THE DERIVE - NEWSLETTER #60

THE BULLETIN OF THE

DERIVE

USER GROUP

+ CAS-TI

Contents:

December 2005

Hans-Jürgen Kayser publications
published by bk-teachware – "the" CAS-Specialist

- Elektronische Arbeitsblätter zur Stochastik in der Sek I mit Derive, SR-27
- Derive im Mathematikunterricht der Sek 1, SR-35
- Analysis mit Derive, Lehr- und Arbeitsbuch für die Jahrgangsstufe 11, SR-38
- Lineare Algebra und Geometrie mit Derive, SR-42
- Derive im Analysisunterricht der Jahrgangsstufen 12 und 13, SR-45
- Derive im Stochastikunterricht der Sekk II, SR-45

Visit **http://shop.bk-teachware.com**

A Website with a bundle of Derive-materials from Maria Koth, Vienna:
http://www.mathe-online.at/lernpfade/derive/

GREAT NEWS from Virgin Islands
2 new Calculator APPS are now available from the author of Calculus Made Easy:

1) **TRIGONOMETRY MADE EASY -**
   Solves ANY Trigonometry Problem instantly.

   Solve any triangle completely, review all trig formulas, graph and model with sin(x)-functions, Unit Circle coordinates, Convert degree/radian, revolutions/degrees and much more.

2) **MATRICES AND VECTORS MADE EASY -**
   Solves ANY Matrix and Vector problem instantly.

   Edit and View Matrices, do 20+ matrix computations such A+B, A*B, Inverse, Determinant, Eigenvalues,
   LU-Factorization, Solve systems of equations by viewing steps, Apply Matrices. Add and Graph Vectors,
   Compute Dot/Cross Products, UnitVectors, and much more.

Calculus Made Easy 6.0 is now available.

Major Upgrades make CME now even more powerful and easier to use:

1) NEW: Perform Logarithmic Differentiation. Find Average Rate of Change. Simpson Rule for area approximation.
2) NEW: Do both Fundamental Theorems of Calculus.
3) NEW: Graph ...functions, ...Average Value, ...Area between Functions, Under Curves, two Areas of Equal Size, Net and Total Area.
4) NEW: Display setup of Integrals before actual Integration. More Information on Differential Equations: Logistic Growth, Exponential Growth, etc.
5) NEW: Module to find Extrema of 2-dim. functions significantly improved.
6) NEW: Take a break from Calculus and play a game of Yahtzee.
7) And much more...i.e. scroll answers too wide for screen.

Available at www.ti89.com . Trial Versions available.

Dear DUG-Members,

First of all I´d like to apologize once more for the delay of publishing this DNL#60. As I wrote in my last information mail we were in Namibia for three weeks in December. But this was not a mathless time for me because when my travel companions had heard that I was a math teacher they and I discovered a lot of mathematics during the trip. So I remembered David Halprin´s great "Sand Dunes" article climbing up the dune in the Namib desert,

quiver trees reminded me on fractal structures,

a window of the church in Windhoek invites to be modeled by geometric figures

and even the horns of a rhino let us think on mathematical curves.

I intended to include Peter Schofield´s and Don Phillips´ articles but as you can see this DNL is overfull again. I am very proud that Bernhard Kutzler delivered a lecture in written form which he gave last summer in Atlanta and that Josef Lechner presents his rose in winter time. His rose inspired me to add the respective TI-92/V 200 treatments and a slider bar activity for Derive.

Johann Wiesenbauer provides a special tool for DNL#60. His Sodoku-Solver is once more an excellent example for Derive programming.

We have an extended User Forum. A few challenging requests asked for being answered.

Let me finally wish you all a great 2006 – and see you in Dresden!!

Don´t forget DES-TIME, the DERIVE & CAS-TI & ACDCA-Conference is approaching. 15 February is deadline for submitting papers.

www.des-time-2006.de

Download all *DNL-DERIVE*- and TI-files from

http://www.austromath.ac.at/dug/

http://www.derive-europe.com/support.asp?dug

The *DERIVE-NEWSLETTER* is the Bulletin of the *DERIVE* & CAS-*TI User Group*. It is published at least four times a year with a contents of 44 pages minimum. The goals of the *DNL* are to enable the exchange of experiences made with *DERIVE* and the *TI*-89/92/*Titanium*/*Voyage* 200 as well as to create a group to discuss the possibilities of new methodical and didactical manners in teaching mathematics.

As many of the *DERIVE* Users are also using the CAS-*TIs* the *DNL* tries to combine the applications of these modern technologies.

Editor: Mag. Josef Böhm
A-3042 Würmla
D´Lust 1
Austria
Phone/FAX: 43-(0)2275/8207
e-mail: nojo.boehm@pgv.at

Contributions:
Please send all contributions to the Editor. Non-English speakers are encouraged to write their contributions in English to reinforce the international touch of the *DNL*. It must be said, though, that non-English articles will be warmly welcomed nonetheless. Your contributions will be edited but not assessed. By submitting articles the author gives his consent for reprinting it in the *DNL*. The more contributions you will send, the more lively and richer in contents the *DERIVE* & CAS-*TI Newsletter* will be.

Next issue: March 2006
Deadline 28 February 2006

Preview: Contributions waiting to be published

Two Stage Least Squares, M. R. Phillips, USA
Some simulations of Random Experiments, J. Böhm, AUT & L. Kopp, GER
Wonderful World of Pedal Curves, J. Böhm
Another Task for End Examination, J. Lechner, AUT
Tools for 3D-Problems, P. Lüke-Rosendahl, GER
ANOVA with *DERIVE* & *TI*, M. R. Phillips, USA
Hill-Encription, J. Böhm
Farey Sequences on the *TI,* M. Lesmes-Acosta, COL
Simulating a Graphing Calculator in *DERIVE*, J. Böhm, AUT
Henon & Co, J. Böhm
Challenges from Fermat, Bj. Felsager, DEN
Are all Bodies falling equally fast, J. Lechner, AUT
Modelling Traffic Density, Th. Himmelbauer, AUT
Do you know this? Cabri & CAS on PC and Handheld, W. Wegscheider, AUT
An Interesting Problem with a Triangle, Steiner Point, P. Lüke-Rosendahl, GER
Mathematics and Design, Hubert Weller, GER
Diophantine Polynomials, Duncan E. McDougall, Canada
Financial Mathematics III, M. R. Phillips
Contour Plots and Implicit Plots, Peter Schofield, UK

and Setif, FRA; Vermeylen, BEL; Leinbach, USA; Koller, AUT; Baumann, GER; Keunecke, GER, ...and others

Impressum:
Medieninhaber: *DERIVE* User Group, A-3042 Würmla, D´Lust 1, AUSTRIA
Richtung: Fachzeitschrift
Herausgeber: Mag.Josef Böhm
Herstellung: Selbstverlag

Peter Wehling, Dierhagen P.Wehling@bafz.de

Sehr geehrter Herr Böhm,

Ihre Kontaktadresse habe ich von der Internetseite der DERIVE User Group.

Seit zwei Wochen bin ich Nutzer von DERIVE 6, und ich finde es - zumal als Nichtmathematiker - hilfreich, fesselnd und inspirierend. Meine Mathematikkenntnisse sind indes wohl eher als durchschnittlich zu bezeichnen; ihr Erwerb liegt bereits gut 2 Jahrzehnte zurück, und ich benutze sie als Nichtmathematiker - ich bin in der Züchtungsforschung an Pflanzen tätig - im wesentlichen auch nicht regelmäßig und systematisch, sondern bedarfsbedingt. Als DUG-Mitglied wäre ich daher kaum von Wert für andere.

Deshalb wage ich es, Sie auf diesem Wege direkt anzusprechen und Sie um Rat zu einem mathematischen Problem zu fragen.

Es geht, allgemein gesprochen, um die graphische Darstellung von dreidimensionalen Variablenbereichen, für welche a <= f(x,y,z)) <= b gilt.

Im konkreten Fall ist f(x,y,z) eine Funktion, welche LOD-Werte für gemeinsame Maximum-Likelihood-Schätzwerte von x,y,z angibt, und zu bestimmen ist ein Konfidenzintervall der x,y,z-Schätzwerte, in der Form LODmax-1 <= f(x,y,z) <= LODmax. x,y und z sind ihrerseits auf definierte Wertebereiche beschränkt.

Die Frage ist also, wie die für obige Beziehung gültigen x/y/z-Variablenkombinationen gefunden und geplottet werden können, sei es als Flächenkörper oder als Wolke diskreter x,y,z-Punkte mit vordefinierten Schrittweiten.

Ein Arbeitsblatt mit einigen Details ist angefügt; für Rückfragen stehe ich jederzeit gern zur Verfügung.

Ich würde mich freuen und wäre dankbar, wenn Sie oder ein Kollege mir einen Tipp geben könnten, der mir weiterhilft. (Übrigens, über Programmierkenntnisse verfüge ich noch nicht. Ihr Buch "Programmieren mit DERIVE" ist bereits geordert, wird aber wg. Urlaubszeit erst in zwei Wochen ausgeliefert.)

Mit freundlichen Grüßen

Peter Wehling
D-18347 Dierhagen

Mr Wehling is a DERIVE "freshman" and wants to use DERIVE for presentating 3D-confidence regions. Mr Wehling is not a mathematician but he uses math-tools for his work as a researcher in the field of plant breeding. So his request refers to results from his research work with genes. He describes the source of his problem in the following DERIVE file. This is too specific for me, so I didn´t even try to translate the background into English.

It is a statistic problem dealing with maximum likelihood estimation for three parameters.

In a few words: The problem is to find the body which is given by all points (c,d,s) which give function values z(c,d,s) between 4.37 and 5.37. $0 \le c, d \le 0.5$ and $0 \le s \le 1$.

The DERIVE-file

Hintergrund: Es handelt sich um ein Problem aus der Genetik: Vier Genotypenklassen für zwei segregierende Gene A und B mit gleichen erwarteten Häufigkeiten (1/4) und mit beobachteten Häufigkeiten M, N, O, P. Die Genotypenklassen sind Funktionen von drei Variablen, die zu schätzen sind: c und d für Rekombination zwischen den Genen A–Q und Q–B im Intervall A–Q–B; s für den selektierenden Einfluss von Q auf die Häufigkeiten der A/B–Genotypenklassen. c und d können jeweils zwischen 0 und 0.5, s kann zwischen 0 und 1 variieren.

Die Funktion für den LOD–Wert zur Maximum–Likelihood–Schätzung von c, d und s bei beobachteten Klassenhäufigkeiten M, N, O, P ist:

$$\#1:\quad z(M, N, O, P, c, d, s) := M\cdot LOG\left(\frac{4\cdot(c\cdot d + (1 - c)\cdot(1 - d)\cdot(1 - s))}{2 - s}, 10\right) + N\cdot LOG\left(\frac{4\cdot(c\cdot(1 - d) + (1 - c)\cdot d\cdot(1 - s))}{2 - s}, 10\right) + O\cdot LOG\left(\frac{4\cdot((1 - c)\cdot d + c\cdot(1 - d)\cdot(1 - s))}{2 - s}, 10\right) + P\cdot LOG\left(\frac{4\cdot((1 - c)\cdot(1 - d) + c\cdot d\cdot(1 - s))}{2 - s}, 10\right)$$

Mit folgendem Ausdruck habe ich den maximalen LOD–Wert ermittelt. Für das u.g. Beispiel mit gegebenen M, N, O, P ist z–max ungefähr 5.37. Der dazu gehörige gemeinsame ML–Schätz–wert für die c/d/s–Kombination ist 0.33/0.07/0.84:

```
#2:   MAX(MAX(MAX(VECTOR(VECTOR(VECTOR(z(5, 13, 4, 26, c, d, s), c, 0.01, 0.5, 0.01), d,
      0.01, 0.5, 0.01), s, 0, 1, 0.01))))
```

Nun ist es wünschenswert, für diesen gemeinsamen c/d/s–Schätzwert ein Konfidenzintervall anzugeben. Für meine Zwecke ausreichend wäre z.B. das "LOD–1–Intervall", welches alle c/d/s–Kombinationen mit LOD–Werten **zwischen 4.37 und 5.37** enthält.

Man könnte sich zunächst damit behelfen, zweidimensionale Konfidenzintervalle (z.B. für c/d) über den (schrittweise abgetasteten) Variationsbereich der verbliebenen dritten Variablen zu bestimmen und graphisch mit 2D darzustellen als Menge der Teilfunktionen, welche die betr. BOOLE'sche Verknüpfung erfüllen. Das habe ich versucht, und es klappt gut (wenn auch ein bisschen Programmierkenntnis das schrittweise Abtasten der dritten Variablen wesentlich eleganter gestalten könnte):

```
#3:   (z(5, 13, 4, 26, c, d, 1) ≥ 4.37 ∧ 0 ≤ c ≤ 0.5 ∧ 0 ≤ d ≤ 0.5) ∨ (z(5, 13, 4, 26,
      c, d, 0.99) ≥ 4.37 ∧ 0 ≤ c ≤ 0.5 ∧ 0 ≤ d ≤ 0.5) ∨ u·s·w
```

Bei dem Versuch, auch das **dreidimensionale** c/d/s–Konfidenzintervall graphisch darzustellen – eine über Schrittweiten regulierbare Punktewolke im 3D–Koordinatensystem würde ja schon genügen –, bin ich dann aber gescheitert. Zwar konnte ich die c/d/s–Kombinationen, für welche die betr. BOOLE'sche Verknüpfung wahr (oder falsch) ist, indirekt mit Hilfe von MAP_LIST oder TABLE darstellen (wenn auch nicht in einem Rutsch für die u.g. Schrittweiten und Wertebereiche, weil dann mein PC wg. mangelnden Speichers streikt):

```
#4:   MAP_LIST(MAP_LIST(MAP_LIST(4.37 ≤ z(5, 13, 4, 26, c, d, s) ≤ 5.37, c, 0.001, 0.5,
      0.001), d, 0.001, 0.5, 0.001), s, 0, 1, 0.01)
```

Nur, wie können die betr. c/d/s–Kombinationen, für welche 4.37 ≤ z ≤ 5,37 "true" ist, in **Punktevektoren für eine 3D–Graphik** ausgegeben werden???

DNL: You can imagine that this problem was a real challenge for me and it took me one evening to find a first – more or less – satisfying solution:

Lieber Herr Wehling, Dear Mr Wehling,

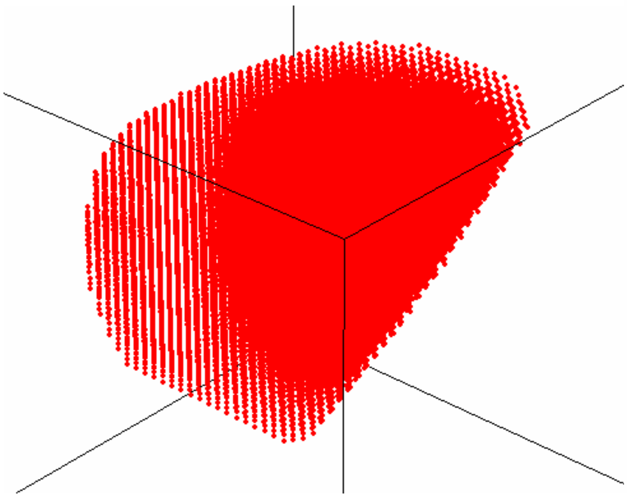
I tried to apply some tricks:

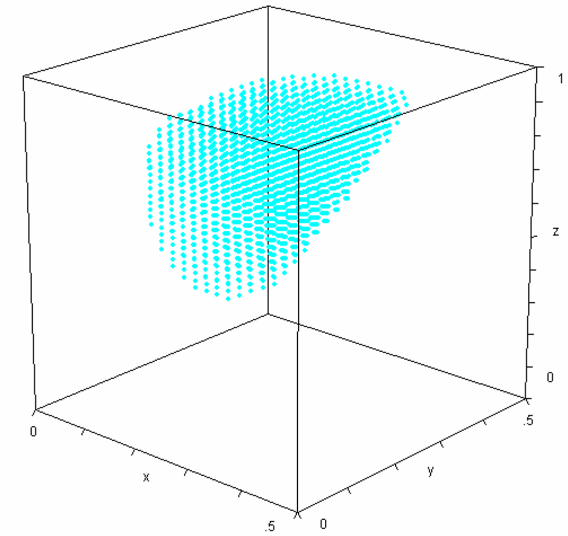
I wrote a program `pkte(lower bound, upperbound)` containing loops for values c, d and s. If a triple of co-ordinates fulfills the condition for "function value" z the respective point will be added to a list.
Taking your parameters (small increments) calculation needs a long time and can lead to memory problems. It might be possible to extend the DERIVE memory, but I believe that the result would not be much better than mine with larger increments.

The increments are given within the program using in the assigments i:+..., j:+... and k:+.... The left graph was created using smaller increments the cloud of points is more compact.

```
        pkte(u, o, i, j, k, liste) :=
          Prog
            liste := []
            k := 0
            Loop
              j := 0
              Loop
                i := 0
                Loop
#6:               If u ≤ z(5, 13, 4, 26, i, j, k) ≤ o
                     liste := APPEND([[[i, j, k]]], liste)
                  WRITE([i, j, k])
                  i :+ 0.02
                  If i > 0.5 exit
                j :+ 0.02
                If j > 0.5 exit
              k :+ 0.02
              If k > 1 exit
            RETURN liste
```

```
#7:     pkte(4.37, 5.37)
```





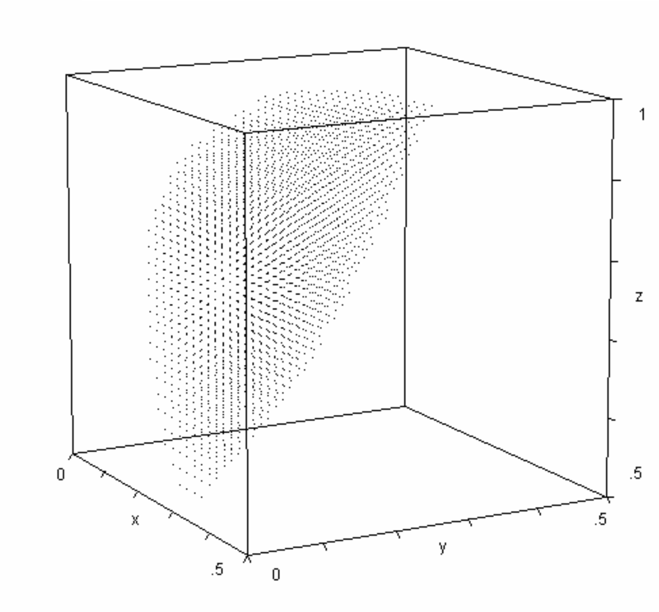
The `WRITE`-command shows progress of the calculation in the status line (bottom left). Point Size should be set to `SMALL` (issue `Insert Plot` in the 2D-Plot Window).

I was not really satisfied with my result. The points appeared too large, I wanted to distinguish the single points in the cloud. Peter Schofield gave an advice: don´t plot a single point, but plot a segment with length zero.
(By the way I remembered Euklid´s classic definition of a point .....)

I had only to exchange one line in the program from above to receive the points as dots:

```
Loop
  If u ≤ z(5, 13, 4, 26, i, j, k) ≤ o
     liste := APPEND(liste, [[[i, j, k; i, j, k]]])
  WRITE([i, j, k])
```

```
pkte_small(4.37, 5.37)
```



Two days later I sent another mail to Mr Wehling:

Lieber Herr Wehling, Dear Mr Wehling,

As announced in my last mail, I was not fully satisfied with my point diagram. I developed two other models which should accomplish visualization in space:

I plot the horizontal levelplanes in 0.02 z-increments and I plot the contour curves of the body built by the points.

