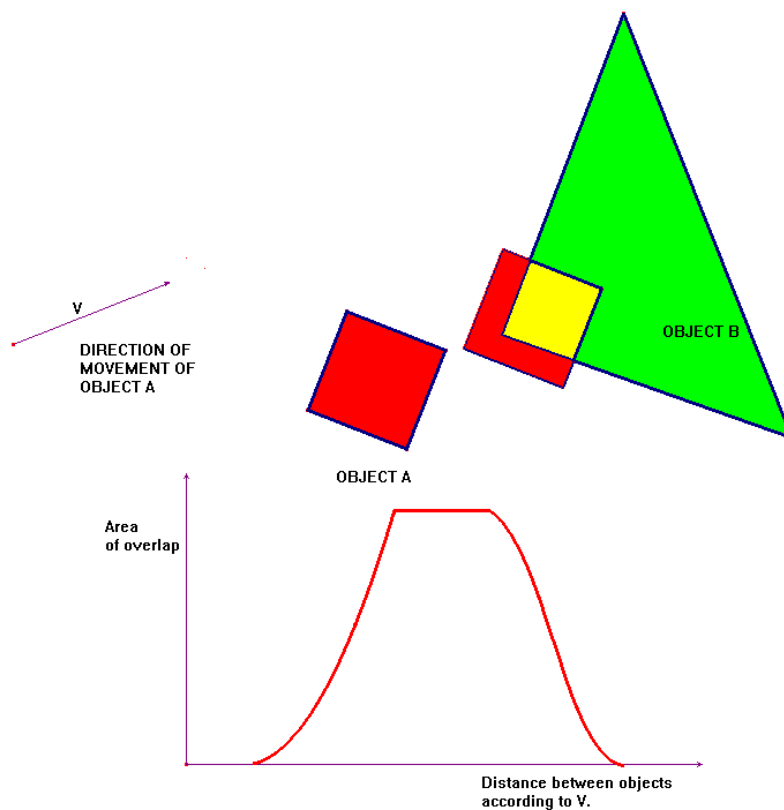


Compare this with the measure of the overlap of two real integrable functions $f, g$ defined over the interval $[0, t]$ and the expression $\int_0^t f(\tau)g(t-\tau)d\tau$

## 2. Linear Time Invariant Systems.

The notion of convolution appears naturally in Linear Time Invariant Systems.

In the distribution theory the representation

$$f(t) = \int_{-\infty}^{+\infty} f(\tau)\delta(t-\tau)d\tau$$

is a tool to explain why the convolution integral appears in this context. The signal $f$ is processed by a LTI System $H$ affecting the Dirac delta distribution because of the linearity

$$H(f(t)) = H\left(\int_{-\infty}^{+\infty} f(\tau)\delta(t-\tau)d\tau\right) = \int_{-\infty}^{+\infty} f(\tau)H(\delta(t-\tau))d\tau$$

Let the response of the System to the Dirac delta distribution be $H(\delta(t)) = h(t)$ then if the LTI System is invariant in time (that is the system always responses in the same way no matter the instant of time) it follows that $H(\delta(t-\tau)) = h(t-\tau)$, then we get

$$H(f(t)) = \int_{-\infty}^{+\infty} f(\tau)h(t-\tau)d\tau$$

which is known as the convolution between $f$ and $h$

$$f * h = \int_{-\infty}^{+\infty} f(\tau)h(t-\tau)d\tau$$

### 3. Convolution and *DERIVE*

For a *Derive* user in many cases it is easy to calculate this integral, for example if

$$f(t) := t.chi(0,t,1)$$

$$g(t) := 2.chi(0,t,2)$$

then

$$INT(f(\tau) \cdot g(t-\tau), \tau, -\infty, +\infty)$$

is the convolution represented in the following picture
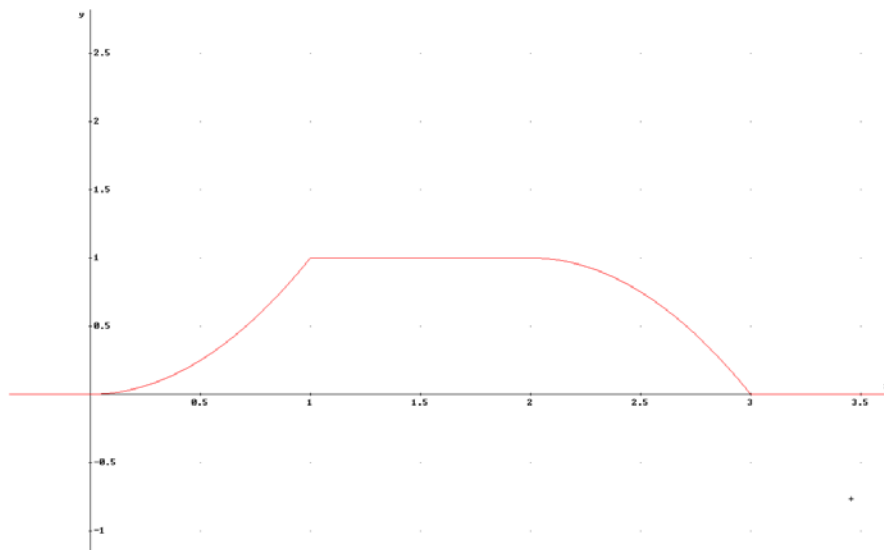


### 4. The discrete convolution.

The convolution of two discrete signals is given by

$$f(m) * g(m) = \sum_{k=-\infty}^{+\infty} f(k)g(m-k)$$

For two discrete signals $f, h$ defined over the set $\{1, 2, 3, ..., n\}$ we have $[f(1), f(2), f(3), ..., f(n)]$ and $[h(1), h(2), h(3), ..., h(n)]$ the derive user can get the convolution $f * h$ as the coefficients of the product

$$\left(f(1)x + f(2)x^2 + f(3)x^3 + ... + f(n)x^n\right).\left(h(1)x + h(2)x^2 + h(3)x^3 + ... + h(n)x^n\right)$$

Maybe it is important to remember the DNL solutions to Rainer Wonisch´s question in Newsletter #49 p 4. and the findings of some DERIVER´s. Of course, you will recognize the z-transform in this process and the interested reader must see the dfw files of Terence Etchells.

One way to get the coefficient of $x^m$ of the polynomial p(x) is:

define p(x), calculate

$$\frac{DIF(p(x), x, m)}{m!}$$

and evaluate at x:=0.

## 5. The sum of two independent random variables and convolution.

In the random experiment consisting of tossing a fair die, for the random variable X "the number of the point in the die", the probability mass function is shown in the following figure
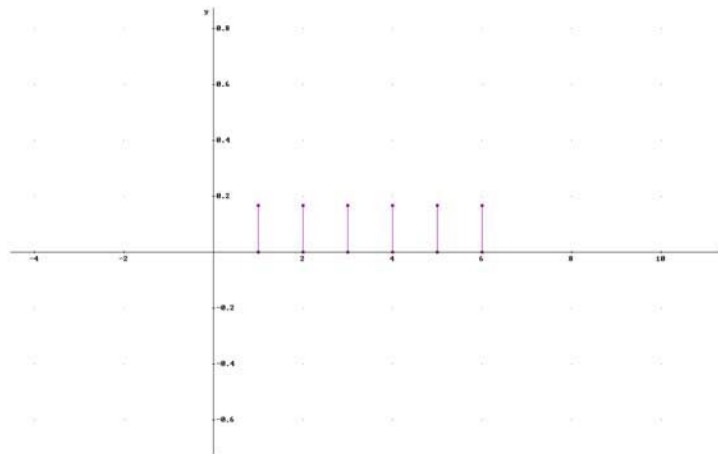


The random experiment of tossing two fair dice has a probability mass function which can be calculated as a convolution to get
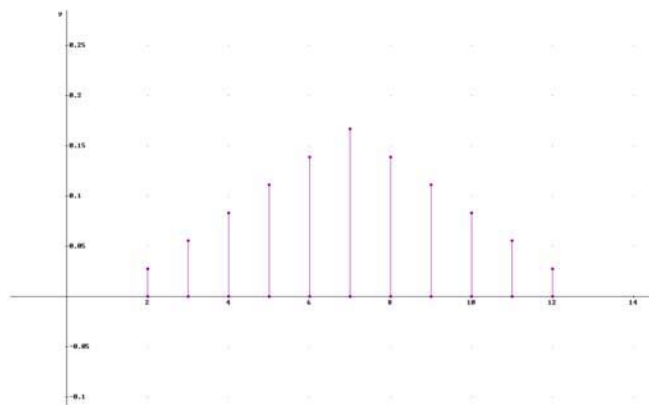
This was done with the coefficients of the product

$$\left(\frac{1}{6}x+\frac{1}{6}x^2+\frac{1}{6}x^3+\frac{1}{6}x^4+\frac{1}{6}x^5+\frac{1}{6}x^6\right)\cdot\left(\frac{1}{6}x+\frac{1}{6}x^2+\frac{1}{6}x^3+\frac{1}{6}x^4+\frac{1}{6}x^5+\frac{1}{6}x^6\right)$$

In the continuous case with the uniform distribution that has the probability mass function

$$f_X(x)=\begin{cases}1, & for \quad 0\le x\le 1\\ 0, & otherwise\end{cases}$$

the probability mass function of the sum of the random variables each distributed uniform is the con-

volution $f = f_X * f_X$, the distribution function $F(x) = \int_{-\infty}^{x} f(t)dt$ is shown in the following picture
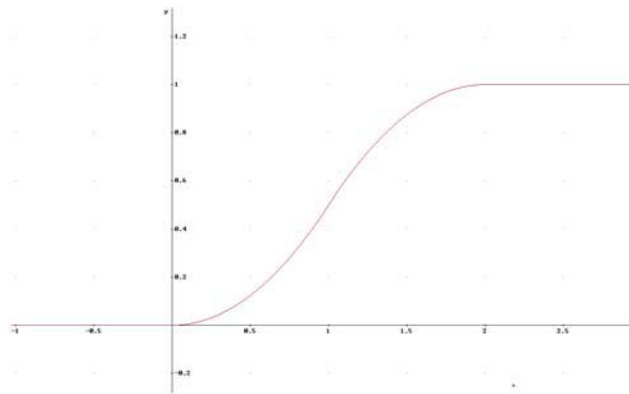


As it is seen it is needed to calculate a convolution and a integral to get the distribution function. Of course that for the integral we could use a Monte Carlo method.

