THE DERIVE - NEWSLETTER #64

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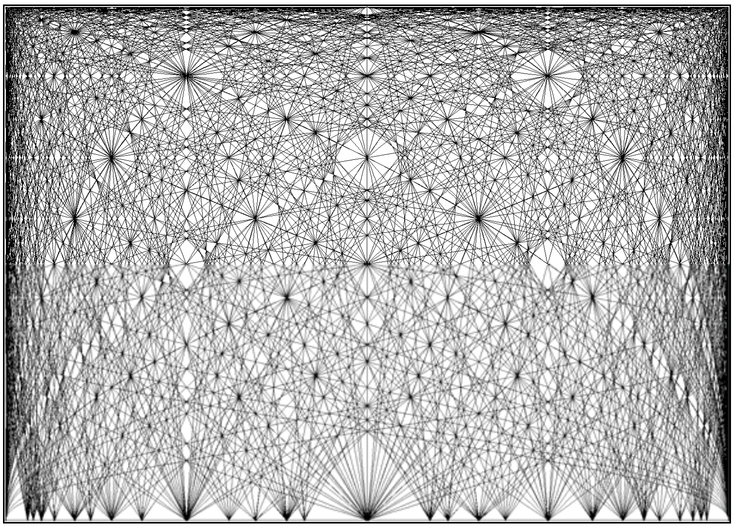
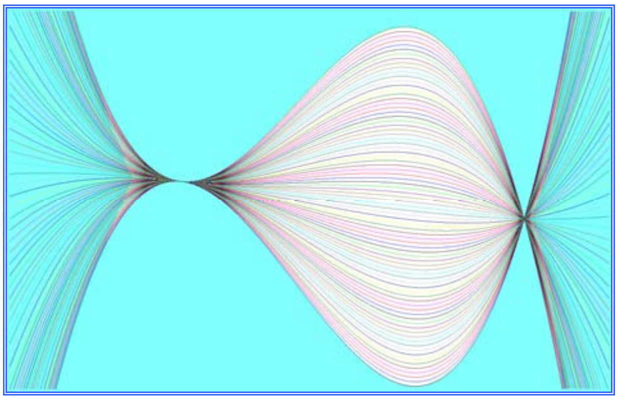
THE BULLETIN OF THE DERIVE USER GROUP + CAS-TI

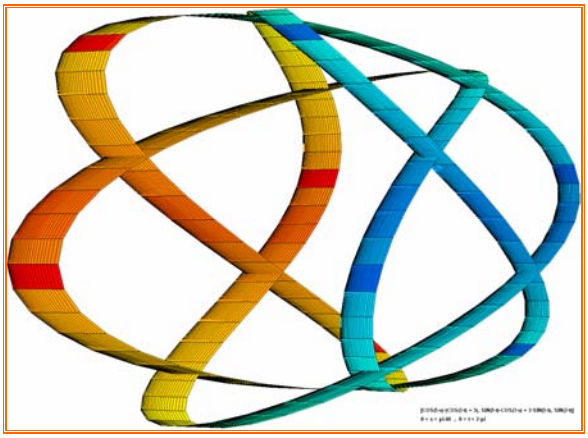
Contents:

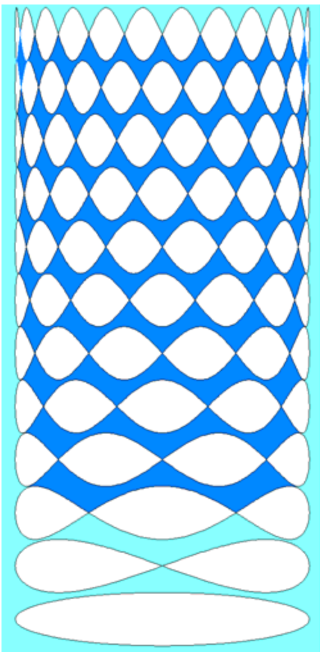
December 2006/January 2007

Pierre Charland, a new DUG Member from Canada sent a couple of "Graphics bundles" containing about 200 *DERIVE* plots. This is a tiny mouth watering selection:









What a wonderful design for a water glass. (It reminds me on Youth Style, Josef)

Dear DUG Members,

this is not the first issue of 2007 but should be considered as last issue of 2006. Anyhow, it does not contain so many contributions but it shows an amount of 46 full pages. I didn´t want to split Bjørn Felsager´s article into parts and I wanted to have a series of three articles on a fine use of CAS in classroom. As you can see there are only two of them – Traffic Density and Morley Triangle –, the third one about a pretty property of quartics must be postponed for the next DNL. When I was ready copying and pasting Peter Lüke-Rosendahl´s paper *Morley in the Mirror* I had in mind to leave it for the next issue because the newsletter was full, but then I decided to keep it as a penitence for my delay delivering DNL#64 – and, to be honest I found it too nice to leave it unread two months in my computer.

Johann Wiesenbauer prepared a new Titbits-contribution about semigroups and groups, (which will appear in DNL#65) and he promised to revise his Titbits from earlier issues. Next revised version will be DNL#13 with the first Titbits-column. (By the way, do you remember Titbits#1 with the Extended Euklidean Algorithm.

There is no information in this DNL on the first page but instead of it you can find a first selection of DERIVE graphs which reached me some days before Christmas. Pierre Charland from Canada composed many many terrific graphs, some of them in black and white, some of them with only a few colours and some of them with plenty of colours. I´ll insert a selection of them in the next DNLs.

Koen Stulens from Belgium announced a couple of short articles on the use of TI-Nspire to demonstrate the features of this new CAS-tool.

Finally I don´t want to forget wishing you a healthy and successful year 2007. We are all looking forward what 2007 will unhide for the CASers from all over the world. Let´s make the best of it.

Best regards until March (or April?)

The *DERIVE-NEWSLETTER* is the Bulletin of the *DERIVE* & CAS-*TI User Group*. It is published at least four times a year with contents of 44 pages minimum. The goals of the *DNL* are to enable the exchange of experiences made with *DERIVE* and the *TI*-89/92/*Titanium*/*Voyage* 200 as well as to create a group to discuss the possibilities of new methodical and didactical manners in teaching mathematics.

As many of the *DERIVE* Users are also using the CAS-*TIs* the *DNL* tries to combine the applications of these modern technologies.

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Contributions:
Please send all contributions to the Editor. Non-English speakers are encouraged to write their contributions in English to reinforce the international touch of the *DNL*. It must be said, though, that non-English articles will be warmly welcomed nonetheless. Your contributions will be edited but not assessed. By submitting articles the author gives his consent for reprinting it in the *DNL*. The more contributions you will send, the more lively and richer in contents the *DERIVE* & CAS-*TI Newsletter* will be.

Next issue: March 2007
Deadline 15 February 2007

Preview: Contributions waiting to be published

Two Stage Least Squares, M. R. Phillips, USA
Some simulations of Random Experiments, J. Böhm, AUT, Lorenz Kopp, GER
Wonderful World of Pedal Curves, J. Böhm
Another Task for End Examination, J. Lechner, AUT
Tools for 3D-Problems, P. Lüke-Rosendahl, GER
ANOVA with *DERIVE* & *TI*, M. R. Phillips, USA
Financial Mathematics 4, M. R. Phillips
Hill-Encription, J. Böhm
Farey Sequences on the *TI,* M. Lesmes-Acosta, COL
Simulating a Graphing Calculator in *DERIVE*, J. Böhm
Henon & Co, J. Böhm
Are all Bodies falling equally fast, J. Lechner
Do you know this? Cabri & CAS on PC and Handheld, W. Wegscheider, AUT
An Interesting Problem with a Triangle, Steiner Point, P. Lüke-Rosendahl, GER
Mathematics and Design, H. Weller, GER
Diophantine Polynomials, D. E. McDougall, Canada
Challenger Matrix Problems, G P Speck, New Zealand
MuPAD, a special CAS-Guest, Th. Himmelbauer, AUT
Graphics World, Change Count, P. Charland, CAN
Precise Recurring Decimal Notation, P. Schofield, UK
Problems solved using the TI-Nspire, K. Stulens, BEL
and Setif, FRA; Vermeylen, BEL; Leinbach, USA; Koller, AUT; Baumann, GER; Keunecke, GER, ...and others

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From: DERIVE computer algebra system
On **Behalf Of Fred J. Tydeman**
Sent: Thursday, December 28, 2006 8:31 AM
To: DERIVE-NEWS@JISCMAIL.AC.UK
Subject: 7.5 digit log

How do I find the 7 digit number in the range of 10 to 20 that has a base-10 log that is closest to the midpoint between two 7 digit numbers?

For example,
 log(10.00706,10) is 1.000306_5037...
 log(10.00918,10) is 1.000398_4994...

Fred J. Tydeman   Tydeman Consulting

December 29

In your example the midpoint is exact.
10.00812
Jim FitzSimons

December 29

I believe not:

#1: LOG(10.00812, 10)

#2: 1.000352_504022033086493433326745 62

Fred J. Tydeman

December 29

10^((LOG(10.00706,10)+LOG(10.00918,10))/2)=
SQRT(250406162027)/50000 is the midpoint on the log scale.
This is 10.00812 rounded to 7 digits.
Jim FitzSimons

December 29

While that may be, it is not what I am asking for.

I am asking for an exact 7 digit number (not a rounded number) whose log is closest to the midpoint between two 7 digit numbers.

Fred J. Tydeman

December 29

10.00812 is the exact midpoint. That is a fact.
The logarithm function is one to one so the midpoint on the linear scale is in the middle on the log scale, but not exactly the midpoint on the log scale.

You have to explain the problem better.
I do not understand what you are asking.

Is 10188730000 what you are looking for?
That is not between 10 and 20, but that is not possible given 7 digit numbers 10.00706 and 10.00918.

[LOG(10188720000,10),10.00812,LOG(10188730000,10)]
[10.00811962739663,10.00812,10.00812005364671]

Jim FitzSimons

December 30

How do I find the 7 digit number in the range of 10 to 20 that has a base-10 log that is closest to the midpoint between two 7 digit numbers?

That is, I want the log(x,10) to be of the form:
 1.abcdef_4999...
or
 1.ghijkl_5000...

I do not care what values a,b,c,d,e,f,g,h,i,j,k, or l have.
But, I do want, mod(10^6 * log(x,10)) to be as close as possible to 0.5 with x being a 7 digit number.

For example,
 log(10.00706,10) is 1.000306_5037...
 log(10.00918,10) is 1.000398_4994...

Some more examples:
 log(10.00418,10) is 1.000181_4971...
 log(10.00653,10) is 1.000283_5017...

Fred J. Tydeman   Tydeman Consulting

December 30

```
a1:=LN(2)/LN(10)*RANDOM(1)+1
a1:=1.172936545387103
a2:=(FLOOR(a1*10^6)+0.5)/10^6
a2:=1.1729365
a3:=ROUND(10^(a2+5))/10^5
a3:=14.89143
LOG(VECTOR(a3+x*(0.00001),x,-1,1),10)
LOG([14.89142,14.89143,14.89144],10)
[1.172936112712562,1.172936404353209,1.172936695993659]
```

Jim FitzSimons

January 3, 2007

From Johann Wiesenbauer [j.wiesenbauer@TUWIEN.AC.AT]

Hi Fred,

Looks like

APPROX(LOG(15.51499, 10), 20) = 1.1907515000009954495

is the best solution to your problem for any x in [10,20]. At least that's what my attached DERIVE-program says.