It would not be too difficult to include all parameters (boundaries and increments for c, d and s) into the variable list. In the existing form of the program one has to change the respective values in the program which is not very user friendly and might be too difficult for the unexperienced.

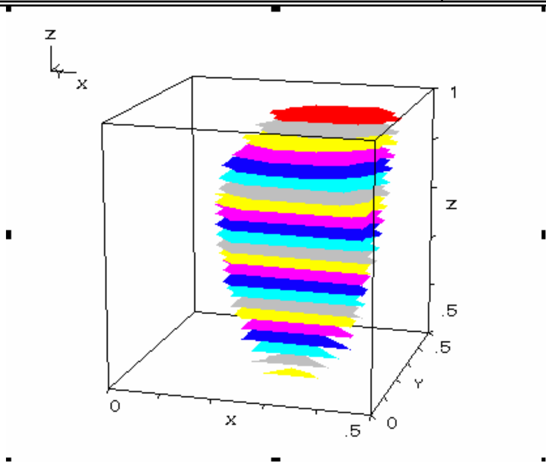
With my best regards
Josef Böhm

If you agree I´d like to publish the result of our inspiring communication in one of the next Derive Newsletters.

See my final results on the next page:

We produce level planes with 0.02 increments in z-direction:

```
pkte_cont(4.37, 5.37)
```

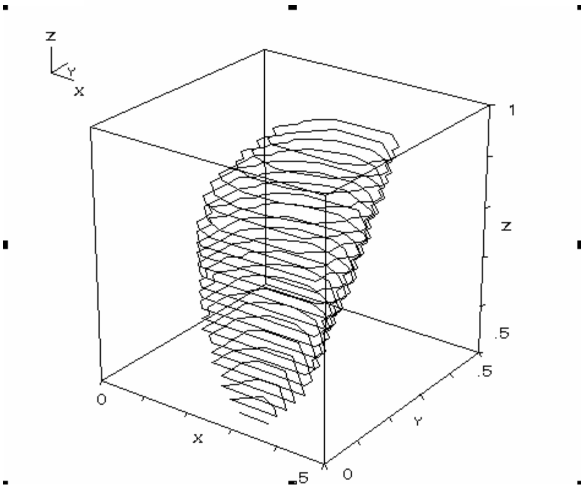


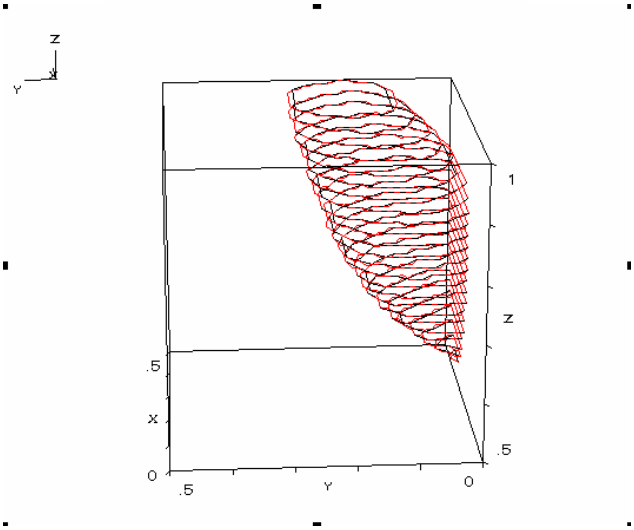
Add 1 as third parameter to obtain only the contour lines plotted (below left).

Scanning with 0.01-increments in y-direction we obtain an result of better accuracy (below right).

```
pkte_cont(4.37, 5.37, 1)
```

```
pkte_cont2(4.37, 5.37, 1)
```





Lieber Herr Böhm,

von Ihren beiden neuen Entwicklungen bin ich begeistert; sie erleichtern die räumliche Visualisierung des Konfidenzbereichs ganz ungemein!

Die Eingabe der Parameter direkt in das Programm ist überhaupt kein Problem für mich (mein PC ist gerade dabei, noch engere Schrittweiten zu errechnen) - eher schon, was genau Ihr Programm mathematisch eigentlich tut; aber zu diesem Verständnis werde ich mit der Zeit schon noch gelangen.

Selbstverständlich können Sie das Erreichte für den DNL nutzen.

Noch einmal meinen herzlichen Dank für Ihre unschätzbare Hilfe.

Mit herzlichen Grüßen

Peter Wehling

Dear Mr Böhm,
I am really enthusiastic about your two new programs. They support 3D-visualization of confidence regions in a high degree. Entering parameters into the program is no problem for me (my PC is just busy calculating smaller increments) – it is a bit more difficult to understand how the program works, but I am sure that I will find it out very soon.

You can use the results for the DNL, of course. Once more many thanks for your unvalueable support.

With best regards
Peter Wehling

Verschiedene Blätter der Rosenkurve
Various Leaves of the Rose-Curve

Josef Lechner, Viehdorf, Austria, lejos@aon.at

Abstract

Ebene Kurven eignen sich besonders für experimentell-heuristische Zugänge zu Problemstellungen. Dies soll am folgenden Beispiel verdeutlicht werden.

Plane curves can be used in a meaningful way for experimental-heuristic approaches for problem solving. This shall be demonstrated by the following example:

Beispiel 1 *Konstruktion von Astroide und Rosette /* Construction of astroid and rose
(mittels Dynamischer-Geometrie-Software (DGS))

Wir wollen mit einer Gleitstrecke experimentieren. Dazu gehen wir von folgender Vorstellung aus: Eine Leiter AB (= Gleitstrecke) sei an eine Wand angelehnt und wird an ihrem unteren Auflagepunkt waagrecht - zwischen den beiden Extrempositionen „stehend“ und „liegend“ - hin- und her bewegt. Eine solche Konstruktion lässt sich mit einem DGS leicht bewerkstelligen. Anschließend können wir bereits mit Experimenten beginnen: Es werden dazu drei Punkte auf der Gleitstrecke markiert, der Halbierungspunkt, und die beiden Lotpunkte (siehe Abb. 1). Frage: „Auf welchen Bahnen bewegen sich die drei Punkte?“. Es ist eine sehr interessante Übung, den Schülern zuerst die gesuchten Bahnen skizzieren zu lassen. Die Ausführung mit DGS zeigt, dass ein Viertelkreis, eine Viertelastroide und eineViertelrosette entstehen.

We start experimenting with a gliding segment. Ladder AB (= gliding segment) is moving between poitions “standing and lying“. This construction can easily be performed using a dynamic geometry program. We mark three points on the segment – the middle point and the two petal points (see fig. 1). Question: “What is the locus of the three points?“. It is a very interesting exercise for the students to let them sketch the loci before “asking“ the DGS. The program shows the results immediately: there are appearing a quarter of a circle, a quarter of an astroid and a quarter of a rosette (if A and B are moving on the positive axes).

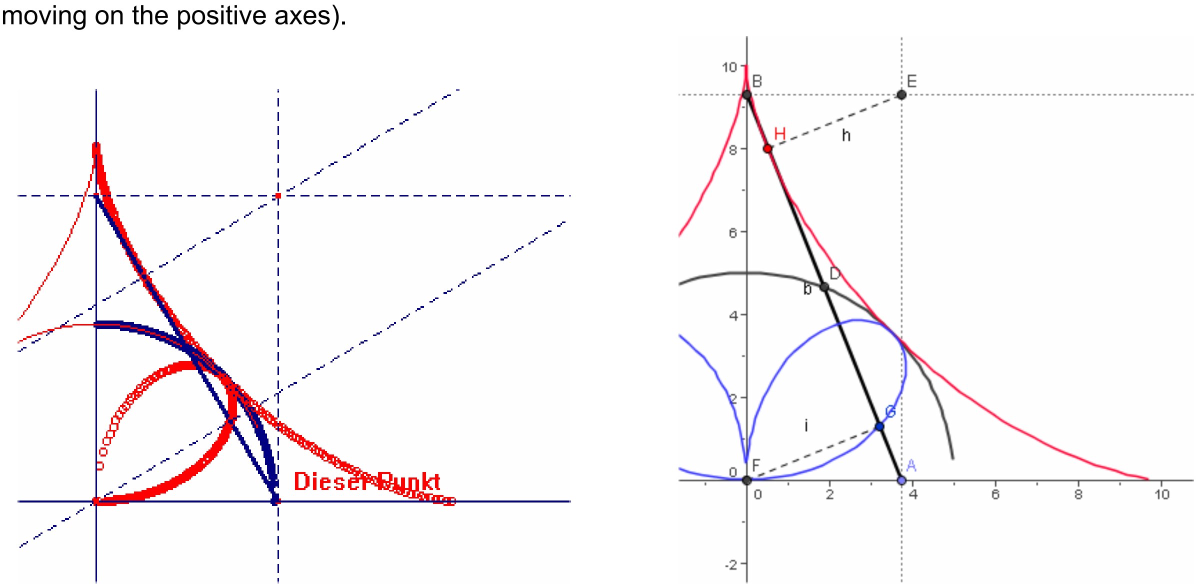


Abb. 1 / fig 1 Experiments with the gliding segment (*performed with Cabri and Geogebra*)

Durch entsprechende Spiegelungskonstruktionen ergeben sich schließlich die vollen Kurven (Abb. 2). Die Bewegung der Gleitstrecke kann zudem mit Hilfe eines Schiebereglers gesteuert werden.

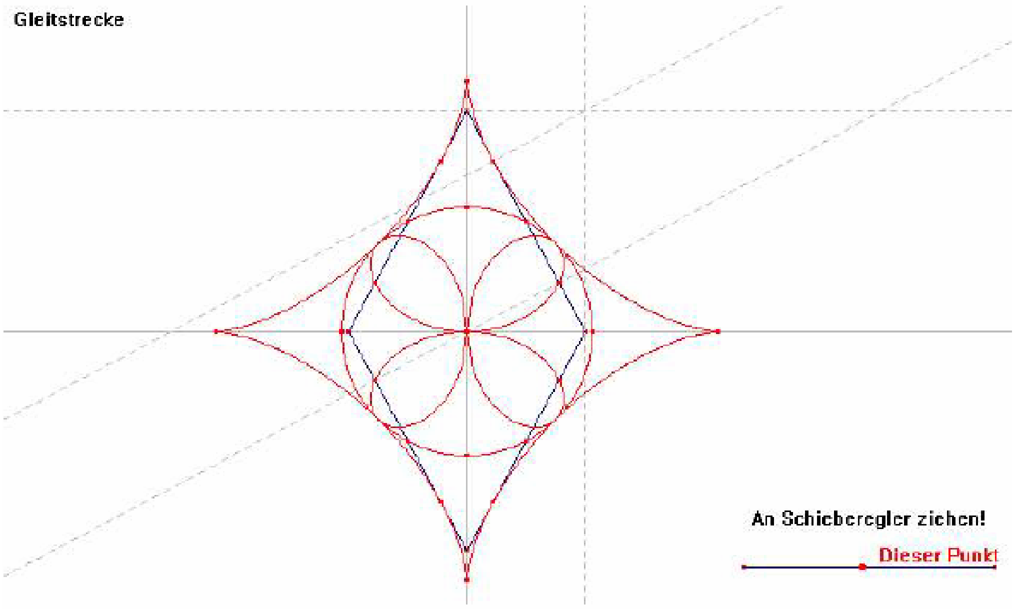


Abb. 2 / fig 2 The moved segment delivers astroid, circle and 4-leaved rose

Der Einsatz von Technologien (wie DGS) verstärkt experimentelles Arbeiten im Sinne von „Was passiert, wenn ... ?" Es begünstigt das „Fragen-Stellen" und „In-Frage-stellen". („Warum ist das so?", „Ist das die einzige Möglichkeit?", „Ist das die günstigste Lösung?", „Stimmt das immer?") Technologische Werkzeuge begünstigen das experimentelle Arbeiten besonders deswegen, weil mit konkreten Objekten gearbeitet werden kann, weil Bilder Assoziationen hervorrufen und bei der *Entwicklung von Grundvorstellungen* helfen.

Experimentelle Mathematik ist Mathematik im Vorfeld einer systematischen Fachbetrachtung. Man sucht eine Vermutung durch systematisches Probieren. Es kann vereinfachend (nicht verfälschend) gearbeitet und gesprochen werden. Die Arbeit am konkreten Bespiel ist wichtiger als der Drang zur Verallgemeinerung, das unmittelbare Testen einer Vermutung (mit Hilfe der zur Verfügung stehenden Werkzeuge) ist wichtiger als der exakte Beweis, Intuitionen und präformales Vorgehen wichtiger als mathematische Strenge.

Es wird seitens der Didaktik oft beklagt, dass die experimentelle Seite im Unterricht oft viel zu kurz kommt; dass viel zu rasch fertige Modelle, Ableitungen, Verfahren den Schülern präsentiert werden, ohne dass diesen die Möglichkeit gelassen wird, selbst Entdeckungen zu machen. Technologie-unterstützter Unterricht bietet hier vielfältige Chancen. Natürlich kann Experimentelle Mathematik ohne theoretische Untermauerung relativ nutzlos sein; wenn nämlich die entsprechenden Begriffsbildungen, kurz die zu Grunde liegende Mathematik, nicht angeschlossen wird. Experimentelle Mathematik kann in keinem Fall die klassische *ersetzen*, aber sie kann z.B. zu Beweisbedürfnis *anregen*, zum Bedürfnis, die zum Problem und Vermutungsformulieren nötige Sprache zu entwickeln u.a.m. Kurz: sie kann helfen „die Weichen richtig zu stellen".

Die experimentell erzeugten, geometrisch vorliegenden Kurven mathematisch-formal zu fassen soll im nächsten Beispiel versucht werden. Dabei soll auf die vielfältigen Darstellungsmöglichkeiten und auf eine Gegenüberstellung der verschiedenen Darstellungsformen eingegangen werden.

Using technologies (like dynamic geometry systems) affirm experimental working in the sense of "What if?". It inspires "posing questions" and "doubting" ("Why is this so?", "Is this the only one possible solution?", "Is this the best solution of all?", "Is this always true?") Technological tools support experimental working especially because of working with concrete objects, because pictures call provoke and support developing *basic concepts*.

Experimental mathematics can be a initial point of systematic view of the subject. One searches for a conjecture by systematic experimenting. One is able to work – and to talk - in a simplified (but not in a falsified way). Working on the given concrete problem ist more important than the wish for generalisation, immediate checking of a conjecture (using the available tools) is more important than the exact proof, intuition and preformal acting is more important than mathematical strictness.

Didactic often complains that math education lacks the experimental view, that too quick ready made models, derivations, methods are presented to the students, without leaving them a chance to make their own discoveries. Technology supported teaching offers many chances. Of course, experimental mathematics can be useless without an appropriate theoretical base, if the underlying mathematical concepts are not following. Experimental mathematics can in no case *substitute* classical mathematics, but it can eg incite the wish for a proof, incite to develop an appropriate language for communication about the findings a.o. To make it short: it can help to "shift the right switch".

In the following we will show various representations of the rose curve (4-leaved rose). This is not only a "Window Shuttle" between algebraic and graphic representation, but also a "Shuttle" between different algebraic representation forms (parameter form, polar form, implicit form, 3D-representation, function form and complex form). We hope that the mathematics behind is self explanatory, so we will minimize additional translations.

Beispiel 2 *Die verschiedenen Darstellungsformen der Rosenkurve*
The various representation forms of the rosette (4-leaved rose)

a) Parameterform (parameter form)

Definition: Wird eine Kurve k in der Form $k \,....\, \vec{x}(t) = \begin{cases} x = x(t) \\ y = y(t) \end{cases}$ mit $t \in [t_1, t_2]$ angeschrieben,

wobei $x(t)$ und $y(t)$ (stetige) Funktionen des Parameters t sind, so spricht man von der *Parameterdarstellung der* (*stetigen*) *Kurve k*. Der Parameter t repräsentiert dabei zumeist die (stetig) verfließende Zeit oder den Winkel, der überstrichen wird.

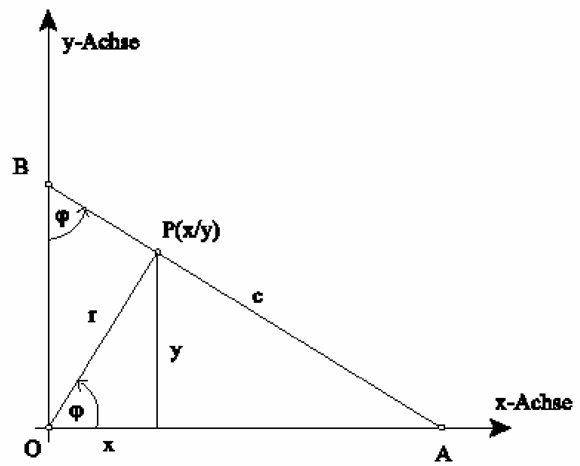


Abb. 3 / fig 3 Sketch for deriving a parameter form

Die Parameterdarstellung ergibt sich aus folgenden geometrischen Überlegungen (Abb. 3):

Für die Streckenlänge $\overline{BO}$ ergibt sich aus $\cos\varphi = \frac{\overline{BO}}{c}$: $\overline{BO} = c\cos\varphi$ und weiter für die Länge r aus

$$\sin\varphi = \frac{r}{\overline{BO}} \text{ und weiter } r = \overline{BO}\sin\varphi = c\cos\varphi\sin\varphi = \frac{c}{2}\sin(2\varphi).$$

Damit lässt sich für die x-Koordinate die folgende Beziehung angeben: $x = r\cos\varphi = \frac{c}{2}\sin(2\varphi)\cos\varphi$.

Mit analogen Überlegungen erhält man für die y-Koordinate: $y = r\sin\varphi = \frac{c}{2}\sin(2\varphi)\sin\varphi$

Damit können wir nun in Parameterdarstellung anschreiben:

$$\text{Rosette:} \quad \vec{x}(\varphi) = \begin{cases} x(\varphi) = \frac{c}{2}\sin(2\varphi)\cos\varphi \\ y(\varphi) = \frac{c}{2}\sin(2\varphi)\sin\varphi \end{cases} \quad \text{mit } \varphi \in [0, 2\pi]$$

Wenn wir für $c = 2$ wählen, so ergibt sich:

```
#1:   [SIN(2 φ) COS(φ), SIN(2 φ) SIN(φ)]
```

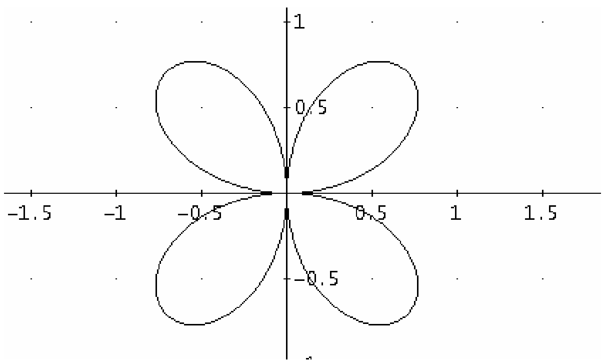


Abb. 4 / fig 4 Rose in parameter form (Derive)

How to do it on the TI handheld:

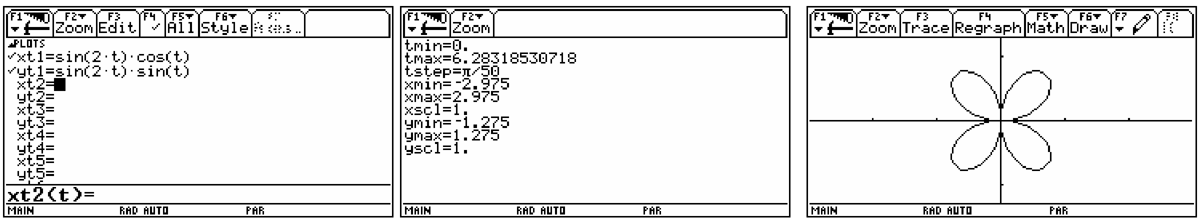


Abb. 5 / fig 5 Rose in parameter form (TI)

b) Polardarstellung (polar form)

Definition: Wird eine Kurve k in der Form

$$r = r(\varphi) \text{ mit } \varphi \in [\varphi_1, \varphi_2]$$

angeschrieben, so spricht man von der *Polardarstellung der Kurve k.* Polardarstellungen eignen sich besonders für Kurven, die symmetrisch um einen Punkt (meist den Ursprung) liegen.