### 6. The Derive, avoiding Convolution and Monte Carlo Method.

First it is described the method to be used: Take samples of the random variable. Sort this samples. Distribute this samples over the interval (0,1)

This is done with the derive instructions

$$W := sort(vector([s(k)], k, 1, 10000))$$

$$vector\left(\left[\det([W_i]), \frac{i}{10000}\right], i, 1, 10000\right)$$

Where s(k) is the k-value of sample of the random variable, it is needed to implement a method to generate s(k) depending of the random variable (or the experiment).

Example: Choose two independent numbers of the interval $[0,1]$ calculate the sum (that is the continuous random variable of example of 5.), The 10000-sample of this random variable is in

$$W := sort(vector([random(1)+random(1)], k, 1, 10000))$$

and the distribution $vector\left(\left[\det([W_i]), \frac{i}{10000}\right], i, 1, 10000\right)$ is in the following graph
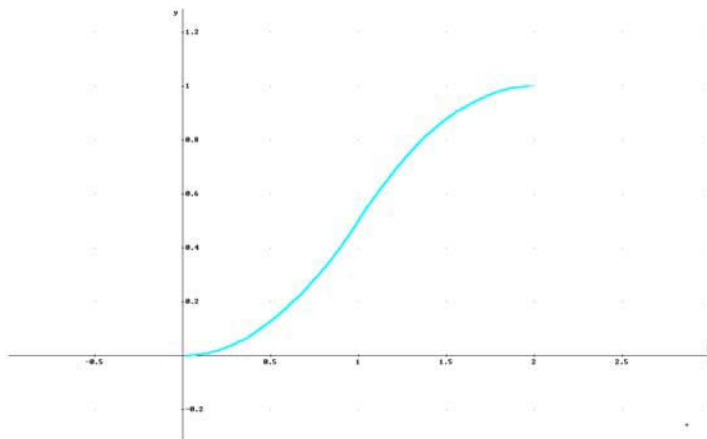
this is a nice result, and it is seen how the convolution and the counting of the generated points (to apply the Monte Carlo method) are avoided.

The following figure shows a distribution of 10000 points result of the instruction

$$vector\left(\left[random(1)+random(1),\frac{k}{10000}\right],k,1,10000\right)$$
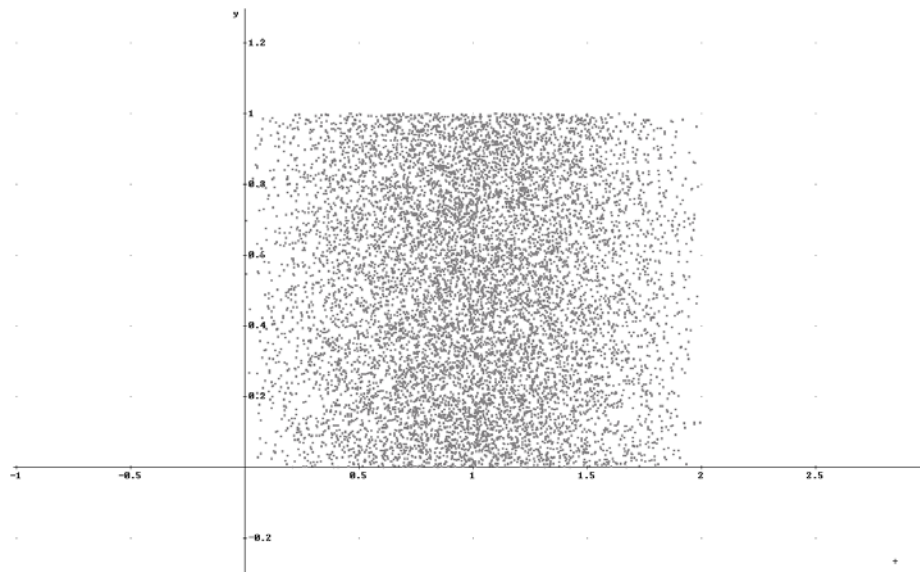


## 7. Trying with discrete cases.

Finally it is shown the result of the formula $vector\left(\left[\det([W_i]),\dfrac{i}{10000}\right],i,1,10000\right)$ in case where the random variable is discrete.

$$W := sort\left(vector\left([random(6)+1+random(6)+1],k,1,10000\right)\right)$$

correspond to a 10000-sample of the random experiment of tossing two fair dice, the following figure shows the result of applying $vector\left(\left[\det([W_i]),\dfrac{i}{10000}\right],i,1,10000\right)$ to this sample.
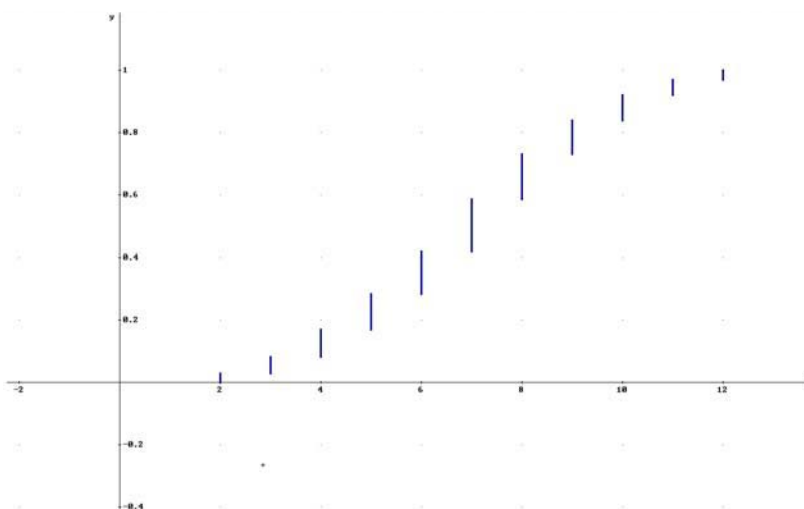
**Martin Lindsay**                                                    MLindsay@distance.vic.edu.au

Dear Joseph Bohm

I have attached my DUG membership form.

I was wondering whether you could help me. My university has just purchased Derive 6.

My problem is this. I'm doing a first year Calculus unit. The usual stuff that you find in any US $130 textbook(!): Derivatives, Applications of, integrals, techniques of, Inverse Trig, Hyperbolics, etc. I thought, given my background and research (I have a PhD in mathematics education) I could put together some Derive labs on this stuff, but it is proving a little harder than I thought.

For a start, there seem to be no Derive/CAS books as such at this level, most of it is pre-calculus and Year 12 calculus. Maybe I'm looking in the wrong places? I've got a couple of UK texts for the old Derive versions which I can adapt for things like applications of Calculus, etc. But that only gets me to week 3 in the course.

My class is once a week, 3 hours late afternoon/early evening. I do a lecture up till 6, then a 1 hour lab after a break. Ideally I'm looking for a couple of lab problem solving exercises each week that will consolidate the theory stuff after the tea break and stop them sneaking out of the lab when my back is turned! Have you got any ideas? I've looked at the Derive User Group discussion but it all seems so esoteric and difficult!

I'd appreciate any feedback and suggestions as to what I could do.

