THE DERIVE - NEWSLETTER #62

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THE BULLETIN OF THE DERIVE USER GROUP + CAS-TI

Contents:

June 2006

New publications (http://shop.bk-teachware.com)

[1] *H-J Kayser*: SR-52: Lehr- und Lernvideos zur Mathematik mit Derive, bk-teachware

[2] *H-J Kayser*: SR-53: Projekte im Mathematikunterricht mit Derive, bk-teachware

[3] *Bernhard Kutzler*: SR-B: Die Sprache der Zahlen, bk-teachware

Announcement of a new publication:

Gutiérrez, A., Boero, P. (Eds.). (2006). ***Handbook of Research on the Psychology of Mathematics Education.*** **Rotterdam, The Netherlands: Sense Publishers.**
This volume is a compilation of the research produced by the *International Group for the Psychology of Mathematics Education* (PME) since its creation, 30 years ago. It has been written to become an essential reference for Mathematics Education research in coming years.

The chapters offer summaries and synthesis of the research produced by the PME Group, presented to let the readers grasp the evolution of paradigms, questions, methodologies and most relevant research results during last 30 years. They also include extensive lists of references. The chapters raise also the main current research questions and suggest directions for future research.

The volume is the result of the effort of 30 authors and 26 reviewers. Most of them are recognized leading international researchers, members of the PME Group, with great expertise on the topic of their chapter. It includes 15 chapters, divided into five sections, devoted to the main research domains of interest to the PME Group. The first to third sections summarize cognitively oriented research on learning and teaching specific content areas (algebra, geometry and measurement, numerical thinking), transversal areas (ATM, proof, visualization, young children's mathematical thinking), and based on technology rich environments (use of technology for teaching and learning algebra, calculus, and geometry). The fourth section is devoted to the research on social, affective, cultural and cognitive aspects of Mathematics Education (affectivity, constructivism, equity, socio-cultural practices). Finally, the fifth section includes two chapters summarizing the PME research on teachers training and professional life of mathematics teachers.

This handbook shall be of interest to both experienced researchers and doctoral students needing detailed synthesis of the advances and future directions of research in Mathematics Education, and also to mathematics teacher trainers who need to have a comprehensive reference as background for their courses on Mathematics Education. You can get more information of the book, and download a nonprintable PDF version of it, from Sense Publishers' web page at
http://www.sensepublishers.com/books/otherbooks/90-77874-19-4.htm.

Josef Böhm, here is a link to tables of elliptic integrals generated with DERIVE.

http://www.getnet.net/~cherry/derive/index.html

Regards, Jim FitzSimons

Download all *DNL-DERIVE*- and TI-files from

http://www.austromath.ac.at/dug/

http://www.derive-europe.com/support.asp?dug

Dear DUG Members,

I am happy that I can publish DNL#62 some days before DES-2006 in Dresden will start. This is a very special Newsletter and it deserves its name "Bulletin of the DERIVE User Group", because all contributions developed from User requests and the respective answers.

The extended exchange of emails concerning COS(π/17) induced "Mr Titbits" Johann Wiesenbauer to bring Derive on its calculation and manipulation boundaries in his Titbits #32. Jan Vermeylen sent a challenge for applying a slider bar to animate the motion of a particle in a magnetic field given by some thousand positions in space. An international cooperation between Sweden, England and Austria found a satisfying solution. Many thanks to David Sjöstrand and Peter Schofield. Finally, Heinz Rainer Geyer reminded me on my career as a teacher with his problem finding a generalised rule to present the frequency distribution of the sum of *n* rolled dice. Here again Johann Wiesenbauer found the shortest and most clever solution.

Unfortunately we are very poor in TI-contributions. Wolfgang Pröpper announced a TI-paper based on a lecture which he gave in Vienna some months ago. Many thanks in advance. I´d like to demonstrate that in many cases it would not be too difficult to transfer the problems and their solutions onto the handheld (dice problem, random walk).

There are other interesting problems (together with their solutions) remaining on my desk: how to load an external data set to DERIVE and assigning it simultaneously to a variable name. Klaus Rohe raised this question and he promised a contribution on the "2 and 3 Body Problem" applying the proposed solution. Another request was on applying slider bars to Peter Schofield sophisticated "ARROW"-tool (a user contributed DERIVE utility file) and there was a also question from a University lecturer on a special representation of Taylor series.

By the way, do you now how to "paint" in and from Derive? A student of Tania Koller found this out by experimenting with the program.

The two pictures are from a workshop with 50 (!!!) students on the Vienna International School. I was invited by Marlene Torres-Skoumal to hold a "Modelling Session" with the 10-graders of this highly reputated institution. I felt very fine among students from so many countries and we agreed to have another workshop in the next term.

Finally I´d like to wish you all a wonderful summer and I hope to meet many of you in Dresden.

Invitation

DERIVE User Group Meeting

DES-TIME 2006 in Dresden

Saturday 22 July 2006, afternoon

Please follow the announcement given on the Conference

The *DERIVE-NEWSLETTER* is the Bulletin of the *DERIVE* & CAS-*TI User Group*. It is published at least four times a year with contents of 44 pages minimum. The goals of the *DNL* are to enable the exchange of experiences made with *DERIVE* and the *TI*-89/92/*Titanium*/*Voyage* 200 as well as to create a group to discuss the possibilities of new methodical and didactical manners in teaching mathematics.

As many of the *DERIVE* Users are also using the CAS-*TIs* the *DNL* tries to combine the applications of these modern technologies.

Editor: Mag. Josef Böhm
A-3042 Würmla
D´Lust 1
Austria
Phone/FAX: 43-(0)2275/8207
e-mail: nojo.boehm@pgv.at

Contributions:
Please send all contributions to the Editor. Non-English speakers are encouraged to write their contributions in English to reinforce the international touch of the *DNL*. It must be said, though, that non-English articles will be warmly welcomed nonetheless. Your contributions will be edited but not assessed. By submitting articles the author gives his consent for reprinting it in the *DNL*. The more contributions you will send, the more lively and richer in contents the *DERIVE* & CAS-*TI Newsletter* will be.

Next issue: September 2006
Deadline 31 August 2006

Preview: Contributions waiting to be published

Two Stage Least Squares, M. R. Phillips, USA
Some simulations of Random Experiments, J. Böhm, AUT
Wonderful World of Pedal Curves, J. Böhm
Another Task for End Examination, J. Lechner, AUT
Tools for 3D-Problems, P. Lüke-Rosendahl, GER
ANOVA with *DERIVE* & *TI*, M. R. Phillips, USA
Financial Mathematics 4, M. R. Phillips
Hill-Encription, J. Böhm
Farey Sequences on the *TI,* M. Lesmes-Acosta, COL
Simulating a Graphing Calculator in *DERIVE*, J. Böhm
Henon & Co, J. Böhm
Challenges from Fermat, Bj. Felsager, DEN
Are all Bodies falling equally fast, J. Lechner
Modelling Traffic Density, Th. Himmelbauer, AUT
Do you know this? Cabri & CAS on PC and Handheld, W. Wegscheider, AUT
An Interesting Problem with a Triangle, Steiner Point, P. Lüke-Rosendahl, GER
Mathematics and Design, Hubert Weller, GER
Diophantine Polynomials, Duncan E. McDougall, Canada
Contour Plots and Implicit Plots, Peter Schofield, UK

and Setif, FRA; Vermeylen, BEL; Leinbach, USA; Koller, AUT; Baumann, GER; Keunecke, GER, ...and others

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From cos(π/17) to acot(x)

An interesting exchange of emails

Starting the discussion:

14 May 2006

How does DERIVE find:

$$\cos\left(\frac{\pi}{17}\right) = \frac{\sqrt{\sqrt{38\cdot\sqrt{17}+170}+3\cdot\sqrt{17}+17}}{8} + \frac{\sqrt{34-2\cdot\sqrt{17}}}{16} - \frac{\sqrt{17}}{16} + \frac{1}{16}$$

Regards, Jim FitzSimons

15 May 2006

Hello Jim,

A quick way to get an approximative value of cos(pi/17) is to first simplify

```
APPROX(NSOLVE(CHEBYCHEV_T(17, x) = -1, x, 0.9, 1), 30)
```

and then to approximate the output, which should lead to the approximation

`x = 0.9829730996` for cos(π/17).

In order to get your radical expression I'm afraid you can't avoid following in Gauss tracks, when he computed it as a 19 year old.
Unfortunately it is too long to describe it a few sentences, but I have given a detailed description of his approach in my Titbits series in the Derive Newletter #20.

Hope this helps.

Cheers,
Johann (Wiesenbauer)

Jim asked for a copy of Johann´s Titbits from DNL#20 and I sent him this contribution. Jim joined the DERIVE User Group and we give him a warm welcome in our group, Josef

15 May 2006

Hello Jim,

solve the equation z^17 = –1 in Polarform!
You get z = cos(pi/17 + 2/17*k*PI) + *i* sin(PI/17 + 2/17*k*PI). For k=0 the Real-part of the solution leads to cos(PI/17).

I don't know if Derive does it this way! But I would!

Greetings,
Manuel

21 May 2006

Hello Manuel et al.,

Sorry for the delayed answer, but it took me some time to "fill in the details" as to your proposal of a computation of cos(π/17) using Derive. It is essentially following Gauss' solution and the outline I gave of it in my Titbits(6) in the DNL #20. (For Jim: This is the Newletter of the Derive User Group, some issues of which can be found online using the keywords dug derive in a Google search.)

If you (or anybody else reading this) can find a shortcut somewhere, I would be very much interested in it.

Cheers,
Johann

File `cospi17.dfw`

Computation of cos(pi/17) using Derive

((c) Johann Wiesenbauer, Vienna University of Technology, 2006)

We start with the observation that cos(pi/17) is a solution of the polynomial equation

chebychev_T(17, x)+1 =0 **(*)**

due to the defining relation chebychev_T(n,cos φ) = cos(nφ). In the following we use Dickson polynomials D(n,x) though, which are closely related to Chebychev polynomials of the first kind. More precisely, the relation

Dickson(n,x) = 2*chebychev_T(n, x/2)

holds and therefore 2cos(pi/17) is a solution of the polynomial equation

Dickson(17,x)+2=0 **(**)**

The polynomials Dickson(n,x) can be also computed in Derive using the definition

```
#1:   Dickson(n, x) := V_MOD(n, x, 1)
```

Hence the LHS of (**) is given by the polynomial

```
#2:   p1 := Dickson(17, x) + 2
```

$$\#3:\quad p1 := x^{17} - 17\cdot x^{15} + 119\cdot x^{13} - 442\cdot x^{11} + 935\cdot x^{9} - 1122\cdot x^{7} + 714\cdot x^{5} - 204\cdot x^{3} + 17\cdot x + 2$$

Factoring over the field Q of rational numbers yields

#4: FACTOR(p1, Rational, x)

#5: $(x + 2)\cdot(x^8 - x^7 - 7\cdot x^6 + 6\cdot x^5 + 15\cdot x^4 - 10\cdot x^3 - 10\cdot x^2 + 4\cdot x + 1)$

In fact, this is the factorization of p1 into irreducible factors and the 8-degree factor p2 is the minimal polynomial of 2cos(pi/17) over Q (as well as of its conjugates 2cos(3pi/17),2cos(5pi/17),...,2cos(15pi/17)).

#6: $p2 := x^8 - x^7 - 7\cdot x^6 + 6\cdot x^5 + 15\cdot x^4 - 10\cdot x^3 - 10\cdot x^2 + 4\cdot x +1$

In order to show this we define the vector cosines consisting of all those cosine values. For example, 2cos(pi/17) = z - z^16 with z = exp(#i pi/17). (Note that we never define z explicitly, but only use (z^17+1)/(z+1)=0 in the following!)

#7: $\text{reduce}(u) := \text{REMAINDER}\left(u, \dfrac{z^{17} + 1}{z + 1}\right)$

#8: $\text{cosines} := \text{VECTOR}(z^k - z^{17 - k}, k, 1, 16, 2)$

#9: $\text{cosines} := [z - z^{16}, z^3 - z^{14}, z^5 - z^{12}, z^7 - z^{10}, z^9 - z^8, z^{11} - z^6, z^{13} - z^4, z^{15} - z^2]$

#10: $\text{VECTOR}(\text{reduce}(\text{SUBST}(p2, x, \text{cosines}_k)), k, 1, 8) = [0, 0, 0, 0, 0, 0, 0, 0]$

By the way, Derive thinks that p2 cannot be factored using radicals, i.e.

#11: $\text{FACTOR}(p2, \text{Radical}, x) = x^8 - x^7 - 7\cdot x^6 + 6\cdot x^5 + 15\cdot x^4 - 10\cdot x^3 - 10\cdot x^2 + 4\cdot x + 1$

In the following we'll show that this is wrong. In fact, it will turn out that p2 can be split up into two polynomials of degree 4 in the ring of integers of Q[sqrt(17)].

In order to show this we first observe that the sum of all cosine values must be the negative coefficient of x^7 of p2, i.e. it must be 1.

#12: $\text{reduce}(\sum(\text{cosines})) = 1$

Now we face the following problem: Is it possible to break up this sum into two partial sums with 4 summands each such that their product is in Q? Do you think that Derive can help us out here?