Cheers,
Johann

```
      fred(x := 10, x1 := 20, p := 20, r_ := 0.01, s_, t_ := []) :=
        Loop
          APPROX(WRITE(x))
          s_ := APPROX(ABS(MOD(10^6·LOG(x, 10)) - 0.5), p)
          If s_ < r_
#1:          Prog
               r_ := s_
               t_ := ADJOIN(APPROX([STRING(x), STRING(LOG(x, 10))], p), t_)
          x :+ 1/10^5
          If x > x1
             RETURN ADJOIN(["x", "LOG(x, 10)"], REVERSE(t_))

#2:   fred()
```

#3:

x	LOG(x, 10)
10.00038	1.0000165028767596509
10.00418	1.0001814971631705636
10.006	1.0002604985473903468
10.00918	1.0003984994511117080
10.01439	1.0006245005390073631
10.02999	1.0013005000247068514
10.03632	1.0015744999939723413
10.17615	1.0075835000036467535
10.49058	1.0207994999987194000
15.51499	1.1907515000009954495

#4: APPROX(LOG(15.51499, 10), 20) = 1.190751500

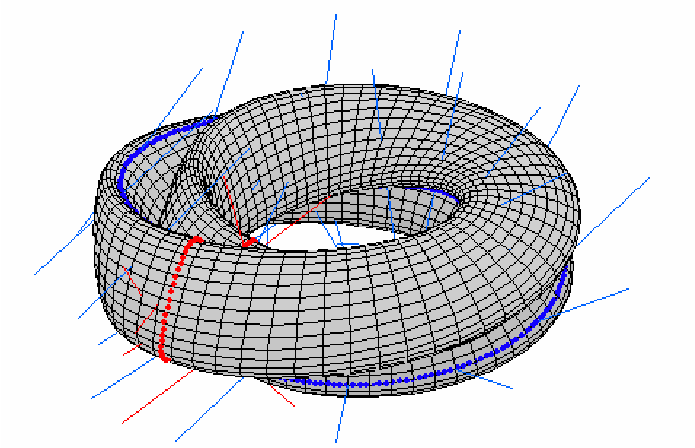
January 3, 2007

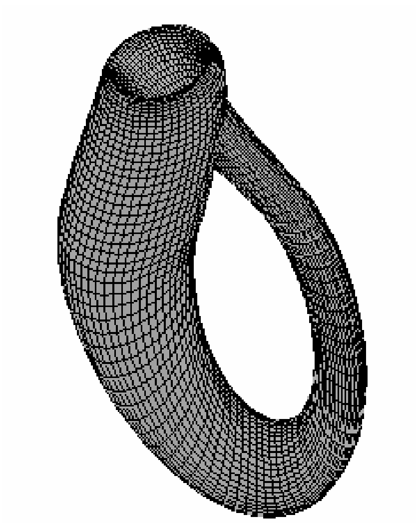
Thank you. That is what I was looking for.
I see that your program did a brute force try all numbers.
Now, to see if I can come up with some faster search method.

Fred J. Tydeman Tydeman Consulting

Just before finishing this DNL an interesting question came in how to representing the "Klein Bottle" together with normals to the surface:

... This is quite a complicated modelling task as not only do I need to determine the normals to the surface at a given z value, but also to make the normal vectors a small enough length to be instructive. Hopefully someone can provide some guidance on this issue as I would be looking to apply the technique to other surfaces.





I´ll come back to the two forms of the Klein Bottle in the next DNL, Josef

Solving some Challenges from Fermat Using Symbolic Algebra

See how to use Newton's geometrical methods to deal with some famous number theoretical problems investigated by Fermat.

Bjørn Felsager; Haslev Gymnasium & HF; Denmark

Fermat and Diophantine equations

Back in the 17'th century Fermat acquired an edition of Diophantus, a classical text on number-theory. While reading it he made a lot of comments in the margin, notes that were finally published by his son in another famous edition of Diophantus. Apart from these notes, Fermat's work with number theory can be followed through his correspondence with many contemporary mathematicians. But these letters seldom contained many details. In fact only one proof has survived from Fermat – at the very end of his copy of Diophantus he found sufficient room to disclose at least some of the details.

This does not mean that Fermat did not pay attention to proofs. In a famous critique from 1657 of a contemporary English mathematician Wallis he comments upon Wallis use of ordinary induction, i.e. establishing a hypothesis by gathering a lot of examples supporting a definite pattern, as opposed to mathematical induction:

> *'One might use this method if the proof of some proposition were deeply concealed and if, before looking for it, one wished first to convince one self more or less of its truth. But one should place only limited confidence in it and apply proper caution. Indeed, one could propose such a statement, and seek to verify it in such a way, that it would be valid in several special cases, but none the less false and not universally true, so that one has to be most circumspect in using it. No doubt it can still be of value if applied prudently, but cannot serve to lay the foundations for some branch of science, as Mr. Wallis seeks to do, since for such a purpose nothing short of a demonstration is admissible.'*

Similarly Fermat writes to Cherselier in 1662:

'The essence of a proof is that is compels belief'

So clearly Fermat was aware of the importance of proofs. This did not prevent him from making mistakes, some obvious, some subtler. But by and large, if Fermat corresponded about something he claimed he knew how to prove, it should be taken seriously – despite the fact, that only one proof was actually worked out in writing in sufficient detail, to make a reconstruction of his reasoning possible.

Fermat is most famous for his negative propositions, i.e. theorems that certain problems have no solutions. E.g. he challenges Sainte/Croix in a letter from 1638 to find

- A right-angled triangle, whose area is a square.
- Two cubes whose sum is a cube.
- Two fourth powers whose sum is a fourth power.

And he repeats these challenges to Frenicle in 1640. These problems were solved using the method of infinite descent, which he described very briefly in the following way:

> *'There is no right angled triangle in numbers whose area is a square ... if the area of such a triangle were a square, then there would be a smaller one with the same property, and so on, which is impossible.'*

The idea is very ingenious, but without details it is certainly not easy to fill in the gaps in the argument!

But Fermat was also involved in *affirmative problems*, which he actually found harder. In solving such problems he used the *method of ascent*, which was an extension of a method already described in Diophantus and known as *'Methodus vulgaris'*. He was very fond of this method:

'which has astonished the greatest of experts, in particular Mr. Frenicle'.

In fact he illustrated his method using two examples he learned from Frenicle, who sent them to him in 1641:

1) Find a right-angled triangle such that $a > b$ and such that $(a - b)^2 - 2b^2$, is a square.
2) Find a right-angled triangle such that both c and $a + b$ are squares.

We shall comment on these problems in considerable detail later on, but for the moment we just make the remark, that at first Fermat was unable to do anything about them. As he wrote to Carvaci:

'I saw no way of even tackling them'.

But a few years later he can solve them both. In 1643 he communicated the latter solution to Mersenne, the triangle being given by:

$$a = 1061652293520 \quad b = 456486027761 \quad c = 4687298610289$$

In 1644 he reports the solution of the first problem to Frenicle

$$a = 1517, \ b = 156 \ \text{ and } c = 1525$$

So the years from 1641 to 1643 were very productive for Fermat, and that's where he laid the foundation for his favorite and impressive method of ascent. It is this method that we will try to explain in the following, but apart from the fact that Fermat revealed no details, it seemed that Fermat solved them using purely algebraic techniques, which can be somewhat difficult to follow. Especially since Fermat did not possess an efficient algebraic notation and therefore had to spell his arguments in long chains of words. In stead we will follow an idea from Newton, and show how to solve such problems using geometrical insight.

Newton's insight

Newton acquired an edition of Diophantus in 1670, which may in fact be the very one published by Fermat's son thus containing the marginal notes of Fermat. Newton wrote his own comments upon the method of Diophantus, and especially he showed how to interpret these methods in geometrical terms. He established thereby that:

- Many number theoretical problems can be converted to a geometrical form dealing with *rational points* on simple algebraic curves.
- The simplest such curves, i.e. the *conics*, possess rational parameterizations, which can be used to summarize all the rational points.
- The next simplest curves, the so called *elliptic curves*, had rules for generating new rational points from old ones, which were summarized in the so-called *chord-tangent*-method.

These methods we will try to throw some light upon in what follows.

Pythagorean triples

Diophantine equations have a long story. One of the oldest problems in number theory is no doubt the problem about *Pythagorean triples*, i.e. triples of whole numbers (a, b, c) which fulfill the Pythagorean relation $a^2 + b^2 = c^2$, i.e. a, b and c are the legs and the hypotenuse of a right angled triangle. Specific solutions like the famous (3, 4, 5) has apparently existed back to the earliest records of mathematical activities. General rules for findings such triples go back at least to Pythagoras and Plato, but they are presumably much older. E.g. the Pythagoreans knew that the sum of the first n odd numbers is a square. In modern notation we therefore have:

$$1 + 3 + 5 + \ldots + (2n-1) = n^2 .$$

But it then follows that two subsequent square numbers have to satisfy the relation:

$$n^2 + (2n+1) = (n+1)^2$$

Provided we can come up with an odd number $2n+1$ which is also a square t^2 we therefore generate a triple, since we now get:

$$n^2 + t^2 = (n+1)^2, \quad \text{with } n = ½(t^2 - 1) .$$

Notice, that t is necessarily an odd number.

When $t = 3$ we get $n = 4$, i.e. we generate the triple $3^2 + 4^2 = 5^2$.
When $t = 5$ we get $n = 12$, i.e. we generate the triple $5^2 + 12^2 = 13^2$.
When $t = 7$ we get $n = 24$, i.e. we generate the triple $7^2 + 24^2 = 25^2$.

But there are other triples, where the hypotenuse does not differ by one unit from the longest leg, e.g. $8^2 + 15^2 = 17^2$. So how do we generate *all* Pythagoreans triples? That's the problem we will now solve using Newton's method!

Rational points on the unit circle

The first observation is that we can rearrange the Pythagorean identity in the following way:

$$a^2 + b^2 = c^2 \quad \Rightarrow \quad \left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1 \quad .$$

Pythagorean triples (a, b, c) therefore generate *rational points* $(x, y) = (a/c, b/c)$ on the *unit circle*:

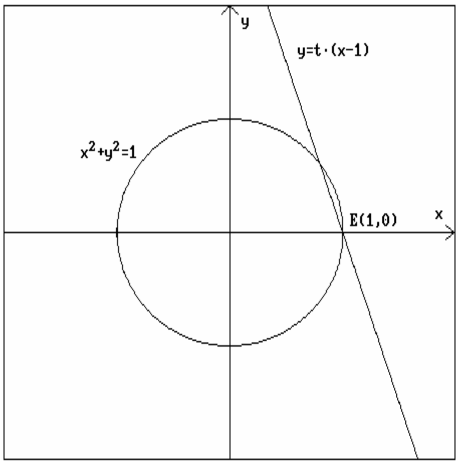
$$x^2 + y^2 = 1 \quad .$$

But this also goes the other way, since we can always find a common denominator for the rational coordinates of a rational point. I.e. we can always write it in the form (a/c, b/c), with a, b and c consequently constituting a Pythagorean triple. This is a very important observation: Typically a Diophantine problem dealing with integers can in this way be turned into a problem dealing with rational points on a simple algebraic curve.