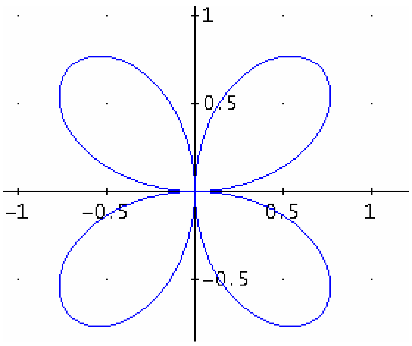
Die Polardarstellung ergibt sich sehr einfach unter Verwendung der oben hergeleiteten Parameterdarstellung:

$$r^2 = x^2 + y^2 = \frac{c^2}{4}\sin^2(2\varphi)\cos^2\varphi + \frac{c^2}{4}\sin^2(2\varphi)\sin^2\varphi = \frac{c^2}{4}\sin^2(2\varphi).$$

Rosette: $r(\varphi)=\frac{c}{2}\sin(2\varphi)$ mit $\varphi\in[0,2\pi]$

Wenn wir wieder $c = 2$ wählen, ergibt sich die bekannte Rosette, diesmal in Polardarstellung:

#2: r = SIN(2 φ)



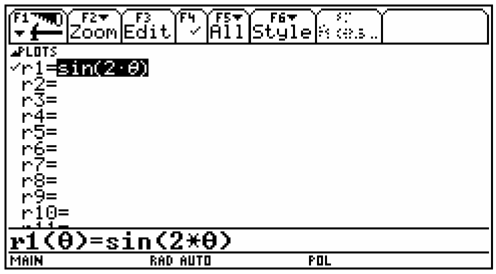


Abb. 6 / fig 6 Rose in polar form

c) implizite Darstellung (implicit form)

Definition: Wird eine Kurve k in der Form

$$f(x,y)=0$$

angeschrieben, so spricht man von der *impliziten Darstellung der Kurve k*. Treten in dieser Darstellung nur Potenzen und Produkte von x und y auf, so spricht man von einer *algebraischen* Kurve: $\sum_{k=0}^{n}\sum_{m=0}^{n-k}a_{km}\cdot x^k\cdot y^m=0$.

Die implizite Darstellung lässt sich problemlos aus der Polardarstellung gewinnen:

Unter Berücksichtigung der beiden Transformationsbeziehungen $x=r\cos\varphi$ und $y=r\sin\varphi$ erhalten wir

$$r^2=c^2\cos^2\varphi\sin^2\varphi=c^2\frac{x^2}{r^2}\frac{y^2}{r^2} \text{ und weiters } \left(r^2\right)^3=c^2x^2y^2.$$

Da aber $r^2=x^2+y^2$ ergibt sich

Rosette: $(x^2+y^2)^3-c^2x^2y^2=0$

#3: $(x^2+y^2)^3 = 4\cdot x^2\cdot y^2$

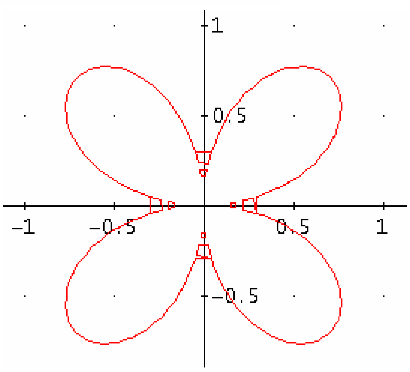
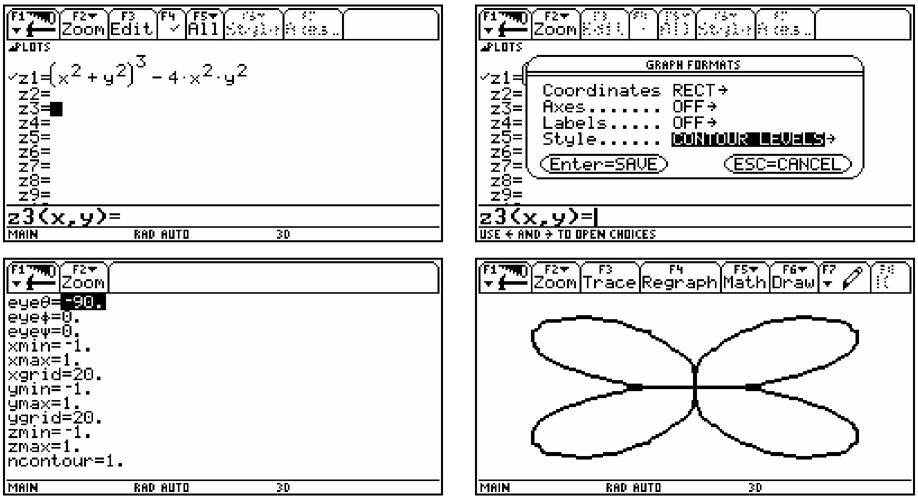


Abb. 7 / fig 7 Rose in implicit form

On the TI produce a contour plot of level 0 (see below).



Or do immediately an implicit plot.

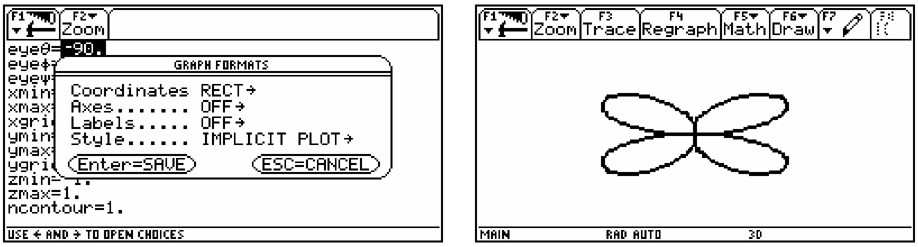


Abb. 8 / fig 8 Rose in implicit form on the TI

d) Darstellung als Funktion in zwei Variablen (function of two variables)

Die implizite Darstellung einer Kurve *k* lässt sich interpretieren als Schnitt einer Funktion $z(x,y)$ in 2 Variablen mit der Ebene $z = 0$.

$$\underbrace{(x^2+y^2)^3 - c^2x^2y^2}_{z(x,y)} = 0$$

Die Kurve erscheint in dieser Betrachtungsweise als Höhenlinie (die Kurve bekommt damit sozusagen ein „dreidimensionales Leben“). Mit CAS-Unterstützung lässt sich dies gut visualisieren. Die Schüler können diese Darstellung auch leicht drehen und aus verschiedenen Blickwinkeln beobachten.

#4: $z(x, y) := (x^2 + y^2)^3 - 4\cdot x^2\cdot y^2$

#5: $z(x, y) = 0$

#6: $Z = 0$

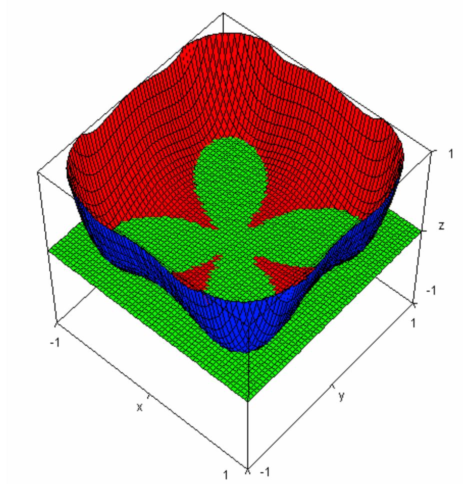
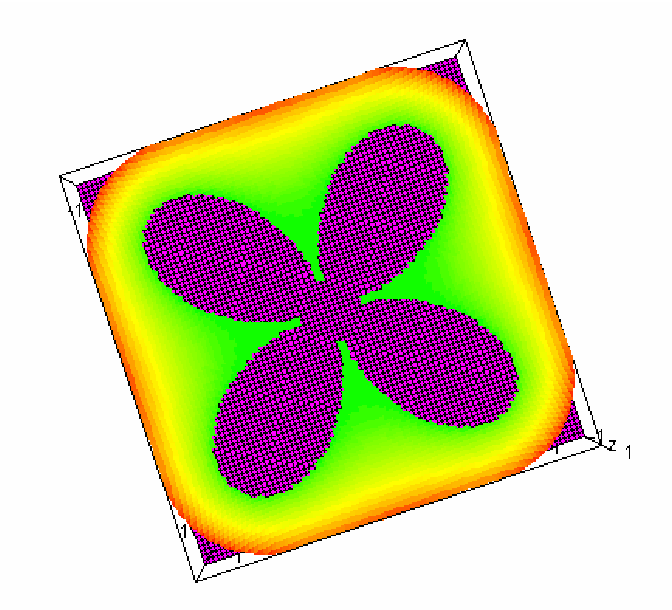


Abb. 9 / fig 9 Function of two variables – 3D plot

The right sequence of screeshots show an attempt to visualize the surface together with the intersecting plane z = 0.



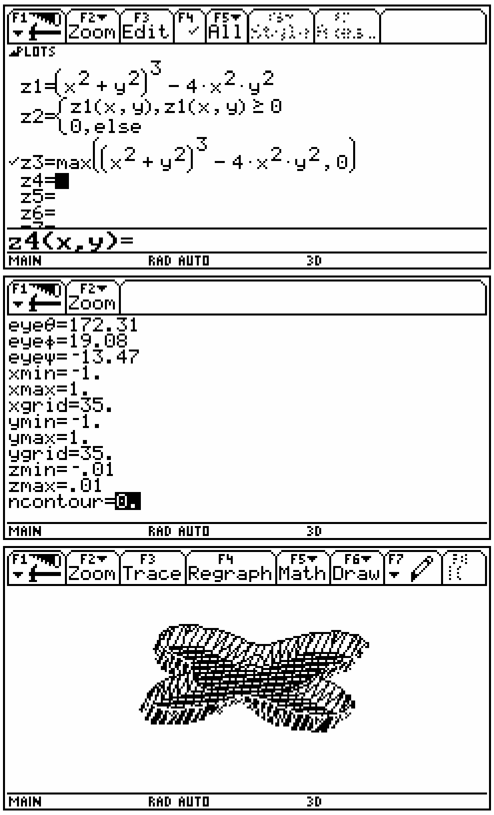
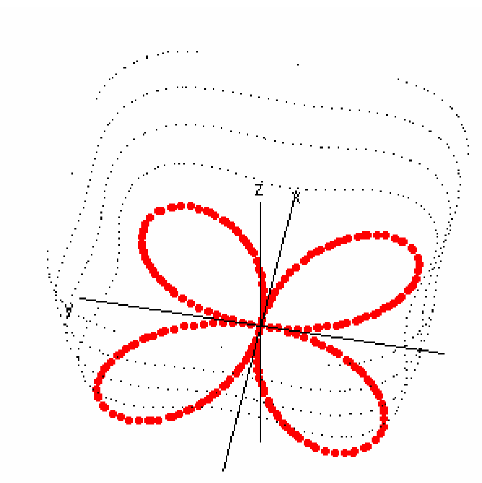


Abb. 10 / fig 10 Function of two variables in Derive and on the TI

DERIVE verfügt über keine implementierte Funktion zur Darstellung von Konturkurven (Höhenschichtenlinien). Auch in den USER-Dateien kann man nichts derartiges finden. Ein Artikel im naächsten DNL wird helfen, diese Lücke zu schließen. Abbildung 11 zeigt einige Konturkurven mit der extra betonten 0-Kurve (links) und die Fläche mit der 0-Niveau-Kurve (rechts).

Derive does not offer an implemented Contour Plot function. There is no such function among the USER files. Refer to a contribution in the next DNL and learn how to fill this gap. Figure 11 shows some level curves with the 0-level curve (left) and the surface with the 0-level curve (right).



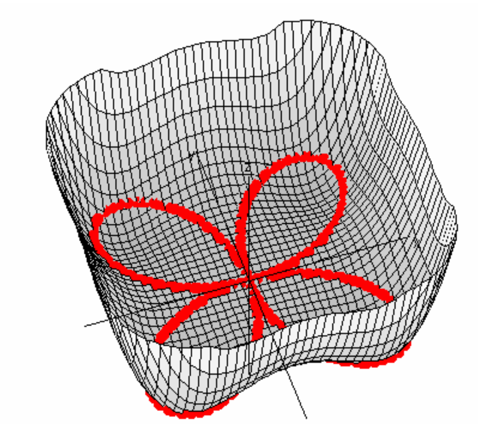


Abb. 11 / fig 11 Contour plot with Derive

d) Darstellung mit Hilfe von Funktionen (function form)

Viele Kurven lassen sich stückweise aus Funktionen aufbauen bzw. sich in diese zerlegen. Mit CAS-Unterstützung ist das auch hier möglich, wenngleich die sich ergebenden Funktionen dann durch wahre „Termmonster“ (siehe die folgenden Terme) dargestellt werden.

Many curves can be composed of functions or can be decomposed in functions. CAS support makes this possible in our case, too. However, the resulting functions are really "monsters of expressions" as can be seen in the following.

#7: $\mathrm{SOLVE}((x^2 + y^2)^3 = 4x^2y^2, y)$

#8:

$$y = -\frac{3^{3/4}\sqrt{x\left(2\sqrt{3}\,\mathrm{SIGN}(x)\,\mathrm{COS}\left(\frac{\mathrm{ASIN}\left(\frac{3\sqrt{3}\,x}{4}\right)}{3}\right) - 2\,\mathrm{SIN}\left(\frac{\mathrm{ASIN}\left(\frac{3\sqrt{3}\,x}{4}\right)}{3}\right) - \sqrt{3}\,x\right)}}{3} \vee y =$$

$$\frac{3^{3/4}\sqrt{x\left(2\sqrt{3}\,\mathrm{SIGN}(x)\,\mathrm{COS}\left(\frac{\mathrm{ASIN}\left(\frac{3\sqrt{3}\,x}{4}\right)}{3}\right) - 2\,\mathrm{SIN}\left(\frac{\mathrm{ASIN}\left(\frac{3\sqrt{3}\,x}{4}\right)}{3}\right) - \sqrt{3}\,x\right)}}{3} \vee y = -$$

$$\frac{3^{3/4}\sqrt{-x\left(2\sqrt{3}\,\mathrm{SIGN}(x)\,\mathrm{COS}\left(\frac{\mathrm{ASIN}\left(\frac{3\sqrt{3}\,x}{4}\right)}{3}\right) + 2\,\mathrm{SIN}\left(\frac{\mathrm{ASIN}\left(\frac{3\sqrt{3}\,x}{4}\right)}{3}\right) + \sqrt{3}\,x\right)}}{3} \vee y =$$

$$\frac{3^{3/4}\sqrt{-x\left(2\sqrt{3}\,\mathrm{SIGN}(x)\,\mathrm{COS}\left(\frac{\mathrm{ASIN}\left(\frac{3\sqrt{3}\,x}{4}\right)}{3}\right) + 2\,\mathrm{SIN}\left(\frac{\mathrm{ASIN}\left(\frac{3\sqrt{3}\,x}{4}\right)}{3}\right) + \sqrt{3}\,x\right)}}{3} \vee y = -$$

$$\frac{3^{3/4}\sqrt{x\left(4\,\mathrm{SIN}\left(\frac{\mathrm{ASIN}\left(\frac{3\sqrt{3}\,x}{4}\right)}{3}\right) - \sqrt{3}\,x\right)}}{3} \vee y = \frac{3^{3/4}\sqrt{x\left(4\,\mathrm{SIN}\left(\frac{\mathrm{ASIN}\left(\frac{3\sqrt{3}\,x}{4}\right)}{3}\right) - \sqrt{3}\,x\right)}}{3}$$

Plottet man diese sechs verschiedenen Funktionen (#8 – auf einmal plotten), so erhält man wieder das gewohnte Bild:
(Die offensichtlichen Lücken in der graphischen Darstellung sollten Fragen provozieren!)

Plotting expression #8 – as a whole – gives the plot. The gaps in the graphic representation should provoke questions!