Regards
Martin Lindsay

If you have any advice for Martin Lindsay then please contact him. Many thanks, Josef

January, February and March were very busy with receiving, answering and forwarding *DERIVE*-related questions. I bothered and urged Albert Rich and Theresa Shelby with lots of questions. I hope that you all will find the questions and the answers interesting enough to be published in the DNL. Many problems address differences between *DERIVE* 5 and *DERIVE* 6. I recommend to check if you need to adapt functions and programs of your one which work properly in *DERIVE* 5 for their functionality in *DERIVE* 6. I can tell from my own experience that I sometimes had very unexpected results working with old DfW5-functions in DfW 6. Josef

Hello Josef,

I apologize for the delay in responding to the questions raised in your emails. Today I answered the question in your 18 January email via separate email. The following is in response to your numbered questions in the file `questions_for_Albert.dfw`:

## Question 1 (presented by Josef)

$$\#1: \quad 14 \cdot n^2 - 13 \cdot n < 24 \cdot n^2 + 5 \cdot n + 9$$

I want to simulate solving this inequality step by step manually;
#1 - (14n^2-13n)
and I expect 0 < 10n^2+18n+9 (as I ever found working with DERIVE 5).

$$\#1: \quad 14 \cdot n^2 - 13 \cdot n < 24 \cdot n^2 + 5 \cdot n + 9$$

$$\#2: \quad (14 \cdot n^2 - 13 \cdot n < 24 \cdot n^2 + 5 \cdot n + 9) - (14 \cdot n^2 - 13 \cdot n)$$

$$\#3: \quad 0 < 10 \cdot n^2 + 18 \cdot n + 9$$

$$\#2: \quad (14 \cdot n^2 - 13 \cdot n < 24 \cdot n^2 + 5 \cdot n + 9) - (14 \cdot n^2 - 13 \cdot n)$$

$$\#3: \quad n \cdot (5 \cdot n + 9) > - \frac{9}{2}$$

and another attempt is even worse:

$$\#4: \quad (14 \cdot n^2 - 13 \cdot n < 24 \cdot n^2 + 5 \cdot n + 9) - 14 \cdot n^2 + 13 \cdot n$$

$$\#5: \quad n > - \frac{10 \cdot n^2 - 164 \cdot n + 9}{182}$$

## Answer 1

In Derive 6 when an expression containing inequalities is simplified, like all other functions and operators (except equalities as discussed in Question 5 below), the inequalities are actively simplified as well. For example,

$$2 + 3$$

simplifies to the scalar value 5 and

$$2 < 3$$

simplifies to the Boolean truth-value "true". Therefore,

$$(2 < 3) - 2$$

simplifies to "true - 2", since it makes no sense to add scalars to Boolean truth-values.

## Question 2 (presented by Jan Vermeylen)

Why is `vector(factor(k),k,2005,2006)` simplifying without factoring 2006 into its prime factors?

#6:  FACTOR(2006) = 2·17·59

but:

#7:  VECTOR(FACTOR(k), k, 2005, 2006) = [5·401, 2·1003]

or

#8:  VECTOR(FACTOR(10^k), k, 1, 3) = [2·5, 10^2, 10^3]

#9:  FACTOR([10, 100, 1000]) = 10·[1, 10, 100]

*Factoring out the gcd of the elements of the vector is a nice new feature!!*

## Answer 2

The problem factoring integers other than at the top-level has been resolved for the next release of Derive 6 as discussed in my email to the Derive Newsgroup.

## Question 3 (presented by Andre Schmidt)

The *DERIVE*-calculation $\int e^{m \cdot x} \, dx = \dfrac{e^{m \cdot x}}{m} - \dfrac{1}{m}$ shows a strange (incorrect?) result.

Are there any explications for this behaviour?

## Answer 3

The antiderivative INT (#e^(x*m), x) simplifies to  #e^(m*x)/m - 1/m  instead of just #e^(m*x)/m  because the limit of the latter as m approaches 0 is infinite.  Whereas the limit of the former as m approaches 0 is x, which is the same as the result of substituting m = 0 before integrating the above expression.  This is why Derive includes the constant -1/m in the antiderivative.  Note that similar remarks apply to the antiderivative returned by INT(x^m, x) where the problem occurs at m = -1.

## Question 4 (presented by Fritz Mohr)

#11:  M := $\begin{bmatrix} 0.4 & 0.2 \\ 0.5 & 0.3 \end{bmatrix}$

#12:  [M·M, M·M·M] = $\left[ \begin{bmatrix} 0.26 & 0.14 \\ 0.35 & 0.19 \end{bmatrix}, \begin{bmatrix} 0.174 & 0.094 \\ 0.235 & 0.127 \end{bmatrix} \right]$

but defining n as integer and positiv

#13:  n :∈ Integer (0, ∞)

#14:  M^n = $\begin{bmatrix} 0.4^n & 0.2^n \\ 0.5^n & 0.3^n \end{bmatrix}$

*This is obviously wrong! DERIVE 5 works correct! Do you have any advice?*

using the VECTOR-command gives no problems:

#15:  VECTOR(M^n, n, 2, 3) = $\left[ \begin{bmatrix} 0.26 & 0.14 \\ 0.35 & 0.19 \end{bmatrix}, \begin{bmatrix} 0.174 & 0.094 \\ 0.235 & 0.127 \end{bmatrix} \right]$

## Answer 4

Even if the degree of a matrix power is symbolic, the next release of Derive will not distribute the power over the elements of the matrix.

## Question 5 (again presented by Josef)

I am co-author of a series of technology based textbooks for secondary schools and wanted to support a colleague in treating simultaneous linear equations. See what happens:

#16:   eq1 := 3·x − 2·y + z = 10

#17:   eq2 := 4·x − y + 2·z = 20

#18:   eq3 := 2·x + y − 5·z = −10

#19:   [eq4 := 4·eq1 − 3·eq2, eq5 := 2·eq1 − 3·eq3]

#20:   eq4 = (− 5·y − 2·z = −20)

#21:   eq5 = (17·z − 7·y = 50)

This is ok. But collecting the new equations in a vector - strange output!!

$$\#22: \quad \begin{bmatrix} eq4 \\ eq5 \end{bmatrix} = \begin{bmatrix} 7·x − 3·y + 3·z = 30 \\ 5·x − y − 4·z = 0 \end{bmatrix}$$

This does not happen in *DERIVE* 5. I am sure that this might confuse many teachers and students.

## Answer 5

Derive 6 artificially suppresses active simplification of equalities entered at the top-level, making it possible to manually solve them by adding and multiplying constants.  However, this suppression leads can lead to confusing results like those raised in Question 5.  For version 7 of Derive, some other method will have to be found allowing equations and inequalities to be solved manually *without* suppressing the simplification of these expressions.

## Question 6 (presented by Gary Turner, Rochester College, and others)

I have a question concerning rectangular coordinates and spherical coordinates.

Using my calculus text, I convert the spherical point $(9, \pi/4, \pi)$ to the rectangular point $(0,0,9)$. However, when I graph these two points using *DERIVE*, the points do not match up. That is, I'm seeing two points in space when I should be seeing only one point. I'm graphing $(9, \pi/4, \pi)$ in the spherical system and I'm graphing $(0,0,9)$ in the rectangular system.

Any help would be greatly appreciated

> The next minor release of Derive will convert 3D data point plots using the current coordinate specified by the user.  I apologize for the inconvenience this bug may have caused.

Is it possible to avoid simplification of hyperbolic functions. A colleague wants to apply
$\sinh(x+y) = \sinh(x)\cosh(y)+\cosh(x)\sinh(y)$ in both directions and expand $\sinh(x)–\sinh(y)$ into a product without simplifying the expressions in expontial functions.
Is this possible? Are there any tricks or settings?

> No, in the current version of Derive hyperbolic functions are always converted to exponential form and are not converted back to hyperbolic functions for display.  However, I hope to do this conversion for the next major release of Derive.

By the way, a user from Austria asked, if it is possible to programm the graphic commands (plot or draw ....)
You know that this is one of our great wishes for the future - it is just a reminder

> Yes, the ability to program plot commands is high on the priority list for the next major release of Derive.
> Hope this helps,
> Albert Rich

**Question 7** (presented by Eberhard Lehmann, Berlin)

Dear Josef, once more the best for 2005 together with a question to the *DERIVE*-specialist:

Why does it need so much time for calculating:

```
        dreia(n) :=
          If n = 1
              19
#23:        If MOD(dreia(n - 1), 2) = 0
                dreia(n - 1)/2
                3·dreia(n - 1) + 1
```

```
#24:  [TABLE(dreia(n), n, 1, 10)]'
```

```
#25:  [[  1   2   3   4   5   6   7   8   9  10 ]]
       [[ 19  58  29  88  44  22  11  34  17  52 ]]
```

takes 0.05 sec

```
#26:  [TABLE(dreia(n), n, 1, 22)]'
```

takes 220 sec!!

## Answer 7

Hello Josef,

The time required to compute `dreia(n)` using the definition

dreia(n) := IF(n = 1, 19, IF(MOD(dreia(n - 1), 2) = 0, dreia(n - 1)/2, 3·dreia(n - 1) + 1))

increases dramatically with n because of what I call the "Fibonacci Fan-out Problem".

This problem is discussed in the Derive 6 on-line help in the section on Derive programming called "Recursive Functions". Consider the following definition of a function for computing Fibonacci numbers:

FIB(n) := IF(n < 2, n, FIB(n - 1) + FIB(n - 2))

FIB is slow because to compute FIB(n), FIB(n-2) is computed twice: once for FIB(n) and once for FIB(n-1). Similarly, FIB(n-3) is computed three times: once for FIB(n-1) and once for each of the two computations of FIB(n-2). In general, FIB(n-i) must be computed the $i^{th}$ Fibonacci number of times. For example, FIB(26) re-computes FIB(1) 75025 times! Thus the number of redundant calculations, and hence the time required to compute the nth Fibonacci number using FIB increases rapidly (Fibonaccially?) with n.

Such redundant computation is typical for recursive definitions that naively implement recurrences, like Fibonacci, that are doubly recursive. In the case of the dreia function, the solution to the problem is to simply save the value of dreia(n-1) so it can be reused as follows:

dreia(n) := PROG(IF(n = 1, RETURN 19), n := dreia(n - 1), IF(EVEN?(n), n/2, 3·n + 1))

Using this definition, it takes less than a 1/10 of a second to compute [TABLE(dreia(n), n, 22)]`. An even more efficient definition of `dreia` can be implemented using the LOOP programming construct.