The next observation deals with rational points on a quadratic curve, i.e. a conic section. It follows the following principle, known as the principle of *all or nothing*:

Either there are no rational points or they are all given by a rational parameterization.

We will show this in detail for the unit circle, and then comment upon the general principle afterwards.



The unit circle contains four trivial rational points, the four corner points on the axes: (±1, 0) and (0, ±1). We will select one of them as the unit point E for the unit circle. Traditionally the corner point E = (1,0) is selected. Now consider the straight lines passing through E. They give rise to the following nice observation:

> A secant passing through E will intersect the unit circle in another rational point precisely when the slope of the secant is rational.

Proof: First the *easy direction*. Assuming the secant will intersect in the additional rational point (x_0, y_0) the slope will be given by

$$t = \frac{y_0}{x_0 - 1}$$

which will therefore also be rational.

Next the *difficult direction*. If the slope t is rational it follows that the equation of the secant has rational coefficients. In fact the equation is given by $y = t\,(x–1)$ $(= tx – t)$. The secant is forced to intersect the unit circle in yet another point since it is already passing through the unit point *E*. The *x*-coordinate for this additional intersection point can be found by solving the quadratic equation obtained by substitution of the secant equation in the equation for the unit circle. But this quadratic equation must have rational coefficients, since the coefficients from both the secant equation and the equation for the unit circle are rational. Furthermore we already know one of the solutions, $x = 1$, since the secant passes through the unit point $E = (1, 0)$. But if one solution is rational so is the other. There are several ways to see this. It follows immediately from the general formula for solving the quadratic equation, since the discriminant is forced to be a square number. It also follows from the following important observation: The sum of the roots is given by $–b/a$ in terms of the coefficients *a*, *b* and *c* from the quadratic equation. But the coefficients are rational, so the same must be true of the sum. As a consequence either both roots are rational or none of them are rational. Substituting this value for x in the equation for the secant it follows that the *y*-coordinate is also rational.

In fact it is trivial to solve for the intersection required using the solve command:

$$\text{Solve}(x^2 + y^2 = 1 \text{ and } y = t*x – 1,\ \{x,y\})$$

which gives

$$x = \frac{t^2 - 1}{t^2 + 1} \quad \text{and} \quad y = \frac{-2t}{t^2 + 1} \ .$$

So the value for the *x*- and *y*-coordinate is rational as claimed, since *t* is assumed rational. ☺

Notice that we have now produced the following *rational parameterization* for the unit circle:

$$(x, y) = \left(\frac{t^2 - 1}{t^2 + 1}, \frac{-2t}{t^2 + 1} \right) \ .$$

Substituting simple rational values for the parameter t produces the classical Pythagorean triples. Since they correspond to points in the first quadrant we have to select a *negative* slope less than -1:

$$t = -2 \ \Rightarrow \ (x, y) = \left(\frac{3}{5}, \frac{4}{5} \right) \qquad t = -3 \ \Rightarrow \ (x, y) = \left(\frac{8}{10}, \frac{6}{10} \right) = \left(\frac{4}{5}, \frac{3}{5} \right)$$

$$t = -4 \Rightarrow (x, y) = \left(\frac{15}{17}, \frac{8}{17} \right) \qquad t = -\frac{3}{2} \Rightarrow (x, y) = \left(\frac{5}{13}, \frac{12}{13} \right)$$

$$t = -\frac{4}{3} \ \Rightarrow \ (x, y) = \left(\frac{7}{25}, \frac{24}{25} \right) \ldots$$

In fact we can recognize the rule for generating the Pythagorean triples by substituting the rational value $t = – u/v$, where *u* and *v* are integers, such that $u > v$:

$$t = -\frac{u}{v} \ \Rightarrow \ (x, y) = \left(\frac{\frac{u^2}{v^2} - 1}{\frac{u^2}{v^2} + 1}, \frac{-2 \cdot -\frac{u}{v}}{\frac{u^2}{v^2} + 1} \right) = \left(\frac{u^2 - v^2}{u^2 + v^2}, \frac{2uv}{u^2 + v^2} \right) \ .$$

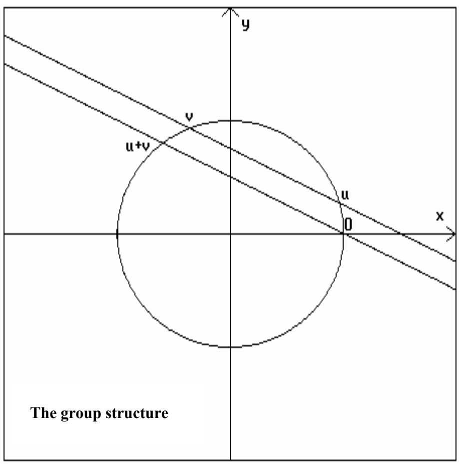
This corresponds to the Pythagorean triple: $(u^2 - v^2, \quad 2uv, \quad u^2 + v^2)$.

Remark: Notice that Pythagorean triples that have a common factor, i.e. are reducible will correspond to the same rational points. So essentially we can only hope to produce primitive Pythagorean triples, i.e. irreducible triples without a common factor. Furthermore the generators u and v – besides from being relative prime –have to have opposite parity. If both are odd they will generate a triple, where all numbers are even and hence reducible. We summarize this observation in the following famous structure formula for the primitive Pythagorean triples:

> A triple (a, b, c) is a *primitive Pythagorean triple* precisely when it can be written in the form
>
> $$(a,b,c) = (u^2 - v^2, \quad 2uv, \quad u^2 + v^2)$$
>
> with the generators (u, v) being relative prime, with $u > v$ and the generators having opposite parity.

The Group structure for the unit circle

The existence of a rational parameterization is closely associated with the group structure of the rational points on the unit circle. On the unit circle you can add angles and hence you can add points. The interesting point is that you can define this adding procedure in a *purely algebraic fashion*, that furthermore can be generalized to other families of simple algebraic curves, the so called elliptic curves. Notice first that the secant connecting two points P_u and P_v has to be parallel with the secant connecting the unit point E and the sum of the points P_{u+v}. In particular they have to have the same slope:



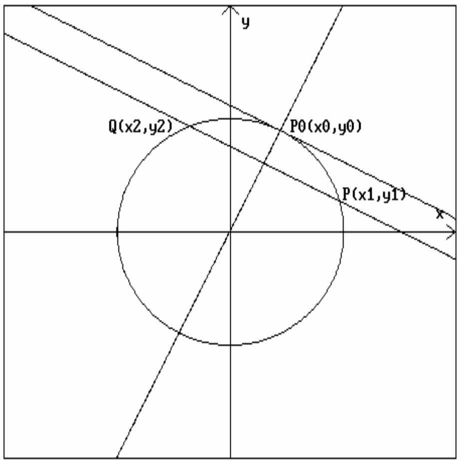
Hence we can define the sum of the two points P and Q as the intersection point between the unit circle and the secant passing through the unit point E, having the same slope as the secant PQ.

This is where the rational parameterization comes into the game. The sum $P + Q$ will have the slope $\frac{y_2 - y_1}{x_2 - x_1}$ relative to the unit point E, i.e. the parameter t from the rational parameterization has precisely the rational value:

$$t = \frac{y_2 - y_1}{x_2 - x_1}.$$

So the group structure on the unit circle is respecting the rational points.

Substituting the above parameter into the equations for the parameterizations actually makes it possible to compute the coordinates for the point $P + Q$ in a purely algebraic fashion, although CAS-programs will have great difficulties reducing the end result to its most simple form.



We will therefore restrict ourselves to a discussion of the *duplication formula*, where we simply double the angle for a point $P(x_0,y_0)$ on the unit circle. The tangent is perpendicular to the radius, so the slope of the tangent is given by

$$\alpha_{\tan gent} = -\frac{x}{y} \quad .$$

But since the slope of the tangent at the point $P = (x_0, y_0)$ is given by $-x_0/y_0$ the duplicate point $2P$ must correspond to the parameter value $t = -x_0/y_0$. Consequently the duplicate point $2P$ is given by:

$$(x,y) = \left(\frac{t^2-1}{t^2+1}, \frac{-2t}{t^2+1}\right) = \left(\frac{\frac{x_o^2}{y_0^2}-1}{\frac{x_o^2}{y_0^2}+1}, \frac{-2\cdot(-\frac{x_0}{y_0})}{\frac{x_o^2}{y_0^2}+1}\right) = \left(\frac{x_0^2-y_0^2}{x_0^2+y_0^2}, \frac{2x_0y_0}{x_0^2+y_0^2}\right) \quad .$$

But since the point P is a point on the unit circle we know that $x_0^2 + y_0^2$ reduces to 1. Hence the formula for $2P$ simplifies to:

$$2P = (x, y) = (x_0^2 - y_0^2, 2x_0y_0)$$

Once again we can throw in a rational point $(x, y) = (a/c, b/c)$ corresponding to a primitive Pythagorean triple (a, b, c). Consequently the duplicate point $2P$ gets the coordinates:

$$\left(\frac{a^2-b^2}{c^2}, \frac{2ab}{c^2}\right)$$

This corresponds to the primitive Pythagorean triple $(a^2-b^2,\ 2ab\ , c^2)$ and thus we have also found a duplication formula for primitive Pythagorean triples! E.g. the primitive Pythagorean triple (3, 4, 5) is duplicated to $\left(4^2-3^2, 2\cdot3\cdot4, 5^2\right)=(7, 24, 25)$. Notice the similarity with the structure formula: The generators for the duplicate triple are precisely the legs of the original triple. It follows that you can reverse the process precisely when the hypotenuse is a square.

This observation makes primitive Pythagorean triples with a *square hypotenuse* so much more interesting!

Fermat's cubic problem

Let us now turn to *Fermat's cubic problem* dealing with splitting two cubes into a cube, i.e. this time we are looking for integers a, b and c fulfilling the relation

$$a^3 + b^3 = c^3 \quad .$$

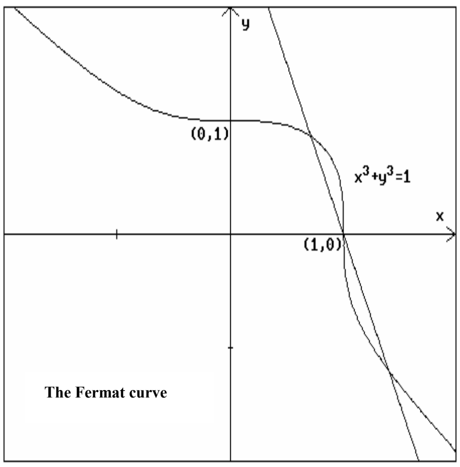
We will try solving this problem with a similar technique. The first step is a rearrangement of the equation to the form

$$a^3+b^3=c^3 \Rightarrow \left(\frac{a}{c}\right)^3+\left(\frac{b}{c}\right)^3=1 \quad .$$

Non-trivial solutions (a, b, c) to the cubic Fermat equation thus corresponds to non-trivial rational points $(x, y) = (a/c, b/c)$ on the cubic curve

$$x^3 + y^3 = 1 \quad .$$

If it were possible to find a rational parameterization of this curve we would of course control all the rational solutions. We could therefore try the same idea as before. There are two trivial rational points on the Fermat curve: (1,0) and (0,1).