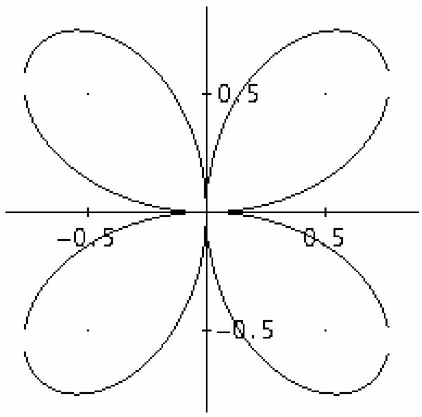


Abb. 12 / fig 12 Contour plot with Derive

I failed trying to perform the same procedure on the TIs because the handheld device is unable to solve cubic equation. What we can do is to transfer the DERIVE solutions to the TI and then plot the rosette as composition of functions.
See below Figure 13

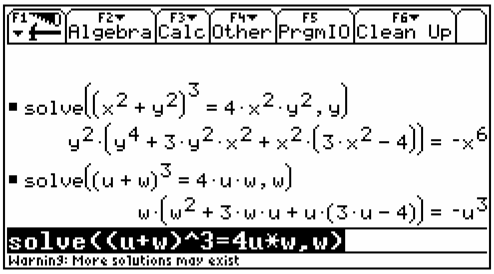


Abb. 13 / fig 13 Try with the TI

I send expressions #7 and #8 from above (using `SOLUTIONS` instead of `SOLVE`) to the V200. In the TI-Text editor I accomplish the script storing the elements of the list of solutions as functions y1(x) through y6(x). Then I execute the script. Inspecting the table I find that two branches are containing only imaginary points, so they can be omitted for the plotting procedure. (By the way, try plotting expression #8 from above with Option "`Plot real and imaginary parts`" activated – Derive, ofcourse!) The screen shots below illustrate how it was done:

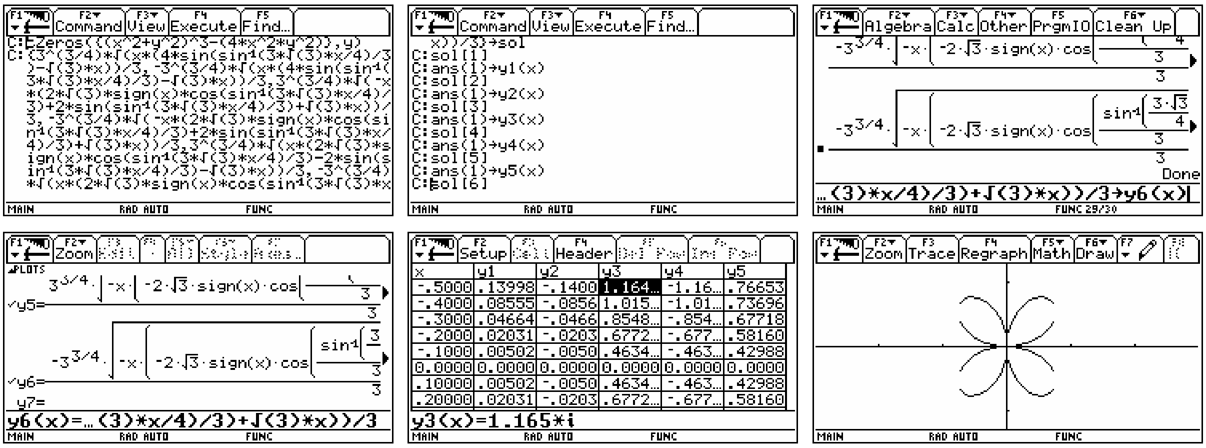


Abb. 14 / fig 14 InterConnectivity between Derive and the Voyage 200

f) **Darstellung in der komplexen Zahlenebene** (complex form)

Die Rosenkurve besitzt – kaum überraschend – auch eine Darstellung im Komplexen. Diese erhalten wir, wenn wir von der Parameterdarstellung ausgehen und die beiden Koordinatenfunktionen als Real- und Imaginärteil interpretieren.

$$\left.\begin{aligned} x(t) &= \sin(2t)\cos(t) \\ y(t) &= \sin(2t)\sin(t) \end{aligned}\right\} \Rightarrow z(t) = \sin(2t)\left(\cos(t) + i\sin(t)\right)$$

Unter Verwendung des EULERSCHEN Satzes erhalten wir schließlich:

Rosette:	$z(t) = \sin(2t)\, e^{it}$

Mit einem kleinen Zeichenmodul „`cplot(z_)`“, das wir uns definieren, erhalten wir die Rosette in der GAUSZschen Zahlenebene.

```
#9:   cplot(z_) := [RE(z_), IM(z_)]

#10:  z_rosette(φ) := SIN(2 φ) COS(φ) + î SIN(2 φ) SIN(φ)

#11:  cplot(z_rosette(φ))
```

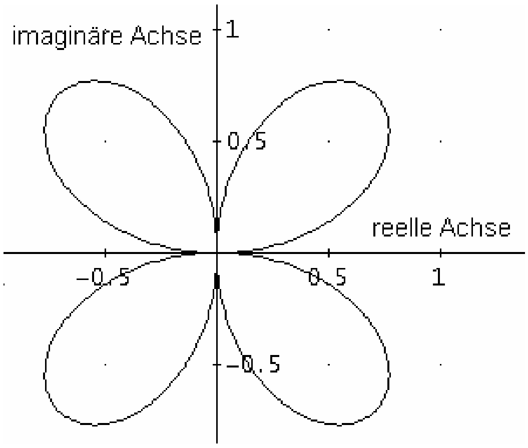


Abb. 15 / fig 15 Complex form

Doesn´t need any additional comments:

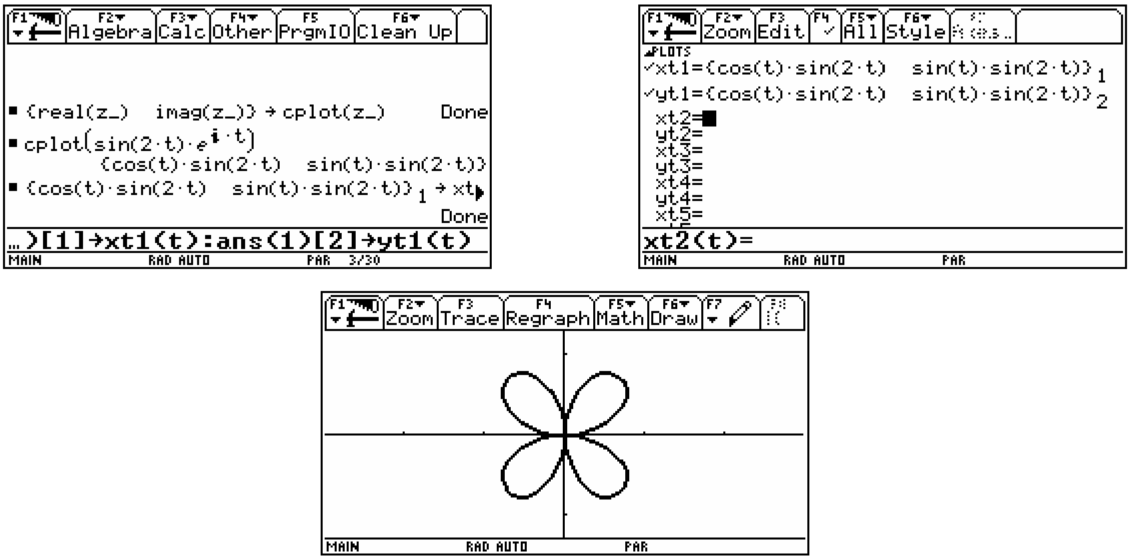


Abb. 16 / fig 16 Complex form on the TI

Weitere Aufgaben könnten gestellt werden, wie zB:

- Lässt sich die Astroide auf ähnliche Weise so erschöpfend behandeln?
- Welches ist die Ortslinie eines beliebigen – festen – Punktes auf dem gleitenden Segment? (Damit gelangen wir zur Ellipsenbewegung und zur punktweisen Konstruktion einer Ellipse = Papierstreifenkonstruktion).
- Erzeugung der Rosenkurve und der Astroide als Radlinie.

Follow up tasks could be:

- Can you treat the Astroid in a similar way?
- What is the locus of any fixed point of the gliding segment? (We find the ellipse motion and a a way to sketch an ellipse point by point.)
- Create the rose curve and the astroid as a trochoid.

I was vey much inspired by Josef´s contribution, so that I did not only add the TI-solutions – with his agreement, of course, but had additional ideas to open another "Presentation WINDOW".

Mit Hilfe der Schieberegler lässt sich mit DERIVE eine Lücke zwischen dynamischer Geometrie und Computeralgebra zumindest teilweise schließen. Die Konstruktion unter Verwendung von Parametern lasst einerseits die Animation der Grafik zu, zwingt aber andererseits sofort zu einer notwendigen Verallgemeinerung der „konkreten Objekte". Die angestellten Überlegungen werden sofort verifiziert oder falsifiziert. Dies eröffnet eine ganz neue Dimension im Mathematikunterricht.

Presentation of the "gliding ladder" and the interesting three points using two slider bars helps to close the gap between dynamic geometry and computeralgebra. I insert two slider bars, one for the length r of the ladder (with $0 \le r \le 6$) and the other one to move the base point A(t,0) of the ladder (with $-6 \le t \le 6$). We calculate and plot simultaneously in two open windows (Algebra and 2D-Plot). Parallel geoemtric and generalized algebraic representation opens new dimensions in math teaching.

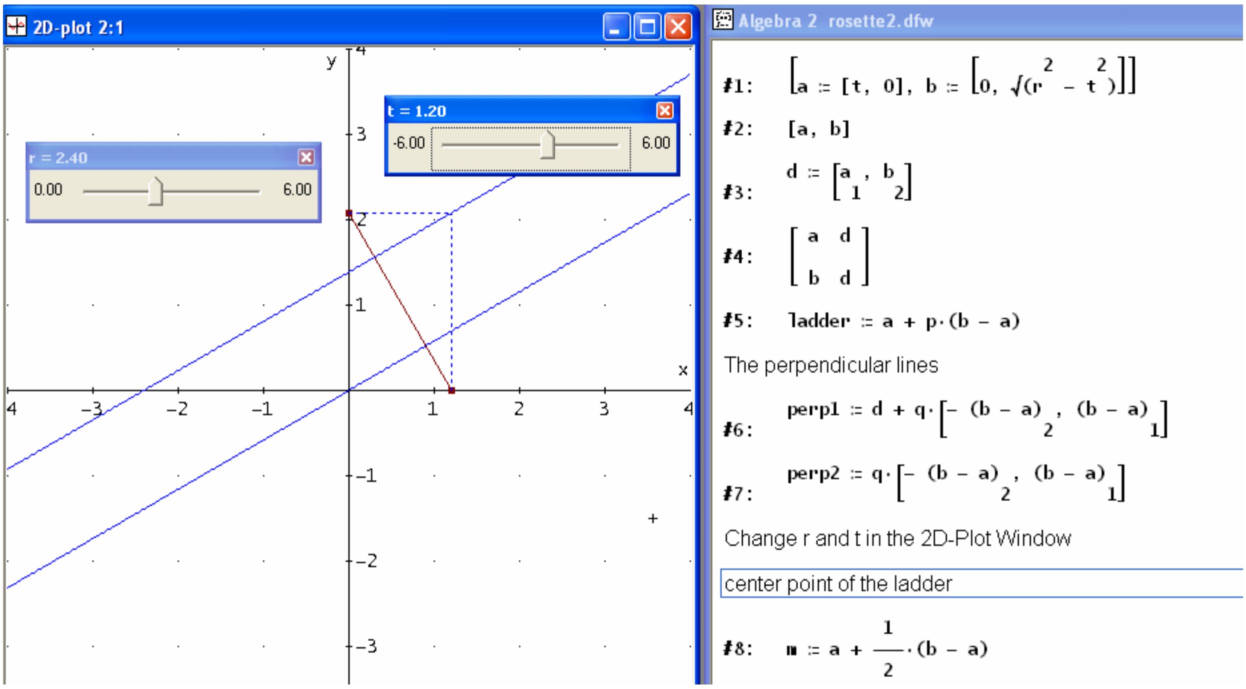


Abb. 17 / fig 17 Complex form on the TI

Midpoint and the intersection points of the perpendicular lines with the segment are calculated – and plotted.

The co-ordinates of midpoint m represent the parameter form of the locus of m.

For plotting the locus we have to change the variable for the parameter, because variable t is now preserved for the slider bar.

(Are the students able to recognize the curve from its parameter form?)

#8: $m := a + \frac{1}{2}\cdot(b - a)$

#9: $m := \left[\frac{t}{2}, \frac{\sqrt{(r^2 - t^2)}}{2}\right]$

#10: $\left[\frac{t\_}{2}, \frac{\sqrt{(r^2 - t\_^2)}}{2}\right]$

#11: SOLVE(ladder = perp1, [p, q])

#12: $p = \frac{r^2 - t^2}{r^2} \wedge q = \frac{t\cdot\sqrt{(r^2 - t^2)}}{r^2}$

#13: $\text{SUBST}\left(\text{ladder}, p, \frac{r^2 - t^2}{r^2}\right)$

#14: $\left[\frac{t^3}{r^2}, \frac{(r^2 - t^2)^{3/2}}{r^2}\right]$

#15: $\left[\frac{t\_^3}{r^2}, \frac{(r^2 - t\_^2)^{3/2}}{r^2}\right]$

#16: SOLVE(ladder = perp2, [p, q])

#17: $p = \frac{t^2}{r^2} \wedge q = -\frac{t\cdot\sqrt{(r^2 - t^2)}}{r^2}$

#18: $\text{SUBST}\left(\text{ladder}, p, \frac{t^2}{r^2}\right)$

#19: $\left[t - \frac{t^3}{r^2}, \frac{t^2\cdot\sqrt{(r^2 - t^2)}}{r^2}\right]$

#20: $\left[t\_ - \frac{t\_^3}{r^2}, \frac{t\_^2\cdot\sqrt{(r^2 - t\_^2)}}{r^2}\right]$

The plot shows the three loci – but not as a static result but as a "living" object of our own creation. By the way we found another parameter form for the three curves (without trig functions – only using analytic geometry!).

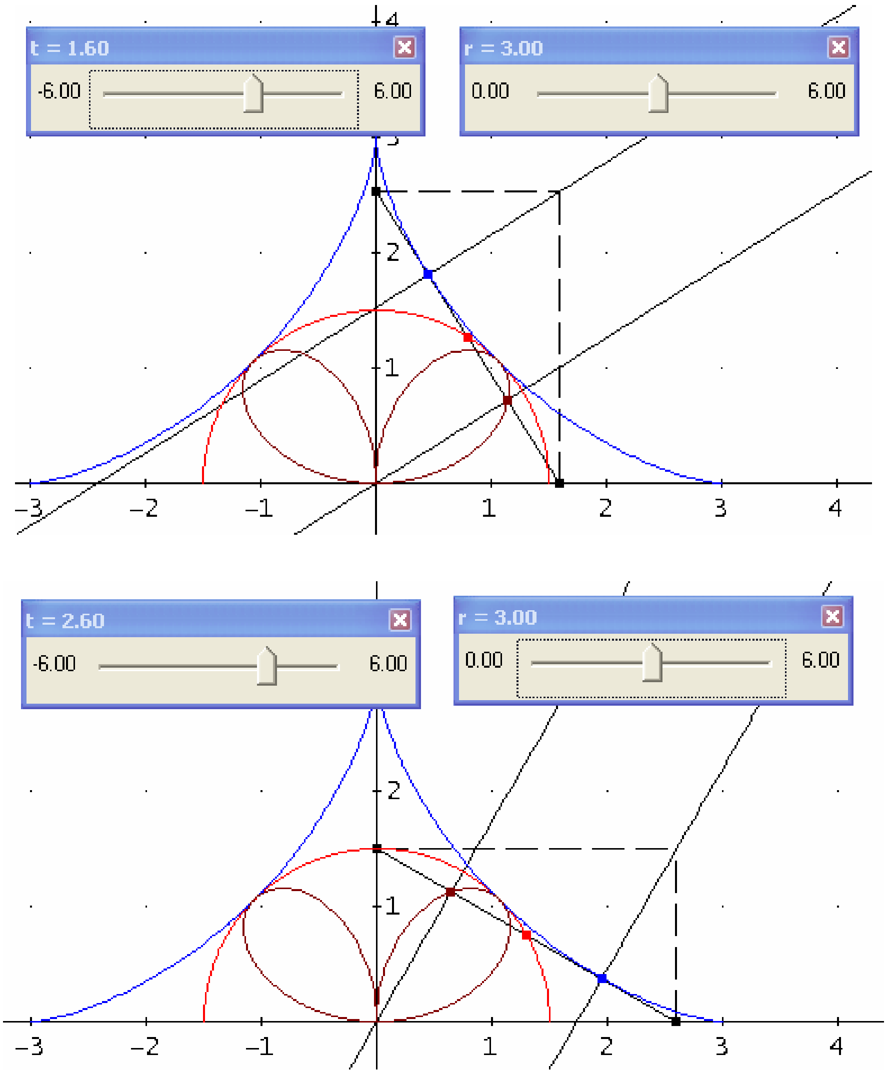


Abb. 18 / fig 18 All loci and slider bars for r and t

#21 creates the whole curves in one step:

$$\#21:\ \begin{bmatrix} \dfrac{t\_}{2} & -\dfrac{\sqrt{(r^2 - t\_^2)}}{2} \\ \dfrac{t\_^3}{r^2} & -\dfrac{(r^2 - t\_^2)^{3/2}}{r^2} \\ t\_ - \dfrac{t\_^3}{r^2} & -\dfrac{t\_^2 \cdot \sqrt{(r^2 - t\_^2)}}{r^2} \end{bmatrix}$$

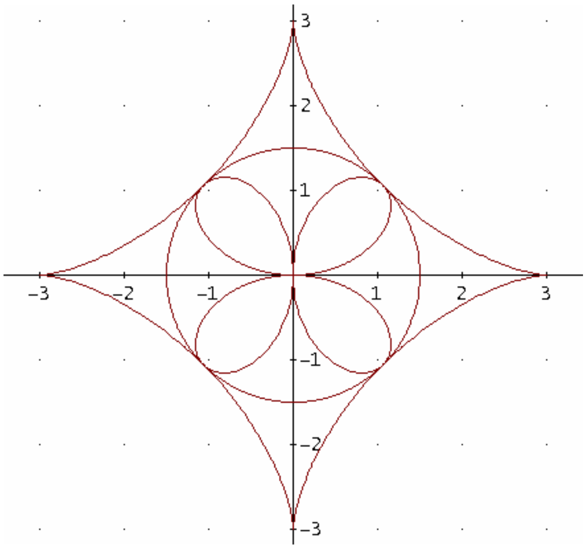


Abb. 19 / fig 19 The parameter forms and the respective plots

Eine nette Aufgabe ist es auch, aus dieser Parameterdarstellung die implizite Form der Rosenkurve zu gewinnen. Zuerst wird überprüft, ob die Parameterdarstellung in die vorliegende implizite Form passt. Dann ist es wichtig zu erkennen, dass auch in der Komponenten für x der Term $r^2 - t^2$ enthalten ist.

Einige kleine Schritte werden im Kopf – oder am Papier – durchgeführt, die restliche Prozedur erfordert auch eine gehörige Kompetenz im Umgang mit dem Programm, aber auch im Erkennen der notwendigen Strategie, nur Verrechnen und Schreibfehler sind Geschichte.

It is a nice exercise to derive the implicit form of the curve from this parameter form. At first we check if the parameter form "fits" into the given implicit expression. Then it is important to recognize that expression $r^2 - t^2$ is hidden in the first component of the parameter form.

Some minor steps are performed mentally, the remaining procedure requires a good deal of competence using the tool, but also deciding the right strategy. But Calculating and writing errors are "history".

$$\mathrm{SUBST}\left((x^2 + y^2)^3 = r^2\ x^2\ y^2,\ [x,\ y],\ \left[t - \frac{t^3}{r^2},\ \frac{t^2\ \sqrt{(r^2 - t^2)}}{r^2}\right]\right)$$

true

$$\#40:\ \frac{r^2\ t - t^3}{r^2}$$

$$\#41:\ \left[x = \frac{t\ (r^2 - t^2)}{r^2},\ y = \frac{t^2\ \sqrt{(r^2 - t^2)}}{r^2}\right]$$

$$\#42:\ \left(y = \frac{t^2\ \sqrt{(r^2 - t^2)}}{r^2}\right)^2 = \left(y^2 = \frac{t^4\ (r^2 - t^2)}{r^4}\right)$$

#43: $\left[x = \frac{t\,w}{r^2},\ y^2 = \frac{t^4\,w}{r^4}\right]$

#44: $\text{SUBST}\left(y^2 = \frac{t^4\,w}{r^4},\ w,\ \frac{r^2\,x}{t}\right) = \left(y^2 = \frac{t^3\,x}{r^2}\right)$

#45: $\text{SUBST}\left(x = \frac{t\,(r^2 - t^2)}{r^2},\ t,\ \left(\frac{y^2\,r^2}{x}\right)^{1/3}\right) = \left(x = \frac{\text{SIGN}(x^{2/3})\,|r\,y|^{2/3}}{x^{1/3}} - \frac{y^2}{x}\right)$

#46: $\left(x = \frac{\text{SIGN}(x^{2/3})\,|r\,y|^{2/3}}{x^{1/3}} - \frac{y^2}{x}\right) + \frac{y^2}{x} = \left(x + \frac{y^2}{x} = \frac{\text{SIGN}(x^{2/3})\,|r\,y|^{2/3}}{x^{1/3}}\right)$

#47: $\left(x + \frac{y^2}{x} = \frac{\text{SIGN}(x^{2/3})\,|r\,y|^{2/3}}{x^{1/3}}\right)^3 = \left(\frac{(x^2 + y^2)^3}{x^3} = \frac{r^2\,y^2}{x}\right)$

#48: $\left(\frac{(x^2 + y^2)^3}{x^3} = \frac{r^2\,y^2}{x}\right)\,x^3$

We did it!

#49: $(x^2 + y^2)^3 = r^2\,x^2\,y^2$

It might be easier for students to work with $r = 2$.