#13: $\text{SELECT}(\text{RATIONAL?}(\text{reduce}(\sum(\text{cosines}_{\text{SORT}(s)})\cdot\sum(\text{cosines}_{\text{SORT}(\{1,\ ...,\ 8\}\ \backslash\ s)}))),\ s,\ \text{POWER\_SET}(\{1,\ ...,\ 8\},\ 4))$

#14: $\{\{1, 5, 7, 8\}, \{2, 3, 4, 6\}\}$

Wow! There is exactly one such partition and Derive found it in splits of a second! In fact, we have

#15: $[s1 := \sum(\text{cosines}_{[1,\ 5,\ 7,\ 8]}),\ s2 := \sum(\text{cosines}_{[2,\ 3,\ 4,\ 6]})]$

#16: $\text{reduce}(s1\cdot s2) = -4$

which means that the real values corresponding to s1 and s2 must be the among the roots of

#17: $\text{SOLUTIONS}([u + v = 1,\ u\cdot v = -4],\ [u,\ v])$

#18: $$\begin{bmatrix} \frac{\sqrt{17}}{2}+\frac{1}{2} & \frac{1}{2}-\frac{\sqrt{17}}{2} \\ \frac{1}{2}-\frac{\sqrt{17}}{2} & \frac{\sqrt{17}}{2}+\frac{1}{2} \end{bmatrix}$$

There is a small problem though: We don't know the correspondence between solutions u,v and s1,s2. One way to find this out is to use their numerical values:

#19: $$\text{APPROX}\begin{bmatrix} \frac{\sqrt{17}}{2}+\frac{1}{2} & \frac{1}{2}-\frac{\sqrt{17}}{2} \\ \frac{1}{2}-\frac{\sqrt{17}}{2} & \frac{\sqrt{17}}{2}+\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 2.561552812 & -1.561552812 \\ -1.561552812 & 2.561552812 \end{bmatrix}$$

#20: $$\text{APPROX}\left[2\cdot\sum\left(\text{COS}\left(\frac{(2\cdot k - 1)\cdot\pi}{17}\right),\ k,\ \{1,\ 5,\ 7,\ 8\}\right),\ 2\cdot\sum\left(\text{COS}\left(\frac{(2\cdot k - 1)\cdot\pi}{17}\right),\ k,\ \{2,\ 3,\ 4,\ 6\}\right)\right]$$

#21: $[-1.561552812,\ 2.561552812]$

Hence, as from these numerical values, we have got the following equations

#22: $\left[s1 = \frac{1}{2} - \frac{\sqrt{17}}{2},\ s2 = \frac{\sqrt{17}}{2} + \frac{1}{2}\right]$

Furthermore, the partition {{1, 5, 7, 8}, {2, 3, 4, 6}} of {1,2,3,4,5,6,7,8} tells us exactly the grouping of linear factors of p2 for the factorization of p2 mentioned above. In the first place, it's easy to get the approximative representation of the factors by

#23: $\text{APPROX}\left(\text{EXPAND}\left(\prod\left(x - 2\cdot\text{COS}\left(\frac{(2\cdot k - 1)\cdot\pi}{17}\right), k, \{1, 5, 7, 8\}\right)\right)\right)$

#24: $x^4 + 1.561552812\cdot x^3 - 3.561552812\cdot x^2 - 6.123105625\cdot x - 1$

#25: $\text{APPROX}\left(\text{EXPAND}\left(\prod\left(x - 2\cdot\text{COS}\left(\frac{(2\cdot k - 1)\cdot\pi}{17}\right), k, \{2, 3, 4, 6\}\right)\right)\right)$

#26: $x^4 - 2.561552812\cdot x^3 + 0.5615528128\cdot x^2 + 2.123105625\cdot x - 1$

But what about their exact representations? Here we make use of the fact that the exact coefficients are linear combinations of s1 and s2. (For the cognoscenti, {s1,s2} is a basis for the field extension Q(sqrt(17)) over Q.) Hence, we could make a list of "small" linear combinations and simply compare its entries in the first colums with numerical values of the coefficients of our polynomials above, e.g.

#27: $s := \left[\frac{1}{2} - \frac{\sqrt{17}}{2}, \frac{\sqrt{17}}{2} + \frac{1}{2}\right]$

#28: $\text{APPROX}(\text{SORT}(\text{VECTOR}([s\cdot x, x], x, \{-3, ..., 3\}^2)))$

-12.36931687	[3, -3]
-10.80776406	[2, -3]
-9.807764064	[3, -2]
-9.246211251	[1, -3]
-8.246211251	[2, -2]
-7.684658438	[0, -3]
-7.246211251	[3, -1]
-6.684658438	[1, -2]
-6.123105625	[-1, -3]
-5.684658438	[2, -1]

#29:

-5.123105625	[0, -2]
-4.684658438	[3, 0]
-4.561552812	[-2, -3]
-4.123105625	[1, -1]
-3.561552812	[-1, -2]
-3.123105625	[2, 0]
-3	[-3, -3]
-2.561552812	[0, -1]
-2.123105625	[3, 1]
-2	[-2, -2]
-1.561552812	[1, 0]
-1	[-1, -1]
-0.5615528128	[2, 1]
-0.4384471871	[-3, -2]
0	[0, 0]
0.4384471871	[3, 2]
0.5615528128	[-2, -1]
1	[1, 1]
1.561552812	[-1, 0]
2	[2, 2]
2.123105625	[-3, -1]
2.561552812	[0, 1]
3	[3, 3]
3.123105625	[-2, 0]
3.561552812	[1, 2]
4.123105625	[-1, 1]
4.561552812	[2, 3]
4.684658438	[-3, 0]
5.123105625	[0, 2]
5.684658438	[-2, 1]
6.123105625	[1, 3]
6.684658438	[-1, 2]

7.246211251	[-3, 1]
7.684658438	[0, 3]
8.246211251	[-2, 2]
9.246211251	[-1, 3]
9.807764064	[-3, 2]
10.80776406	[-2, 3]
12.36931687	[-3, 3]

Looking up this table we get the following "exact" factors of p2:

#30: $p21 := x^4 + (s\cdot[-1, 0])\cdot x^3 + (s\cdot[-1, -2])\cdot x^2 + (s\cdot[-1, -3])\cdot x + s\cdot[-1, -1]$

#31: $p21 := x^4 + x^3\cdot\left(\frac{\sqrt{17}}{2} - \frac{1}{2}\right) - x^2\cdot\left(\frac{\sqrt{17}}{2} + \frac{3}{2}\right) - x\cdot(\sqrt{17} + 2) - 1$

#32: $p22 := x^4 + (s\cdot[0, -1])\cdot x^3 + (s\cdot[-2, -1])\cdot x^2 + (s\cdot[-3, -1])\cdot x + s\cdot[-1, -1]$

#33: $p22 := x^4 - x^3\cdot\left(\frac{\sqrt{17}}{2} + \frac{1}{2}\right) + x^2\cdot\left(\frac{\sqrt{17}}{2} - \frac{3}{2}\right) + x\cdot(\sqrt{17} - 2) - 1$

Indeed, Derive can verify this factorization now (though only after one of those strange "double simplifications"!)

#34: $p2 - p21\cdot p22$

#35: $x^2\cdot((\sqrt{17} - 2)\cdot(\sqrt{17} + 2) - 13)$

#36: 0

Let's continue by setting

#37: $p3 := p21$

#38: $p3 := x^4 + x^3\cdot\left(\frac{\sqrt{17}}{2} - \frac{1}{2}\right) - x^2\cdot\left(\frac{\sqrt{17}}{2} + \frac{3}{2}\right) - x\cdot(\sqrt{17} + 2) - 1$

Now by the same reasoning as above, we know that the sum of its zeros, namely

#39: $\sum(\text{cosines}_{[1, 5, 7, 8]}) = - z^{16} + z^{15} + z^{13} + z^9 - z^8 - z^4 - z^2 + z$

is an element of Q(sqrt(17)), but how can we break up this sum into two sums of equal size such that their product is in Q(sqrt(17)) too? The following computation shows that the partition {{1,7},{5,8}} works:

#40: $\text{reduce}(\sum(\text{cosines}_{[1,\ 7]})\cdot\sum(\text{cosines}_{[5,\ 8]})) = -1$

It should be clear by now how the story goes on.

#41: $\text{SOLUTIONS}\left(\left[u + v = \frac{1}{2} - \frac{\sqrt{17}}{2},\ u\cdot v = -1\right],\ [u,\ v]\right)$

#42: $$\begin{bmatrix} \frac{\sqrt{(34 - 2\cdot\sqrt{17})}}{4} - \frac{\sqrt{17}}{4} + \frac{1}{4} & - \frac{\sqrt{(34 - 2\cdot\sqrt{17})}}{4} - \frac{\sqrt{17}}{4} + \frac{1}{4} \\ - \frac{\sqrt{(34 - 2\cdot\sqrt{17})}}{4} - \frac{\sqrt{17}}{4} + \frac{1}{4} & \frac{\sqrt{(34 - 2\cdot\sqrt{17})}}{4} - \frac{\sqrt{17}}{4} + \frac{1}{4} \end{bmatrix}$$

#43: $$\text{APPROX}\begin{bmatrix} \frac{\sqrt{(34 - 2\cdot\sqrt{17})}}{4} - \frac{\sqrt{17}}{4} + \frac{1}{4} & - \frac{\sqrt{(34 - 2\cdot\sqrt{17})}}{4} - \frac{\sqrt{17}}{4} + \frac{1}{4} \\ - \frac{\sqrt{(34 - 2\cdot\sqrt{17})}}{4} - \frac{\sqrt{17}}{4} + \frac{1}{4} & \frac{\sqrt{(34 - 2\cdot\sqrt{17})}}{4} - \frac{\sqrt{17}}{4} + \frac{1}{4} \end{bmatrix} = \begin{bmatrix} 0.4879283649 & -2.049481177 \\ -2.049481177 & 0.4879283649 \end{bmatrix}$$

#44: $$\text{APPROX}\left[2\cdot\sum\left(\text{COS}\left(\frac{(2\cdot k - 1)\cdot\pi}{17}\right),\ k,\ \{1,\ 7\}\right),\ 2\cdot\sum\left(\text{COS}\left(\frac{(2\cdot k - 1)\cdot\pi}{17}\right),\ k,\ \{5,\ 8\}\right)\right] = [0.4879283649,\ -2.049481177]$$

From the approximations above we can conclude that

#45: $\left[t1 := \sum(\text{cosines}_{[1,\ 7]}),\ t2 := \sum(\text{cosines}_{[5,\ 8]})\right]$

#46: $\left[t1 = \frac{\sqrt{(34 - 2\cdot\sqrt{17})}}{4} - \frac{\sqrt{17}}{4} + \frac{1}{4},\ t2 = - \frac{\sqrt{(34 - 2\cdot\sqrt{17})}}{4} - \frac{\sqrt{17}}{4} + \frac{1}{4}\right]$

Finally we have

#47: $\text{reduce}(\text{cosines}_1 \cdot \text{cosines}_7) = z^{14} + z^{12} - z^5 - z^3$

#48: $\text{reduce}(\text{cosines}_5 \cdot \text{cosines}_8) = - z^{11} + z^{10} - z^7 + z^6$

which can also be written as

#49: $\text{reduce}(\text{cosines}_1 \cdot \text{cosines}_7) = - \sum(\text{cosines}_{[2,\ 3]})$

#50: $\text{reduce}(\text{cosines}_5 \cdot \text{cosines}_8) = - \sum(\text{cosines}_{[4,\ 6]})$

The sums on the RHS of these equation can be computed as follows

#51: $\left[r1 := \sum(\text{cosines}_{[2,\ 3]}),\ r2 := \sum(\text{cosines}_{[4,\ 6]})\right]$

#52: $r1 + r2 = \frac{\sqrt{17}}{2} + \frac{1}{2}$

#53: $\text{reduce}(r1\cdot r2) = -1$

The rest is supposed to need no further comment.

#54: $\text{SOLUTIONS}\left(\left[u + v = \frac{\sqrt{17}}{2} + \frac{1}{2},\ u\cdot v = -1\right],\ [u,\ v]\right)$

#55: $$\begin{bmatrix} \frac{\sqrt{(2\cdot\sqrt{17} + 34)}}{4} + \frac{\sqrt{17}}{4} + \frac{1}{4} & -\frac{\sqrt{(2\cdot\sqrt{17} + 34)}}{4} + \frac{\sqrt{17}}{4} + \frac{1}{4} \\ -\frac{\sqrt{(2\cdot\sqrt{17} + 34)}}{4} + \frac{\sqrt{17}}{4} + \frac{1}{4} & \frac{\sqrt{(2\cdot\sqrt{17} + 34)}}{4} + \frac{\sqrt{17}}{4} + \frac{1}{4} \end{bmatrix}$$

#56: $$\text{APPROX}\begin{bmatrix} \frac{\sqrt{(2\cdot\sqrt{17} + 34)}}{4} + \frac{\sqrt{17}}{4} + \frac{1}{4} & -\frac{\sqrt{(2\cdot\sqrt{17} + 34)}}{4} + \frac{\sqrt{17}}{4} + \frac{1}{4} \\ -\frac{\sqrt{(2\cdot\sqrt{17} + 34)}}{4} + \frac{\sqrt{17}}{4} + \frac{1}{4} & \frac{\sqrt{(2\cdot\sqrt{17} + 34)}}{4} + \frac{\sqrt{17}}{4} + \frac{1}{4} \end{bmatrix} = \begin{bmatrix} 2.905703544 & -0.3441507314 \\ -0.3441507314 & 2.905703544 \end{bmatrix}$$

#57: $$\text{APPROX}\left[2\cdot\sum\left(\text{COS}\left(\frac{(2\cdot k - 1)\cdot\pi}{17}\right), k, \{2, 3\}\right), 2\cdot\sum\left(\text{COS}\left(\frac{(2\cdot k - 1)\cdot\pi}{17}\right), k, \{4, 6\}\right)\right] = [2.905703544, -0.3441507314]$$

#58: $$\left[r1 = \frac{\sqrt{(2\cdot\sqrt{17} + 34)}}{4} + \frac{\sqrt{17}}{4} + \frac{1}{4}, r2 = -\frac{\sqrt{(2\cdot\sqrt{17} + 34)}}{4} + \frac{\sqrt{17}}{4} + \frac{1}{4}\right]$$