Looking for the secant passing through (1,0) we could therefore investigate additional intersection points. But unfortunately there will in general be two additional intersection points. And even if the slope is rational – forcing the resulting cubic equation to have rational coefficients – that only allows us to conclude, that the sum of the three roots is rational – the two additional roots themselves need not be rational. To guarantee an additional rational intersection point, it is therefore necessary to know *two* rational points on the cubic curve in order to compute the additional intersection point with the secant passing through these two points.

But we already have two rational points (1,0) and (0,1). So why not combine these to another rational point? Unfortunately it does not work. The equation for the secant becomes $x + y = 1$. Substituting this into the equation for the Fermat curve the cubic equation unfortunately degenerates into a quadratic equation:

$$x^3 + y^3 = 1 \Rightarrow x^3 + (1-x)^3 = 1 \Rightarrow -3x + 3x^2 = 0$$

(corresponding simply to the two unit points , i.e. $x = 0$ i.e. $x = 1$).

We could also be clever and use one of the unit points as a *double point* on the curve, i.e using the tangent passing through the point. That would also have an equation with rational coefficients and hence intersect in a third point with rational coordinates. In principle we need therefore only know one rational point, because we can the shoot along the tangent to produce another rational point!

But once again the idea fails. Both unit points are actually *inflection points*. As a consequence they correspond to triple points on their tangents, so they do not produce additional solutions. What worked with such a great success for the unit circle completely fails for the cubic Fermat curve!

We can however easily find a *duplication formula* for the cubic Fermat curve. It is done in exactly the same way as for the unit circle. This time the duplicate point $2P$ is defined as the additional intersection point with the tangent through P. It will make it a little easier to symmetrise the problem. So instead of looking at the equation

$$a^3 + b^3 = c^3 \quad .$$

we will move the last term to the same side as the other and *change the sign* of c. The problem then is to find integers a, b and c, such that the sum of their cubes vanishes:

$$a^3 + b^3 + c^3 = 0 \quad .$$

The corresponding rational point then gets the equation

$$x^3 + y^3 + 1 = 0 \quad .$$

In this equation we can isolate y to get the equation on its functional form

$$y = \sqrt[3]{-x^3 - 1} \quad .$$

Using a Taylor expansion we can then compute the equation for the tangent at a point on the cubic curve $P(x_0, y_0)$:

$$y = taylor(\sqrt[3]{-x^3-1},\ x,\ 1, x_0)$$

Computing the intersection point with the cubic curve then produces the additional solution:

$$x = \frac{-x_0 \cdot (x_0^{\ 3}+2)}{2x_0^{\ 3}+1} \quad \text{and} \quad y = \ldots$$

Substituting $x = a/c$ and $y = b/c$ we get the following simple formula for x:

$$x = \frac{-a\cdot(a^3+2c^3)}{(2a^3+c^3)\cdot c}$$

and a similar formula, albeit more complicated for y. But focusing on the formula for $x = A/C$ we can already read off the following part of the duplication formula:

$$\begin{aligned} A &= -a\cdot(a^3+2c^3) = -a\cdot(-b^3-c^3+2c^3) = -a\cdot(c^3-b^3) = a\cdot(b^3-c^3) \\ B &= \ldots \\ C &= (2a^3+c^3)\cdot c = c\cdot(2a^3-a^3-b^3) = c^3\cdot(a^3-b^3) \end{aligned}$$

For symmetry reasons we therefore must have the following duplication formula:

$$(A,\ B,\ C) = (a\cdot(b^3-c^3),\ \ b\cdot(c^3-a^3),\ \ c\cdot(a^3-b^3))$$

So if (a, b, c) is any solution – whether rational or not – to the cubic equation $a^3 + b^3 = c^3$, then the same holds for the duplicate triple (A, B, C). You can check that this is the case by computing $A^3+B^3+C^3$ and show that it factorizes with one of its factors being $a^3+b^3+c^3$. This is something that is fun to do using a CAS-system. But imagine you had to do the factorization using paper and pencil!

A famous challenge

Many of the Diophantine problems that Fermat studied were concerned with right-angled triangles, which lead to Pythagorean triples. As we have seen we can write down an explicit structure formula generating all Pythagorean triples, so we know them all! But you can still look for Pythagorean triples with special properties. As we have seen it is easy to find Pythagorean triples with one of the sides being a square number, such as 3 – 4 – 5 and 7 – 24 – 25 . But remarkably enough it seems like no one will contain *two* squares. You therefore have to relax your requirements. E.g. you could ask for the hypotenuse being a square and then try to impose further restrictions. Thus you can ask for a Pythagorean triple where both the hypotenuse and the difference of the legs is a square – or a Pythagorean triple where both the hypotenuse and the sum of the legs is a square. These are essentially Frenicle's challenges to Fermat, the challenges that Fermat was particularly proud of having solved. You might also ask if there are right-angled triangles, where not only the hypotenuse, but also the area is a square etc.

We are now going to investigate the particular problem of finding a Pythagorean triple *where both the hypotenuse and the sum of the legs is a square*! First we must reformulate the problem into a geometric setting. We start by observing, that we can write both c and $a + b$ as squares, whereas we know nothing about the difference $a - b$:

$$c = p^2 \ , \ a + b = q^2 \text{ and } a - b = r.$$

So the Pythagorean triple (a, b, c) has been replaced by the triple (p, q, r) which satisfies the *extended Pythagorean identity*:

$$(a+b)^2 + (a-b)^2 = 2a^2 + 2b^2 = 2c^2 \quad \Rightarrow \quad q^4 + r^2 = 2p^4 \quad \Rightarrow \quad r^2 = 2p^4 - q^4$$

Dividing by q^4 this can be rearranged in the usual manner:

$$\left(\frac{r}{q^2}\right)^2 = 2\left(\frac{p}{q}\right)^4 - 1 \quad \Rightarrow \quad y^2 = 2x^4 - 1 \qquad \text{with } y = \frac{r}{q^2} \text{ and } x = \frac{p}{q} \ .$$

However from the triangular inequality we also get the following constraints

$$a - b < c < a + b \quad \Leftrightarrow \quad r < p^2 < q^2 \ .$$

From these it follows that both x and y have to be less than 1: $0 < x, y < 1$.

If on the other hand we can find a rational point (x, y) on the algebraic curve $y^2 = 2x^4 - 1$ satisfying the constraints 0 < x, y < 1 then we can put (by a suitable choice of denominators)

$$x = \frac{p}{q} \text{ and } y = \frac{r}{q^2} \ .$$

Hence the corresponding Pythagorean triple (a, b, c) given by

$$p^2 = c \ , \ q^2 = a + b \quad \text{and} \quad r = a - b \quad \textit{i.e.} \quad c = p^2 \ , \ a = \frac{r + q^2}{2} \quad \text{and} \quad b = \frac{q^2 - r}{2} \ .$$

This is the key to finding a solution to the challenge proposed by Frenicle!

If (a, b, c) is a primitive Pythagorean triple satisfying the conditions:
- The hypotenuse c is a square.
- The sum of the legs, $a + b$, is a square.

Then we can find a rational point (x, y) on the elliptic curve $y^2 = 2x^4 - 1$ satisfying the constraint $0 < x,y < 1$.
The correspondence is given by

$$c = p^2, \ a = \frac{q^2 + r}{2}, \ b = \frac{q^2 - r}{2} \qquad \text{and} \qquad x = \frac{p}{q}, \ y = \frac{r}{q^2}$$

Remark: If instead you tried to find a Pythagorean triple (a, b, c) where both the hypotenuse c and the *difference* between the legs $a - b$ are squares you can similarly show that they correspond to a rational point (x, y) on the *same* algebraic curve $y^2 = 2x^4 - 1$. But this time it satisfies the constraint x, y >1 .

Remark: To see why the latter problem is equivalent with Frenicle's first challenge, we notice, that because the hypotenuse is a square the generators (u, v) for the Pythagorean triple (a, b, c) actually form the legs of another right angled triangle (u, v, p) obtained by bisection!):

$$u^2 - v^2 = a, \quad 2uv = b, \quad u^2 + v^2 = c = p^2$$

But the difference between the original legs should also be a square, i.e. the generators must satisfy the relation:

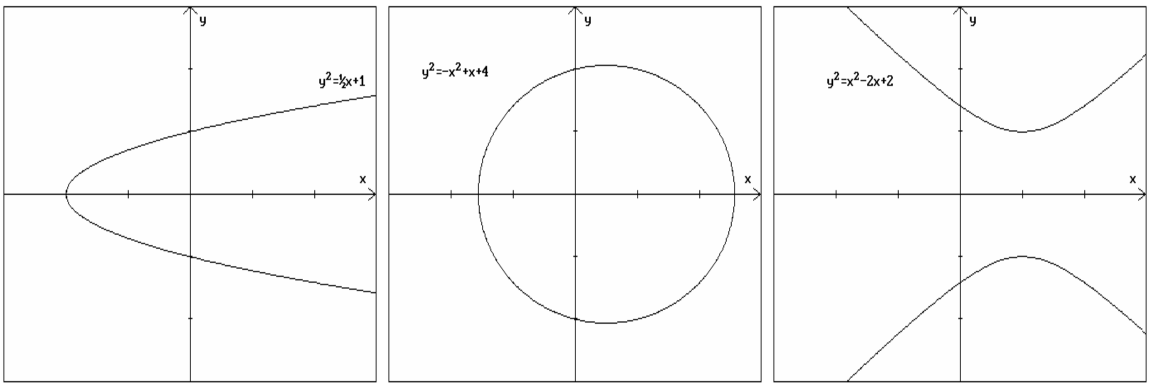
$$q^2 = a - b = u^2 - v^2 - 2uv = (u - v)^2 - 2v^2$$

So the *root triangle* (u, v, p) precisely solves Frenicle's problem!

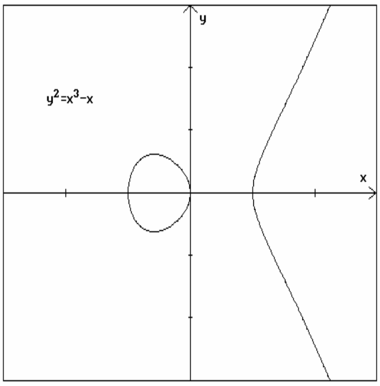
A summary of Newton's chord-tangent method

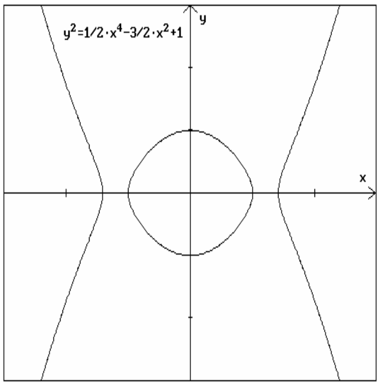
As we have seen several Diophantine problems end up studying rational points on simple algebraic curves. It pays to classify such curves in the following way:

> If the algebraic curve has an equation of the form
>
> $y^2 = p_1(x)$ (where p_1 denotes a polynomial of degree 1)
>
> or
>
> $y^2 = p_2(x)$ (where p_2 denotes a polynomial of degree 2)
>
> it is a *conic section* (in fact it is a parabola in the first case and an ellipse or a hyperbola in the second case).



> If the algebraic curve has an equation of the form
>
> $y^2 = p_3(x)$ (where p_3 denotes a polynomial of degree 3)
>
> or
>
> $y^2 = p_4(x)$ (where p_4 denotes a polynomial of degree 4)
>
> it is called an *elliptic curve* (cubic if the degree is 3 and quartic if the degree is 4).