See finally the Cabri-version performed on the TI:

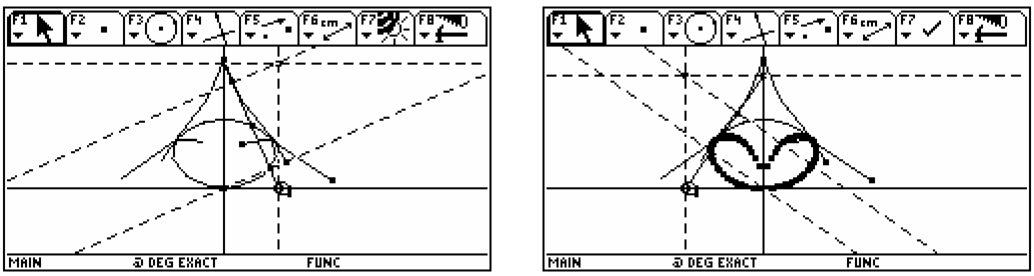


Abb. 20 / fig 20 The TI-Cabri version

Numerics versus Symbolics

B Kutzler (Linz, Austria)

This lecture is a meditation about two concepts which, in the context of computer algebra, sometimes appear as opposing each other.

1. Etymology

We start with the etymology of the two words. The word “numerics” comes from the Latin word “numerus” which means “part”, “number”, where “number” itself is derived from “part” as the result of a counting process.

The word “symbolics” comes from the Greek word “symbolon” which is composed of the two words “sym” (meaning “together”) and “ballein” (meaning “to throw”). Therefore, “symbolics” means “to put together”. “Putting together” can be for two reasons: It can be for constructing something, i.e. for creating a whole from parts. And it can be for putting things next to each other so that they can be compared[1]. For the Greek a “symbolon” was anything that would be comparable to the real thing whose place it took.

It is important to note that, strictly speaking, “5” also is a symbol for, say, the number of fingers of a hand, and “3.2” is a symbolic representation of the number obtained by dividing 32 by 10. But we don’t use “symbolics” in this narrow sense of the word here, because then all mathematics would have to be called symbolic mathematics.[2]

The German mathematician C F Gauss said: *“Mathematics is concerned only with the enumeration and comparison of relations.”* With the above in mind, this means that mathematics is concerned only with numerics (“enumerate”) and symbolics (“compare”).

2. First thoughts

In a computer algebra system (we use Derive 6) enter $\sqrt{24}$, simplify, then approximate.

#1:	$\sqrt{24}$
#2:	$2\cdot\sqrt{6}$
#3:	4.898979485

#1 and #2 are two different symbolic representations of this number. #3 is a numeric (decimal) representation of a ten-digit approximation of the same number[3]. Pragmatically, we can say that numeric mathematics is the mathematics on numbers (typically in decimal notation) such as #3. Symbolic mathematics is mathematics on everything else.

---

[1] The word “compare” is composed of the two Latin words “com” = ”cum” (meaning “together”) and “par” (meaning “equal”).

[2] In fact, in the narrow sense of the word “symbolic” one could consider mathematics the science of symbols.

[3] Compare our respective comment from section 1.

With a calculation such as the above on the screen of a symbolic calculator, Bert Waits once asked: *"How do you recognize a mathematician?"* and suggested the following answer: *"A mathematician considers #2 a beautiful result."* So we mathematicians like symbolics more than numerics ... although we know that numerics has its virtues too, and sometimes we cannot do without numerics. More about this later. For now we quote C F Gauss again, who has said that *"a poor mathematical education often is demonstrated by a highly developed skill of mental arithmetic."* (In the context of this paper this could be rephrased as: *"Poor symbolics often is demonstrated by good numerics."*)

As we said, expression #3 represents only an approximation of $\sqrt{24}$. The precise decimal representation of this number has infinitely many digits and, therefore, cannot be written in a finite amount of time or in a universe with only a finite amount of matter. Therefore, practical numeric mathematics necessarily is an approximative mathematics.

3. Different kinds of mathematics

The integers I, the rational numbers Q, and the real numbers R are three important number sets in (school) mathematics. They possess useful properties. An important property is "closure". It guarantees that one remains inside the domain when performing an arithmetic operations. I is closed w.r.t. addition, subtraction, and multiplication. Q and R are closed w.r.t. addition, subtraction, multiplication, and division (except for 0). The "deficiency" of the integers is that division can take us outside the domain, for example when dividing 3 by 4.

In Q (and R) we can divide 3 by 4. The result, ¾ or 0.75, is an element of Q (and R). If you divide 1 by 3, the result, 1/3 or 0.33333..., also is an element of Q (and R). But the latter example causes "trouble" when it comes to a "material" representation of the number in "practical" numeric mathematics, for example on a computer or a calculator, where numbers are represented in decimal notation with up to *n* digits (sum of digits before and after the comma). On calculators *n* often is 12 or 14.

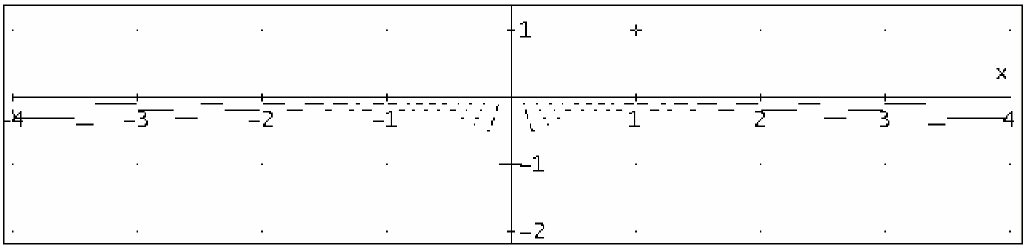
On such a device, the result of the division of 1 by 3 cannot be represented, therefore an approximation obtained by cutting off (infinitely many) decimal places is used. For example 1/3 will be approximated by 0.33333333333333.

Floating point arithmetic uses decimal numbers with up to n digits total. Fix point arithmetic uses decimal numbers with up to n digits following the comma. Let's look at the somewhat simpler case of fix point arithmetic. We formalized this as follows: Define R(n) to be the set of all real numbers with n decimal places. Then:

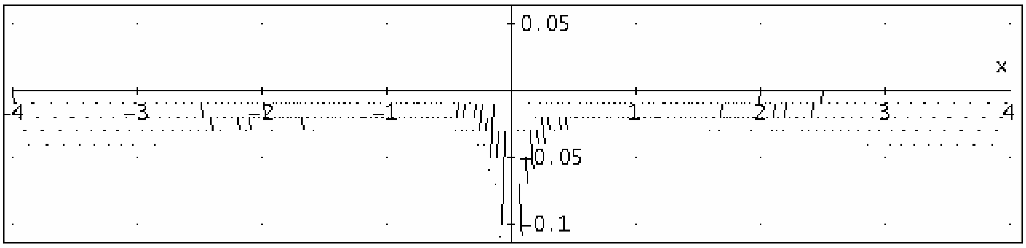
$$I = R(0) \subset R(1) \subset R(2) \subset \ \dots \subset R(n) \subset \ \dots \subset Q \subset R$$

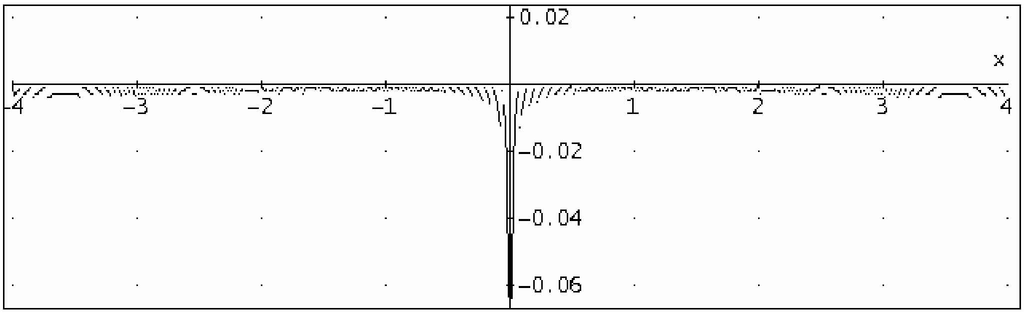
I is closed w.r.t. addition, subtraction, and multiplication
R(*n*) is closed w.r.t. addition and subtraction
Q and R are closed w.r.t. addition, subtraction, multiplication, and division

Closure is an important property when it comes to the validity of identities. For example, the simple identity $\frac{1}{x} \cdot x = 1$ is valid in Q and R, but it is not valid in any R(*n*). On most calculators this fact is hidden in "obvious" cases because numbers such as 0.999999999 are rounded to 1. What really happens can easily be visualized in Derive. The following is a graph of $y = \frac{1}{x} \cdot x - 1$ in R(1).



As you can see, the graph mostly is different from 0, hence for most values of x $\frac{1}{x}\cdot x \neq 1$. Screen images for R(2) and R(3) follow. Appendix 1 gives the code of the function used to produce these graphs.





Another example is the identity $(x^2 - y^2) = (x+y)\cdot(x-y)$. In R(1) we have

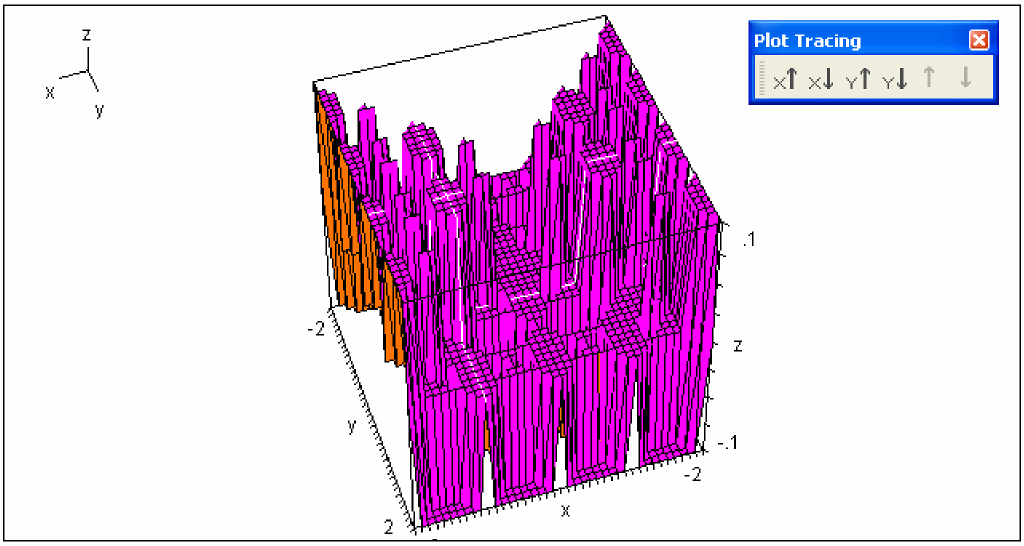
$$(1.1^2 - 0.2^2) = 1.2(1) - 0.0(4) = 1.2$$

and

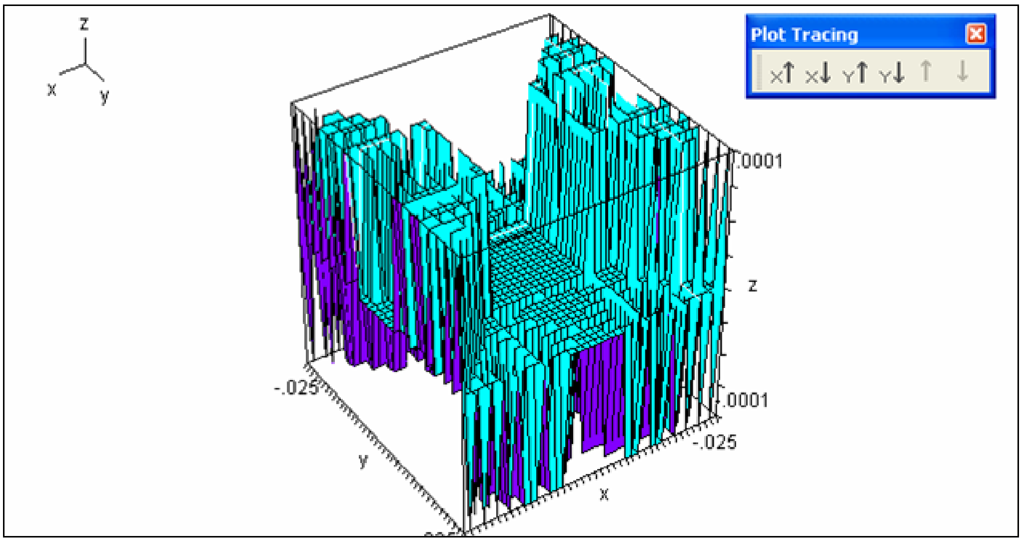
$$(1.1+0.2)\cdot(1.1-0.2) = 1.3\cdot 0.9 = 1.1(7).$$

```
#8:   test(1.1, 0.2)

#9:                        [1.2, 1.1]
```

Also for this example, falsity of the identity can be visualized in Derive. The following picture shows the graph of $z = (x^2 - y^2) - (x+y)\cdot(x-y)$ in R(1). The result is different from zero for many values of *x* and *y*.



With the help of Derive's trace function one can easily find concrete examples in any R(n). We find $x = 1$, $y = 0.5$ as another example in R(1) (above picture) and $x = 0.02$, $y = 0.0075$ in R(4) (below picture).



None of the traditional computation tools such as calculators are suitable for the "classical" mathematics in Q or R. Only computer algebra systems with their symbolic representations of rational numbers as quotients of two integers, with fractional powers, with π, *e*, etc., are appropriate.

By the way, the traditional tool for numerics is the abacus. Slide rules, four-function-calculators, scientific calculators and traditional (numeric) computer software are but sophisticated editions of an abacus. Computer algebra systems are a quantum leap. They are for symbolics what the abacus is for numerics.

It is perfectly fine to do mathematics in an "R(n) environment", but students need to understand the consequences. They need to understand, ideally by experiencing it with appropriate examples, what can happen with identities such as the above. In Derive one can do approximate arithmetic with a specified number of digits. This is of great help when studying or demonstrating the effects in an "R(n) environment".

We end this section by looking at the example $x\cdot\sqrt{x}\cdot\left(\sqrt{x+1}+\sqrt{x-1}-2\sqrt{x}\right)$. Enter the expression, substitute 1 million (1 000 000) for x, then approximate.

```
#1:   x·√x·(√(x + 1) + √(x - 1) - 2·√x)

#2:   1000000·√1000000·(√(1000000 + 1) + √(1000000 - 1) - 2·√1000000)

#3:                                     0
```

Looks like a clear zero. By default, Derive uses ten digits for approximations. If, instead, we approximate #2 with an accuracy of 15 digits, the result is very much different:

```
#4:                              -0.250012501243889
```

In [Kutzler/Kokol-Voljc 2003], pages 72ff, we show how to use Derive 6 to investigate this example in detail and demonstrate how it happens that the two results are so different.

4. Numerics helps Symbolics (i): Limitations of Symbolics

Symbolics has its limitations. Some symbolic computations are impossible,
- because an algorithm cannot be found due to theoretic limitations,
- because an algorithm has not been found yet,
- because an implementation of the algorithm does not exists yet,
- because the execution of the algorithm requires too much time or space.

In some of these cases a numeric solution is better than no solution.

Examples are:

(a) It is impossible to find symbolic solutions (using known functions and constants) of general polynomial equations of degree higher than four.

```
        5      4      3      2
#4:   x  + 2·x  - 3·x  + 4·x  - 5·x + 6 = 0

              5      4      3      2
#5:   SOLVE(x  + 2·x  - 3·x  + 4·x  - 5·x + 6 = 0, x)

                          5      4      3      2
#6:                      x  + 2·x  - 3·x  + 4·x  - 5·x = -6

               5      4      3      2
#7:   NSOLVE(x  + 2·x  - 3·x  + 4·x  - 5·x = -6, x)

#8:   x = -0.2389321333 - 1.151781933·i v x = -0.2389321333 +
        1.151781933·i v x = 0.9390814243 - 0.6271970073·i v x =
        0.9390814243 + 0.6271970073·i v x = -3.400298582
```

(b) For certain functions it is impossible to find closed form antiderivatives (using known functions and constants).

#4: $$\int_0^5 \mathrm{SIN}\left(\frac{1}{x}\right) dx$$

#5: 2.035618287

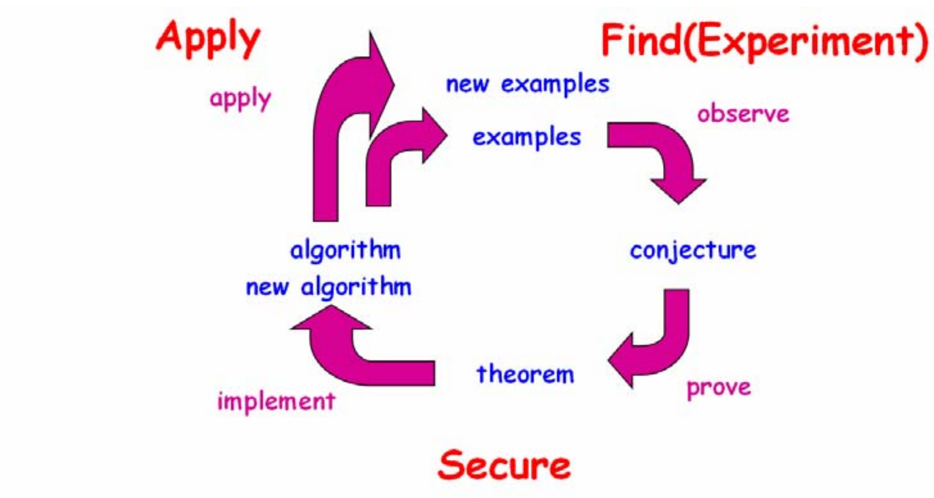
(c) It is impossible to find, for certain classes of expressions, an algorithm which can decide the equivalence of two expressions.

(d) Constructng a Groebner basis has double exponential complexity.

5. Numerics helps Symbolics (ii): The Creativity Spiral

When asked how he came upon his theorems, C F Gauss answered: *"... through systematic, palpable experimentation."*

According to one of the epistemologically oriented theories one can visualize the main steps of (mathematical) discoveries as follows: Applying known algorithms produces examples. From the examples we observe properties, which are expressed as a conjecture. Proving the conjecture yields a theorem, i.e. guaranteed knowledge. The theorem's algorithmically usable knowledge is implemented in a new algorithm. Then the algorithm is applied to new data, yielding new examples, which lead to new observations, ...



This picture of a spiral which demonstrates the path of discovery of (mathematical) knowledge was proposed by Bruno Buchberger. A detailed description of *Buchberger's Creativity Spiral* and references to related models can be found in the highly recommended (German language) book [Heugl/Klinger/Lechner 1996].

In this spiral we find three phases. During the *phase of finding/experimenting* one uses known algorithms to generate examples, then obtains conjectures through observation. During the *phase of securing* conjectures are turned into theorems through the method of proving, then algorithmically useful knowledge is implemented as algorithms. During the *phase of applying* one applies algorithms to new data.

In the finding phase often we have to find patterns in sequences of numbers. A well known example from the topic "proof by induction" is to find a closed form expression for $\sum_{i=1}^{n} i$. For this particular example there exists an elegant solution which is based on an observation made by C F Gauss when he was still very young. Here we use a method which may be helpful also for other such problems. Clearly, there are many alternative approaches which will lead to the solution.