#59: $$\text{SOLUTIONS}\left(\left[u + v = \frac{\sqrt{(34 - 2\cdot\sqrt{17})}}{4} - \frac{\sqrt{17}}{4} + \frac{1}{4}, u\cdot v = -\left(\frac{\sqrt{(2\cdot\sqrt{17} + 34)}}{4} + \frac{\sqrt{17}}{4} + \frac{1}{4}\right)\right], [u, v]\right)$$

#60: $$\begin{bmatrix} \frac{\sqrt{(\sqrt{(38\cdot\sqrt{17} + 170)} + 3\cdot\sqrt{17} + 17)}}{4} + \frac{\sqrt{(34 - 2\cdot\sqrt{17})}}{8} - \frac{\sqrt{17}}{8} + \frac{1}{8} & -\frac{\sqrt{(\sqrt{(38\cdot\sqrt{17} + 170)} + 3\cdot\sqrt{17} + 17)}}{4} + \frac{\sqrt{(34 - 2\cdot\sqrt{17})}}{8} - \frac{\sqrt{17}}{8} + \frac{1}{8} \\ -\frac{\sqrt{(\sqrt{(38\cdot\sqrt{17} + 170)} + 3\cdot\sqrt{17} + 17)}}{4} + \frac{\sqrt{(34 - 2\cdot\sqrt{17})}}{8} - \frac{\sqrt{17}}{8} + \frac{1}{8} & \frac{\sqrt{(\sqrt{(38\cdot\sqrt{17} + 170)} + 3\cdot\sqrt{17} + 17)}}{4} + \frac{\sqrt{(34 - 2\cdot\sqrt{17})}}{8} - \frac{\sqrt{17}}{8} + \frac{1}{8} \end{bmatrix}$$

#61: $$\begin{bmatrix} 1.965946199 & -1.478017834 \\ -1.478017834 & 1.965946199 \end{bmatrix}$$

#62: $$\text{APPROX}\left(2\cdot\text{COS}\left(\frac{\pi}{17}\right)\right) = 1.965946199$$

#63: $$\text{COS}\left(\frac{\pi}{17}\right) = \frac{\sqrt{(\sqrt{(38\cdot\sqrt{17} + 170)} + 3\cdot\sqrt{17} + 17)}}{8} + \frac{\sqrt{(34 - 2\cdot\sqrt{17})}}{16} - \frac{\sqrt{17}}{16} + \frac{1}{16}$$

And what does Derive think of this final equation? Let's conclude with

#64: $$\text{APPROX}\left(\text{COS}\left(\frac{\pi}{17}\right) = \frac{\sqrt{(\sqrt{(38\cdot\sqrt{17} + 170)} + 3\cdot\sqrt{17} + 17)}}{8} + \frac{\sqrt{(34 - 2\cdot\sqrt{17})}}{16} - \frac{\sqrt{17}}{16} + \frac{1}{16}\right) = \text{true}$$

21 May 2006

Johann, thank you for your help. I have a copy of TITBITS6.
Here are links to my first try.

http://www2.pvc.maricopa.edu/~fitzsimons/cosine.doc
http://www2.pvc.maricopa.edu/~fitzsimons/cosine.mth

The college web site is down now, but it will be back up soon.
Regards, Jim

22 May 2006

Hello all,

Is there any prime p>17 such that COS(π/p) or SIN(π/p) is expressible in terms of radicals?

Aloha,
Albert D. Rich

22 May 2006

Well, there are two more Fermat primes known: 257 and 65537.

Cheers,
David W. Cantrell

22 May 2006

Hello Albert,

As David has already mentioned such a prime must be of Fermat type, hence p=257 and p=65537 are certainly further solutions. It is not even known whether are finitely many Fermat primes, although this is probably true for heuristic reasons.

This also gives me the opprortunity to correct a mistake in my presentation (near line #40) in the attached file below, which went undetected due to a strange coincidence and didn't affect the computations thereafter.

Cheers, Johann

(*Comment of the publisher: the printed file cospi17.dfw is the latest version.*)

22 May 2006

Hello all,

Thank you for your positive response to my question.

Why does a Fermat prime p of the form 2^2^n+1, where n is a nonnegative integer, guarantee that COS(pi/p) is expressible in terms of radicals?

Has anyone ever bothered to determine the radical representation of COS(π/257)? It must be huge...

Aloha, Albert

23 May 2006

Hello Albert,

First a word of caution: "Expressible in terms of radicals" means here that a term can be achieved starting with rational numbers and applying the operations +,-,*,/, SQRT() finitely many times in any order. In particular, radicals other than square roots are not allowed. Having said this what follows is an outline of the proof:

Suppose that u = cos(pi/p) is expressible in terms of radicals for some odd prime p. Then the minimal polynomial m(x) of cos(pi/p) over the field Q of rational numbers is given by

m(x) = (x^p+1)/(x+1)

as can be seen by applying Eisenstein's criterion to the polynomial m(x-1). Hence, the algebraic field extension Q(u) is of degree p-1. On the other hand, each operation when "building" the term for u corresponds to a field extension of degree 1 or 2 and multiplying all those degrees together we get a power 2^k, which must be equal to p-1. Hence, p is a prime of the from 2^k+1, which implies that k = 2^n for some n and p is a Fermat prime. Conversely, if p is a Fermat prime F_n, then the Galois group of the field extension Q(u):Q is cyclic and of order 2^2^n. This means that this extension can be broken up into a series of extensions of degree 2, leading to a radical expression for u.

Hope this helps.

Cheers,Johann

PS. As for the radical expression of cos(pi/257), it is huge, indeed. I'll try to achieve it in Derive, but this may take some time...

23 May 2006

Thank you, Johann, for making that explicit. However, that raises a question I had, but did not mention, when I first replied to Albert:

He had asked if there were any primes p > 17 for which cos(pi/p) is expressible in terms of radicals. I knew there were, and replied, merely based on my knowledge of constructibility of regular polygons, that there are two Fermat primes > 17 known. I realized that the radicals required in those two cases would be just square roots.

You said "As David has already mentioned such a prime must be of Fermat type", but I didn't say quite that. In fact, I was careful to avoid saying that such a prime **must be** of Fermat type because I thought that Albert was asking about expressibility in terms of general radicals, not necessarily square roots.

What I don't know is whether allowing other radicals (cube roots, fifth roots, etc., and thus taking us out of the realm of constructible regular polygons) would allow us to express cos(pi/p) in terms of radicals for other primes p. Do you know the answer to that?

BTW, Albert later asked if anyone had ever determined the radical expression for COS(π/257). Those interested in that question might like to see

<http://mathworld.wolfram.com/257-gon.html> and

<http://mathworld.wolfram.com/65537-gon.html>.

Regards,
David

23 May 2006

Ah, silly me! I just hadn't thought about the question much beforehand, but I now see that the answer is YES. For example, COS(π/7) can be expressed in terms of radicals (specifically, square and cube roots).

David

23 May 2006

For anyone who might be interested, I've now gotten COS(π/7) in a nice form. Letting
u = (7/2 (–1 + 3 sqrt(–3)))^(1/3), we have COS(π/7) = 1/6 (u + 1 + 7/u).

David

23 May 2006

David, I know you can express COS(π/7) in cube roots. COS(π/7) can not be expressed in square roots.
Regards, Jim FitzSimons

23 May 2006

Thanks Johann and David for your insightful responses to my questions.

If u := (7/2 (–1 + 3 √(-3)))^(1/3), Derive simplifies

COS(π/7) = 1/6 (u + 1 + 7/u)

to

COS(π/7) = √(7)·COS(ACOT(–√(3)/9)/3)/3 + 1/6

This leads to the potential simplification rule

COS(ACOT(–√(3)/9)/3) → √(7)·(6·COS(π/7) – 1)/14

That leads me to wonder if there is a generalization of this rule that would simplify expressions of the form COS(ACOT(n)/3) and COS(ATAN(n)/3) where n is an algebraic number to expressions free of inverse trig functions. Any thoughts on this?

Aloha,
Albert

23 May 2006

It all depends, I think, on whether we require that the radicals be real. If not, then we can say, for example,

CAS(ACOT(x)/r) = 1/2 (u + 1/u)

where u = ((x + #i)/ √(x^2 + 1))^(1/r).

Is that the sort of thing you wanted?

Cheers,
David

24 May 2006

Thanks for pointing out the nice identity for COS(ACOT(x)/r). I wasn't aware of it.

Unfortunately, the answer to your question is 'no'. For it to be a transformation rule Derive can use, the expression equivalent to COS(ACOT(n)/3) must simpler in the sense that it consists only of real radicals and COS(π/m) where m is an integer. I doubt there exists such an expression for arbitrary algebraic numbers n; however as the example below shows, there are at least some n for which there is a simpler equivalent.

So my question is: What is the class of algebraic numbers for which there is a simpler equivalent AND what is that equivalent?

Aloha,
Albert

24 May 2006

Hallo Albert,

There must be something wrong with this identity because a simple numerical observation leads to the following contradiction within DERIVE:

(1) COS(ACOT(-SQRT(3)/9)/3 is approximately 0.8326204335

(2) COS(ACOT(SQRT(3)/9)/3 is approximately 0.8959532196

These are the results due to Derive 5.06 and 6.01.

But both results must be equal, because arccot(-x)=-arccot(x) and cos(-x)=cos(x). So there appears a numerical discrepancy.
On the other hand SQRT(7)*(6*COS(pi/7) - 1)/14 evaluates numerically to 0.8326204335.
Other CAS and numerical calculators suggest that both,
COS(ACOT(-SQRT(3)/9)/3) and COS(ACOT(SQRT(3)/9)/3), should evaluate numerically to 0.8959532196, and that SQRT(7)*(6*COS(pi/7) - 1)/14 should evaluate numerically to 0.8326204335.

So I conclude at a first glance that the identity above may be false.

"Aloha", Stefan Welke

24 May 2006

Albert, I searched for solutions for COS(x/3) and I have not found any yet.
Regards, Jim FitzSimons

24 May 2006

Hi Jim,

I am concerned about the special case when your x is of the form ACOT(n) where n is an algebraic number. For example, as shown below COS(ACOT(n)/3) does simplify for n = –SQRT(3)/9.

The question is: What is the class of n for which COS(ACOT(n)/3) simplifies AND what exactly is the simplification rule?

Aloha, Albert

24 May 2006

Hello Stefan,

Your assertion that ARCCOT(–x) = – ARCCOT(x) is incorrect. As plotting quickly shows, the correct identity is

ARCCOT(–x) = π – ARCCOT(x)

Therefore, Derive's approximations of the below expressions are both correct, and the identity I gave is also correct.

Aloha, Albert

24 May 2006

Albert,

Let me try to nip a potential argument in the bud.

You told Stefan that his "assertion that ARCCOT(-x) = –ARCCOT(x) is incorrect." You are undeniably correct, but only because this is DERIVE-NEWS.

Let us not forget that there are **two** common ways of defining ARCCOT(x), each with its own advantages and disadvantages. Derive uses one of those ways.
Done the other way, ARCCOT(–x) = –ARCCOT(x) holds for all nonzero real x.

David

25 May 2006

```
InputMode:=Word

"The question is:  What is the class of n for which COS(ACOT(n)/3) simplifies
 AND what exactly is the simplification rule?"

m=[1,2,3,4,5,6,8,10,15,16,17,20,"..."]

k:epsilonInteger

COS(ACOT(n)/3)

ACOT(n)/3=k*pi/m

;Solve(#6,n)
SOLVE(ACOT(n)/3=k*pi/m,n)

;Simp(Solve(#6,n))
n=IF((3*k-m)/m<0 AND (6*k-m)/m>-1,COT(3*pi*k/m))
```

Regards, Jim FitzSimons

25 May 2006

n = –SQRT(3)/9.
Gives COS(π/7) which can not be expressed as with only sqrt radicals.
Regards, Jim FitzSimons

25 May 2006

Yes, I should have been more precise by adding the phrase "in Derive, at least" to the end of the sentence you quote below. Thanks for nipping this argument in the bud!

However, now I'm curious as to what percentage of DeriveNews readers prefer the Derive definition of ARCCOT(x), and what percentage don't. For simplicity sake, let's stick to real-valued x.

Derive defines ARCCOT(x) as π/2–ARCTAN(x). Therefore ARCCOT(-x) equals π-ARCCOT(x).
How is the alternative definition of ARCCOT(x) defined in terms of ARCTAN(x) so that
ARCCOT(–x) equals –ARCCOT(x)?

Aloha,
Albert

25 May 2006

Hello Albert,

sorry, I should had done a plot of ACOT in DERIVE first. What I had to learn as a mathematician and as a teacher is that most CAS and calculators define arccot(x) as arctan(1/x) which is technically very simple because you only have to implement arctan (or arccot).

One reason may be the following argument: y=arccot(x) is equivalent to x=cot(y). We know that tan(y)=1/cot(y)=1/x, so y=arctan(tan(y))=arctan(1/x). We finally arrive at arccot(y)=arctan(1/x).

In pre-computer times I learned at school and university that the real cotangent function must be first restricted to the open interval]0,pi[where it is strictly monotone. Now the restricted cotangent function has an inverse arccot, which is defined on]-inf,inf[with arccot(]-inf,inf[) =]0,pi[. This explains why arccot(y)=arctan(1/x) is mathematically incorrect. The simple reasoning above ignores the fact that the inverse functions arctan and arccot are defined only for restrictions of the the tan and cot functions. By the way, it is very difficult to teach to average students that they have to interpret the results of their calculators with these facts in mind.

To answer your question, Albert: I am very comfortable with the way, DERIVE defines the arccot function because it is mathematically correct for real numbers.
What I learned once again is the importance to know, how the particular program you are working with, defines its functions. This knowledge is indispensable to obtain correct and reliable results.