Hope these explanations help.

Aloha,
Albert D. Rich
Co-author of Derive

## A Hardware Problem reported by Tania Koller and resolved by Günter Schödl:

Tania had a problem appearing on her brandnew laptop computer. Working with DERIVE she faced problems with the screen and found screen appearances as shown in the right picture .
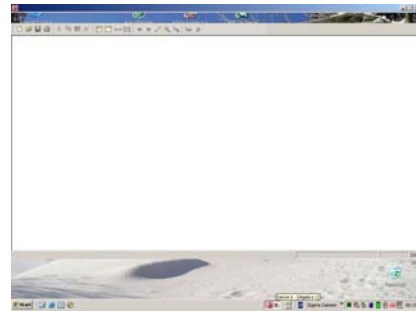


At the occasion of a seminar she asked Günter who is one of the most experienced IT-experts I know and he immediately could help. Tania´s graphic card was too fast. So they set "Hardware acceleration" back using a slider bar. Start navigating through the menus with the "Display Properties" > "Advanced" > "Troubleshoot" and reach finally "Hardware acceleration". (*Thanks to Walter Wegscheider for providing the English "path"!*)
In German: > Eigenschaften >Einstellungen > Erweitert > Problembehandlung > Hardwarebeschleunigung
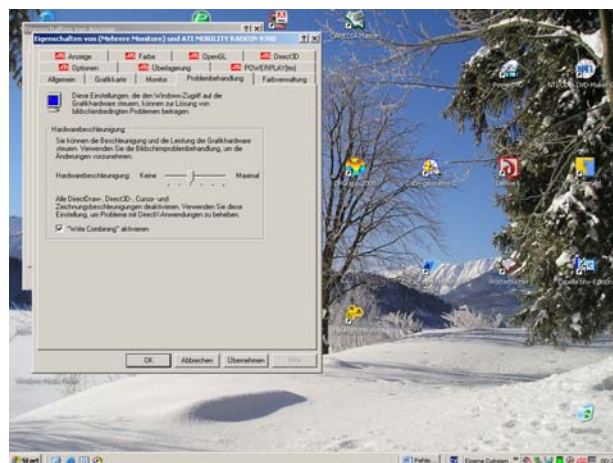


A mail from Canada

**McDougall Duncan**                      duncanemcdougall@hotmail.com

Dear Josef,

   Getting back into my routine and getting things back on schedule took more time than I anticipated this year but things have finally settled down a bit. What I want to tell you is how impressed my associates are with DERIVE. They had worked with MAPLE and couldn't believe how simple and direct using DERIVE was. WE ARE IMPRESSED !!!

A mail from Sweden

*Dear Josef!*
*Thank You for your mail.*
*I've got DNL #37-52 and now I'm happy to get the first 36 issues too.*
*That's great! There is always something interesting in each issue so please continue to publish them*
*on the internet. I'll also appreciate that You rewrits it for Derive 5.*
*Best regards from*
Sture Färnström

**Thierry Dana-Picard**

*Dear Theresa,*

*For a couple of months I have worked with Derive 6.1 and enjoy it very much.*

*I would like to share with you three cases where Derive gives the correct answer when asked to give a result immediately, but the step-by-step indications should perhaps be improved (IMHO):*

*1. the identitites employed here are not trivial; an average student would not understand them.*

*2. the same remark is valid here. In the other direction, the fact that Derive computes this integral for a "general" parameter r is remarkable.*

*In both cases, a shorter way exists.*

*3. here an explanation is needed: what leads the student to the usage of this (non trivial) formula?*

*The files are attached in the corresponding order.*

*May I once again thank you for the wonderful work you and your team are doing.*

*Yours sincerely,*

*Thierry Dana-Picard*

*Dept of Applied Mathematics - Jerusalem College of Technology*

#1: $\displaystyle \int \frac{\sqrt{x}}{\sqrt{x} + \sqrt{(a - x)}}\, dx$

$$\int F(\sqrt{(a + b\cdot x)},\ x)\, dx \rightarrow \frac{2}{b}\cdot\text{SUBST}\left(\int \left(x\cdot F\left(x,\ \frac{x^2 - a}{b}\right)\right)\, dx,\ x,\ \sqrt{(a + b\cdot x)}\right)$$

#2: $\displaystyle - 2\cdot\text{SUBST}\left(\int \frac{x\cdot\sqrt{(- x^2 + a)}}{\sqrt{(- x^2 + a)} + x}\, dx,\ x,\ \sqrt{(-x + a)}\right)$

#5: $\displaystyle \int_{0}^{\pi/2} \frac{1}{1 + e^{r\cdot(- \text{LN}(\text{COS}(x)) + \text{LN}(\text{SIN}(x)))}}\, dx$

$$\int_{a}^{b} F(x)\, dx \rightarrow \frac{\displaystyle\int_{a}^{b} (F(x) + F(a + b - x))\, dx}{2}$$

#6: $\displaystyle \int_{0}^{\pi/2} \frac{1}{2}\, dx$

Dear Thierry,

Thank you for your enthusiastic response to Derive 6.1. We are very glad that you enjoy it and appreciate your suggestions for improvement. I have forwarded these particular suggestions to Albert Rich, the author of the display step feature.

It is a pleasure hearing from you.

Regards,

Theresa

Hello Dr. Dana-Picard,

I also want to thank you for your kind words and suggestions for the Derive 6 Display Step feature. First some general comments about this new feature:

The transformation rules Derive displays are those *it* uses to simplify an expression. They may or may not be the same as those currently taught to students. However, if teachers see an advantage to an unfamiliar rule used by Derive, they may want to ask their students to verify the validity of the rule and then the students will have an additional tool in their arsenal.

Thanks in part to your examples and suggestions, numerous improvements to the Display Step rules have been made for the next release of Derive. New algebra and calculus simplification rules significantly improve the understandability of the steps used to simplify your example 1 (i.e. the double integration of the arccosine of x). The resulting dfw file is attached to this email.

Your original complaint for example 1 is that "the identities employed here are not trivial; an average student would not understand them". Instead of re-deriving integration rules for each new problem (e.g. using integration by parts or substitution), Derive uses general purpose rules. Rather than being a mysterious "black art", integration is reduced to determining the right rule to apply based on the form of the integrand.

As I suggest above, students encountering a rule with which they are not familiar provides the teacher with the perfect opportunity to ask the students to verify the rule. It seems to me that the ability of students to derive general purpose rules is preferable to their re-deriving special cases of those rules each time a new problem is encountered. Also the recognition that there *are* general purpose rules may be enlightening to some.

For your examples 2 and 3 the definite integration rule in question is

$$INT(F(x), x, a, b) \to INT(F(x)+F(a+b-x), x, a, b)/2$$

While apparently not yet widely known, it may be a rule that teachers should know about. As to your question as to when it should be applied for a given integrand $F(x)$, simply compute $F(x)+F(a+b-x)$ and see if it is simpler than $F(x)$. Note that before simplifying $F(x)+F(a+b-x)$, the domain of x should be declared to be the open interval $(a,b)$ . In both your examples, the $F(x)+F(a+b-x)$ simplifies to 1 which is certainly easier to integrate than the original integrands.

The symbolic power r in your example 3 unfortunately results in the display of the steps required to convert the expression to the exponential form used internally by Derive to represent symbolic powers. I hope to avoid the display of these unnecessary steps in a future release of Derive.

Hope this discussion has helped.

Aloha,

Albert


*Dear Albert,*

*Thank you very much for your detailed answer and for the dfw file; it is helpful.*

*For the rule*

$$INT(F(x), x, a, b) \to INT(F(x)+F(a+b-x), x, a, b)/2$$

*I agree with you that it is important, the point was really that it is not so well known (I use such transformations in one of my papers, but a friend of mine, in another institution, told me that he would not hope his students to make such work).*

*For the parametric example, it was a good surprise that Derive computes the integral for general r. We used this example as a core example in a joint paper with J. Steiner.*

*I´ll try to continue being in touch.*

*Yours,*

*Thierry.*


Dear all,
this is a very interesting discussion. I'd like to publish it in the next DNL. I am sure that many of our members - especially the teachers among them - would appreciate this. It is one of the special features of our bulletin, that we can spread such "Insider Information" among our world wide community.
Best regards to Hawaii and Israel,
Josef

Hello Josef,

**I would be delighted if this discussion of the Derive 6 Display Step feature was republished (especially considering the amount of time it took to write the original email :=). However, as educators with experiences broader than mine, I think you and the other recipients of this email should give your own insights into the full potential of the display step feature. In my own self interest, I volunteer you, Josef, to over-see this 'discussion group'.**

**To start things off, in a slightly altered state of consciousness at the beach today, I came up with the idea of a Display Step option for Derive 7 that would only show the antecedent (i.e. left side) of the transformation rule being applied. This would then be a great opportunity for a teacher to ask 'Ok students, what is the consequent (i.e. right side) of this rule?' When the class proposes a rule, the teacher asks them to prove its validity and the domain to which it applies. If the proposed rule turns out to be invalid, the students will not be able to prove it and they will have to come up with another proposed rule.**

**When the correct rule is finally proved, the teacher can send a command to all (or some) copies of Derive on the local area network to mark this rule as "available for use". Then the Display Step feature will use and display the rule without bugging the students any more. I think this is a good example of Bernhard's scaffolding paradigm for teaching mathematics. Students taught in this way would not be forced to clutter their minds memorizing rules (ug!). Rather, they would learn how (_if_ they felt it necessary) to re-derive rules encountered when solving real-world problems in the future.**

**Just a thought.**

Aloha,

Albert

P.S. As far as the rule

$$INT(F(x), x, a, b) \rightarrow INT(F(x)+F(a+b-x), x, a, b)/2$$

not being well known: My being forced to use it to automate definite integration seems like a pretty good indication of its utility to me. It should be taught to students if it is not already. I know it sounds kind of spooky, but maybe we should listen more to what computers have to 'say' about how the Universe works...