We compute the sum for a few (consecutive) values of n, for example $n = 5,6,7,8,9,10$. The results are stored in the second column of the below table. In looking for a pattern we perform a simple factorization of the sums into two "obvious" factors – and store the result in the third column. The pattern is striking, but still a little hard to describe. By doubling the first factor we make it closer to the second factor (fourth column). Now we see that we always have products of two consecutive numbers. In the fifth column the two factors are arranged in ascending order, in the sixth column we divide by two again to compensate for the earlier doubling.

n	$\sum_{i=1}^{n} i$	simple factorization	double the first factor	sort factors	divide by two
5	15	$3\cdot 5$	$6\cdot 5$	$5\cdot 6$	$(5\cdot 6)/2$
6	21	$3\cdot 7$	$6\cdot 7$	$6\cdot 7$	$(6\cdot 7)/2$
7	28	$4\cdot 7$	$8\cdot 7$	$7\cdot 8$	$(7\cdot 8)/2$
8	36	$4\cdot 9$	$8\cdot 9$	$8\cdot 9$	$(8\cdot 9)/2$
9	45	$5\cdot 9$	$10\cdot 9$	$9\cdot 10$	$(9\cdot 10)/2$
10	55	$5\cdot 11$	$10\cdot 11$	$10\cdot 11$	$(10\cdot 11)/2$

Now the pattern is obvious (at least for these 6 values of n) and we can come up with the conjecture $\sum_{i=1}^{n} i = \frac{n\cdot(n+1)}{2}$, which can easily be proved by induction.

This is an inductive process which leads from the special to the general, i.e. from numeric to symbolic.

The other two phases in the above spiral, securing and applying, are both deductive processes leading from the general to the specific. The applying phase is the reverse of the finding phase's inductive process: You go from symbolic to numeric, for example by using the closed form expression to compute the sum of the first 100 natural numbers as $\sum_{i=1}^{100} i = \frac{100\cdot 101}{2} = 5050$ (tribute to C F Gauss!).

The securing phase also is a deductive process, i.e. it leads from the general to the specific, but this is on the level of mathematical logic, where the general are the (inference) rules of logics and the specific is the mathematical theory in which the conjecture is formulated.

6. Symbolics helps Numerics (i): Preprocessing

Say we need to calculate the perpendicular bisector of two points a and b. The steps of using the so called normal vector form are simple.

First we compute the midpoint of a and b. This point, we call it p, is a point on the line we are looking for. Then we compute the vector from a to b. This vector, we call it n, is normal to the line we are looking for. Using n and p we can write the equation of the line using the normal vector form(ula) $n \cdot \begin{pmatrix} x \\ y \end{pmatrix} = n \cdot p$. These mostly numeric computations are easily performed with paper and pencil, we use Derive:

#1: $a := [1, 3]$

#2: $b := [-2, 5]$

#3: $\dfrac{a + b}{2}$

#4: $\left[-\dfrac{1}{2}, 4\right]$

#5: $b - a$

#6: $[-3, 2]$

#7: $[x, y]\cdot[-3, 2] = \left[-\dfrac{1}{2}, 4\right]\cdot[-3, 2]$

#8: $3\cdot x - 2\cdot y = -\dfrac{19}{2}$

If we need perpendicular bisectors very often, for example because we do analytic geometry, the above procedure becomes tedious. It is always the same steps, only the numbers (i.e. the four coordinates of the two points) are different.

To save us from these many numeric applications of the above procedure we can apply it one time to a pair of symbolic points, i.e. points *a* and *b* with symbolic coordinates: $a = \begin{pmatrix} x_a \\ y_a \end{pmatrix}$ and $b = \begin{pmatrix} x_b \\ y_b \end{pmatrix}$.

In a paper and pencil environment this "symbolic application" of the procedure is much more work than its meric application (and most students don't like this kind of calculations). In a computer algebra system the extra effort of a symbolic application is done by the machine, so it is not any more difficult for the user. Below is the computation performed in Derive.

```
#9:   InputMode := Word
#10:  a := [xa, ya]
#11:  b := [xb, yb]
```

#12: $\frac{a + b}{2}$

#13: $\left[\frac{xa + xb}{2}, \frac{ya + yb}{2}\right]$

#14: $b - a$

#15: $[xb - xa, yb - ya]$

#16: $[x, y]\cdot[xb - xa, yb - ya] = \left[\frac{xa + xb}{2}, \frac{ya + yb}{2}\right]\cdot[xb - xa, yb - ya]$

#17: $x\cdot(xa - xb) + y\cdot(ya - yb) = \frac{xa^2 - xb^2 + (ya + yb)\cdot(ya - yb)}{2}$

#18: $x\cdot(xa - xb) + y\cdot(ya - yb) = \frac{xa^2 - xb^2 + (ya^2 - yb^2)}{2}$

We obtain $(x_a - x_b)\cdot x + (y_a - y_b)\cdot y - \frac{x_a{}^2 - x_b{}^2 + y_a{}^2 - y_b{}^2}{2} = 0$ as the equation of the perpendicular bisector of a and b. Now for any new points a and b we can simply substitute the coordinates into this expression and obtain the resulting equation of the line.

The one-time investment of a symbolic application saves a potentially infinite number of numeric applications. This method can be considered a “preprocessing”.

7. Symbolics helps Numerics (ii): The Pentium-Bug

On Oct 19, 1994 Dr. Thomas R Nicely, professor of mathematics at Lynchburg College discovered what later has become known as the Pentium Bug. After lots of testing he found that 824,633,702,441 divided by itself gave 0.999999996274709702 – instead of 1. Later it was found that this is true for all numbers between 824,633,702,418 and 824,633,702,449.

This is the well known part of the story. Here comes the unknown part[4]: Shortly after the Pentium Bug was found, the authors of Derive, David Stoutemyer and Albert Rich from Soft Warehouse, Inc., sent a complimentary copy of Derive to Prof. Nicely. After several weeks Thomas Nicely phoned up David. He apologized for taking so long to respond, but after he became famous for having found the Pentium Bug he received special protection from the police and all mail sent to him had to undergo a screening by a security team. Therefore he got the package with some delay. He thanked David for

---

[4] Personal communication with David Stoutemyer.

the copy of Derive and said that, in fact, he owns a copy of Derive and that Derive's ability to do approximate arithmetic to a specified number of digits (our above "R(n)-arithmetic") has helped him to locate the cause of the problem in the Pentium chip.

Therefore, the bug in the numeric methods of the Pentium chip was discovered with the help of a symbolic tool.

8. More from etymology

In section 1 we said that "numerics" comes from "numerus" (= "part" (of a whole), "number") and "symbolics" comes from "symbolon" (= "to put together", "to compare"). In mathematics we use only the meaning "numbers" for "numerics". "Symbolics", on the other hand, can denote either a composition (of parts) or something which takes the place of something else (and, hence, is comparable to it).

Working with decimal representations of numbers is what we call "numeric". A variable, for example x, which takes the place of something, we call "symbolic". Another example of something "symbolic" is $y = x^2$. $y = x^2$ stands for a whole which is composed of (infinitely many) parts, namely the pairs (x, x^2) for x taken from a certain set. When we take x from the real numbers, these pairs can be interpreted as points in the plane forming a parabola.

A famos quote from the Greek philosopher Aristoteles says: *"The whole is more than the sum of its parts"*. In the above example, the points of the parabola are the parts, the parabola is the whole. The parabola is composed of infinitely many points, so it is the sum of its parts – but it is more than that, because it has properties, which none of the points has or which none of all possible subsets of the points has. Such properties are continuity, symmetry, or simply the fact that the points of the parabola are exactly those points of the plane whose coordinates satisfy the equation $y = x^2$.

Instead of "symbolic computation" we often see "algebraic computation". How are these two words related? The word "algebra" comes from the title of a work written around 825 by the Arabic mathematician known as al-Khowarizmi entitled "al-jebr w'al-muqabalah". In Arabic "al-" is the definite article "the". The first noun in the title is "jebr", which means "reunion of broken parts", from the verb "jabara" which means "to reunite, to consolidate".[5] The second noun in the book title is from the verb "qabala", with meanings that include "to place in front of, to balance, to oppose, to set equal."

Together these two words describe some of the manipulations so common in algebra.

9. Pythagoras' View of Mathematics

Pythagoras of Samos (580 – 496 BC) was a contemporary of Confuzius (551 – 479 BC) and Prince Gautama, the Buddha, (560 – 480 BC). He is considered one of the wisest men of antiquity. Pythagoras lived on Samos, a Greek island. The tyrant Polycrates took power in 538 BC (first with his brother, then alone in 532 BC). Pythagoras disagreed with his rule and left the island 532 BC for South Italy, where he started a school in Croton.

---

[5] This corresponds to the one meaning of the Greek word "symbolon" = "to put together".

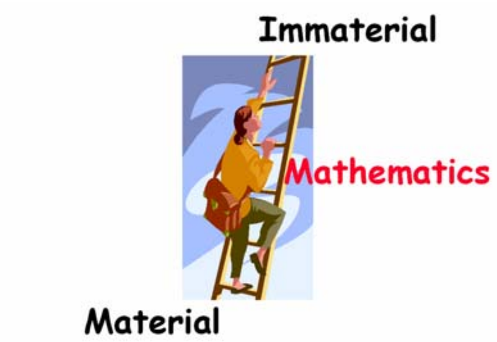
Pythagoras used three disciplines of teaching:
- Nutrition for cleaning and developing the physical body
- Music for cleaning and developing the emotional body (the soul)
- Mathematics for cleaning and developing the mental body (the mind)

Pythagoras had two ideals: Freedom and philosophy. For him, mathematics was the discipline to acquire both as I will explain in the sequel. Pythagoras regarded the three bodies (physis, emotio, ratio) to be closely connected with each body influencing the other two. Therefore he cleaned and developed all three bodies for best results.

“Philosophy” comes from the Greek words “philein” (meaning “to love”) and “sophia” (meaning “wisdom”). “Mathematics” comes from the Greek word “mathema” (meaning “science”), which originates from “mathesis” (meaning “knowledge”). Therefore, in the true sense of the word, mathematics is the only science we have.[6] Pythagoras said: *“Every man has been made by God in order to acquire knowledge and contemplate.”* So we should do mathematics in order to acquire knowledge and we should contemplate in order to transform our knowledge into wisdom. This makes mathematics the path to philosophy.

Next about freedom. According to Pythagoras, the true world of our mind is non-material. However, because of our physical body our mind collects only material experiences. Therefore, our mind is “imprisoned” by the material. Physically we never can be free, nor can we be free emotionally. But mentally we can be free, if we break out of the chains of our mind’s material imprisonment. Mathematics can help with this, because the objects of mathematics are between material and immaterial.

Look at the example of a point. As a mathematical object, a point is infinitely small. But there is nothing like this in the natural (material) world. However, there are objects (such as a point drawn with a pen, or an atom or subatomic particle) which come close to it for they are very, very small. Another example is a line, which, in mathematics, is infinitely thin and infinitely long. Also such a line does not exist in the material world, but there are objects (like a line drawn with a pencil and a ruler on a piece of paper) which comes somehow close to it. When we talk about a triangle in mathematics, we have something “in mind”, which can be considered an abstraction of triangular objects in natur or drawn on paper. Alternatively we can say that a triangle which we draw is a realization of a mathematical triangle.



Mathematical objects are like a rung on a ladder from the material world to the immaterial world. Therefore, doing mathematics helps our mind to shake off the confinements of the material experiences and raise above the material world.

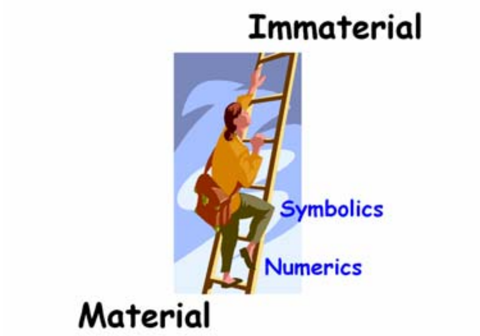
---

[6] Natural sciences are sciences insofar as they use the methods of mathematics.

Now you probably wonder what this has to do with “numerics” and “symbolics”.

Pythagoras said: *“Number is the within of all things.”* The “things” are the objects in the natural (material) world. For Pythagoras the numbers are representations of these natural objects. This is why we call them “natural numbers”. The “natural numbers” are the most material of all mathematical objects insofar as they are very close to material things. (“5” is a mathematical object very close to the five fingers of a hand.) From the natural numbers we construct new objects which may be less material, such as the negative numbers. (“-5” is not so easily recognized in the material world.)

Putting parts (remember: “part” = “number”) together as a new whole (remember: “whole” = “symbol”) is a basic technique in mathematics which leads to “less material” = “less numeric” = “more symbolic” = “more immaterial” objects.



If we look at mathematics as (part of) a ladder from the material world to the immaterial world, “numerics” is the lower end (the first rung) on it, “symbolics” are the higher rungs. This means that the higher a person’s mathematical education, the higher on the ladder (s)he reaches. If you are high on the ladder there is the danger to lose contact to he material world. The picture of an absentminded mathematics professor appears ... However, the really great minds are those who are “tall” enough to stand with both feet on the material ground **and** reach with their hands high on the ladder.

This picture is also supported by the following quote from Nikolaus of Cues Cusanus, a German theologian and humanist who lived 1401-1464: *“If there is no other path to the divine then through symbols, we should use mathematical symbols for they possess indestructible certainty. Knowledge about the divine is out of reach for the mathematically illiterate.”*

The following table summarizes our philosophical mediation:

number	***symbol***
more material	less material
material	immaterial
finite	infinite
physical	spiritual

When *“number is the within of all things”* (Pythagoras), then symbolics can be considered the within of mathematics. The Scottisc mathematician Eric Temple Bell (1883 –1960) said: *“Any impatient student of mathematics or science or engineering who is irked by having algebraic symbolism thrust upon him should try to get along without it for a week.”*

We can't do mathematics without symbolics, but also we shouldn't do mathematics without numerics either for the numbers connect mathematics with the real world.

Closing Remark

I do hope that you found the contents of this presentation trivial, because, speaking with C F Gauss: *"When a philosopher says something that is true then it is trivial. When he says something that is not trivial then it is false."*

References

H Heugl, W Klinger, J Lechner, 1996: *Mathematikunterricht mit Computeralgebra-Systemen (Ein didaktisches Lehrerbuch mit Erfahrungen aus dem österreichischen DERIVE-Projekt)*. Bonn:Addison-Wesley, 307 pages, ISBN 3-8273-1082-2.

B Kutzler, V Kokol-Voljc, 2003: *Introduction to Derive 6*. Hagenberg: Soft Warehouse GmbH&CoKG, 268 pages, ISBN 3-9500364-5-8.

Appendix 1

```
#1:   nodi := 1

      intgr(x) :=
        If INTEGER?(x)
           x
#2:        If x ≥ 0
              FLOOR(x)
              FLOOR(x) + 1
```

$$\#3:\quad r(x) := \frac{\mathrm{intgr}(x\cdot 10^{nodi})}{10^{nodi}}$$

$$\#4:\quad \mathrm{test1}(n) := r\left(r\left(\frac{1}{n}\right)\cdot r(n)\right) - 1$$

```
#5:   test1(n)
```

Appendix 2

```
#1:   nodi := 1

      intgr(x) :=
        If INTEGER?(x)
           x
#2:        If x ≥ 0
              FLOOR(x)
              FLOOR(x) + 1
```

$$\#3:\quad r(x) := \frac{\mathrm{intgr}(x\cdot 10^{nodi})}{10^{nodi}}$$

$$\#4:\quad \mathrm{test}(x,\ y) := \left[r(r(x)^2) - r(r(y)^2),\ r((r(x) + r(y))\cdot(r(x) - r(y)))\right]$$

$$\#5:\quad \mathrm{testdiff}(x,\ y) := r(r(x)^2) - r(r(y)^2) - r((r(x) + r(y))\cdot(r(x) - r(y)))$$

```
#6:   testdiff(x, y)
```

Ein Brief von Mandala / **A Letter from Mandala**

Sehr geehrter Herr Böhm
Vielen Dank für Ihr E- mail und Ihre Glückwünsche für meine Aufnahme in der Open University. Auch Ihre Hinweise zu MathCad sind sehr nützlich! Ich werde sie gerne beherzigen. Wie es scheint, ist diese Software integraler Bestandteil des Mathematik-Kurses der Undergraduate-Stufe der OU, so dass ich damit ein neues Programmiersystem kennen lerne. Umso besser!

Meine Bewerbung bei der OU hat sich in der ersten Phase mithilfe von Korrespondenz ohne Kopie vollzogen, da aufgrund der OU- Satzung ich mir nicht im Entferntesten - das Aufnahme-Mindestalter beträgt 18 Jahre- eine Chance ausgerechnet hatte, Student der OU zu werden. Das zweieinhalbstündige Kollogium, dass sich um meine Englisch - und Mathematikkenntnisse drehte (insbesondere um meine Fertigkeit, rekursive Programme mit dem TI- 92 und der Turtle- Geometry zu erstellen) wurde per E- mail anberaumt. Der leitende Mathematik- Tutor versicherte mir alsdann, dass mein Paper im DERIVE- Newsletter über die Turtle- Geometrie ihn sehr beindruckt und dieses auch wesentlich zu meiner Aufnahme in die OU beigetragen habe.

Ich kann jedoch gerne bei nächster Gelegenheit eine schriftliche Bestätigung des Vorgangs für Sie bei der OU anfordern.

Auch ich wünsche Ihnen einen schönen Sommer
Mandala

You might remember Mandala von Westenholz´s contribution on TURTLE Graphics with the TI-92 from DNL#57. Mandala writes in her mail that she applied for acceptance in Open University. She did not expect being accepted because minimum age must be 18. But she was invited for a 2 1/2 hour colloquium testing her English and Mathematics knowledge (recursive programming a.o.). The leading math-tutor told her that especially her paper published in the DNL had impressed him very much and that this paper and that this paper was responsible for a great deal that Mandala is now student of the Open University. Much luck, Mandala for your future and it would be great to hear from you. Thanks also to Bernhard Kutzler who forwarded Mandala´s contribution and suggested publication. Josef

<u>Von Josef an Hans-Jürgen Kayser</u>

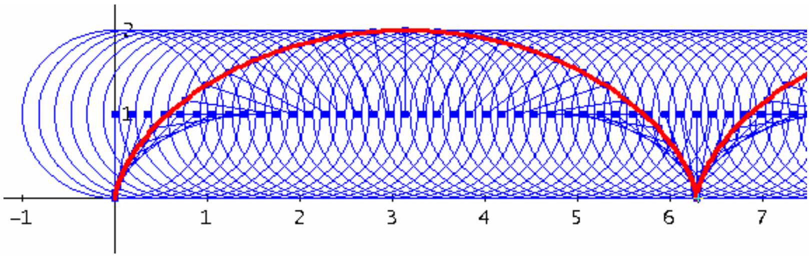
At the occasion of revising and rewriting DNL#6 I wrote a letter to Hans-Jürgen Kayser concerning his contribution to "Graphic Differentiation". In DOS-DERIVE times it was possible to slow down the plotting procedure by setting Plot Accuracy. So we could produce "mathematical movies". DERIVE and the PCs of our days are so fast, that it is not possible to reproduce this nice effect by means of 1992. I didn´t want to leave this "movie effect" and found a little trick to implement a "DELAY-function".

Hans-Jürgen Kayser answered very soon and thanked for "saving" the movie effect. He wrote that he also dealed with this effect by writing his Calculus Book (find a lot of H-J's publications on the book-shelf). He found a similar way to resolve this problem, but uses now a mixture of his idea and my idea (using the `RANDOM`-function). He added some files to experience `DELAY` in Action.

See one example on the next page.