Aloha,
Stefan Welke

25 May 2006

Albert, ARCCOT(x) as π/2 – ARCTAN(x) is the correct definition since it gives ARCCOT(x) the correct range.
I teach trigonometry and in my books the domain of arccot(x) is all real and the range is 0 to π.
Regards, Jim FitzSimons

25 May 2006

Hello Albert,

"For simplicity sake" indeed! Perhaps you don't realize it, but choosing to "stick to real-valued x" is highly prejudicial in this discussion. AFAIK, **any** sane person would choose Derive's definition if we are to deal only with real values. And that is surely why Jim thinks that Derive's is "the correct definition". But, for better or worse, there simply is no single "correct" definition if we go beyond the reals. Out of the infinitude of correct ways that we could restrict the multivalued inverse cotangent relation in order to get a single valued function, there are two that stand out as most convenient. Alas, we mathematicians have no international standardizing body (or at least none that I'm aware of!), and so two common conventions persist.

I should also note perhaps, for anyone not familiar with it, that the only trig functions which have just a single convention for their inverse function are sine, cosine and tangent. The other three all have two distinct conventions. For this reason, I tend to avoid using arccot, arcsec and arccsc. This is always easily enough done. After all, anything that can be expressed using arccot can also be expressed using arctan instead. And done the latter way, there can be no possible ambiguity in the expression.

Hello Stefan,

I agree with almost everything you said.

> sorry, I should had done a plot of ACOT in DERIVE first. What I had to learn as a mathematician
> and as a teacher is that most CAS and calculators define arccot(x) as arctan(1/x) which is techni
> cally very simple because you only have to implement arctan (or arccot).

First, I'm not sure that most define it that way. I haven't done a poll. But I suspect that you're correct.

Second, in order to have a complete definition, one cannot simply say "define arccot(x) as arctan(1/x)" because that leaves arccot(0) undefined. It gives us no idea whether we should choose arccot(0) = π/2 or arccot(0) = –π/2.

BTW, Stefan (assuming, perhaps incorrectly, that you're German), does DIN specify a definition of arccot as being the only "correct" one?

Regards,
David

25 May 206
On Thu, 25 May 2006 15:31:15 +0100, David W. Cantrell wrote:

>Alas, we mathematicians have no international standardizing body (or at
>least none that I'm aware of!), and so two common conventions persist.

ISO/IEC 10967-2:2001 (Language Independent Arithmetic, part 2:
Elementary numerical functions; (LIA-2)), section 5.3.8.12 Radian arc cotangent has two arccot functions.
One is sign symmetric, but discontinuous at 0.
The other is continuous, but not sign symmetric.

That international standard is aimed at computer language standards bodies; so it may not meet your need for mathematicians.

Fred J. Tydeman   Tydeman Consulting

The discussion continued in June but at the moment I´ll refer to Johann´s Titbits #32 published in this issue which deals with the problem – and we have too many other interesting requests which deserve to be presented.

Jan Vermeylen, Belgium

Hello Josef,

I have a question of one of my science colleagues.

He has a set of some thousand quadruples (t,x,y,z) which show the position (x,y,z) of a particle at a certain time t.

Is it possible in Derive to produce a 3D graph of the path of this particle with an animation of the moving particle? Or should I use DPGraph?

Thank you for giving me advice.

Best regards,

Jan

Peter Schofield

Hello Josef,

It is nice to hear from you. Hope you are keeping well. I'm keeping myself busy teaching part-time at College.

Re. your problem. I'm attaching a short Derive6 mth. file that might suggest a way of plotting a vector of 3D-points over a "time sequence".

Open the File and set up a 3D-plot Slider Bar 1<= t <= 1000 with 999 intervals.
Then aproximate and 3D-plot #3 or #4 using InsertPlot> Medium Points> Custom (Black Colours)
(Might be better to turn Auto Change plot colours OFF as well). Takes about 60sec. on my laptop.

The only drawback I can see is that the origin will also be plotted as well as the moving point, and so these plots look better with the Axes ON. I don't know how to get round this (minor) problem.

Note that, in List3, I have stripped out the "time" coordinate of List4, since Derive can only plot 3D-points. However, there is still a notion of time steps in the successive points of List 3, and this is what #3 and #4 are using.

All the best,

Peter

DNL

Dear Peter,

many thanks for your very valueable advice. It works.
I attach a modified file – I don´t need the 4-element vector and I defined two functions.
The origin is nasty but it might happen that the range of the points does not contain the origin, then it does not appear.
Best regards
Josef

File **move_pts.dfw**

This is the first attempt provided by Peter:

#1: $\text{list4} := \text{VECTOR}\left(\frac{[x, x, x, x]}{200}, x, -500, 500\right)$

#2: $\text{list3} := \text{list4}\cdot\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

#3: $\text{VECTOR}([\text{list3}_i]\cdot\text{IF}(t = i, 1, 0), i, 1, 1000)$

#4: $\text{VECTOR}([\text{list3}_i]\cdot\text{IF}(t \geq i, 1, 0), i, 1, 1000)$

See the demonstration applied on 10 random points:

I produce 10 randompoints in a 10 by 10 by 10 cube and
trace the path using two functions derived from your algorithm:

#5: VECTOR([[10·RANDOM(1) – 5, 10·RANDOM(1) – 5, 10·RANDOM(1) – 5]], i, 10)

#6: $\begin{bmatrix} [-4.415, 1.901, -1.113] \\ [-1.956, -0.6297, -4.017] \\ [-3.966, 2.134, -2.530] \\ [-4.789, -3.450, -2.002] \\ [-1.688, -0.1096, 1.452] \\ [3.686, -4.949, 1.404] \\ [3.425, -4.847, 3.268] \\ [0.3028, -1.573, -0.5067] \\ [-1.002, -2.783, 2.610] \\ [-3.980, -2.955, -3.426] \end{bmatrix}$

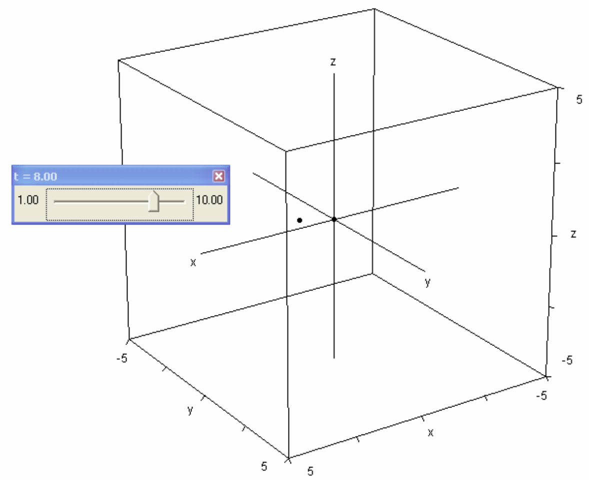
#7: $\text{pts} := \begin{bmatrix} [-4.415, 1.901, -1.113] \\ [-1.956, -0.6297, -4.017] \\ [-3.966, 2.134, -2.530] \\ [-4.789, -3.450, -2.002] \\ [-1.688, -0.1096, 1.452] \\ [3.686, -4.949, 1.404] \\ [3.425, -4.847, 3.268] \\ [0.3028, -1.573, -0.5067] \\ [-1.002, -2.783, 2.610] \\ [-3.980, -2.955, -3.426] \end{bmatrix}$

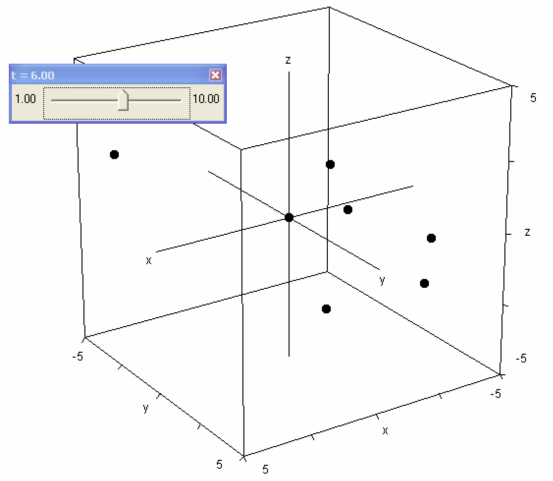
#8: $\text{movepts(list)} := \text{VECTOR}(\text{list}_i\cdot\text{IF}(t = i, 1, 0), i, \text{DIM(list)})$

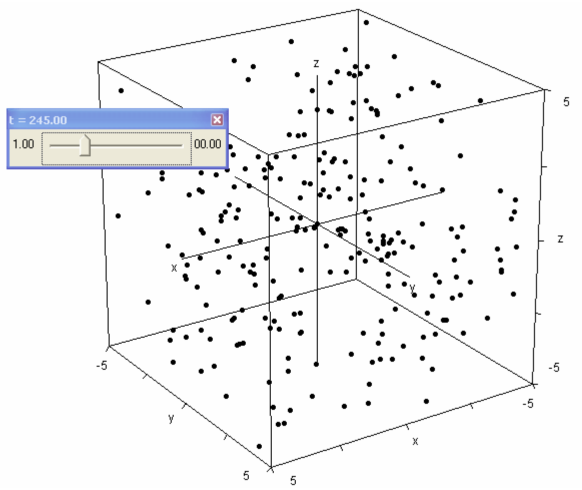
#9: $\text{tracepts(list)} := \text{VECTOR}(\text{list}_i\cdot\text{IF}(t \geq i, 1, 0), i, \text{DIM(list)})$

#10: movepts(pts)

#11: tracepts(pts)







David Sjöstrand

Hi,

I have spent some time thinking on Jan´s problem having a point moving along a curve using a slider bar. In the attached file I have done it for a matrix containg 3 points. If tstep is constant you can do the same with matrices up to 1000 points but not more due to the limitation of the number of intervals in the slider bar.

If you send me the actual matrix I can try to test my idea.

Best regards,
David

File **move_pts0.dfw**

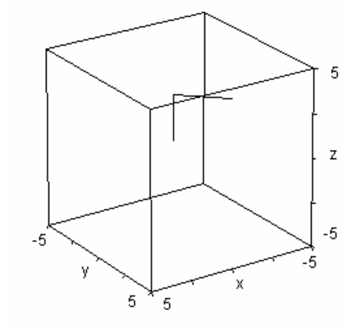
A is the matrix containing the quadruples. If you plot #3 you receive the path of the particle. Insert a sliderbar with the below settings. Then plot #4 and then use the sliderbar to animate the movement of the particle.

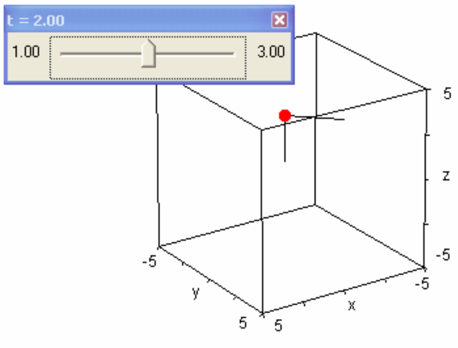
#1: $A := \begin{bmatrix} 1 & -2 & 2 & 3 \\ 2 & 3 & 4 & 5 \\ 3 & 1 & 1 & 1 \end{bmatrix}$

#2: $f(t) := \text{VECTOR}(A_{t,i}, i, 2, 4)$

#3: VECTOR(f(t), t, 1, 3)

#4: f(t)





DNL:

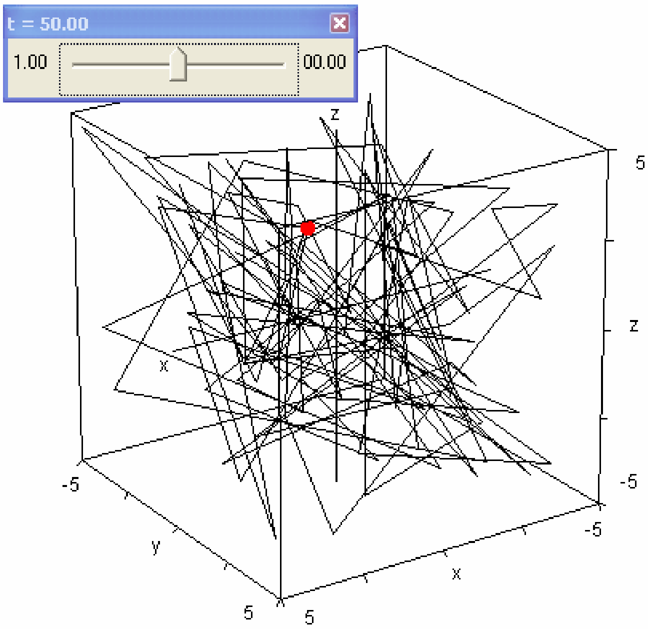
Your PPP-routine (Point – Path – Plotting) seems to work, that´s great. I have produced a data set of a lot of random points and I´ll test your idea.

Thank you on behalf of the DUG. (I thought to do something with a function – and you realized it in a very clear way!!, Congratulations).

I was too enthusiastic about your idea and couldn´t resist to make a try. I attach the file with a 100 point path using two functions based on your idea.

It works!!

File move_pts1.dfw



Dear David,

Dear Jan,

this is another version of „Voyage of the Point“.

The first column (“time column”) seems to be not necessary!

Regards as ever,

Josef

File **move_pts2.dfw**

#1: NotationDigits := 4

#2: $\text{mv(list, t) := VECTOR}(\text{list}_{t,i}, i, 3)$

#3: VECTOR([10·RANDOM(1) − 5, 10·RANDOM(1) − 5, 10·RANDOM(1) − 5], i, 100)

$$\begin{bmatrix} 0.3547 & 4.269 & -2.278 \\ -3.425 & -3.422 & 3.532 \end{bmatrix}$$

Result of #3 is assigned to pts2.
Plot first the complete path by plotting pts2 point size small and points connected, then insert a slider bar with 1 <= t <= 100 (99 intervals) and plot with point size large

In the meanwhile I received another mail from Belgium:

Hello David and Josef,

Sorry that I did not answer the last few weeks, but it has been (again) a bit too hectic in my neighbourhood -☺.