*Dear All,*

*Thanks to all the people who related to my original e-mail. This discussion is very interesting.*

*I totally agree with Albert Rich's final remark. Even in education, too many people expect from a CAS to make computations where they cannot or where they do not want.*
*A step-by-step feature is often understood as an assistent for "a posteriori understanding"; the proposed new feature for Derive 7 could incite educators to have another usage of Derive, for an "a priori study". For integration, this can provide ways towards a more profound insight.*

*As far as I know, the formula under consideration does not appear, neither in textbooks nor in websites. Please tell me if I'm wrong.*

*Enjoy sun and beach. Here it's real winter.*
*Yours, Thierry.*

Question forwarded by a *DERIVE* User:

```
#1:    z := 0.7 + 0.7·i
```

$$\#2: \quad \text{APPROX}\left(\sum_{n=1}^{1000} \frac{\text{EULER\_PHI}(n)\cdot z^{n}}{1 - z^{n}}, 30\right) = -1.695953134 + 0.005234424543\cdot i$$

$$\#3: \quad \text{APPROX}\left(\sum_{n=1}^{1001} \frac{\text{EULER\_PHI}(n)\cdot z^{n}}{1 - z^{n}}, 30\right) = -1.675277045 + 0.02591170059\cdot i$$

$$\#4: \quad \text{APPROX}\left(\sum_{n=1}^{1002} \frac{\text{EULER\_PHI}(n)\cdot z^{n}}{1 - z^{n}}, 30\right) = 189.6955839 - 1.188037185\cdot i$$

Why does this jump so much?

$$\#5: \quad \text{APPROX}\left(\sum_{n=1001}^{1001} \frac{\text{EULER\_PHI}(n)\cdot z^{n}}{1 - z^{n}}, 30\right) = 0.02067608847 + 0.02067727604\cdot i$$

$$\#6: \quad \text{APPROX}\left(\sum_{n=1001}^{1002} \frac{\text{EULER\_PHI}(n)\cdot z^{n}}{1 - z^{n}}, 30\right) = 0.02067555186 + 0.03402483983\cdot i$$

$$\#7: \quad \text{APPROX}\left(\sum_{n=1002}^{1002} \frac{\text{EULER\_PHI}(n)\cdot z^{n}}{1 - z^{n}}, 30\right) = -5.366188532\cdot 10^{-7} + 0.01334756378\cdot i$$

There's nothing in these areas to justify the jump in the above #4.
The Euler_Phi() function seems to be working fine, the problem is
somewhere in the summation.  What's up here?

**Albert Rich answers:**

If the difference between the upper and lower limits is less than a 1000, Derive approximates definite sums by simple iteration.  If the difference is greater than a 1000, Derive attempts to perform the summation by finding the antidifference of the summand (for details, see the on-line help topic "Calculus > Sum command").  Unfortunately, in your example, this results in huge round-off errors.

The optional fifth argument of the SUM function specifies the step size to be used for the summation. The default step size is 1.  If an explicit step size is given, simple iteration is used for the summation no matter how large the difference in the limits.  Therefore, you can obtain an accurate sum by including a step size of 1 in the call on SUM as follows:

APPROX(SUM(EULER_PHI(n)·z^n/(1 - z^n), n, 1, 1002, 1), 30)

which simplfies to

-1.6752775822038766521133561807 + 0.039259264377691130383805233962*#i

We apologize for the inconvenience.

Hope this helps.

Aloha,
Albert

Jaime Marcos

**Subject:** Unexpected Complex Values in Factorial

Dear Derivians,

With the DfW 5.06 if you try to get, for instance, (–40.625)! in exact mode, you get a rational expression times (3/8)!, that simplifies in turn (working with 25 significant digits) to 1.6 669898 143291 70810 E-47. Fine.

But if you try to get (directly) the approximate value of (–40.625)!, you get an ugly complex,
  $2.845729616210276126327153 \cdot 10^{-47} - 1.178739801881079045325915 \cdot 10^{-47} \cdot \hat{\imath}$

Does it happen the same thing in DfW 6.1?

Best wishes for the 2005,
Jaime Marcos


Hello Jaime,

Thank you for pointing out this defect in the implementation of the algorithm Derive uses for approximating the factorial operator and Gamma function. Investigating the problem revealed that it only occurs when approximating the factorial of negative numbers greater than a certain magnitude. The magnitude is dependent on the digits of precision being used. For example, at 10 digits of precision, it occurs for numbers less than -19.6. At 25 digits of precision, it occurs for numbers less than -40.4.

The defect has been found and corrected for the next release of Derive 6. Until then, when approximating the factorial of negative numbers, increase the digits of precision sufficiently to avoid complex results. I apologize for the inconvenience.

Aloha,
Albert D. Rich
Co-author of Derive


We have a new member from Spain who started programming with *DERIVE*. He worked through the JOSEPHUS Problem (first presented by Rüdeger Baumann) from earlier Newsletters (DNL#52, #53) and found a QUICKER-JOSEPHUS-Solution. You can compare calculation times. Angel sent also a small program dealing with the famous QUEENS Problem on a checker board (next DNL).

Angel Aguirre

Josephus problem: my solution (borrowing some ideas from Welke)

```
QuickerJosephus(n, s, p_, l_, u_, v_) :=
  Prog
    [p_ := s, l_ := n, u_ := [1, ..., n]]
    Loop
      v_ := [p_, p_ + s, ..., l_]
      u_ := APPEND(DELETE(u_, v_), u_↓v_)
      p_ := s − MOD(l_ − p_, s)
      l_ :− DIM(v_)
      If l_ = 0
          RETURN u_
```

*Finally we have a TI-related question:*

Lieber Josef,
Einige Fragen zum TI:
- Kann man vom Homefenster auf eine Zelle im Data-Matrix-Fenster irgendeiner Datei zugreifen?
- Wenn ich in einer Spalte einer Data-Matrix Funktionen stehen habe, kann ich die dann elegant plotten?

*Dear Josef,*
- *is it possible to address one single cell of a DATA-Matrix in the HOME-Screen?*
- *having functions in a column of a DATA-Matrix, is there an easy way to plot the function graphs?*

I create a data-sheet heiner (what else) and fill the first two columns with functions and their derivatives



Unfortunately it is not possible to plot the graphs directly. Defining a function in the Y=-Editor doesn´t help.

The function is recognized, because y1(2) returns the correct value 36 in the home screen.



You must explicitly call the element and then assign it to a function. Now it works!



The columns can be transferred into the home screen as lists.

Single elements can be selected using a second index (in brackets!).



This is the error message after switching to the GRAPH-Window.
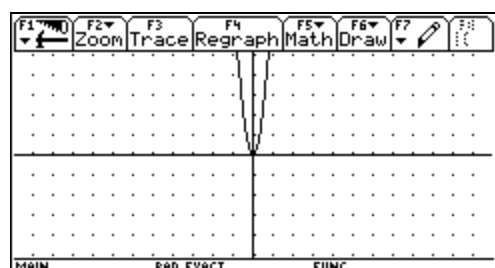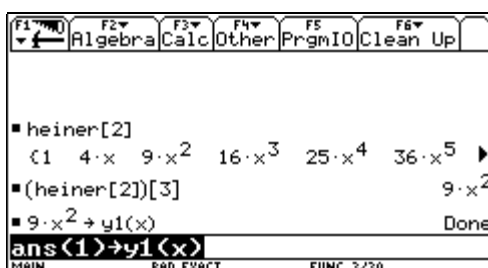
GEOMETRIE MIT TI-92 UND TURTLE-GRAPHIK
Geometry with TI-92 and Turtle Graphic

Mandala von Westenholz, Osnabrück

Im Anschluss an die Ergebnisse der TIMSS (Third International Mathematics and Science Study) und der PISA-Studie hat das **Zentrum für Kreativitätsförderung** in Osnabrück mit jungen Schülern eine Untersuchung gemacht, NEUES LERNEN in Mathematik zu erproben. Hier sind einige Programme, die ich als jüngste Teilnehmerin an diesem Projekt (ich bin 11 Jahre alt und besuche ab Herbst 2003 die gymnasiale Klasse 8), erstellt habe und die illustrieren, wie mit Hilfe der LOGO-Schildkrötengeometrie (TI-92PLUS - Turtle Grafik [1]) das Grundverständnis für die Geometrie sowie die geometrische Anschauung entwickelt und gefördert werden können.

After presentation of the results of TIMSS and PISA the **Creativity Support Center** in Osnabrück started an investigation to test NEW LEARNING in mathematics. Here are a few programs which I – as the youngest participant in this project (I am 11 years old and will attend the 8th form in a gymnasium next fall) produced and which shall demonstrate how basic understanding of geometry can be developed and supported by the use of LOGO-Turtle geometry).