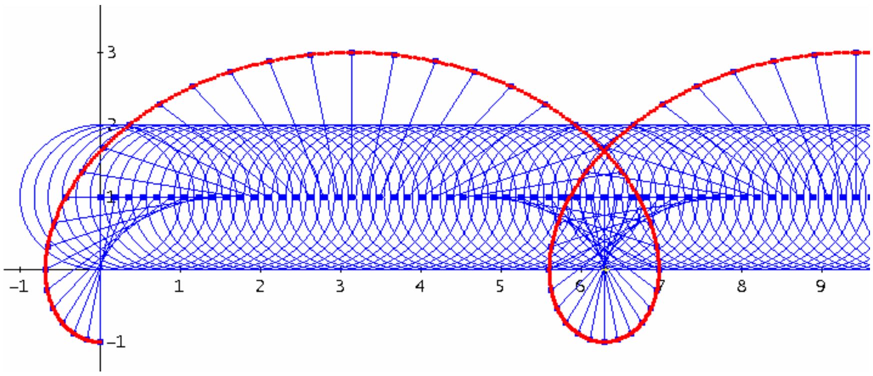
You can find detailed instructions in "Derive im Analysisunterricht der Jahrgangsstufen 12 und 13", bk-teachware, SR-45

```
#1:   [CaseMode := Sensitive, InputMode := Word]

#2:   [K(r, t) := [r·COS(t), r·SIN(t)], M(r, t) := [r·t, r]]

#3:   DELAY(dur) := VECTOR((x - RANDOM(1))^2 + (y - 200)^2 = k__·0.02^2, k__, 1, dur)

#4:   Zykloide_a(r, a, t) := [r·t - a·SIN(t), r - a·COS(t)]

#5:   GRAFIK(r, a, d := 1) := VECTOR([K(r, T) + M(r, t), [M(r, t), Zykloide_a(r, a, t)], DELAY(d)], t, 0, 5·π, π/18)

#6:   ZYKLOIDE(r, a) := VECTOR(Zykloide_a(r, a, t), t, 0, 5·π, π/180)

#7:   GRAFIK(1, 1)

#8:   ZYKLOIDE(1, 1)
```

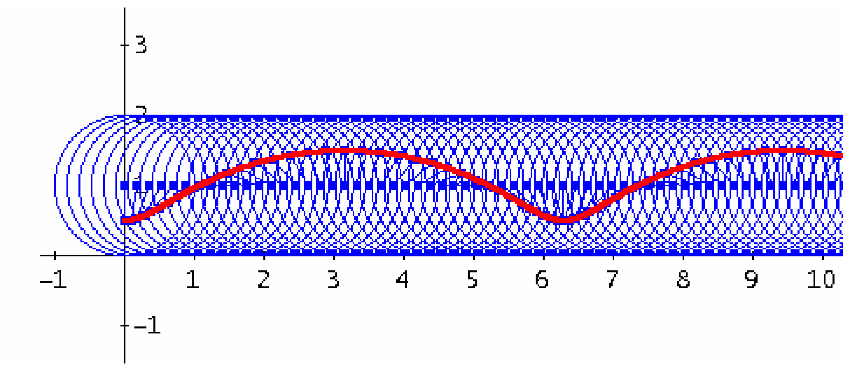


The third parameter 3 delays the animation (default = 1)

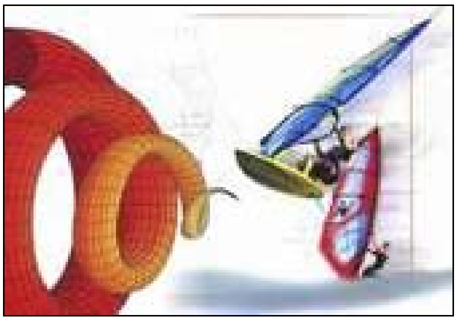
```
#10:  GRAFIK(1, 2, 3)

#11:  ZYKLOIDE(1, 2)
```



```
#12:  GRAFIK(1, 0.5)

#13:  ZYKLOIDE(1, 0.5)
```



Walter Wegscheider



Lieber Josef,
ich bin gefragt worden, wie die Funktion der Spirale lautet, die DERIVE im Cover der CD verwendet. Ist anscheinend eine Art Mischung aus Torus und Spirale. Weißt du, wie die Funktion dazu lautet?

Walter was asked for the function to create the "DERIVE-SPIRAL" which is shown on the CD-Cover and DERIVE Manual

Here is a collection of snails:

#1: $\text{NORMAL\_VECTOR}(v, t) := \text{SIGN}\left(\left(\frac{d}{dt}\right)^2 v\right)$

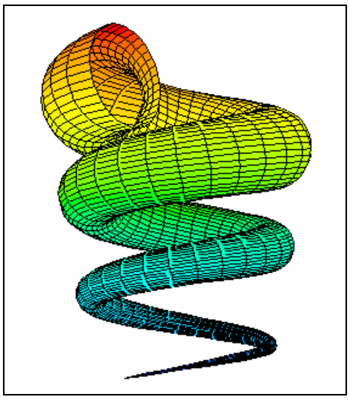
#2: $\text{BINORMAL}(v, t) := \text{SIGN}\left(\text{CROSS}\left(\frac{d}{dt} v, \left(\frac{d}{dt}\right)^2 v\right)\right)$

#3: $\text{snail}(v, t, r, \phi) := v + r\cdot t\cdot(\text{SIN}(\phi)\cdot\text{BINORMAL}(v, t) + \text{COS}(\phi)\cdot\text{NORMAL\_VECTOR}(v, t))$

Dabei ist v die Parameterdarstellung einer Schraublinie mit dem Parameter t
v is the parameter form of the helix.

#4: $\text{snail}([5\cdot\text{COS}(t), 5\cdot\text{SIN}(t), 0.9\cdot t], t, 0.16, s)$

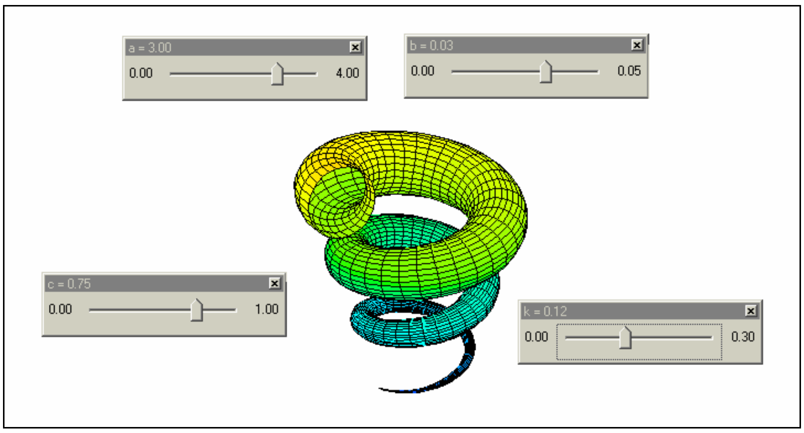
#5: $\text{snail}([5\cdot\text{COS}(t), 5\cdot\text{SIN}(t), 0.6\cdot t], t, 0.1, s)$



Let´s have an animated "spiral snail" with slider bars for a, b, c and k:

#6: $\text{spir} := \left[a\cdot e^{b\cdot t}\cdot\text{COS}(t), a\cdot e^{b\cdot t}\cdot\text{SIN}(t), c\cdot t\right]$

#7: $\text{snail}(\text{spir}, t, k, s)$



Here is another mail from our colleague from Düsseldorf, Hans-Jürgen Kayser. Once more I´d like to draw your attention on his inspiring publications (Information page). Josef

Lieber Herr Böhm,

anbei eine kleine Datei, die das Ziehen mit Zurücklegen (vgl. DNL # 52, Seite 42) m. E. etwas einfacher löst. Ich hoffe, dass Sie sich hierfür interessieren.

HJK

Dear Mr. Böhm,
I attach a short file, which in my opinion solves simulation of "Drawing without Repetitions" (DNL#52, page 42) a bit easier. I hope that you will find it interesting.
Hans-Jürgen Kayser

Lotto / Lottery
Simulation des Zahlen-Lottos "*k* aus *n*"

© HJK 6/2005

Kurzbeschreibung / Short Description

Es ist kein Problem, Derive aus einer Menge von Zahlen (z.B: {1, 2, ..., 49}) 6 Zufallszahlen auswählen zu lassen. It is no problem for Derive to choose 6 random numbers from a set of numbers, eg from {1, 2, ..., 49}.

```
#1:   AUSWAHL(n, k) := VECTOR(1 + RANDOM(n), j, 1, k)
#2:   AUSWAHL(49, 6)
#3:   [17, 29, 12, 26, 46, 48]
#4:   [19, 26, 29, 31, 19, 29]
```

Wie die zweite Serie zeigt, können bei den so definierten "Ziehungen" Wiederholungen auftreten.
Die Funktion AUSWAHL simuliert das "Ziehen *mit* Zurücklegen", da bei jeder neuen "Ziehung" wieder alle n Zahlen zur Auswahl stehen.
Beim Zahlenlotto wird aber ohne Zurücklegen gezogen. Dieses "Ziehen *ohne* Zurücklegen" soll im Folgenden simuliert werden.
As the second sequence of numbers shows repetitions of numbers can occur in the drawings defined by function #1. In the following we will simulate "Drawings *without* Repetitions".

Vorübung / Preparatory Exercise:

```
#5:   LOTTO1 := PROG(POOL := {1, ..., 49}, LOOP(IF(DIM(POOL) = 43,
          RETURN {1, ..., 49} \ POOL), POOL := POOL \ {1 + RANDOM(49)}))
```

Zwei Beispiel-Ziehungen / Two sample drawings:

```
#6:   LOTTO1 = {12, 17, 26, 29, 46, 48}
#7:   LOTTO1 = {22, 27, 29, 40, 41, 45}
```

Lotto "*k* aus *n*":

```
         LOTTO(n, k, POOL) :=
           Prog
             POOL := {1, ..., n}
#8:          Loop
               If DIM(POOL) = n - k
                  RETURN {1, ..., n} \ POOL
               POOL := POOL \ {1 + RANDOM(n)}
```

Hinweis: #8 ist folgendermaßen einzugeben (vgl. # 5) / Hint: #8 must be authored as follows:

```
LOTTO(n,k,POOL):=PROG(POOL:={1,...,n},LOOP(IF(DIM(POOL)=n-k,RETURN{1,...,n}\
                 POOL),POOL:=POOL\{1+RANDOM(n)}))
```

Beispiele / Samples: Lotto "6 aus 49", "3 aus 6" und "5 aus 12"

```
#9:   LOTTO(49, 6) = {3, 12, 15, 47, 48, 49}

#10:  LOTTO(6, 3) = {3, 4, 6}

#11:  LOTTO(12, 5) = {1, 2, 6, 11, 12}
```

Eine Liste mit Tippreihen zum Lotto "6 aus 49" / A Sequence of drawings "6 out of 49":

```
#12:  VECTOR([LOTTO(49, 6)], k, 1, 10)

       ┌ {12, 17, 26, 29, 46, 48} ┐
       │ {22, 27, 29, 40, 41, 45} │
       │  {1, 11, 26, 33, 46, 48} │
       │  {4, 7, 22, 32, 36, 44}  │
       │  {1, 10, 14, 20, 22, 45} │
#13:   │ {11, 14, 22, 24, 28, 31} │
       │ {30, 35, 37, 39, 42, 48} │
       │  {1, 15, 20, 22, 24, 31} │
       │  {4, 8, 13, 18, 30, 31}  │
       └  {2, 9, 14, 21, 25, 49}  ┘
```

Alternative (aus: DERIVE-NEWSLETTER #52, p 42):

```
         LOTTERY(k, list, t_, newlist, counter) :=
           Prog
             newlist := []
             counter := 1
             Loop
#14:           If counter > k
                  RETURN SORT(newlist)
               t_ := RANDOM(DIM(list)) + 1
               newlist := ADJOIN(list↓t_, newlist)
               list := DELETE(list, t_)
               counter :+ 1
```

Beispiel-Ziehungen / Sample Drawings:

```
#15:  LOTTERY(6, [1, ..., 49]) = [11, 17, 26, 30, 46, 48]

#16:  VECTOR(LOTTERY(6, [1, ..., 49]), j, 1, 10)
```

#17:

11	17	26	30	46	48
20	26	29	40	44	45
1	26	32	40	46	48
4	8	11	33	37	45
1	10	14	20	22	46
10	14	22	24	29	32
22	31	35	38	42	48
1	19	22	24	34	42
12	14	17	23	31	33
4	8	10	15	23	30

Simulation des Zahlen-Lottos "*k* aus *n*"/ Ende

I´d like to add a tip: Include simplifying **RANDOM(0)**, either explicitly as first command or built in in the frame of the program (eg. **dummy:=RANDOM(0)**), to obtain different simultations at each new session. Josef

Bill Martin bill_e_martin@HOTMAIL.COM

Apologies if this is the wrong forum in which to pose this question.

I use Derive 5 and sometimes need the Kronecker product of two matrices. Matlab has a "kron" function but I believe Derive does not possess an equivalent.

The Kronecker product can be formed in Derive using a longwinded manual method involving the commands “ELEMENT”, “APPEND_COLUMNS” and “APPEND”.

I would much prefer to have a simple command function but do not possess the requisite skills to program in Derive.

I would be most grateful if anyone could offer assistance, suggestions and, best of all, the Derive code for a Kronecker product function.

Thank you.

Bill,

I don't remember if Derive 5 has the MAP_LIST function, but it if does you could use something like this.

```
KroneckerProduct(a, b) := APPEND(MAP_LIST(APPEND_COLUMNS(x_), x_, MAP_LIST
                              (MAP_LIST(v_·b, v_, w_), w_, a)))
```

Don Phillips

Titbits from Algebra and Number Theory (31)

by Johann Wiesenbauer, Vienna

Have you ever used sets and set-theoretic operations in your programs? If your answer is no, then you may have missed something. In fact, they come in handy on many occasions, where most people don't think of sets in the first place.

Let's start with a very simple example of this sort. Imagine 30 hunters aiming at 10 ducks in front of them and shooting at exactly the same time. We assume here that each hunter selects the target independently of the others and that there is no miss. What do you think is the chance that all 10 ducks were hit after that shooting? Would you dare to bet on this event?

Well, I'm sure most mathematicians will immediately translate this question into the following one: What is the probability that a random mapping from a 30-element set into a 10-element set is surjective? Now some will try to answer this question in a purely theoretical way, which is not too difficult, namely

$$\text{APPROX}(\text{STIRLING2}(30, 10)\cdot 10!\cdot 10^{2-30}) = 62.91371892$$

Here, STIRLING2(30,10) is the number of ways to decompose a 30-element set into exactly 10 nonempty subsets. These correspond to partitions induced on the 30-element set by any surjective mapping onto the 10-element set, where elements belong to the same class if and only if they have the same image under the mapping at issue. For each of these partitions you can select images of its members in 10! ways, hence the factor 10! above. Finally, one must divide STIRLING2(30,10) 10! by the number of all mappings, i.e. 10^{30}, and multiply it by 10^2 to get the percentage.

Many Derivers will prefer an "experimental" approach though, which is sketched below. First of all, we need a routine that determines whether a mapping from an n-element set into a k-element set is surjective or not. This is where our sets come into play, as you can see in the following implementation.

```
surjective?(n, k, s_ := {}) :=
  Loop
    If n = 0
       RETURN DIM(s_) = k
    s_ := s_ ∪ {RANDOM(k)}
    n :- 1
```

Here and in the following, we make use of the basic fact about sets that every element in a set occurs only once and adding an element already contained in the set doesn't change it.

The next routine performs a number of tests w.r.t. surjectivity (hence its unimaginative name!) and after that computes the proportion of surjective mappings as a percentage.

```
test(m, n, k, m_ := 0, t_ := 0) :=
  Loop
    If m_ = m
       RETURN 100·t_/m
    If surjective?(n, k)
       t_ :+ 1
    m_ :+ 1
```

```
APPROX(test(10000, 30, 10)) = 62.86
```

In particular, we can see a good coincidence with the theoretical result and they both show that is worthwhile to bet on all ducks getting hit. Is this what you guessed?

Our next example is from coding theory. A binary linear code *C* could be given by a matrix *G* with entries in $\mathbf{Z}_2$ along with the rule that a vector c is in *C* if and only c can be written as a sum of certain row vectors of *G*. For example, if

$$G := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

then $c = (1,1,0,1,0,1)$ belongs to the code, because it is the sum of the first two row vectors. How could a program look like that computes the whole code *C*?

Would you believe that this small problem usually gives most Derivers quite a headache, even if I give as a hint that set-theoretic functions should be used? Ok, I don't want to put you on tenterhooks no more, here is the surprisingly short solution using Cartesian products.

```
C := VECTOR(MOD(v·G, 2), v, {0, 1}^3)
```

$$C := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

If you prefer the representation of *C* as a set rather than a list you should use

```
C := MAP_LIST(MOD(v·G, 2), v, {0, 1}^3)
```

```
C := {[0, 0, 0, 0, 0, 0], [0, 0, 1, 0, 1, 0], [0, 1, 0, 1, 0, 0], [0, 1, 1, 1, 1, 0], [1, 0, 0, 0, 0, 1], [1, 0, 1,
  0, 1, 1], [1, 1, 0, 1, 0, 1], [1, 1, 1, 1, 1, 1]}
```

After these warm-up exercises, it's high time we tackled a more ambitious project dealing with sets. I'm speaking of a routine that helps you a lot when solving Sudokus, the well-known Japanese cult puzzles. For members of the supposedly minority, who never heard of it, here a short explanation. Given is a 9x9-matrix like the matrix A below with some nonzero entries in the range 1-9.

$$A := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 7 & 5 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 6 & 3 & 0 & 0 \\ 0 & 0 & 2 & 8 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 8 & 0 & 0 & 0 & 0 & 0 & 0 \\ 5 & 3 & 0 & 0 & 0 & 0 & 0 & 2 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 9 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 & 1 & 0 & 0 \\ 0 & 0 & 7 & 9 & 0 & 0 & 0 & 8 & 0 \\ 0 & 0 & 6 & 2 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