Remark: The name is a little peculiar, since the elliptic curves are not directly related to ellipses. It turned out however that curves from the above family naturally pops up if you investigate the arc length of an ellipse – hence the name!

Diophantine problems related to rational points on a conic section are considered *trivial*. There is a complete theory for how to solve such problems: Either there are no rational points or the conic section has a rational parameterization comprising all the rational points. It all follows along the same line as we have been discussing in the case of the unit circle.

Diophantine problems related to rational points on an elliptic curve are considered *hard* – but within reach. There exists a huge theory for the behavior of such points although it is by no means complete. Provided the elliptic curve is not degenerate (i.e. does not contain cusps or double points or isolated points) it will not have a rational parameterization. But there do exist duplication formulas as well as addition formulas which will allow us to systematically build up all the rational solutions. It is precisely this structure that is revealed in the chord tangent method going back to Newton.

Diophantine problems related to rational points on algebraic curves of even higher degree (or rather higher genus) are considered extremely difficult. Almost nothing is known of their general behavior.

Cubic elliptic curves

We will start by examining cubic elliptic curves $y^2 = ax^3 + bx^2 + cx + d$. We can then try to combine rational points on the elliptic curve thus producing new rational points. This requires two rational points to begin with: $P_1(x_1,y_1)$ and $P_2(x_2,y_2)$. Looking at the chord passing through these two points $y = Ax + B$, we notice that it gets a *rational slope*

$$A = \frac{y_2 - y_1}{x_2 - x_1}$$

The usual formula for the intercept B

$$B = y_1 - ax_1$$

shows that B is rational as well. So the equation for the chord has rational coefficients. We now intersect the chord with the elliptic curve $y^2 = ax^3 + bx^2 + cx + d$. This will lead us to a cubic equation

$$(Ax + B)^2 = ax^3 + bx^2 + cx + d$$

and since we already know two solutions x_1 and x_2 there has to be and additional root of this equation x_3. But the sum of the roots is clearly rational. Hence the third root is also rational. Substituting x_3 into the chord equation we find the y-coordinate $y_3 = Ax_3 + B$. Hence y_3 is also rational. So using the chord we have succeeded in *combining* the two rational points P_1 and P_2 into another rational point $P_3 = P_1 \& P_2$.

Should the two rational points be identical i.e. represent a double point all we have to do is to replace the chord with the tangent passing through this point.

An example from Diophantus

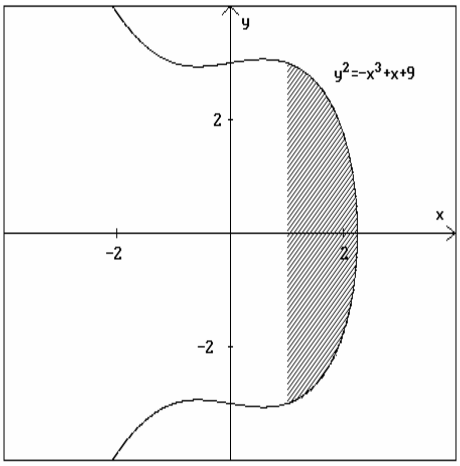
We will exemplify Newton's method using the following problem from Diophantus (but stating it in a modern language!)

Decompose 6 as a sum of two rational numbers, such that their product is a cube minus its root.

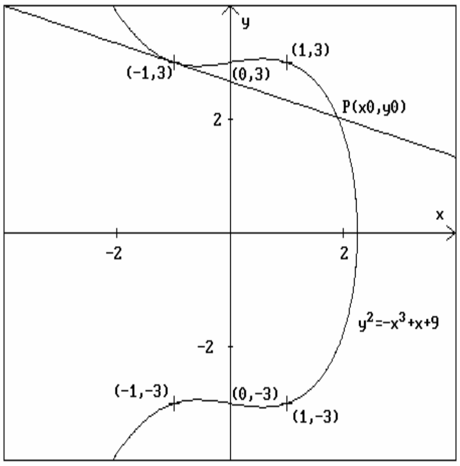
Decomposing 6 as the sum of $3 + y$ and $3 - y$, we must therefore be able to write their product as $x^3 - x$, i.e.:

$$(3 + y)(3 - y) = x^3 - x \quad \Rightarrow \quad 9 - y^2 = x^3 - x \quad \Rightarrow \quad y^2 = -x^3 + x + 9$$

We are thus looking for non-trivial rational points satisfying the equation $y^2 = -x^3 + x + 9$ as well as the constraints $x > 1$ and $|y| < 3$.



Notice the trivial solutions: (±1,±3) and (0,±3). None of these are admissible by themselves, but they can be used to generate admissible solutions. For this purpose we have to use chords or tangents. In this case chords will not work, so we will use tangents to generate solutions satisfying the appropriate constraint. Looking at the tangent associated with the rational point (–1,3) we succeed:



Clearly the equation for the tangent is found using the Taylor command:

$$y = taylor(\sqrt{-x^3+x+9},\ x,\ 1,\ -1) = -\frac{1}{3}\cdot x + \frac{8}{3}$$

Solving for the intersection between the tangent and the elliptic curve we then get: $x = \dfrac{17}{9}$ and $y = \dfrac{55}{27}$. Hence we can decompose 6 as the sum of the following two rational numbers

$$3 + y = \frac{136}{27} \quad \text{and} \quad 3 - y = \frac{26}{27}$$

whose product has the required structure

$$\frac{3536}{729} = \left(\frac{17}{9}\right)^3 - \frac{17}{9} \qquad ☺$$

Quartic elliptic curves

Finally we turn to the case of quartic elliptic curves

$$y^2 = ax^4 + bx^3 + cx^2 + dx + e \ .$$

This time a chord is of no use since we will need *three* rational points if we want to generate an additional rational point. In stead we use the parabola passing through the three rational points

$$P_1(x_1,y_1),\ P_2(x_2,y_2) \text{ and } P_3(x_3,y_3).$$

As is well known there will be precisely one parabola $y = Ax^2 + Bx + C$ passing through three points provided these points do not fall on a straight line. Ignoring these exceptions we proceed with the parabola. The coefficients A, B and C can be found solving the following linear system of equations

$$y_1 = Ax_1^2 + Bx_1 + C$$
$$y_2 = Ax_2^2 + Bx_2 + C$$
$$y_3 = Ax_3^2 + Bx_3 + C$$

Hence the coefficients are rational.

Intersecting this parabola with the elliptic curve $y^2 = ax^4 + bx^3 + cx^2 + dx + e$ *we* this time obtain a quartic equation $(Ax^2 + Bx + C)^2 = ax^4 + bx^3 + cx^2 + dx + e$. Since we already knows three solutions x_1, x_2 and x_3 to this equation, there has to be a *fourth solution* x_4. And since the sum of the roots is rational, the same must be true for the last root. Substituting x_4 into the equation for the parabola we get the corresponding y-coordinate: $y_4 = Ax_4^2 + Bx_4 + C$. Hence y_4 is also rational. Proceeding in this way we have succeeded in *combining* the three rational points P_1, P_2 and P_3 into a fourth rational point $P_4 = P_1 \,\&\, P_2 \,\&\, P_3$.

Should some of the given points be identical we just need to interpret the parabola suitably. With a *double point*, the parabola will have the *tangent* in common with the elliptic curve, i.e. they will have the *same first derivative*. With a *triple point* the parabola will further more have the *curvature* in common with the elliptic curve, i.e. they will have the *same second derivative*.

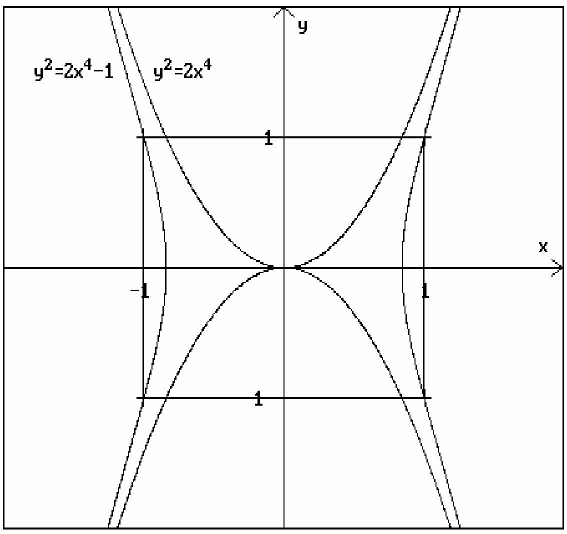
So if we only know a *single* rational point $P_0(x_0, y_0)$ we use it as a *triple point*. Provided the elliptic curve has the tangent slope y_0' (necessarily being rational) and the second derivative y_0'' (also necessarily being rational) at this point P_0 we thus use the parabola with the equation:

$$y = \tfrac{1}{2}y_0''\,(x - x_0)^2 + y_0'\,(x - x_0) + y_0$$

Intersecting this with the elliptic curve we thus find *the triplication point* $P_0 \,\&\, P_0 \,\&\, P_0$. In this way we can build up a chain of solutions – unless we have bad luck and just keep generating the same points over and over.

Solving Fermat's challenge

As we have seen this deals with the quartic elliptic curve $y^2 = 2x^4 - 1$, having the trivial solutions $(\pm 1, \pm 1)$. Due to the high degree of symmetry any solution is actually related to three other solutions through reflections in the x- and/or y-axis. This is because the equation is even in both coordinates x and y, so that we are free to change the signs. The curve furthermore has the double parabola $y = \pm\sqrt{2}\,x^2$ as an asymptote:



But we are really only interested in rational points *within the unit square*! However it will be necessary to combine our way throughout the whole curve. We have four related starting points, so nothing will come out of combining them with each other. Instead we will use one of them as a triple point $P(1,1)$. So we have to calculate the Taylor polynomial of degree 2, i.e. the quadratic curve passing through the same point with the same slope and curvature. But this is easy using a CAS-system. We start by isolating y and then use the Taylor command:

$$y = taylor(\sqrt{2x^4 - 1}, x, 2, 1) \ .$$

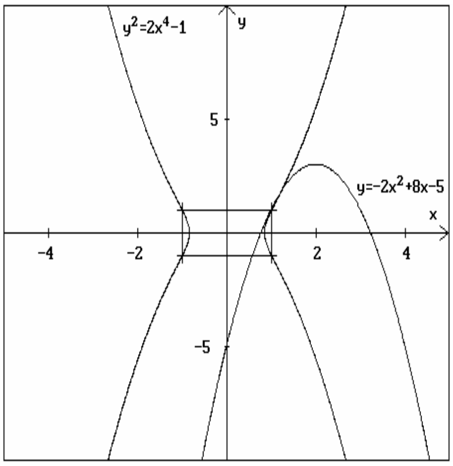
The result is the quadratic curve

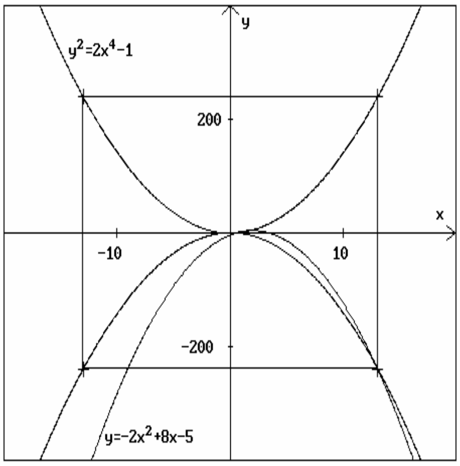
$$y = -2(x-1)^2 + 4(x-1) + 1 = -2x^2 + 8x - 5 \ .$$

We must therefore find the additional intersection point between this parabola and the quartic elliptic curve using a solve command. The result is

$$x = 13 \text{ and } y = -239 \ .$$

So the additional point is $Q(13,-239)$ and we now know four more rational points on the curve.