Hauptdarstellerin ist die hier abgebildete Schildkröte TURTLE-FIX. Unter ihrer Mitwirkung sollen die folgenden Probleme gelöst werden:
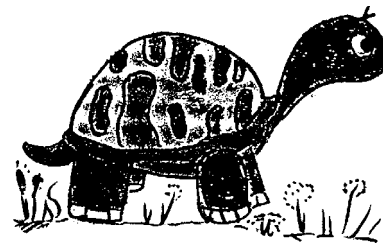
Turtle TURTLEFIX is our main actor and it will help solving the following problems.

**Problem 1:** Definiere eine Prozedur „square", die ein Quadrat der Seitenlänge $x$ erzeugt. Wie verwendet man dieses Programm, um eine Familie von „Wachstumsquadraten" zu erzeugen?

**Problem 1:** Define a procedure "square", which generates a square with side $x$. How to use this program to create a family of "Growth-Squares"?



Die abgebildete Familie von Wachstumsquadraten habe ich mit `quadrate(80)` aufgerufen.

I produced the family of Growth-Squares by executing `quadrate(80)`.

(Before leaving the turtle alone on the screen One has to set the initial values for the little creature: `xturtle` and `yturtle` are the coordinates of the starting point and θ`turtle` is the initial direction, Josef)

Diskussion: Das entsprechende Programm in der Programmiersprache LOGO lautet:
This is the respective program written in LOGO:

```
PR Quadrate :x
wenn :x < 10 dann rk
Quadrat :x
Quadrate :x – 5
ENDE
PR Quadrat :x
wh [ vw :x re 90 ]
ENDE
```

(mit den Schildkrötenbefehlen wh = wiederhole, vw = vorwärts, re = rechts, x = Seitenlänge)
(Turtle commands wh = repeat, vw = forwards, re = turn right, x = side length)

Als nächstes stellt sich die Frage: Wie kann ich mit dem TI-92 und Turtlefix ein regelmäßiges Fünfeck konstruieren? Mit der Programmiersprache LOGO untersuche ich hierzu, wie Turtlefix für eine entsprechende Schildkrötenrundreise zu programmieren ist.

Next question is how to guide Turtlefix to plot a regular pentagon on the TI-92? At first I used LOGO to investigate how to program Turtlefix for a respective turtle roundtrip.

Die Figur zeigt: Die Schildkröte dreht über die **Außenwinkel**. Die Winkel, die wir normalerweise mit dem Winkelmesser (Geodreieck) messen, sind **Innenwinkel.** Insgesamt muss sich Turtlefix bei einem Umlauf um 360 Grad drehen, also bei jeder Ecke um

$$\beta = \frac{360°}{n} = 72° \text{ für } n = 5.$$

Ein Außenwinkel ergibt sich somit als Quotient aus 360° durch die Eckenanzahl $n$.
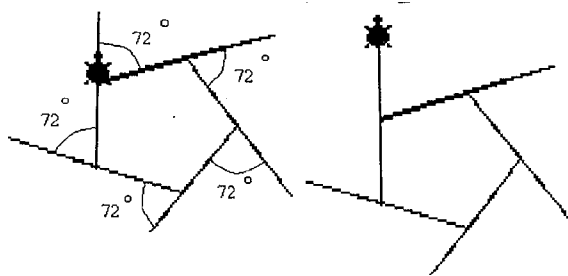


The figure shows: the turtle turns according to the outer angle (left picture). Angles measured by our tool are usually inner angles. Turtlefix must turn about 360 degrees making one complete pentagon, so at each vertex turn about 360°/5 = 72°.
The right picture shows the wrong programmed Turtlefix.

Ich erzeuge die Turtlefix-Pentagonfigur mit einem iterativen LOGO-Programm (iterativ = sich wiederholend) dann so:

```
PR PENTAGON :x :n
wh :n [ vw :x re 360/n ]
ENDE
```
This is the LOGO program.

**Problem 2:**  Wie ist mit dem TI-92 und der Turtle-Grafik ein interaktives Programm zu erzeugen, dass Turtlefix ein Pentagon zeichnet?

**Problem 2:**  How to design a program with the TI-92 and Turtle-Graphics to make Turtlefix drawing a pentagon?

**Lösung / Solution:**
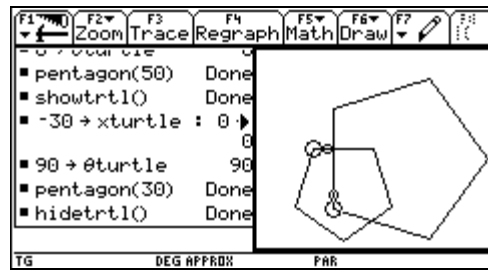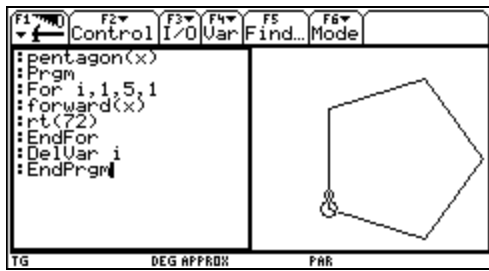




**Beispiel:** Das Netz eines Pentagon-Dodekaeders lässt sich nun programmieren (Turtlefix ist versteckt!).

**Example:** The net of a pentagon dodekaeder can be programmed. (Turtlefix is hidden!)
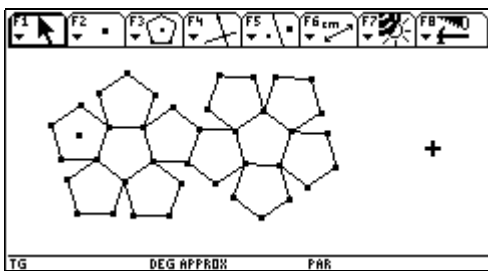


This net is not produced by a program. As you can see it is made using Cabri on the TI-92.
Mandala did it this way too. It is up to you to write a program using `pentagon()`.

Aus dem regelmäßigen Fünfeck ensteht durch Verlängerung seiner Seiten das **Pentagramm**. Diese Figur war schon im Griechenland der Antike von Bedeutung, insbesondere als Symbol der Bruderschaft der Pythogoreer, also der Anhänger des berühmten Mathematikers **Pythagoras**.

Lengthening the sides we obtain a pentagram. This figure had its importance in antique Greece as symbol of brotherhood of Pythagoreans.



links: Griechische Münze mit eingeprägtem Pentagramm (420 v.Chr)
rechts: Griechische Münze mit Porträt von Pythagoras (430 v. Chr.)

left: Greek coin with pentagram (420 BC)
right: Greek coin with picture of Pythagoras (430 BC)



**Problem 3:** Ein Archäologe findet eine antike griechische Münze. Das eingeprägte Pentagramm ist beschädigt. Um die Figur dennoch im Internet archivieren zu können, programmiert er das Pentagramm mit dem TI-92 und Turtlefix. Wie muss das Programm beschaffen sein?

**Problem 3:** An archaeologist finds an antique greek coin showing a damaged pentagram. He wants to archive the figure in Internet and programs the pentagram on his TI-92 supported by Turtlefix. How will his program look like?

**Lösung / Solution:**

The instruction how to draw a pentagram leads to a program (fixed side length 60), star(-30,-20,60) is slightly improved.
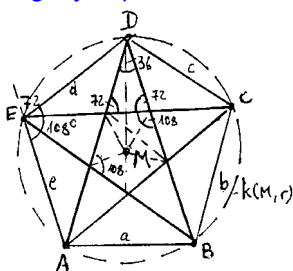


```
star()
Prgm
For i,1,5,1
forward(6Ø)
rt(144)
EndFor
EndPrgm
```
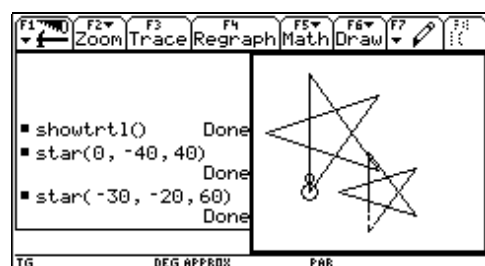
Für einen Ausflug in die Welt der **fraktalen Geometrie** von **Benoit Mandelbrot** mache ich von **rekursiven Programmen** Gebrauch. Was versteht man hierunter?

„Rekursiv" = „zurücklaufend", das bedeutet: Eine geometrische Konstruktion oder eine bestimmte Beweisführung wird verschiedene Male wiederholt.

**Problem 4:** Wie ist eine rekursive Prozedur beschaffen (eine Prozedur, die sich selbst aufruft), die das Fraktal eines binären Baums erzeugt?

**Problem 4:** Which is the recursive procedure (a procedure which calls itself), which creates the fractal of a binary tree?

**Lösung / Solution:**

Für den Binärbaum gilt: ein Teil ist eine exakte Kopie.

Each part of the binary tree is an exact copy of the whole.