For the solution of a sudoku it is required to replace all zeros, which are actually empty entries in a real sudoku, again by numbers in the range 1-9, such that every row, every column and nine 3x3-submatrices contain all numbers 1-9. As for the nine 3x3-submatrices, you will see them in the solution below, which was obtained by the program on the next page and the input below.

```
A := sudoku()
```

$$A := \begin{bmatrix} \begin{bmatrix} 8 & 6 & 1 \\ 9 & 4 & 5 \\ 3 & 7 & 2 \end{bmatrix} & \begin{bmatrix} 3 & 4 & 7 \\ 1 & 2 & 6 \\ 8 & 5 & 9 \end{bmatrix} & \begin{bmatrix} 5 & 9 & 2 \\ 3 & 7 & 8 \\ 6 & 1 & 4 \end{bmatrix} \\ \begin{bmatrix} 6 & 1 & 8 \\ 5 & 3 & 9 \\ 7 & 2 & 4 \end{bmatrix} & \begin{bmatrix} 7 & 9 & 2 \\ 4 & 6 & 1 \\ 5 & 3 & 8 \end{bmatrix} & \begin{bmatrix} 4 & 3 & 5 \\ 8 & 2 & 7 \\ 9 & 6 & 1 \end{bmatrix} \\ \begin{bmatrix} 2 & 8 & 3 \\ 4 & 5 & 7 \\ 1 & 9 & 6 \end{bmatrix} & \begin{bmatrix} 6 & 7 & 4 \\ 9 & 1 & 3 \\ 2 & 8 & 5 \end{bmatrix} & \begin{bmatrix} 1 & 5 & 9 \\ 2 & 8 & 6 \\ 7 & 4 & 3 \end{bmatrix} \end{bmatrix}$$

```
sudoku(n := ∞, a_, b_, c_ := 1, i_ := 1, j_ := 1, k_ := 1, l_ := 1, s_, t_, u_) :=
  Loop
    If DIM(A) = 9
       Prog
         A :+ VECTOR(VECTOR(123456789·0^A↓i↓j, j, 9), i, 9)
         A := VECTOR(VECTOR(VECTOR(VECTOR(A↓(i + k)↓(j + l), l, 3), k, 3), j, 0, 6, 3), i, 0, 6, 3)
    If n = 0 ∨ c_ = 0
       RETURN A
    n :- 1
    c_ := 0
    d_ := 1
    Loop
      a_ := ABS(A↓i_↓j_↓k_↓l_)
      If a_ = 0
         RETURN "Computation has been stopped due to zero entries!"
      s_ := MAP_LIST(a_↓m_, m_, {1, ..., DIM(a_)})
      t_ := TERMS(Σ(x^ABS(A↓i_↓(m_↓1)↓k_↓(m_↓2)), m_, {1, 2, 3}^2 \ {[j_, l_]}))
      u_ := SUBST(t_, x, 1)
      t_ := x·∂(LN(t_), x)
      Loop
        If u_ = [] exit
        If DIM(FIRST(t_)) = FIRST(u_)
           s_ := s_ \ MAP_LIST((FIRST(t_))↓m_, m_, {1, ..., DIM(FIRST(t_))})
        u_ := REST(u_)
        t_ := REST(t_)
      t_ := TERMS(Σ(x^ABS(A↓(m_↓1)↓j_↓(m_↓2)↓l_), m_, {1, 2, 3}^2 \ {[i_, k_]}))
      u_ := SUBST(t_, x, 1)
      t_ := x·∂(LN(t_), x)
      Loop
        If u_ = [] exit
        If DIM(FIRST(t_)) = FIRST(u_)
           s_ := s_ \ MAP_LIST((FIRST(t_))↓m_, m_, {1, ..., DIM(FIRST(t_))})
        u_ := REST(u_)
        t_ := REST(t_)
      t_ := TERMS(Σ(x^ABS(A↓i_↓j_↓(m_↓1)↓(m_↓2)), m_, {1, 2, 3}^2 \ {[k_, l_]}))
      u_ := SUBST(t_, x, 1)
      t_ := x·∂(LN(t_), x)
      Loop
        If u_ = [] exit
        If DIM(FIRST(t_)) = FIRST(u_)
           s_ := s_ \ MAP_LIST((FIRST(t_))↓m_, m_, {1, ..., DIM(FIRST(t_))})
        u_ := REST(u_)
        t_ := REST(t_)
      t_ := 0
      Loop
        If s_ = {} exit
        t_ := 10·t_ + FIRST(s_)
        s_ := REST(s_)
      b_ := IF(a_ = t_)
      c_ :+ 0^b_
      A↓i_↓j_↓k_↓l_ := (2·b_ - 1)·t_
      l_ :+ 1
      If l_ > 3
         [l_ := 1, k_ :+ 1]
      If k_ > 3
         [k_ := 1, j_ :+ 1]
      If j_ > 3
         [j_ := 1, i_ :+ 1]
      If i_ > 3
         [i_ := 1, exit]
```

I don't want to explain this program in detail, but only point out some important points. First of all, the first parameter is sort of a switch, by which you can control how many steps are performed before the program stops. (Note that, if there are no changes anymore as to the entries of the matrix, it will stop anyway!) In particular, you can set up a step-by-step mode by choosing n=1. In this case, changed entries w.r.t to the last matrix will be indicated by a minus sign.

For example, setting n:=1 we get for the same matrix A as above

A := sudoku(1)

$$A := \begin{bmatrix} \begin{bmatrix} -13689 & -689 & -139 \\ -1789 & 4 & -159 \\ -13679 & -5679 & 2 \end{bmatrix} & \begin{bmatrix} -134 & -12349 & 7 \\ -15 & -1259 & 6 \\ 8 & -13459 & -1359 \end{bmatrix} & \begin{bmatrix} 5 & -149 & -124689 \\ 3 & -179 & -1289 \\ -467 & -1479 & -1469 \end{bmatrix} \\ \begin{bmatrix} -24679 & 1 & 8 \\ 5 & 3 & -49 \\ -247 & -27 & -4 \end{bmatrix} & \begin{bmatrix} -34567 & -2345679 & -2359 \\ -146 & -14689 & -189 \\ -1357 & -123578 & -12358 \end{bmatrix} & \begin{bmatrix} -4 & -35 & -35 \\ -8 & 2 & 7 \\ 9 & 6 & -1 \end{bmatrix} \\ \begin{bmatrix} -2389 & -2589 & -359 \\ -1234 & -25 & 7 \\ -13489 & -589 & 6 \end{bmatrix} & \begin{bmatrix} -3567 & -35678 & 4 \\ 9 & -1356 & -135 \\ 2 & -13578 & -1358 \end{bmatrix} & \begin{bmatrix} 1 & -3579 & -23569 \\ -26 & 8 & -23456 \\ -7 & -3459 & -3459 \end{bmatrix} \end{bmatrix}$$

For sudoku solvers, who want rather a "hint" than the whole solution, this could also be what they want. Furthermore, you can observe, as it were, how the program "thinks", which might be interesting as well.

Let's have a look at the very first entry -13689. It says that there was a change, namely only the digits 1,3,6,8,9 are possible in this place instead of all digits before. Scanning the first row of the original matrix A, you see that 7 and 5 are excluded, indeed. As for the first column, there is nothing else we can conclude from it. Finally, the first 3x3-submatrix says that 2 and 4 are also impossible in top left corner, leaving only the possibilities 1,2,6,8,9, indeed.

Now let's focus on the two entries -35 at the end of the fourth row. What do they tell us? Of course, experienced sudoku solvers know the answer: The digits 3 and 5 are excluded in the rest of the row and also in the rest of the 3x3-submatrix. The same is true if a number with 3 digits occurs three times in a row, column or 3x3-submatrix and so on.

Actually, this is the only strategy the program pursues. Hence, it could be that the program stops although it has not yet arrived at a solution. In this case it's up to you to edit the output matrix in some places (e.g. by trying out one of two possibilites) and the program will continue from there. Nice, isn't it? I hope you enjoy the program as much as I do.

As you can imagine, I immediately tried this tool – I am not a SODUKOer, but my wife Noor is and I found several Sodukos in the newespapers. Easy and Medium was no problem for the program, but …

EASY

	9	6		5		3	2	
4			8		3			9
1				7				4
	5		6		8		1	
8		2				4		6
	4		9		1		3	
9				1				2
5			7		2			3
	3	8		9		7	5	

MEDIUM

		4				7		
			7	5	1			
1			4	6	3			2
	5	2				1	3	
	9	7				8	4	
	1	3				6	2	
5			1	9	8			3
			2	4	6			
		8				4		

"DIABOLICAL"

		5				6		
			1		2			
	9						2	
3			5		6			1
9								7
2			3		1			4
	8						1	
			4		3			
		6				8		

#2: $A := \begin{bmatrix} 0 & 9 & 6 & 0 & 5 & 0 & 3 & 2 & 0 \\ 4 & 0 & 0 & 8 & 0 & 3 & 0 & 0 & 9 \\ 1 & 0 & 0 & 0 & 7 & 0 & 0 & 0 & 4 \\ 0 & 5 & 0 & 6 & 0 & 8 & 0 & 1 & 0 \\ 8 & 0 & 2 & 0 & 0 & 0 & 4 & 0 & 6 \\ 0 & 4 & 0 & 9 & 0 & 1 & 0 & 3 & 0 \\ 9 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 2 \\ 5 & 0 & 0 & 7 & 0 & 2 & 0 & 0 & 3 \\ 0 & 3 & 8 & 0 & 9 & 0 & 7 & 5 & 0 \end{bmatrix}$

#3: $\text{sudoku}() = \begin{bmatrix} \begin{bmatrix} 7 & 9 & 6 \\ 4 & 2 & 5 \\ 1 & 8 & 3 \end{bmatrix} & \begin{bmatrix} 1 & 5 & 4 \\ 8 & 6 & 3 \\ 2 & 7 & 9 \end{bmatrix} & \begin{bmatrix} 3 & 2 & 8 \\ 1 & 7 & 9 \\ 5 & 6 & 4 \end{bmatrix} \\ \begin{bmatrix} 3 & 5 & 9 \\ 8 & 1 & 2 \\ 6 & 4 & 7 \end{bmatrix} & \begin{bmatrix} 6 & 4 & 8 \\ 5 & 3 & 7 \\ 9 & 2 & 1 \end{bmatrix} & \begin{bmatrix} 2 & 1 & 7 \\ 4 & 9 & 6 \\ 8 & 3 & 5 \end{bmatrix} \\ \begin{bmatrix} 9 & 7 & 4 \\ 5 & 6 & 1 \\ 2 & 3 & 8 \end{bmatrix} & \begin{bmatrix} 3 & 1 & 5 \\ 7 & 8 & 2 \\ 4 & 9 & 6 \end{bmatrix} & \begin{bmatrix} 6 & 8 & 2 \\ 9 & 4 & 3 \\ 7 & 5 & 1 \end{bmatrix} \end{bmatrix}$

3.83 sec

#4: $A := \begin{bmatrix} 0 & 0 & 4 & 0 & 0 & 0 & 7 & 0 & 0 \\ 0 & 0 & 0 & 7 & 5 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 4 & 6 & 3 & 0 & 0 & 2 \\ 0 & 5 & 2 & 0 & 0 & 0 & 1 & 3 & 0 \\ 0 & 9 & 7 & 0 & 0 & 0 & 8 & 4 & 0 \\ 0 & 1 & 3 & 0 & 0 & 0 & 6 & 2 & 0 \\ 5 & 0 & 0 & 1 & 9 & 8 & 0 & 0 & 3 \\ 0 & 0 & 0 & 2 & 4 & 6 & 0 & 0 & 0 \\ 0 & 0 & 8 & 0 & 0 & 0 & 4 & 0 & 0 \end{bmatrix}$

#5: $\text{sudoku}() = \begin{bmatrix} \begin{bmatrix} 3 & 6 & 4 \\ 2 & 8 & 9 \\ 1 & 7 & 5 \end{bmatrix} & \begin{bmatrix} 8 & 2 & 9 \\ 7 & 5 & 1 \\ 4 & 6 & 3 \end{bmatrix} & \begin{bmatrix} 7 & 5 & 1 \\ 3 & 6 & 4 \\ 9 & 8 & 2 \end{bmatrix} \\ \begin{bmatrix} 8 & 5 & 2 \\ 6 & 9 & 7 \\ 4 & 1 & 3 \end{bmatrix} & \begin{bmatrix} 6 & 7 & 4 \\ 3 & 1 & 2 \\ 9 & 8 & 5 \end{bmatrix} & \begin{bmatrix} 1 & 3 & 9 \\ 8 & 4 & 5 \\ 6 & 2 & 7 \end{bmatrix} \\ \begin{bmatrix} 5 & 4 & 6 \\ 7 & 3 & 1 \\ 9 & 2 & 8 \end{bmatrix} & \begin{bmatrix} 1 & 9 & 8 \\ 2 & 4 & 6 \\ 5 & 3 & 7 \end{bmatrix} & \begin{bmatrix} 2 & 7 & 3 \\ 5 & 9 & 8 \\ 4 & 1 & 6 \end{bmatrix} \end{bmatrix}$

4.83 sec

#7: $\text{sudoku}() = \begin{bmatrix} \begin{bmatrix} 1478 & 12347 & 5 \\ 4678 & 3467 & 3478 \\ 14678 & 9 & 13478 \end{bmatrix} & \begin{bmatrix} 789 & 34789 & 4789 \\ 1 & 3456789 & 2 \\ 678 & 345678 & 4578 \end{bmatrix} & \begin{bmatrix} 6 & 34789 & 389 \\ 34579 & 345789 & 3589 \\ 13457 & 2 & 358 \end{bmatrix} \\ \begin{bmatrix} 3 & 47 & 478 \\ 9 & 1456 & 148 \\ 2 & 567 & 78 \end{bmatrix} & \begin{bmatrix} 5 & 24789 & 6 \\ 28 & 248 & 48 \\ 3 & 789 & 1 \end{bmatrix} & \begin{bmatrix} 29 & 89 & 1 \\ 235 & 3568 & 7 \\ 59 & 5689 & 4 \end{bmatrix} \\ \begin{bmatrix} 457 & 8 & 23479 \\ 157 & 1257 & 1279 \\ 1457 & 123457 & 6 \end{bmatrix} & \begin{bmatrix} 2679 & 25679 & 579 \\ 4 & 1256789 & 3 \\ 279 & 12579 & 579 \end{bmatrix} & \begin{bmatrix} 234579 & 1 & 23569 \\ 2579 & 5679 & 2569 \\ 8 & 34579 & 2359 \end{bmatrix} \end{bmatrix}$

1.78 sec

The „DIABOLIC" was not so easy to solve. Johann sent a mail with some hints referring to the last paragraph of his contribution.

Program terminates after 2 iterations. We proceed by systematically distinguishing possible cases: investigate systematically possible cases. So on position (6,3) only 7 or 8 are possible. We issue 7 and simplify A:=sudoku() – without pressing RETURN, but using the = command, either by pressing the =-Button left of the entry-line. 7 on this position leads to a contradiction.

#7: sudoku() =

1478	12347	5	789	34789	4789	6	34789	389
4678	3467	3478	1	3456789	2	34579	345789	3589
14678	9	13478	678	345678	4578	13457	2	358
3	47	478	5	24789	6	29	89	1
9	1456	148	28	248	48	235	3568	7
2	567	78	3	789	1	59	5689	4
457	8	23479	2679	25679	579	234579	1	23569
157	1257	1279	4	1256789	3	2579	5679	2569
1457	123457	6	279	12579	579	8	34579	2359

#8: A :=

1478	12347	5	789	34789	4789	6	34789	389
4678	3467	3478	1	3456789	2	34579	345789	3589
14678	9	13478	678	345678	4578	13457	2	358
3	47	478	5	24789	6	29	89	1
9	1456	148	28	248	48	235	3568	7
2	567	7	3	789	1	59	5689	4
457	8	23479	2679	25679	579	234579	1	23569
157	1257	1279	4	1256789	3	2579	5679	2569
1457	123457	6	279	12579	579	8	34579	2359

#9: sudoku() = Computation has been stopped due to zero entries!

Taking 8 on this position leads to the next step, eg, checking 1 and 2 on position (1,2):

#12:

18	12	5	789	34789	4789	6	34789	389
68	347	347	1	3456789	2	34579	345789	3589
168	9	347	678	345678	4578	13457	2	358
3	47	47	5	289	6	29	89	1
9	56	1	28	248	48	235	3568	7
2	56	8	3	79	1	59	569	4
457	8	23479	2679	25679	579	234579	1	23569
157	127	279	4	1256789	3	2579	5679	2569
1457	12347	6	279	12579	579	8	34579	2359

2 turns out to be the right choice.
So you might go on. You are invited to download the solution. It is among the Derive files.

See two more examples, the first one is “EASY”, the second one is “DIFFICULT”:

$$A := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 7 & 4 & 5 & 0 & 6 & 2 & 8 & 0 \\ 0 & 2 & 0 & 4 & 1 & 3 & 0 & 7 & 0 \\ 0 & 5 & 6 & 0 & 2 & 0 & 7 & 3 & 0 \\ 0 & 0 & 3 & 9 & 5 & 7 & 8 & 0 & 0 \\ 0 & 1 & 8 & 0 & 6 & 0 & 9 & 5 & 0 \\ 0 & 3 & 0 & 2 & 8 & 1 & 0 & 4 & 0 \\ 0 & 9 & 2 & 7 & 0 & 5 & 3 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\#3: \quad A := \begin{bmatrix} \begin{bmatrix} 5 & 6 & 1 \\ 3 & 7 & 4 \\ 8 & 2 & 9 \end{bmatrix} & \begin{bmatrix} 8 & 7 & 2 \\ 5 & 9 & 6 \\ 4 & 1 & 3 \end{bmatrix} & \begin{bmatrix} 4 & 9 & 3 \\ 2 & 8 & 1 \\ 6 & 7 & 5 \end{bmatrix} \\ \begin{bmatrix} 9 & 5 & 6 \\ 2 & 4 & 3 \\ 7 & 1 & 8 \end{bmatrix} & \begin{bmatrix} 1 & 2 & 8 \\ 9 & 5 & 7 \\ 3 & 6 & 4 \end{bmatrix} & \begin{bmatrix} 7 & 3 & 4 \\ 8 & 1 & 6 \\ 9 & 5 & 2 \end{bmatrix} \\ \begin{bmatrix} 6 & 3 & 7 \\ 1 & 9 & 2 \\ 4 & 8 & 5 \end{bmatrix} & \begin{bmatrix} 2 & 8 & 1 \\ 7 & 4 & 5 \\ 6 & 3 & 9 \end{bmatrix} & \begin{bmatrix} 5 & 4 & 9 \\ 3 & 6 & 8 \\ 1 & 2 & 7 \end{bmatrix} \end{bmatrix}$$

$$\#4: \quad A := \begin{bmatrix} 0 & 5 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 4 & 0 & 0 & 9 & 0 & 8 & 0 & 0 & 7 \\ 0 & 0 & 0 & 6 & 0 & 5 & 0 & 0 & 0 \\ 0 & 2 & 7 & 0 & 0 & 0 & 8 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 & 0 & 0 & 9 & 6 & 0 \\ 0 & 0 & 0 & 3 & 0 & 2 & 0 & 0 & 0 \\ 6 & 0 & 0 & 4 & 0 & 7 & 0 & 0 & 1 \\ 0 & 4 & 0 & 0 & 0 & 0 & 0 & 8 & 0 \end{bmatrix}$$

$$\#5: \quad A := \begin{bmatrix} \begin{bmatrix} 2 & 5 & 9 \\ 4 & 1 & 6 \\ 7 & 8 & 3 \end{bmatrix} & \begin{bmatrix} 7 & 4 & 3 \\ 9 & 2 & 8 \\ 6 & 1 & 5 \end{bmatrix} & \begin{bmatrix} 6 & 1 & 8 \\ 5 & 3 & 7 \\ 2 & 4 & 9 \end{bmatrix} \\ \begin{bmatrix} 9 & 2 & 7 \\ 8 & 6 & 4 \\ 5 & 3 & 1 \end{bmatrix} & \begin{bmatrix} 1 & 3 & 6 \\ 2 & 5 & 9 \\ 8 & 7 & 4 \end{bmatrix} & \begin{bmatrix} 8 & 5 & 4 \\ 1 & 7 & 3 \\ 9 & 6 & 2 \end{bmatrix} \\ \begin{bmatrix} 1 & 7 & 8 \\ 6 & 9 & 5 \\ 3 & 4 & 2 \end{bmatrix} & \begin{bmatrix} 3 & 6 & 2 \\ 4 & 8 & 7 \\ 5 & 9 & 1 \end{bmatrix} & \begin{bmatrix} 4 & 9 & 5 \\ 3 & 2 & 1 \\ 7 & 8 & 6 \end{bmatrix} \end{bmatrix}$$

You don´t see any command, because following Johann´s instruction I wrote command `A:=sudoku()` into the entry line and clicked on the =-button left of the entry line. Josef