Remark. The two illustrations use very different scales, since you cannot put all details into one single picture.

Unfortunately they all fall *outside* the unit square. We must therefore try again. We will start by switching to the point $Q'(13,239)$ in the first quadrant. We then need to work our way back to the unit square. Using Q' as a triple point unfortunately does not help. So we will have to combine P and Q'. This can be done in essentially two different ways: Either we can use P as a double point and Q' as a single point or vice-versa. It turns out that the first solves the first of Frenicle's challenges, while the second solves his second challenge. This time the parabola therefore only have to be tangential at the point Q'. We therefore build its equation in the following way

$$y = taylor(\sqrt{2x^4 - 1}, x, 1, 13) + a \cdot (x-13)^2 = a \cdot x^2 + \left(\frac{8788}{239} - 26a\right) \cdot x + 169a - \frac{57123}{239}$$

We then need to adjust the coefficient a so that it also passes through (1,1), i.e. to solve the equation

$$solve(y = a \cdot x^2 + \left(\frac{8788}{239} - 26a\right) \cdot x + 169a - \frac{57123}{239}, a) \mid x = 1 \text{ and } y = 1$$

which produces the solution $a = \dfrac{24287}{17208}$. So this time the parabola gets the equation

$$y = \frac{24287 \cdot x^2}{17208} + \frac{637 \cdot x}{8604} - \frac{8353}{17208} \quad .$$

Computing the intersection points between this parabola and the quartic elliptic curve we this time get:

$$x = -\frac{2165017}{2372159} \quad \text{and} \quad y = \frac{3503833734241}{5627138321281}$$

So we did it! We landed right inside the unit square and we have essentially solved Fermats challenge! The x-value already gives us the hypotenuse as well as the sum of the legs:

$$x = \frac{p}{q} \quad \Rightarrow \quad c = p^2 = (2165017)^2 = 4687298610289$$

$$a + b = q^2 = (2372159)^2 = 5627138321281$$

Notice that the denominator of the y-coordinate precisely coincides with q^2. From the y-value we then obtain the difference between the two legs a and b:

$$a - b = r = 3503833734241$$

Hence the legs are given by

$$a = \frac{q^2 + r}{2} = 1061652293520 \quad b = \frac{q^2 - r}{2} = 4565486027761$$

The final words

We will let Fermat have the last word. In a letter to Huygens he concludes:

> *'Such is in brief the tale of my musings on numbers. I have put it down only because I fear that I shall never find the leisure to write out and expand properly all these proofs and methods. Anyway this will serve as a pointer to men of science for finding for themselves what I am not writing out, particularly if Mr. Carvaci and Frenicle communicate to them a few proofs by descent that I have sent them on the subject of some negative propositions. Maybe posterity will be grateful to me for having shown that the ancients did not know everything ...'*

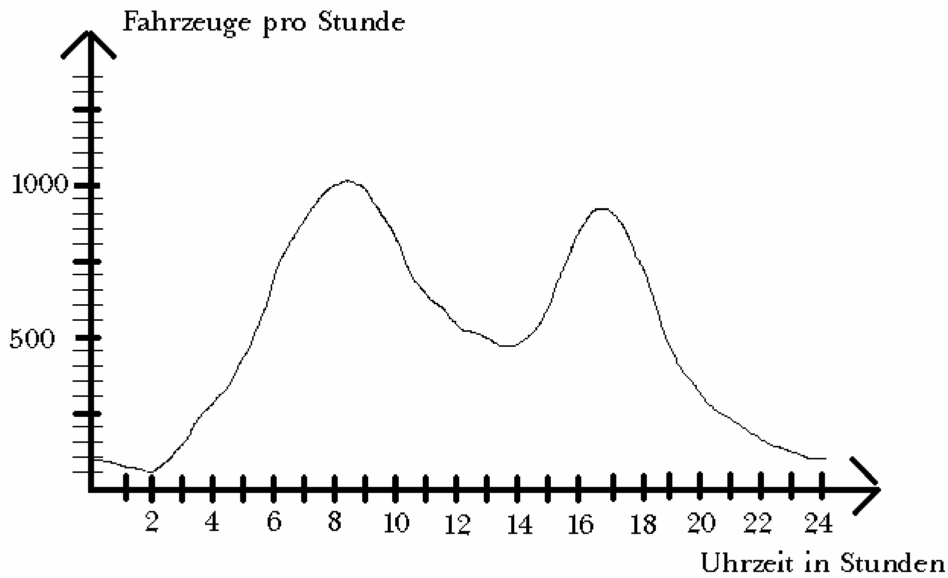
During a seminar for Austrian teachers a group of us tried to collect some stuff for the ACDCA-collection of CAS-problems (http://www.acdca.ac.at/). Thomas Himmelbauer designed a traffic density problem for the TI-92/Voyage200. We exchanged our problems and gave comments. I liked this problem at the first sight and we worked together on modelling the two peaked density function – first on the TI and then I switched to DERIVE using the slider bars and other features. I like this problem because it is a fine example for applying defined and undefined integral. See here our final version, Josef

A Traffic Density Problem

Thomas Himmelbauer (& Josef Böhm), ACDCA, Austria

Before opening the tunnel of the Semmering road (October 2004) the vehicles which passed Spital am Semmering were counted (for a period of 24 hours). The diagram shows the number of cars passing per hour against time of one day during the counting period.

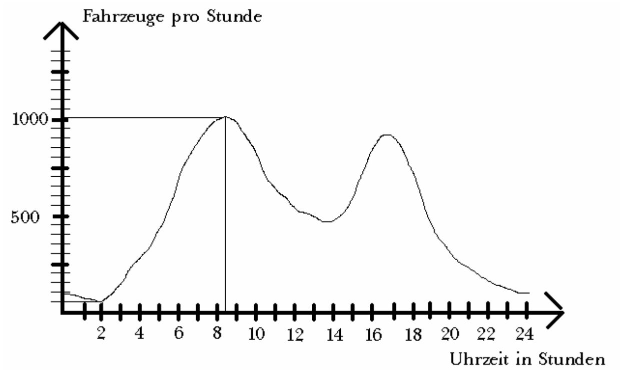
a) Find the time of the day when the maximum/minimum number of cars per hour was counted.
b) Give reasons why you can be sure that the total number of cars passing Spital during these 24 hours must be less than 24 000 and greater than 1 200.
c) Calculate the total number of cars by means of a lower sum and an upper sum. Divide the interval into 24 equal parts. The numerical values must be read off from the graph.
d) Find an appropriate model function $a(t)$ to describe the traffic density (number of vehicles/hour against time of the day t).
e) Using Calculus find the turning points of this function. Can you calculate all extreme values by means of Calculus?
f) Find the value of the 1st derivative of $a(t)$ for $t = 7$ and $t = 21$. Explain the results.
g) Find the total number of vehicles passing this day using the trapezium method, the midpoint method and the Simpson method. Use 60 intervals.
h) Find the total number of all vehicles using the fundamental theorem of Calculus.
i) Which is the function describing the cumulated number of vehicles for each time of the day?
j) Find the average number of vehicles passing per hour.
k) Do you know other distributions which show more than one peak?
l) How to extend the model for a series of days?

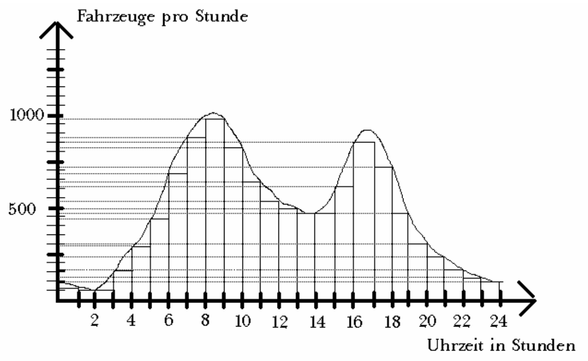


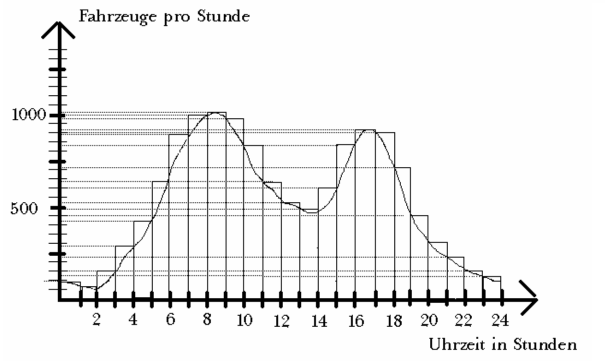
a) The requested times can be read off from the diagram. The maximum number of approximately 1000 cars/h passes at 8 30 am and the minimum number of 50 vehicles/h at 2 am.

b) The total number must be less than $24 \cdot 1000 = 24000$ and the minimum number must be greater than $24 \cdot 50 = 1200$ vehicles.

c) For c) and d) we try to find an approximation drawing and measuring the rectangles.







Lower and upper sum:

Δt	1	1	1	1	1	1	1	1	1	1	1	1
number	60	50	50	160	390	440	675	875	990	820	640	540
Δt	1	1	1	1	1	1	1	1	1	1	1	1
number	500	475	475	610	850	725	475	300	240	160	110	100

$$\sum_{i=1}^{24} f(x_i) \cdot \Delta x_i = 1 \cdot (60 + 50 + 50 + 160 + 390 + 440 + 675 + 875 + 990 + 820 + 640 +$$

$$+ 540 + 500 + 475 + 475 + 610 + 850 + 725 + 475 + 300 + 240 + 160 + 110 + 100) = 10235$$

Δt	1	1	1	1	1	1	1	1	1	1	1	1
Anzahl	100	75	150	280	420	640	940	1000	1020	990	825	640
Δt	1	1	1	1	1	1	1	1	1	1	1	1
Anzahl	525	500	600	825	910	900	610	450	310	240	150	120

$$\sum_{i=1}^{24} f(x_i) \cdot \Delta x_i = 1 \cdot (100 + 75 + 150 + 280 + 420 + 640 + 940 + 1000 + 1020 + 990 + 825 +$$

$$+ 640 + 525 + 500 + 600 + 825 + 910 + 900 + 610 + 450 + 310 + 240 + 150 + 120) = 13220$$

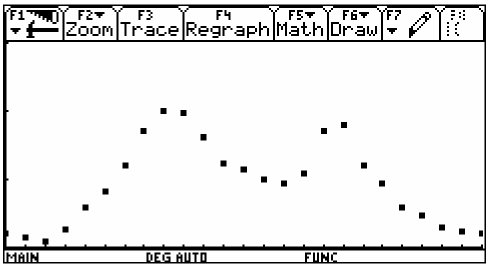
$$\text{Average: } \frac{13220 + 10235}{2} \approx 11728$$

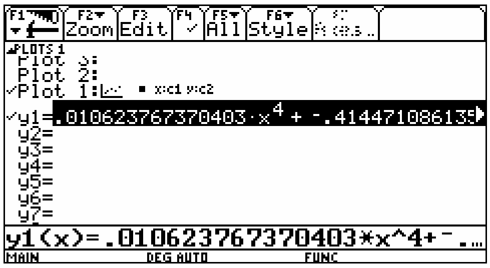
Questions a) – c) didn´t need any technology except ruler, paper and pencil. This is now the moment when technology comes into play. For the following we read off the "traffic density values" from the graph and either put them into a data sheet on the TI or put them into a matrix with DERIVE. I will start with the TI-process:

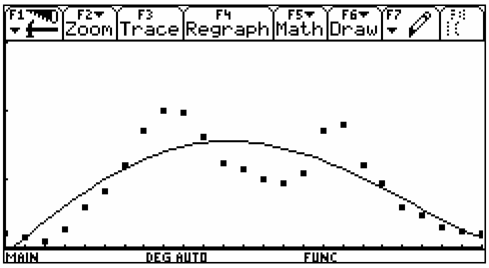
Additional question for the students: Right- and Leftsum have the same value? Why can this happen? How to obtain these to sums with one single command?