- Programmiere den Stamm (der Länge x)
- Programmiere den linken Teil mit halber Länge
- Programmiere den rechten Teil mit halber Länge
- Programmiere die Rückkehr zum Ausgangspunkt

- Program the trunk (length x)
- Program the left part (half length)
- Program the right part (half length)
- Programm the return to the starting point
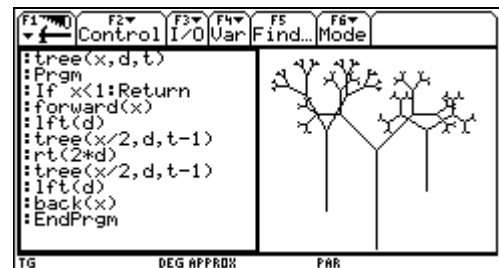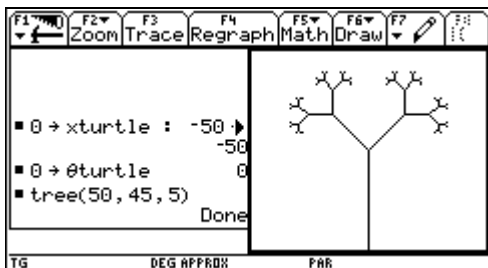


Die Figur habe ich mit `tree(50,45,5)` aufgerufen. Mit $t$ bezeichne ich die Verzweigungstiefe und $d$ ist der Winkel unter dem ein Teilbaum verzweigt.

```
PR TREE :x :t
wenn :t = 0 dann vw :x rw :x rk
vw :x re 45
TREE :x/2 :t-1
li 90
TREE :x/2 :t-1
re 45 rw :x
ENDE
```
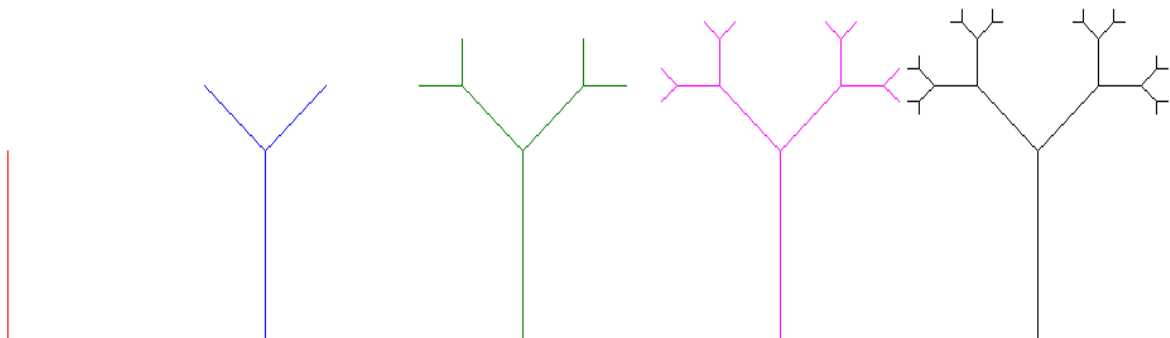


The trees of levels 0 – 4 were created by using the *DERIVE* file logodfw5.mth which has to be adapted for use with *DERIVE* 6 and is now logodfw6.mth. It was produced as a coproduction of Josef Lechner, Eugenio Roanes and Johann Wiesenbauer (DNL#25 and DNL#38). Josef

**Problem 5:** Wie ist ein **Koch-Fraktal** mittels der Turtle-Geometrie zu programmieren?

**Problem 5:** How to program a **Koch-Fractal** applying Turtle-Geometry?

**Lösung / Solution:**

Gegeben ist eine Strecke der Länge x. Ersetze diese durch folgende Figur:

Given is a segment with length x which is substituted by the following figure:

- Strecke x/3, drehe um 60° nach links,
- Strecke x/3, drehe um 120° nach rechts,
- Strecke x/3, drehe um 60° nach links,
- Strecke x/3.

- Segment x/3, left turn by 60°,
- Segment x/3, right turn by 120°.
- Segment x/3, left turn by 60°,
- Segment x/3.

```
PR KOCH :x :t
wenn :t = 0 dann vw :x rk
KOCH :t-1 :x/3
li 60
KOCH :t-1 :x/3
re 120
KOCH :t-1 :x/3
li 60
KOCH :t-1 :x/3
ENDE
```





To make it easier setting the initial conditions of Turtlefix I defined a tiny function tstart(x,y,θ).
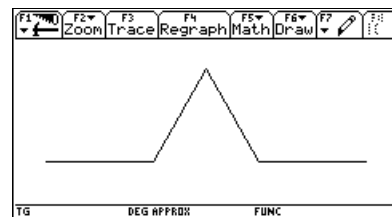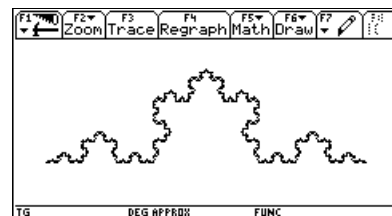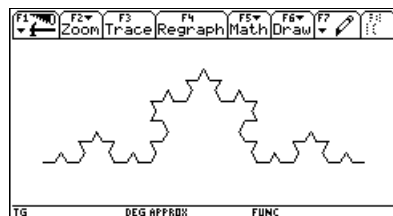












This is the fractal Snowflake:
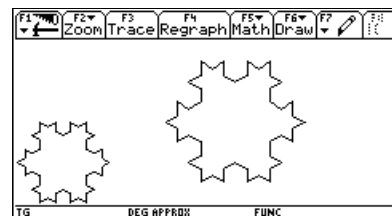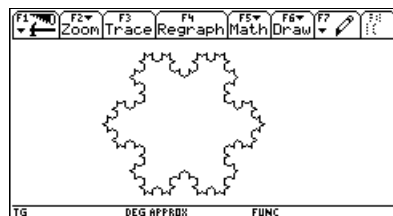






**Problem 6:** Ein **Sierpinski-Dreieck** ist ein Fraktal, das sich durch **Ähnlichkeitsabbildungen** (Geometrie der gymnasialen Klasse 9) so beschreiben lässt: Gegeben sei ein Dreieck ABC mit den Ecken (0,0), (2,0) und (1,1). Ein Sierpinski-Sieb entsteht dann durch die Definition von drei Ähnlichkeitsabbildungen, **zentrische Streckungen mit dem Streckfaktor k = 1/2**, die, verknüpft mit Translationen um die Strecke 1 und 1/2 in *x*- bzw. *y*-Richtung wie folgt gegeben sind.

**Problem 6:** A **Sierpinski-Triangle** is a fractal which can be described by similarity mappings (geometry of form 9 in gymnasium) as follows. Given is a triangle ABC with vertices (1,0), (2,0) and (1,1). A Sierpinski-Sieve is generated by defining three similarity mappings, centric stretching with stretch factor = 1/2 combined with translations by 1 in *x*- and *y*-direction.

Jeder Punkt ($x,y$) des Dreiecks wird in einen Punkt ($x/2,y/2$) abgebildet. Dadurch entsteht das Dreieck #1 von der halben Größe in der linken unteren Ecke. In Matrixform geschrieben:

Each Point ($x,y$) of the triangle is mapped into a point ($x/2,y/2$) giving triangle #1 left on the bottom which has half size. This can be written in matrix form.

$$ f: \quad \begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} $$
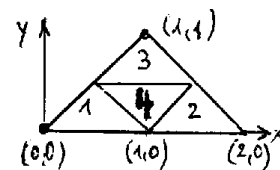
Jeder Punkt ($x,y$) des Dreiecks wird in einen Punkt ($x/2+1,y/2$) transformiert. Es entsteht das auf die Hälfte verkleinerte Abbild #2 des Dreiecks. In Matrixform geschrieben:

Each Point ($x,y$) of the triangle is mapped into a point ($x/2+1y/2$) giving triangle #2 which has half size again. This can be written in matrix form:

$$ g: \quad \begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} $$

Jeder Punkt ($x,y$) des Dreiecks wird in einen Punkt ($x/2+1/2,y/2+1/2$) transformiert. Durch Halbieren aller Koordinaten und anschließender Translation um 1/2 entsteht die auf die halbe Größe verkleinerte Kopie #3 des Dreiecks. In Matrixform geschrieben:

Each Point ($x,y$) of the triangle is mapped into a point ($x/2+1/2,y/2+1/2$) giving triangle #3 which is a half size copy of the initial triangle. This can be written in matrix form:

$$ g: \quad \begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}. $$

Der **iterative Konstruktionsalgorithmus** lautet: Verbinde die Mittelpunkte der Dreiecksseiten, entferne das mittlere Dreieck #4, verfahre in gleicher Weise mit den übrigen Teildreiecken #1, #2 und #3.

The **iterative algorithm** is: Connect the midpoints of all sides of the triangle, remove triangle #4, proceed in the same way with the remaining triangles #1, #2 and #3.

How to program a Sierpinski Triangle applying Turtle Geometry on the TI-92?