Here is the data set that I received from my colleague. It is derived from a scientific experiment where a particle was sent through a whirling magnetic field (I think).

Please note that, when you see the time sequence, there are big time intervals in between where the particle “disappears”.

Can you test your solution with (parts) of the data?

A2113 | fx 27234,1

	A	B	C	D
1	time (ms)	x (mm)	y (mm)	z (mm)
2	11773,4	261,8	34,2	191,9
3	11782,3	260	44,9	189,7
4	11787,6	259,6	54,2	186,6
5	11791,8	260,9	61,1	184,1
6	11797,4	260,1	72,4	186,9
7	11802,8	261,6	81	184,1
8	11806,8	261,1	87,5	184,4
9	11811	261	94,6	185,3
10	11814,6	262,2	101,1	186,6

7382	188310,2	270,9	547	218,6
7383	188315,4	270,6	550,2	212,9
7384	188322,2	271,2	557,2	216,8
7385	188330,3	269,5	563,4	211,9
7386	188338,6	268	568,7	209,8
7387	188349,5	266,7	572,2	223,5
7388				
7389				

Blad1 / Blad2 / Blad3

The problem kept Peter busy, too, improving his “Point Moving Procedure”:

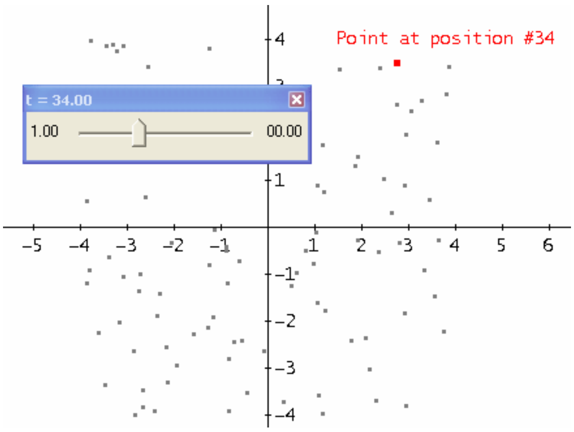
Hello Josef,

I've been thinking some more about avoiding plotting the origin in these sliderbar applications.
Please find attached three Derive 6 files which go some way to solving this problem.
You need "Approx before plotting" ON in both the 2D- and 3D-plot windows.
Also, for the 2D-examples, use Points> Size (Large); Connect (Yes)

File **move_pts3.dfw**

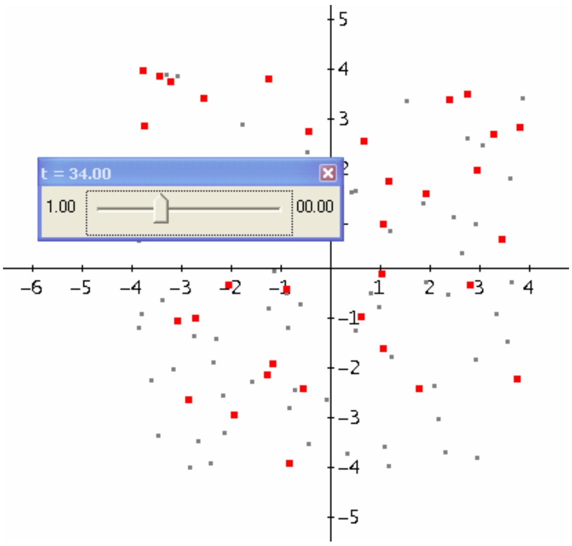
```
In 2D: 100 randompoints in an 8 by 8 square and trace the path using four functions
derived from the improved algorithm:

#1:     movepts(list) := VECTOR(list_i ·IF(t = i, 1, ∞), i, DIM(list))

#2:     tracepts(list) := VECTOR(list_i ·IF(t ≥ i, 1, ∞), i, DIM(list))

#3:     pts := VECTOR([[8·RANDOM(1) - 4, 8·RANDOM(1) - 4]], i, 100)

#4:     pts1 := VECTOR([[[0, 0], pts_i,1]], i, 1, DIM(pts))

Set up a 2D-sliderbar:  1<=t<=100 with 99 intervals.
```

I plot the point list (grey and size Medium), then:



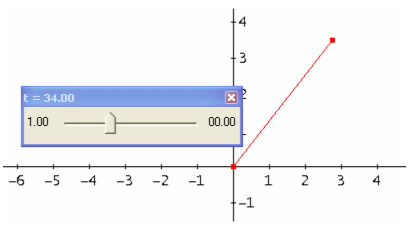
movepts(pts)

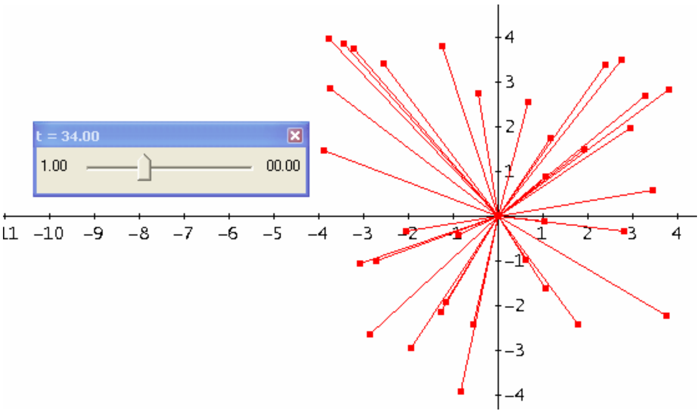
Shows one point after the other by moving the slider.



tracepts(pts)

Shows the first t points by moving the slider.





movepts(pts1) and tracepts(pts2) show the points + pointers to them.

File **move_pts4.dfw**

#1: $\text{movepts(list)} := \text{VECTOR}(\text{list}_i \cdot \text{IF}(t = i, 1, \infty), i, \text{DIM(list)})$

#2: $\text{tracepts(list)} := \text{VECTOR}(\text{list}_i \cdot \text{IF}(t \ge i, 1, \infty), i, \text{DIM(list)})$

#3: movepts(pts)

#4: tracepts(pts)

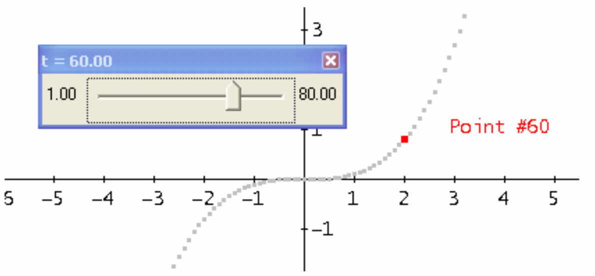
In 2D: 80 points on "y = x^3/10" containing a data point at the origin.
How to adapt the data list so that this point is not plotted
each time when using a slider bar?

#5:
```
nudge(v) :=
  If NUMBER_TYPE?(v)
     If v = 0
        10^(-10)
        v
     VECTOR(nudge(v↓i), i, DIM(v))
```

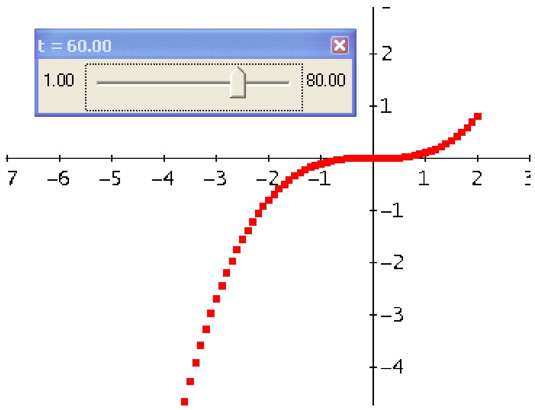
#6: $\text{pts} := \text{VECTOR}\left(\left[\left[\frac{i - 40}{10}, \frac{(i - 40)^3}{10000}\right]\right], i, 80\right)$

Plot #6 as grey mid sized not conntected points.
Set up a 2D-sliderbar: 1<=t<=80 with 79 intervals
and observe the point "Walking the Line".

#7: movepts(nudge(pts))



#8: tracepts(nudge(pts))



Comment: Tracing the points only on a parameter defined curve is easier done by:

$$\left[\frac{i - 40}{10}, \frac{(i - 40)^3}{10000}\right]$$

and applying a sliderbar for i.

I wanted to make the "path" of the point visible and experimented with Peter´s functions.

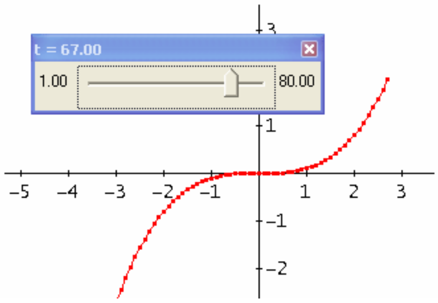
See my results in

tracepts_conn(nudge(pts))

Josef

#9: $\text{tracepts\_conn(list)} := \text{VECTOR}\left(\left[\left[(\text{list}_i \cdot \text{IF}(t \ge i, 1, \infty))_1, (\text{list}_{i+1} \cdot \text{IF}(t \ge i + 1, 1, \infty))_1\right]\right], i, \text{DIM(list)} - 1\right)$

#10: tracepts_conn(nudge(pts))



File **move_pts5.dfw**

100 randompoints in a 10 by 10 by 10 cube and trace the path using four functions derived from the improved algorithm:

#1: $\text{movepts(list)} := \text{VECTOR}(\text{list}_i \cdot \text{IF}(t = i, 1, \infty), i, \text{DIM(list)})$

#2: $\text{tracepts(list)} := \text{VECTOR}(\text{list}_i \cdot \text{IF}(t \ge i, 1, \infty), i, \text{DIM(list)})$

#3: pts := VECTOR([[10·RANDOM(1) – 5, 10·RANDOM(1) – 5, 10·RANDOM(1) – 5]], i, 100)

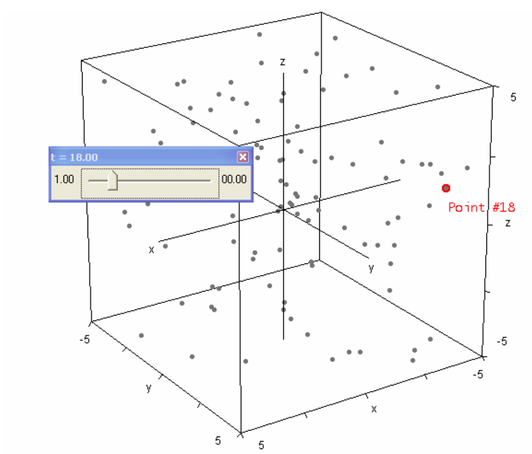
#4: $\text{pts1} := \text{VECTOR}\left(\left[\left[[0, 0, 0], \text{pts}_{i,1}\right]\right], i, 1, \text{DIM(pts)}\right)$

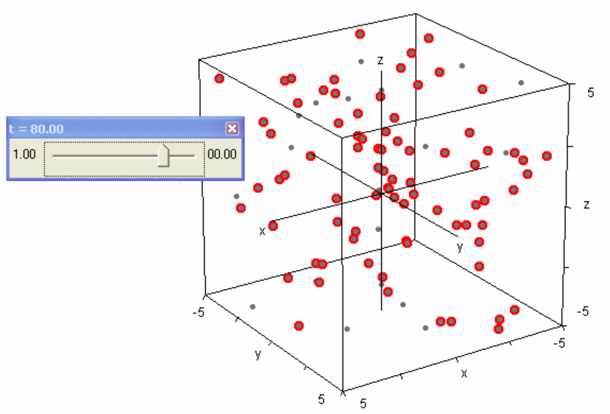
Set up a 3D-sliderbar: 1<=t<=100 with 99 intervals.

#5: pts

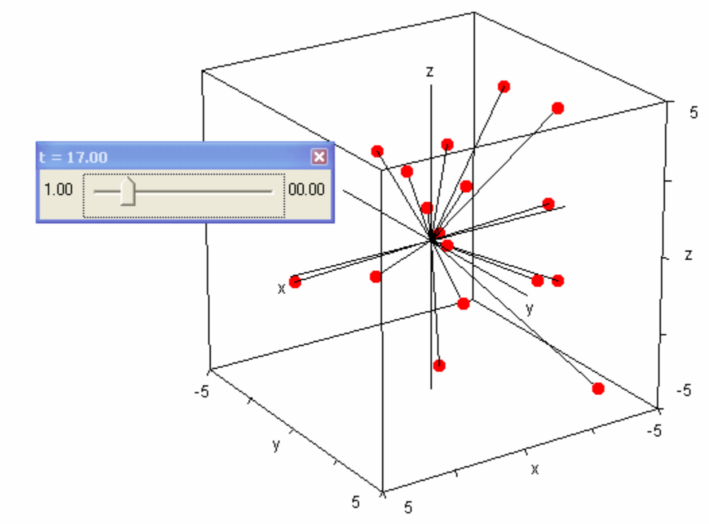
#6: movepts(pts)

#8: tracepts(pts)





```
#10:  [tracepts(pts), tracepts(pts1)]
```

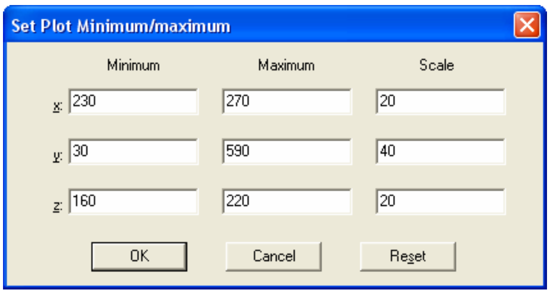


Now it is time to return to the given problem: tracing the particle through a magnetic field. First of all I transferred the first 1000 positions from Excel via Word to DERIVE, taking columns 2, 3 and 4. (See DNL#   how this can be performed. Another contribution (next DNL) will explain how to assign a data matrix to a variable using a DERIVE program.)

File **gegevens.dfw**

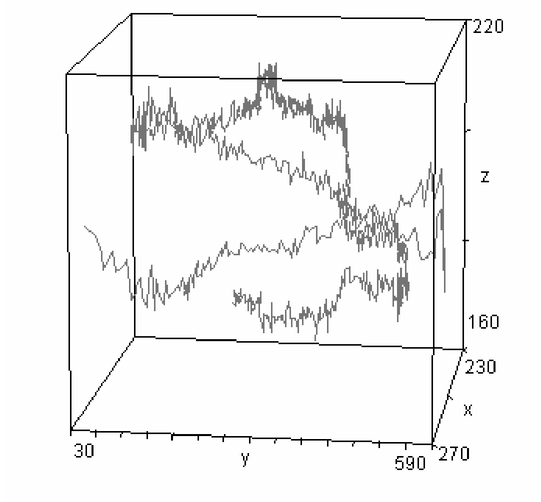
```
#1:   NotationDigits := 4

      mv(list, t) := VECTOR(list   , i, 3)
#2:                             t,i

#3:   gegev1 := [[261.8, 34.2, 191.9], [260, 44.9, 189.7], [259.6, 54.2, 186.6], [260.9, 61.1, 184.1],
        [260.1, 72.4, 186.9], [261.6, 81, 184.1], [261.1, 87.5, 184.4], [261, 94.6, 185.3], [262.2, 101.1,
        186.6], [262.1, 107.1, 179.5], [261.8, 113.5, 180.4], [263.6, 118.6, 182.3], [262, 121.5, 183.2],
      ......
      ......

This is to find appropriate values for the bounding box:

#4:   VECTOR([MIN(gegev1↓↓k), MAX(gegev1↓↓k)], k, 3)
```

$$\#5: \quad \begin{bmatrix} 241.3 & 266 \\ 34.2 & 572.3 \\ 174 & 216.7 \end{bmatrix}$$



next produces the path of the particle (1st 1000 positions)

#6: gegev1

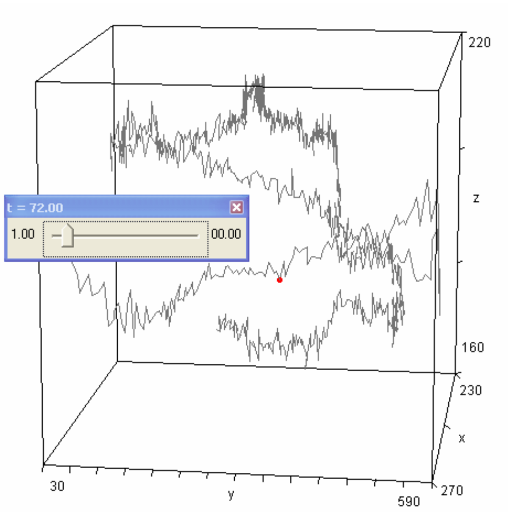


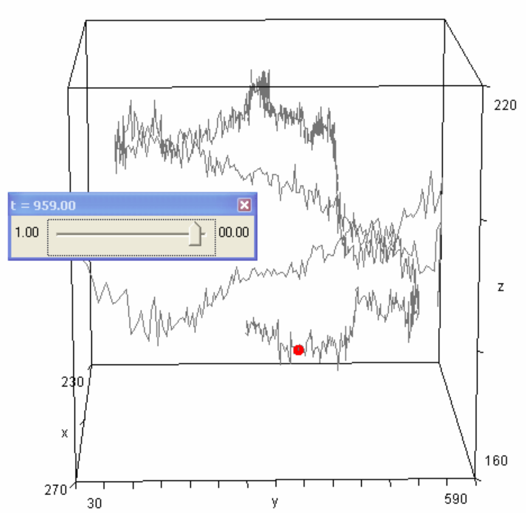
#7: mv(gegev1)

Introduce a slider bar for t and the plot #7.
Highlight #7 and Insert > Plot > point size Medium or Large,
Colors Custom (I set all colors RED).

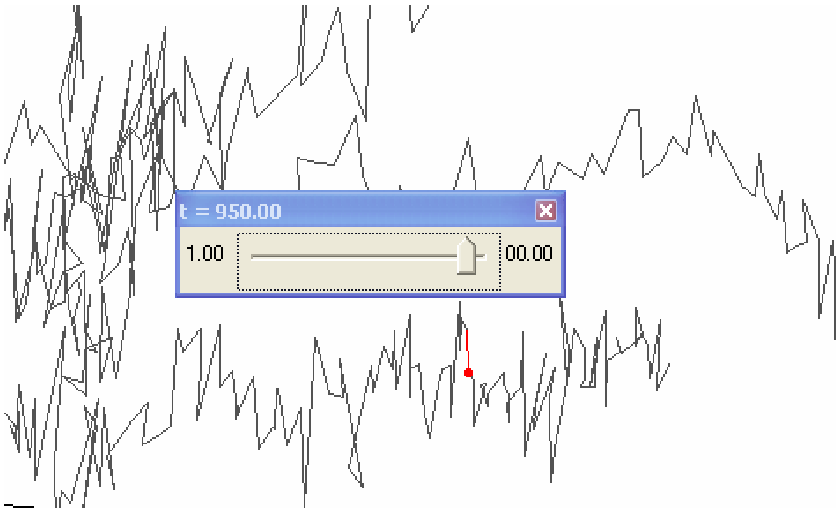
Voila, it works!!

We see the particle on its position #72 and then
just before finishing the first part of its journey through the magnetic field.





```
        pathstep3d(list, t) := [VECTOR(list    , i, 3), VECTOR(list        , i, 3)]
#9:                                      t,i                        t + 1,i

        mv_e(list, t) := VECTOR(list        , i, 3)
#10:                                t + 1,i

#11:    [pathstep3d(gegev1), mv_e(gegev1)]
```

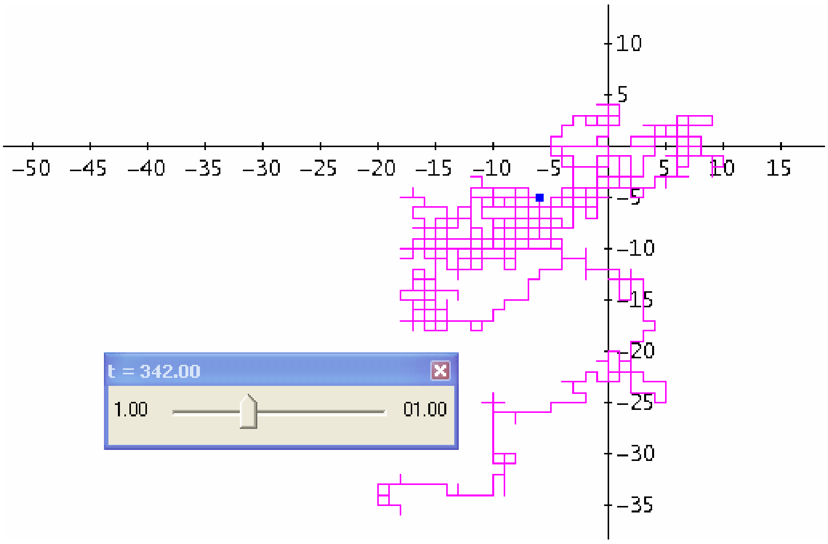


A Random Walk

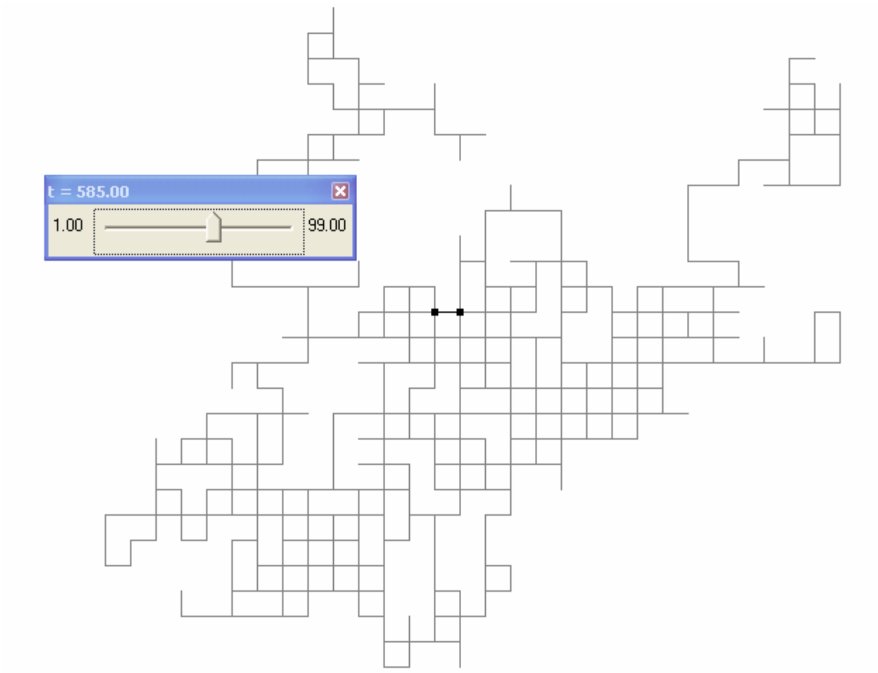
I was so excited about our success that I wanted to produce a simulation of a simple random walk in the 2D-plane:

```
       mv2d(list, t) := VECTOR(list    , i, 2)
#1:                                t,i

       mv3d(list, t) := VECTOR(list    , i, 3)
#2:                                t,i

       pathstep2d(list, t) := [VECTOR(list    , i, 2), VECTOR(list        , i, 2)]
#3:                                      t,i                        t + 1,i

       pathstep3d(list, t) := [VECTOR(list    , i, 3), VECTOR(list        , i, 3)]
#4:                                      t,i                        t + 1,i

        randwalk(n, m, dummy, p, i, k) :=
          Prog
            m := [1, 0; -1, 0; 0, 1; 0, -1]
            dummy := RANDOM(0)
            p := [0, 0]
            trace := [p]
            i := 1
#5:         Loop
              If i > n
                 RETURN "result in matrix trace"
              k := RANDOM(4) + 1
              p := p + m↓k
              trace := APPEND(trace, [p])
              i :+ 1

#6:     randwalk(1000) = result in matrix trace

#7:     trace

#8:     mv2d(trace)
```

```
#6:   randwalk(1000) = result in matrix trace

#7:   trace

#8:   mv2d(trace)
```

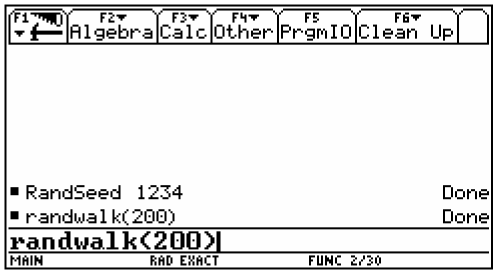


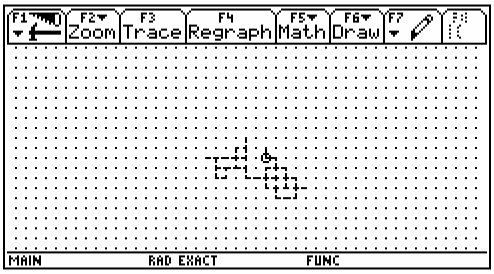

```
#9:   randwalk(1000) = result in matrix trace

#10:  trace

#11:  pathstep2d(trace)
```

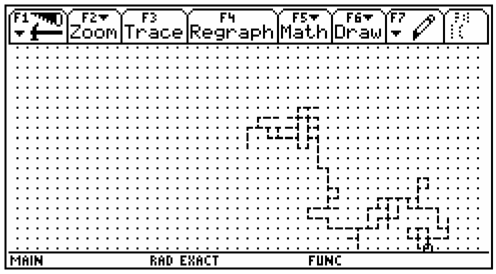


We can perform the walk on the TI, too. Use the following program for observing a point randomly strolling around on the TI-92 or Voyage graph screen:

```
randomwalk(n)
Prgm
Local  dirs,i,j,p,pn
FnOff  :ClrDraw
[[1,Ø][¯1,Ø][Ø,1][Ø,¯1]]→dirs
¯23.8→xmin:23.8→xmax
¯1Ø.2→ymin:1Ø.2→ymax
[[Ø,Ø]]→p:[[Ø,Ø]]→pn
Circle  p[1,1],p[1,2],Ø.5
©Pause
For  i,1,n
Circle  p[1,1],p[1,2],Ø.5,Ø
©Line p[1,1],p[1,2],pn[1,1],pn[1,2]
p+dirs[rand(4)]→pn
Line  p[1,1],p[1,2],pn[1,1],pn[1,2]
For  j,1,5
Circle  pn[1,1],pn[1,2],Ø.5
EndFor
©Pause
pn→p
EndFor
```



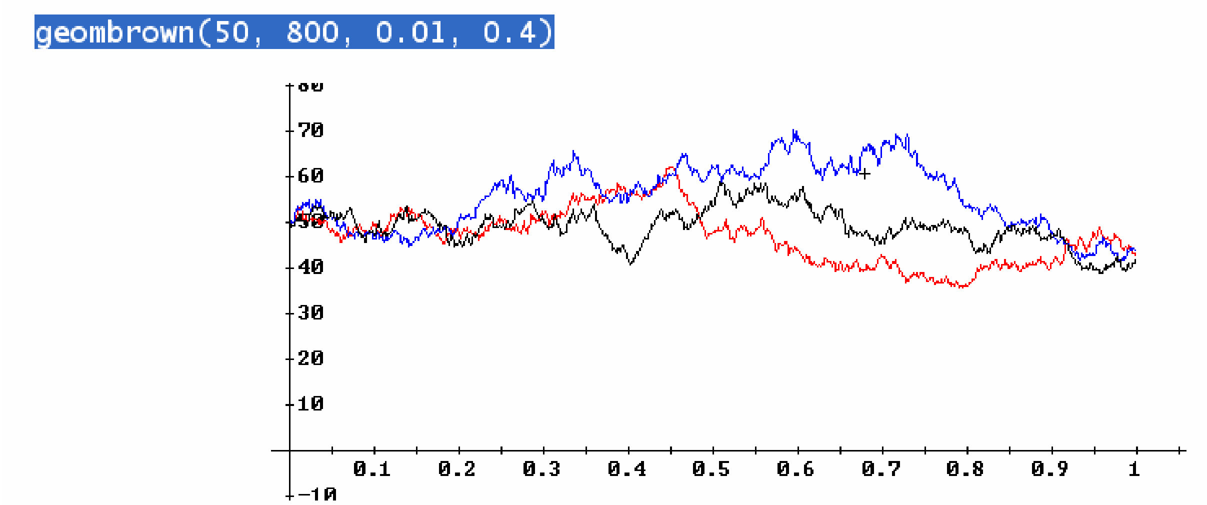




A fine application of a Random Walk is the Brownian Motion. A Random Walk-hypothesis models the share prices based on a Brownian Motion with expected value and volatility of the yield of the share as parameters.