**Lösung / Solution:**

Mit dem Aufruf `sierpin(5Ø)` erhalte ich den folgenden Screenshot:
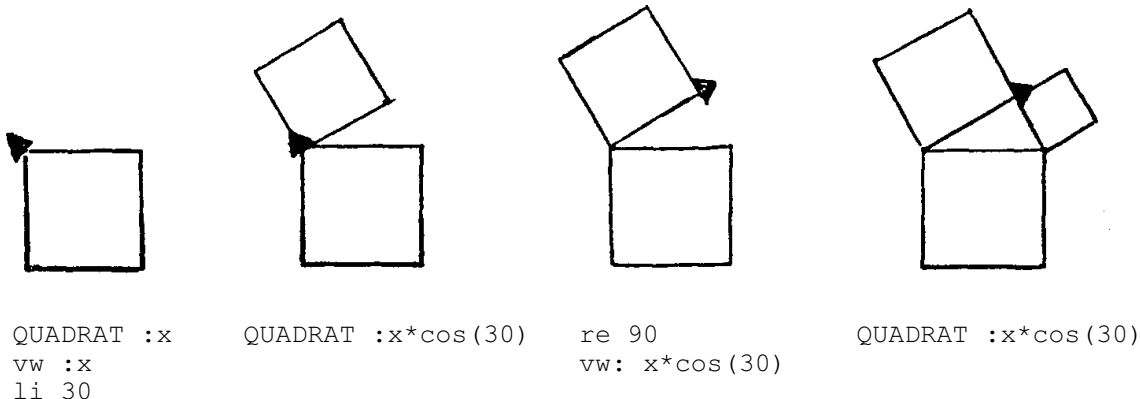Calling `sierpin(5Ø)` I obtain the following screenshot:
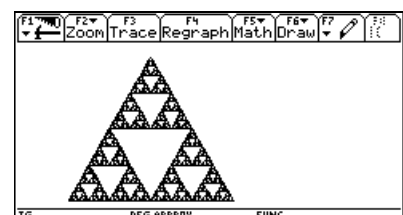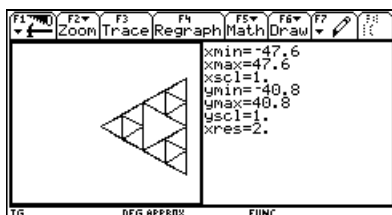
Das letzte Problem befasst sich mit dem

## BAUM DER ERKENNTNIS VON PYTHAGORAS

in der Welt der Turtle-Geometrie. Jedoch: Wie sieht ein solcher Pythagorasbaum überhaupt aus? Zunächst betrachte ich vier verschiedene „Momentaufnahmen", die Turtlefix beim Zeichnen einer Pythagorasfigur zeigen. Man erkennt die Quadrate über den Katheten und der Hypothenuse.

The last problem deals with the **TREE OF KNOWLEDGE OF PYTHAGORAS** in turtle geometry world. But how does this tree look like? At first I inspect four different snapshots which show Turtlefix drawing the figure. One recognizes squares above the sides of a right triangle.

```
QUADRAT :x        QUADRAT :x*cos(30)     re 90             QUADRAT :x*cos(30)
vw :x                                    vw: x*cos(30)
li 30
```

Das einfache LOGO-Programm zeigt, wie mit Turtlefix der Pythagorasbaum zu programmieren ist:
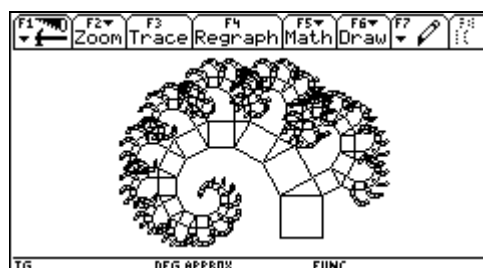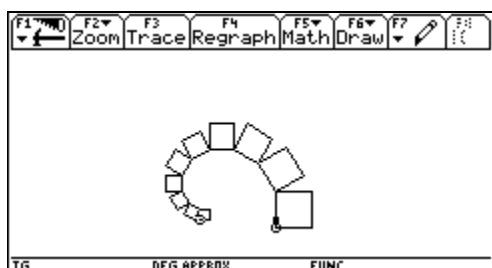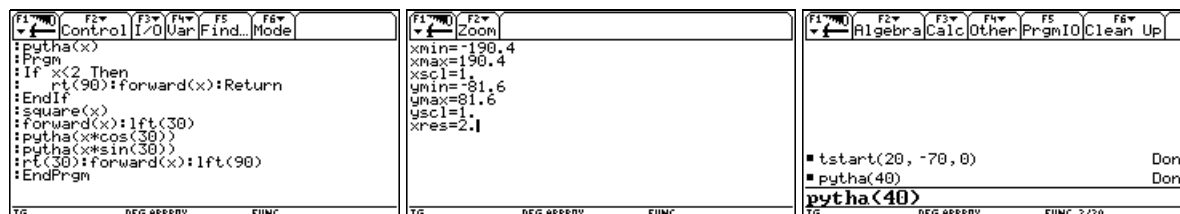
```
LERNE PYTHAGORAS :x
QUADRAT :x
WENN :x<1 DANN re 90 vw :x rk
vw :x li 30
PYTHAGORAS :x*cos(30)
PYTHAGORAS :x*sin(30)
re 30 vw :x li 90
ENDE
```

**Problem 7:** Welches TI-92-Programm erzeugt einen Pythagorasbaum?

**Problem 7:** Which TI-92 program creates a Tree of Pythagoras?

**Lösung / Solution:**

```
:pytha(x)
:Prgm
:If x<2 Then
:   rt(90):forward(x):Return
:EndIf
:square(x)
:forward(x):lft(30)
:pytha(x*cos(30))
:pytha(x*sin(30))
:rt(30):forward(x):lft(90)
:EndPrgm
```

```
xmin=-190.4
xmax=190.4
xscl=1
ymin=-81.6
ymax=81.6
yscl=1.
xres=2.
```

```
■ tstart(20, -70,0)           Done
■ pytha(40)                   Done
pytha(40)
```

Literatur / References

[1]     B. Kutzler et D. Stoutemyer, *Tolle TI-92-Programme – Bd 1*, Hagenberg, 1997

[2]     G. Otto, *Logo für den PC*, Bonn 1994

[3]     W. Quehl/H. Löthe, *LOGO-Materialien*, PH Ludwingsburg 1985

[4]     Texas Instruments, *TI-92 Guidebook* 1995

[5]     N. Kuenzer, Bruneck (Italien)

I would like to add:

[6]     B. Kutzler at D. Stoutemyer, *Great TI-92-Programs – Vol. 1*, Hagenberg 1997

[7]     J. Lechner, E. Roanes L., E. Roanes M., J. Wiesenbauer, *Turtle Graphic in DERIVE*, DNL#25

[8]     Josef Lechner, *A Turtle for DfW5*, Derive Newsletter #38, 2000

Mandalas paper is based on the program package Turtle-Graphic from [1] and [6]. I try to get permis-sion from the authors for including this package to the files which can be downloaded woth *DNL#57*, Josef.

# Online-Tutorials für Voyage 200 und *DERIVE* 6

Haben Sie eine Einstellung vergessen? Fällt Ihnen Befehl nicht gleich ein, dann besuchen Sie die Tu-torials für Voyage 200 (auch für TI-92, TI-92PLUS und TI-89 einsetzbar) und das neue Tutorial für *DERIVE* 6. Beide wurden von Walter Wegscheider zusammengestellt und werden weiter ausgebaut. Vorerst gibt es diese wertvollen Hilfen nur in Deutsch.

http://www.austromath.at/daten/derive/
http://www.austromath.at/daten/voyage200/

Beide Tutorials sind auch hervorragend für den Gebrauch durch Schüler geeignet.

Until now time both Tutorials are available only in German.

## Take care when providing selfmade UTILITY-files

Dear Josef

I have been trying to load mth. Files as utility-files. However in some cases it works perfectly, in other cases, I cannot use the functions defined in the utility file. Are there any requirements to the content of a utility-file? I attach two files, one which can be loaded, the other can't.

Best wishes
Ove

***DNL:***

Many thanks Ove for this interesting question. I am sure that this might be of interest for many members who like to prepare utility-files.

I inspected the mysterious file and indeed I found the message that file `Funktionsun-dersøgelse(f,t,n)` was loaded as utility file, but there was no single function present!! There were no problems to load the MTH-file as a MATH-file.

```
#1:     LOAD(C:\DOKUS\DNL\DNL57\funktionsundersøgelse.mth)
```

As you might know MTH-files are saved in pure ASCII-Code but DERIVE 6 works with Unicode, which is an very much extended code to also include the many mathematical characters and the characters belonging to various languages. Look at the following function definition:

```
Skrå(f, t, n) :=
  Prog
    t := NUMERATOR(f)
    n := DENOMINATOR(f)
    If POLY_DEGREE(t, x) = POLY_DEGREE(n, x) + 1
      ["Skrå asymptote y="; QUOTIENT(t, n)]
      "Ingen skrå asymptote"
```

And then look at the same function in any textprocessor or any editor:

```
Skr□00e5(f,t,n):=PROG(t:=NUMERATOR(f),n:=DENOMINATOR(f),IF(POLY_DEGR
EE(t,x)=POLY_DEGREE(n,x)+1,[["Skr□00e5 asymptote y="],
[QUOTIENT(t,n)]], "Ingen skr□00e5 asymptote"))
```

As you can see, the Danish "**å**" looks very strange in ASCII-Code. So I renamed the file as `Funktionsundersoegelse.mth` and `Skrå(f,t,n)` as `Skra(f,t,n)` and did the same with all the other "Danish" function names. The same problems occur in German using the letters Ä, ä, Ö, ö, Ü, ü or ß. The problems appear giving a "regular" filename and "irregular" function names.

Dear Josef
It works!
I will have to be more careful using all the possibilities of the Danish language (as you probably will using German!)
Still not using files as utility files there seems to be no problems!
Thanks for your quick answer.
Best regards
Ove