See three simulations of 800 steps how a share with a price of 50 will develop (mean and volatility of yield = 0.01 and 0.4)

(Part of a possible future contribution, Josef)



I received an interesting mail from Heinz Rainer Geyer concerning a random experiment which I had used very often as an introductory example for probability theory: I brought three dice with me in class and asked the students what they would like to pay for one game: rolling the dice and win 1 through 6 EURO for a sum from 13 to 18 respectively.
They all made their offers ….
Then we made a number of experiments, noted the incomes and the payments of the bank-holder and finally we calculated the expected value of the payment. What would you like to bet? For calculating the mean we needed the answer to Heinz Rainer´s question:

Heinz Rainer Geyer, Germany

Hallo lieber Josef,

vor einer Stunde wollte ich dich noch mit einem Problem belästigen, das ich jetzt zumindest ansatzweise gelöst habe. Es geht um die alte Aufgabe der Verteilung der Augensummen bei n gleichzeitigen Würfeln.

Ich habe lange keine eindimensionale Liste aller Ergebnisse bei 3 Würfeln aufbauen können.
Die jetzige Lösung ist auch nicht gerade elegant und schreit förmlich nach einem anderen Aufbau.
Hast du dafür einen Tipp?

Das zweite Problem wäre eine Verallgemeinerung der Strukturen, so dass man nur noch die Anzahl der Würfeln als Variable verwenden muss, um die Verteilung der Augensummen zu erhalten.

Kennst du eigentlich einen Algorithmus zur direkten Berechnung der Häufigkeiten der Augensummen, zumindest bei 3 Würfeln? Ich habe bisher nichts gefunden.

(Heinz Rainer wants to automate the well known problem to find the frequency distribution of the sum of three (or more) rolled dice. He had some problems to build a list of all possible cases for three dice and asks for generalisation for any number of dice rolled.

Finally he asks if I knew any algorithm to obtain directly the distribution of the sums. See the last lines of his DERIVE file.

```
#19:  wurf4 := APPEND(VECTOR(APPEND(VECTOR(APPEND(VECTOR(VECTOR([t, u, v, w], w, 1, 6), v, 1, 6)), u, 1, 6)),
         t, 1, 6))
```

#20: $\text{zaehl\_4}(n) := \text{DIM}\left(\text{SELECT}\left(\sum_{i=1}^{\text{DIM}(w\_)} w\_{}_i = n,\ w\_,\ \text{wurf4}\right)\right)$

```
#21:  VECTOR([n, zaehl_4(n)], n, 4, 24)'
```

#22:

4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
1	4	10	20	35	56	80	104	125	140	146	140	125	104	80	56	35	20	10	4	1

DNL:

Lieber Rainer,
ich schicke Dir ein kleines Programm, das für eine allgemeine Zahl von Würfeln alle möglichen Würfe mit den Augensummen und deren Häufigkeiten angibt.

Ich bin (fast) sicher, dass es noch eleganter geht.

I sent a short program which returns for any number of dice the requested distribution, but I was quite sure, that it must be able to find a more elegant way to tackle the problem.

Heinz Rainer:

Hallo lieber Josef,

das ist ein interessantes Programm, da muss ich mich mal einarbeiten. Mit CODES_TO_NAME() usw. hab ich mich bisher noch nicht befasst.

Offensichtlich hast du mit APPEND(list,[wurf]) deine Struktur zusammensetzen können.
Ich wollte ja aus der Ergebnismenge mit 6^n Elementen die Summen mit SELECT abzählen.
Dazu müsste ich zu jedem Element des vorherigen Vectors $[w1,w2,...,w_{n-1}]$ die 6 neuen Würfe anfügen.

Irgendwie vermisst man dabei die FOR Schleifen, aber mit LOOP sollte das ja auch gehen.

Heinz Rainer worked through the program and noted that he had not worked with CODES_TO_NAME() before. He missed the FOR-loops but LOOP should work in a similar way.

Heinz Rainer:

Heureka,
Mann, das lies mich jetzt nicht mehr los! Es hat mich ein ganzes Wochenende und den Nachmittag gekostet, aber es ist der direkte Zugang, den ich mir vorgestellt hatte.
Elegant ist das sicher nicht.

Hast du das Problem schon an Johann gemailt? Er macht bestimmt einen rekursiven 4-Zeiler draus?

Liebe Grüße
Rainer

Danke für deine neue Datei, ich habe sie in der letzten Zeile gerade ein wenig umgestellt, um die Häufigkeiten auch zu plotten. So sieht das natürlich viel kompakter aus als meine vielen LOOPs.

Finally Heinz Rainer worked hard a full weekend to find a direct approach but he admitted, that his solution would not be very elegant. Then he asked: “Did you send the problem to Johann (Wiesenbauer = Mr. Titbits). I am sure that he will answer with a recursive 4 line function”.

Great idea, I asked Johann and received an answer within a few hours.

Lieber Rainer,
und hier ist er, der "Einzeiler" von Johann Wiesenbauer.
Er hat mir diese mail geschrieben, die Funktion dazu zu basteln, war nicht schwierig.
Liebe Grüße und viel Spaß beim Zocken,
Josef

This is Johann´s answer: it is not 4 lines but only one DERIVE function used in a clever way to give the distribution of the sum of three dice. It was not difficult to generalise this function for any number *n* of dice.

Von: Johann Wiesenbauer [mailto:j.wiesenbauer@tuwien.ac.at]

Lieber Josef,

Was Deine Anfrage, so ist die tatsächlich ganz einfach zu beantworten.

Simplifiziere mal

table(poly_coeff((x+x^2+x^3+x^4+x^5+x^6)^3,x,k),k,3,18)`

was eine Tabelle für die absoluten Häufigkeiten der Augensummen bei 3 Würfeln liefern sollte. Du kannst Dir dann leicht selbst überlegen, warum das funktioniert und wie man das allgemein auf n Würfeln verallgemeinern kann.

Herzliche Grüße,
Johann

#1: $\text{TABLE(POLY\_COEFF}((x + x^2 + x^3 + x^4 + x^5 + x^6)^4, x, k), k, 4, 24)'$

#2:

4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
1	4	10	20	35	56	80	104	125	140	146	140	125	104	80	56	35	20	10	4	1

And here is the function of your desire (works in an instant):

#3: $\text{hwsumme}(n) := \text{TABLE(POLY\_COEFF}((x + x^2 + x^3 + x^4 + x^5 + x^6)^n, x, k), k, n, 6\cdot n)'$

#4: hwsumme(4) =

4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
1	4	10	20	35	56	80	104	125	140	146	140	125	104	80	56	35	20	10	4	1

You see Johann´s function as core of a TABLE. It is a wonderful tas for students to investigate this function and find out why it works!!

Finally I´d like to present my – DERIVE program – based on counting the frequencies in its last version. It is nice to plot the transposed results with points connected (file `dice_sum.dfw`).

Additionally I wanted to solve the problem for the TI, too. So I invented the `poly_coeff`-function for the TI. I find that this might be an interesting problem for students. Applying this function it is not difficult to find a function for returning the distribution of the sums `dice_cnt(n)`.

(For those of you who don´t now **poly_coeff**: **poly_coeff(u,v,k)** returns the coefficient of v^k in the polynomial u.)

Lieber Rainer,
ich bin eben dabei, den nächsten DNL zu "komponieren". Bei dieser Gelegenheit habe ich mein Würfelprogramm noch etwas abkürzen können, da ich das Zählen der Häufigkeiten wesentlich einfacher gestaltet habe, ohne die Grundidee des Programms zu ändern.

Es ist nett, die transponierten Ergebnismatrizen "verbunden" zu plotten.

Johanns Trick mit poly_coeff habe ich für den V200/TI-92 programmiert.

```
        dice_sh(n, start, end, wurf, k, wcode, hlist) :=
          Prog
            hlist := VECTOR(0, j, 1, 6·n)
            start := CODES_TO_NAME(VECTOR(49, i, n))
            end := CODES_TO_NAME(VECTOR(54, i, n))
            wurf := start
            Loop
              If wurf > end exit
              wcode := NAME_TO_CODES(wurf)
#22:          k := 1
              Loop
                If wcode↓k < 49 ∨ wcode↓k > 54 exit
                k :+ 1
                If k > n
                   hlist↓(∑(NAME_TO_CODES(wurf)) − n·48) :+ 1
                If k > n exit
              wurf :+ 1
            k := n
            RETURN [VECTOR(i, i, n, 6·n), hlist↓[n, ..., 6·n]]
```

#23: dice_sh(4)

#24:

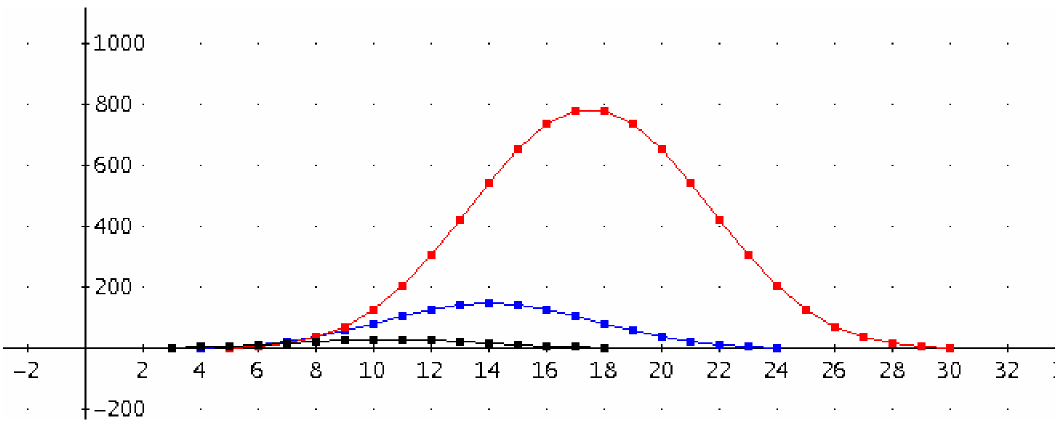
4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
1	4	10	20	35	56	80	104	125	140	146	140	125	104	80	56	35	20	10	4	1

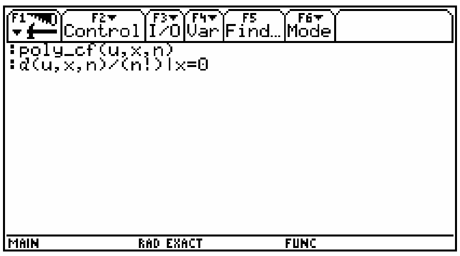
#25: dice_sh(5)

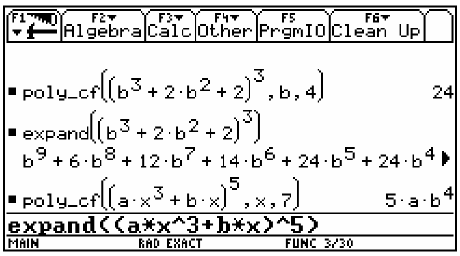
#26:

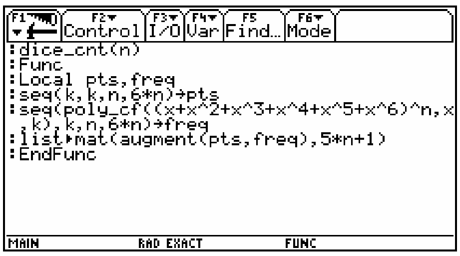
5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
1	5	15	35	70	126	205	305	420	540	651	735	780	780	735	651	540	420	305	205	126	70	35	15	5	1

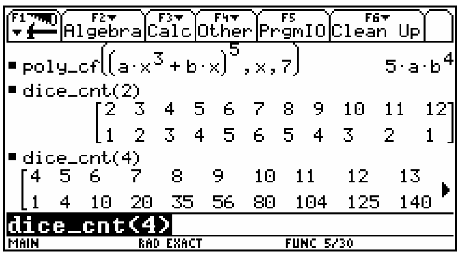
The distribution of the absolute frequencies of the sum of three, four and five dice











Titbits from Algebra and Number Theory (32)

by Johann Wiesenbauer, Vienna

As you may know (or read in this issue), there has been a quite interesting discussion on the Derive newsgroup recently. It was started by Jim FitzSimons, who wanted to know how Derive does find a certain algebraic expression for cos(π/17). Well, as I pointed out in my answer, I had already dealt with a related question in my "Titbits" (in the DNL #20), and later I even repeated the main points of that derivation in an attached Derive-file, but the further discussion made it desirable to look into this topic again from a more general (you could also say more algebraic !) point of view. In particular, when Albert Rich asked for a radical expression for cos(π/257), it was clear for me that I should rewrite that article and have a try at this new challenge. What you read in the following is the outcome of this plan, which is neither a sweeping success nor a total failure though as to this goal.

In order to make everything as transparent as possible, I'll give all basic formulas for a general Fermat prime p, i.e. a prime of the form $2^{2^n}+1$ for some natural number n. On the other hand, as for examples, to save space I'll consider here only the case p = 17 rather than p = 257, which is dealt with though in the Derive-file accompanying this column.

To begin with, we need a so-called primitive root $g \mod p$, i.e. an element g, whose residue class $g \bmod p$ is a generator of the prime residue class group $\mathbf{Z}_p$*. (Note that such an element always exists according to a theorem by Gauss!) In Derive the smallest g with this property can be obtained like this:

```
p := 17

g := PRIMITIVE_ROOT(p)

                    g := 3
```

Next we form a 1-class partition $a0$ containing all odd numbers below p.

$$a0 := \left[\mathrm{VECTOR}\left(\mathrm{IF}(\mathrm{ODD?}(a\_),\ a\_,\ p - a\_),\ a\_,\ \mathrm{ITERATES}\left(\mathrm{MOD}(g\cdot x,\ p),\ x,\ 1,\ \frac{p-3}{2}\right)\right)\right]$$

```
a0 := [[1, 3, 9, 7, 13, 5, 15, 11]]
```

What a strange way to compute those odd numbers, you might be saying! Of course, you are right, but the secret is the very specific order of these elements as will become clear later.

Furthermore, we will need the vector $b0$ of all sums of the terms $2\cos(k\pi/p)$ for all k contained in some list of a. Note that when referring to $2\cos(k\pi/p)$ we will use the term $z^k - z^{p-k}$ in the following, where $z = e^{i\pi/p}$ or z is rather a root of the irreducible polynomial $(z^p + 1)/(z+1)$ over **Q**. (Numerical comparisons below will make sure that the "right" root is taken from the p different roots!)

```
              k    p - k
b0 := VECTOR(Σ(z  - z     , k, a_), a_, a0)
```

$$b0 := \left[-z^{16} + z^{15} - z^{14} + z^{13} - z^{12} + z^{11} - z^{10} + z^{9} - z^{8} + z^{7} - z^{6} + z^{5} - z^{4} + z^{3} - z^{2} + z \right]$$

Furthermore we will need the vector c of the corresponding real numbers represented by those sums. (Note that we deal in the following only with approximative values of c, which is sufficient for our purposes!)

$$c0 := \text{VECTOR}\left(2\cdot\Sigma\left(\text{COS}\left(\frac{k\cdot\pi}{p}\right),\ k,\ a\_\right),\ a\_,\ a0\right)$$

$$\text{APPROX}(c0) = [1]$$

Finally, we introduce a vector $d0$ that contains the "exact" values of the components of $c0$. At the beginning, $d0$ is always [1] as suggested by the approximative value of $c0$ above.

```
d0 := [1]
```

Before going on, we write a routine that does all the steps above automatically, if the parameter ρ is initialized by p.

```
init(ρ) :=
  Prog
    p := ρ
    g := PRIMITIVE_ROOT(p)
    a0 := [VECTOR(IF(ODD?(a_), a_, p - a_), a_, ITERATES(MOD(g·x, p), x, 1, (p - 3)/2))]
    b0 := VECTOR(Σ(z^k - z^(p - k), k, a_), a_, a0)
    c0 := VECTOR(2·Σ(COS(k·π/p), k, a_), a_, a0)
    d0 := [1]
    "ok"
```

Note that all variables in init(ρ) (except for ρ) are global variables, which can be viewed and also used after the call of the routine. (Actually one of the rare cases, where the use of global variables makes sense in Derive!)

We now divide each list in $a0$ into two equally sized parts by taking the oddindexed numbers and the evenindexed numbers thus getting a new list a.

$$a := \text{APPEND}\left(\text{VECTOR}\left(\left[a\_{}_{[1,\ 3,\ ...,\ \text{DIM}(a\_)]},\ a\_{}_{[2,\ 4,\ ...,\ \text{DIM}(a\_)]}\right],\ a\_,\ a0\right)\right)$$

$$a := \begin{bmatrix} 1 & 9 & 13 & 15 \\ 3 & 7 & 5 & 11 \end{bmatrix}$$

Updating $b0$ and $c0$ as well yields the following new values.

$$b := \text{VECTOR}\left(\text{REMAINDER}\left(\sum(z^k - z^{p-k}, k, a\_), \frac{z^p + 1}{z + 1}\right), a\_, a\right)$$

$$b := [z^{14} + z^{12} - z^{11} + z^{10} - z^{7} + z^{6} - z^{5} - z^{3} + 1, - z^{14} - z^{12} + z^{11} - z^{10} + z^{7} - z^{6} + z^{5} + z^{3}]$$

$$c := \text{VECTOR}\left(2\cdot\sum\left(\text{COS}\left(\frac{k\cdot\pi}{p}\right), k, a\_\right), a\_, a\right)$$

$$\text{APPROX}(c) = [-1.561552812, 2.561552812]$$

Now consider two adjacent elements u and v of b such that $u = v_{2i-1}$ and $v = b_{2i}$ for some $i = 1,2,...,\dim(b)/2$. In order to compute d, i.e. the vector with the exact values, we need the crucial fact that both $u + v$ and $u * v$, when reduced mod $(z^p + 1)/(z + 1)$ are linear combinations over **Q** of certain elements of $b0$. (Actually even with integer co-efficients, but we don't make use of this!)

This is trivial as far as $u + v$ is concerned. (Remember, we got the elements of b by "splitting up" the elements of $b0$ into "two halves".) As for the product $u * v$, let's check it here for the only pair of b.

$$\text{VECTOR}\left(\text{REMAINDER}\left(b_{2\cdot i - 1}\cdot b_{2\cdot i}, \frac{z^p + 1}{z + 1}\right), i, 1, \frac{\text{DIM}(b)}{2}\right) = [-4]$$

In other words, the numerical values of u and v can be obtained as solution of some quadratic equation, e.g. in the case at issue:

$$\text{FIRST}(\text{SOLUTIONS}([u + v = 1, u\cdot v = -4], [u, v])) = \left[\frac{\sqrt{17}}{2} + \frac{1}{2}, \frac{1}{2} - \frac{\sqrt{17}}{2}\right]$$

Actually there are always two solutions to a system of equations $u + v = r, uv = s$ with known values of r and s, which only differ by the order though, and to find the "cor-rect order" we must resort to the numerical values stored in c.

```
solution(r, s, c1, c2) :=
  If c1 ≥ c2
     [(√(r^2 - 4·s) + r)/2, (r - √(r^2 - 4·s))/2]
     [(r - √(r^2 - 4·s))/2, (√(r^2 - 4·s) + r)/2]
```

$$d := \text{APPEND}\left(\text{VECTOR}\left(\text{solution}(1, -4, c_{2\cdot i - 1}, c_{2\cdot i}), i, 1, \frac{\text{DIM}(c)}{2}\right)\right)$$

$$d := \left[\frac{1}{2} - \frac{\sqrt{17}}{2}, \frac{\sqrt{17}}{2} + \frac{1}{2}\right]$$

Before going on, let's save the values of a,b,c,d to $a0,b0,c0,d0$, respectively, and com-bine all the assignments above in single routine called nextstep( ):

```
[a0 := a, b0 := b, c0 := c, d0 := d]

nextstep(show := true, s_) :=
  Prog
    a := APPEND(VECTOR([a_↓[1, 3, ..., DIM(a_)], a_↓[2, 4, ..., DIM(a_)]], a_, a0))
    b := VECTOR(∑(z^k - z^(p - k), k, a_), a_, a)
    c := VECTOR(2·∑(COS(k·π/p), k, a_), a_, a)
    s_ := VECTOR(REMAINDER(b↓(2·i - 1)·b↓(2·i), (z^p + 1)/(z + 1)), i, 1, DIM(b0))
    s_ :- (z^p + 1)/(z + 1)·SUBST(s_, z, 0)
    s_ := VECTOR(VECTOR(POLY_COEFF(t_, z, a_), a_, a0 COL 1), t_, s_)
    d := APPEND(VECTOR(solution(d0↓k, s_↓k·d0, c↓(2·k - 1), c↓(2·k)), k, 1, DIM(b0)))
    [a0 := a, b0 := b, c0 := c, d0 := d]
    If ¬ show exit
    FIRST(d)/2
```

The only "tricky" part is to represent the elements of s_ (in the program after the very first assignment) as linear combinations of $(z^p+1)/(z+1)$ and the "old" basis $b0$. This is done in the following two assignments, after which s_ contains the vectors with the integer coefficients of all those linear combination with respect to $b0$. Exactly these coefficients are needed to obtain the new "exact" vector d from $d0$. As an output after each step (if wanted, i.e. show is true), we get $d_1/2$, which is the sum of all eypressions $\cos(k\pi/p)$, where k ranges in that class of the partition of a, which contains 1. At the end, this class contains the number 1 only and hence we get the result, we are longing for, namely $\cos(\pi/p)$.

Let's check it by continuing using this routine, which takes only tenths of a second on a fast PC.

```
nextstep()
```

$$\frac{\sqrt{34-2\cdot\sqrt{17}}}{8}-\frac{\sqrt{17}}{8}+\frac{1}{8}$$

```
nextstep()
```

$$\frac{\sqrt{\sqrt{38\cdot\sqrt{17}+170}+3\cdot\sqrt{17}+17}}{8}+\frac{\sqrt{34-2\cdot\sqrt{17}}}{16}-\frac{\sqrt{17}}{16}+\frac{1}{16}$$

In fact, Derive acknowledges the following equality (in approximate mode though!)

$$\mathrm{APPROX}\left(\cos\left(\frac{\pi}{17}\right)=\frac{\sqrt{\sqrt{38\cdot\sqrt{17}+170}+3\cdot\sqrt{17}+17}}{8}+\frac{\sqrt{34-2\cdot\sqrt{17}}}{16}-\frac{\sqrt{17}}{16}+\frac{1}{16}\right)=\mathrm{true}$$

Now, what about the case p=257? Again the computations are very fast (though taking seconds rather than splits of a second), but as Jim found out independently, Derive can't cope with the huge expressions short before arriving at the final expression. Frankly, if you look at the huge expressions in the Derive-file you really wonder why Derive didn't throw in the towel earlier!

It must be said, that Jim FitzSimmons obtained a result, too. He wrote in his last email: "Johann, I can not get Derive to do the last step. The expression is too long for Derive to handle. The example is attached. (cos257b.mth)

Regards, Jim"

See the last lines of Mr Titbits´ file deriving COS(π/257):

#29: nextstep()

#30: $$\frac{\sqrt{(d1^2 + 4\cdot(d17 + d30))} + d1}{4}$$

The next two assignments (the sense of which will become clear only later!) must be again simplified before going on, although I have delete the putput to save space.

#31: dd1 :≔ SUBST(d_1, ds, d_)

#32: dd8 :≔ SUBST(d_8, ds, d_)

For the last step we must repeat the whole procedure with different notations.

```
#33:  dds :≔ VECTOR(APPEND(dd, STRING(k)), k, 1, DIM(d))

#34:  dds :≔ [dd1, dd2, dd3, dd4, dd5, dd6, dd7, dd8, dd9, dd10, dd11, dd12, dd13, dd14, dd15, dd16, dd17,
        dd18, dd19, dd20, dd21, dd22, dd23, dd24, dd25, dd26, dd27, dd28, dd29, dd30, dd31, dd32, dd33,
        dd34, dd35, dd36, dd37, dd38, dd39, dd40, dd41, dd42, dd43, dd44, dd45, dd46, dd47, dd48, dd49,
        dd50, dd51, dd52, dd53, dd54, dd55, dd56, dd57, dd58, dd59, dd60, dd61, dd62, dd63, dd64]

#35:  d0 :≔ dds

#36:  nextstep()
```

#37: $$\frac{\sqrt{(dd1^2 + 4\cdot dd8)} + dd1}{4}$$

Hence this is the final symbolical formula for cos(pi/257). (Note the difference between "dd1","dd8" and dd1, dd8, respectively, which is not shown on the screen!)

#38: $$\text{cos\_pi\_over\_257} :\equiv \text{SUBST}\left(\frac{\sqrt{(dd1^2 + 4\cdot dd8)} + dd1}{4}, [dd1, dd8], [dd1, dd8]\right)$$

Unfortunately Derive can't carry out this substitution due the size of the expressions involved. (If you are a perfectionist though, you can still go one step further and substitute "dd8" by dd8 alone and leave "dd1" unsubstituted, but I left it at that.) Well, it's like a mountain climber must feel, if he is 100 m below the peak of the Mount Everest, but can't reach it for some reason, isn't it?

At least, we can see that the result is apparently correct. In fact, when approximating this term with an increased precision of 100 digits we get

```
#39:  APPROX(STRING(cos_pi_over_257), 100)

#40:  0.9999252866697325552128053591455562406916298296429212011881377099710071475277558301156004129496104709
```

On the other hand, we have

#41: $$\text{APPROX}\left(\text{STRING}\left(\text{COS}\left(\frac{\pi}{257}\right)\right), 100\right)$$

```
#42:  0.9999252866697325552128053591455562406916298296429212011881377099710071475277558301156004129496103685
```

which shows a remarkable coincidence and is sort of a "proof" of our assertion.