T	0	1	2	3	4	5	6	7	8	9
n	100	75	50	125	290	410	610	860	1000	990
T	10	11	12	13	14	15	16	17	18	19
n	810	620	580	500	475	550	850	900	600	475
T	20	21	22	23	24					
n	300	240	150	120	100					

d) The scatterplot of the data points shows a form with two peaks and we have the idea that a quartic might fit (the TI does not provide polynomial fits of higher degree!).



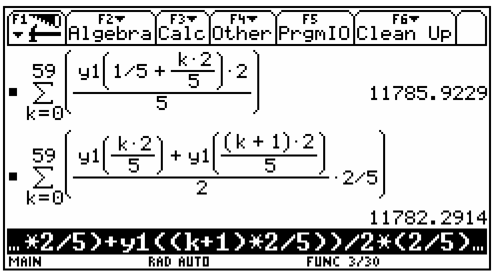


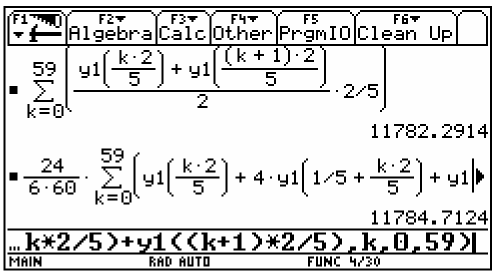


It turns out that the quartic regression does not deliver a good model function, but that the value for the integral seems to give a useful estimation for the total number of vehicles.

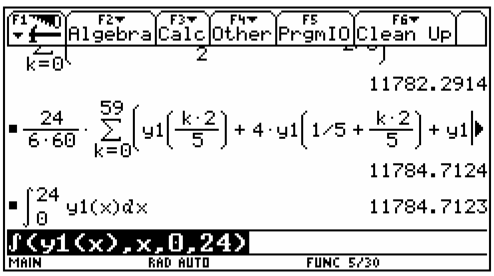
e) and f) Cannot be worked through with this bad model – see the DERIVE-version and the TI-extension.

g) If we accept the quartic being good enough for integrating the sum, then it makes sense to apply numeric methods: Midpoint- and Trapezium Method (left) and Simpson Method (right).

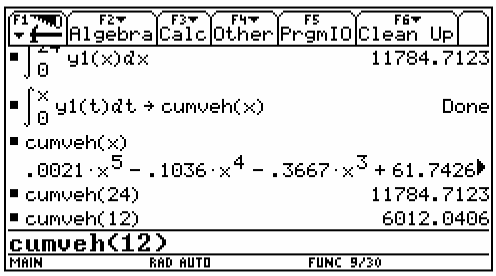




h) This is the defined integral:



i) This is the undefined integral:



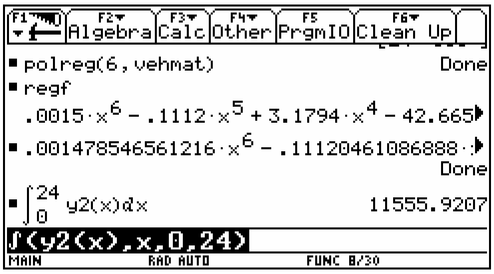
We must still be aware of the fact that the model function is not the best one!!

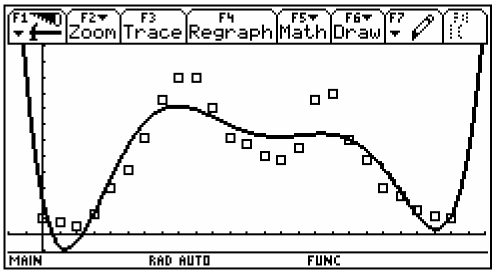
j) The average number per hour is approximately 490 vehicles.

k) What do **you** think?

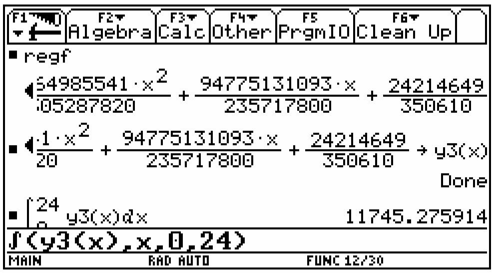
l) See the DERIVE procedure.

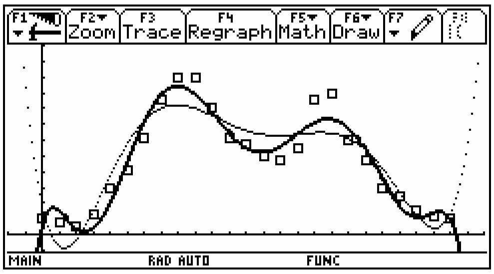
But first I´d like to add a possible extension using the TI-program polreg from DNL#63, which overcomes the quartreg-restriction:





The polynomial regression function of degree 6 looks a bit better, but the estimation of the number of vehicles (area under the curve) is obviously worse. We try a polynomial of degree 8:





This is a again bit better, but it must be said that we did not have polreg at our disposal when we designed the example two years ago ...

Suddenly we had the idea that each of the peaks reminds us on the bell curve of the normal distribution. We took again the TI and tried to find a bell for the morning and another one for the afternoon by varying the parameters ...

In most of the statistics textbooks you can find a sketch of a "bimodal distributions" among the continuous distributions, but without presenting any example! Here is one.

I did the same with DERIVE varying the parameters using the *slider bars*. This is a short report on the DERIVE procedure: You see the last two rows of matrix veh (the data points from above) and the polynomial fit of degree 10:

$$\begin{bmatrix} 23 & 120 \\ 24 & 100 \end{bmatrix}$$

Answers a) - c) are identical to the TI-version

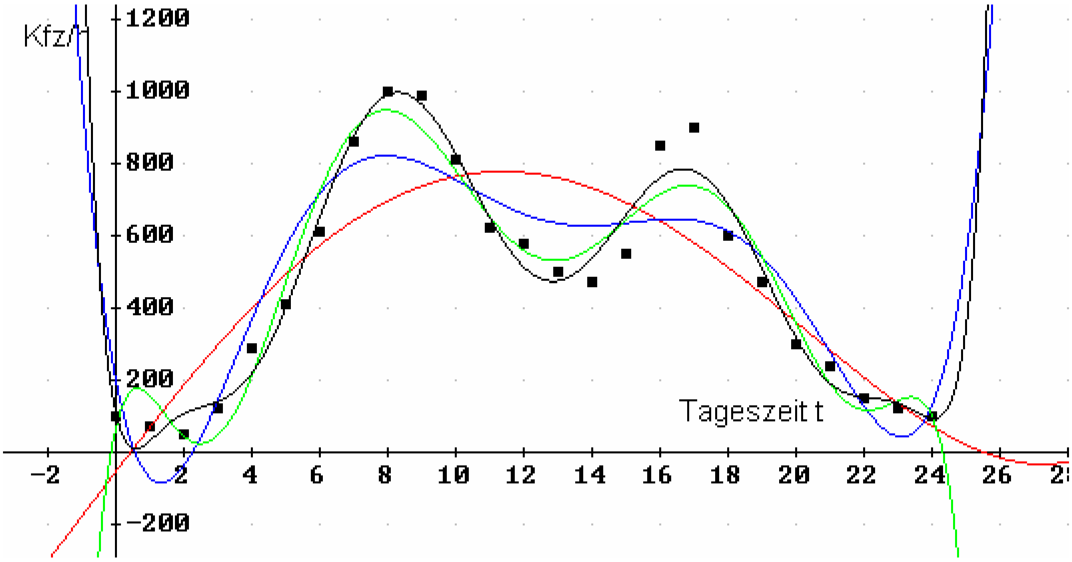
We try a polynomial of degree 10 fit:

#4: $\text{pol10} := \text{FIT}([x,\ a\cdot x^{10} + b\cdot x^9 + c\cdot x^8 + d\cdot x^7 + e\cdot x^6 + f\cdot x^5 + g\cdot x^4 + h\cdot x^3 + i\cdot x^2 + j\cdot x + k],\ \text{veh})$

#5: $\text{pol10} := 4.8431751\cdot 10^{-7}\cdot x^{10} - 6.0325023\cdot 10^{-5}\cdot x^9 + 0.0031706072\cdot x^8 - 0.091285174\cdot x^7 + 1.5637329\cdot x^6 - 16.213440\cdot x^5 + 99.334364\cdot x^4 - 339.55054\cdot x^3 + 600.38404\cdot x^2 - 423.20810\cdot x + 109.25742$

#6: TABLE(pol10, x, 0, 24, 0.01)

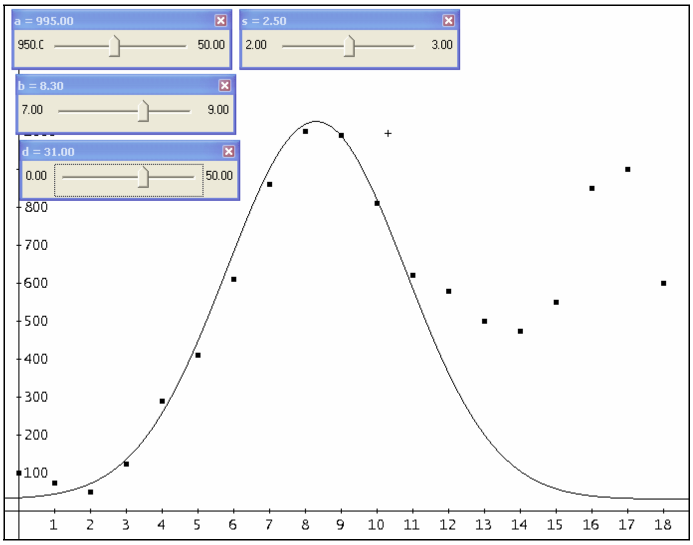
d) The interesting part begins now. First of all see some polynomial regression curves:



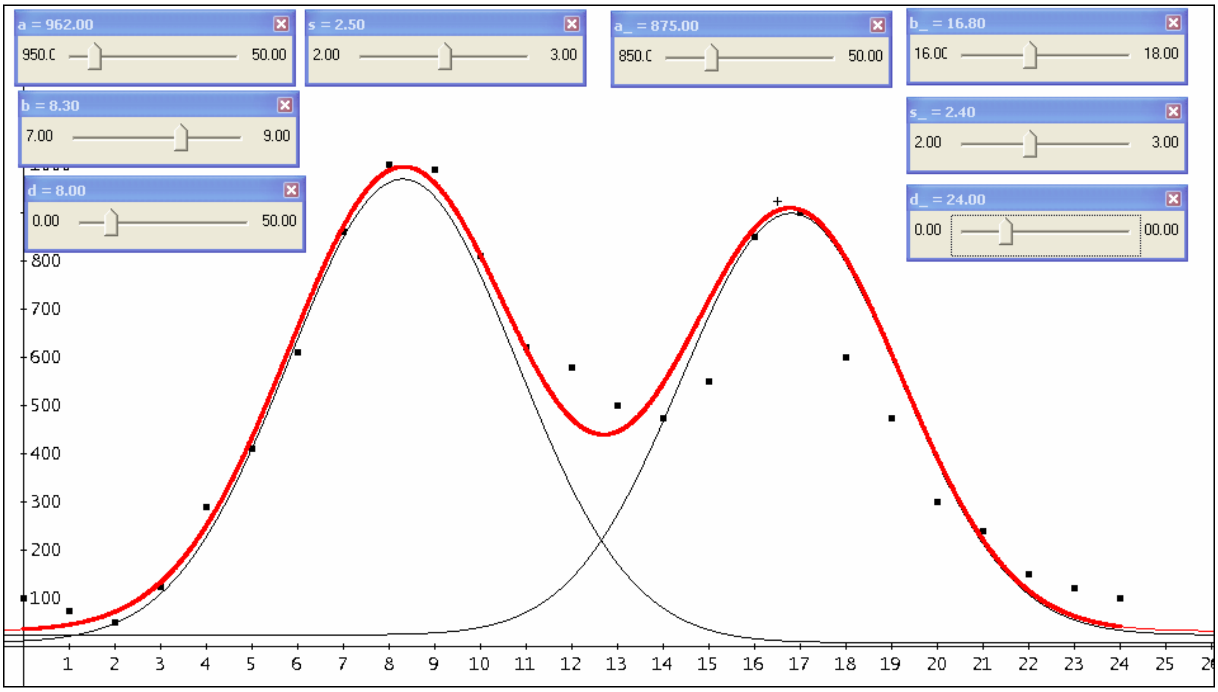
This is the plot of all even polynomial fits from degree 4 to degree 10

Now let´s follow the "Gauß Curve Idea": I define the generalized "Morning Traffic Bell" and introduce slider bars for the parameters *a*, *b*, *s* and *d*. It is a fine question for the students to estimate the values for the parameters (vertical stretch factor *a*, the mean *b*, the standard deviation *s* – they might know that the distance between turning point = mean and inflection point is the standard deviation *s* – and *d* is some vertical shift). Having this in mind it should not be too difficult to install meaningful ranges for the parameters.

$$m\_tr(t) := a \cdot e^{-1/2 \cdot ((x - b)/s)^2} + d$$

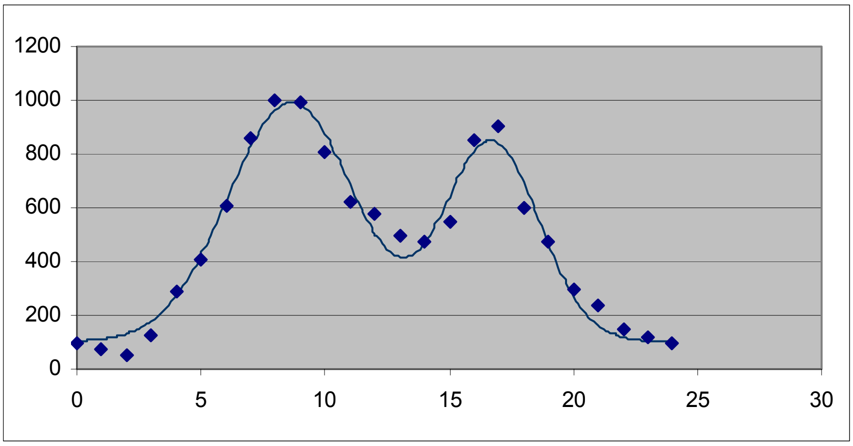


This is the Morning Traffic Curve. In a similar way we can find an Afternoon/Evening Traffic Curve. Then we have two bell curves. How to obtain a curve for the whole day? Simply add the functions. As you can see the afternoon part of the Morning Bell has little influence on the sum curve and the morning part of the Evening Bell has only very little influence on the sum curve, too. So we get the **`vehicle_density`** curve for the whole day. Just play a little with the slider bars to find the best fit and it is done.



#13: $\text{veh\_dens}(t) := 962\cdot e^{-1/2\cdot((x-8.3)/2.5)^2} + 875\cdot e^{-1/2\cdot((x-16.8)/2.4)^2} + 32$

I was just interested how MS-Excel would solve the problem using the solver to minimize the SSE (Sum of Squared Errors). This is the plot created by means of the Excel Solver:



I compared the sum of the squared errors of „my“ double bell shaped-fit-curve and the Excel-Curve of best fit:

#14: $\text{excel\_dens}(t) := 908.16\cdot e^{-((t-8.545)/3.526)^2} + 726.8\cdot e^{-((t-16.674)/3.025)^2} + 88.33$

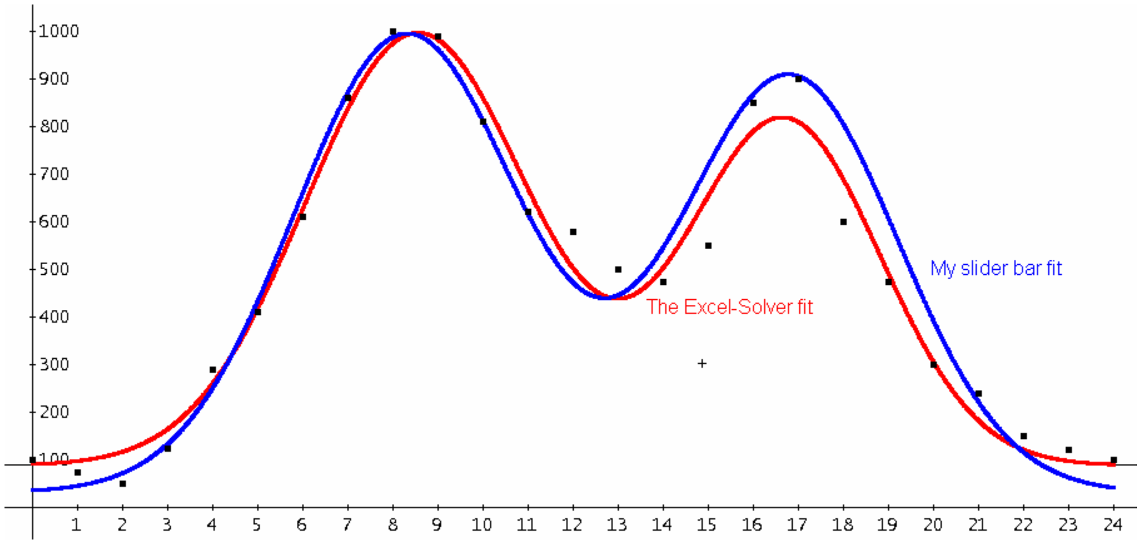
#15: $\sum\left(\text{VECTOR}\left((\text{veh}_{k,2} - \text{veh\_dens}(\text{veh}_{k,1}))^2,\ k,\ 1,\ \text{DIM}(\text{veh})\right)\right) = 135964.24$

#16: $\sum\left(\text{VECTOR}\left((\text{veh}_{k,2} - \text{excel\_dens}(\text{veh}_{k,1}))^2,\ k,\ 1,\ \text{DIM}(\text{veh})\right)\right) = 60312.696$

And I found out that - thanks Bill Gates – that this best fit curve was much better – even without reaching the peak in the afternoon!

I find it very challenging and useful to switch between the tools – even in school. Most students are accustomed to a spread sheet and why not change to it and use its results in CAS and sometimes the other way round? The students should learn to become flexible in using their tools, including brain, paper and pencil!

It is not forbidden to become cleverer, so we will use the Excel-fit-curve for further investigations.



e) We calculate the local extremal values.

We will do the further investigation using the best fit and provide a shorter name for it, trd(t)

#19: $\mathrm{trd}(t) := 908.16\cdot e^{-((t-8.545)/3.526)^2} + 726.8\cdot e^{-((t-16.674)/3.025)^2} + 88.33$

#20: $\left[\mathrm{NSOLVE}\left(\frac{d}{dt}\,\mathrm{trd}(t) = 0,\ t\right),\ \mathrm{NSOLVE}\left(\frac{d}{dt}\,\mathrm{trd}(t) = 0,\ t,\ 10,\ 14\right),\ \mathrm{NSOLVE}\left(\frac{d}{dt}\,\mathrm{trd}(t) = 0,\ t,\ 14,\ 20\right)\right]$

#21: [t = 8.5515292, t = 12.988050, t = 16.635525]

#22: $\mathrm{TABLE}(\mathrm{trd}(t),\ t,\ [8.55,\ 13,\ 16.64]) = \begin{bmatrix} 8.55 & 997.02402 \\ 13 & 438.60933 \\ 16.64 & 819.70609 \end{bmatrix}$

The local maximum values are reached at 8.6 in the morning and 16.6 in the afternoon with a density of approx 1000 and 820 vehicles per hour. Minimum traffic is at 1 pm with about 440 vehicle per hour. The nightly minimum cannot be calculated as a local extremal value. According to the data points it is approximately at 2 am, the model offers a boundary extremal value a 0 o'clock.

f) We find and interpret the values of the 1st derivative for $t = 7$ and $t = 21$

#23: $\mathrm{TABLE}(\mathrm{trd}'(t),\ t,\ [7,\ 21]) = \begin{bmatrix} 7 & 186.33865 \\ 21 & -88.904376 \end{bmatrix}$

At 7 o'clock traffic density increases by 186 cars/per hour and at 21 o'clock it decreases by 89 cars/hour.

g) Total number of all cars passing this day is the area under the curve, i.e. the defined integral. We start using some numerical methods in the same way as with the TIs. We can assume that the respective functions for numerical integration are ready made and the students can load their tool box.

$$\#24:\quad \mathrm{trapsum}(u, v, a, b, n) := \frac{1}{2}\cdot\frac{b-a}{n}\cdot\sum_{i=1}^{n}\left(\mathrm{SUBST}\left(u, v, a+\frac{(i-1)\cdot(b-a)}{n}\right)+\mathrm{SUBST}\left(u, v, a+\frac{i\cdot(b-a)}{n}\right)\right)$$

$$\#25:\quad \mathrm{midptsum}(u, v, a, b, n) := \frac{b-a}{n}\cdot\sum_{i=1}^{n}\mathrm{SUBST}\left(u, v, a+\frac{b-a}{n}+\frac{(i-1)\cdot(b-a)}{n}\right)$$

$$\#26:\quad \mathrm{simps}(u, v, a, b, n) := \sum_{i=1}^{n}\frac{b-a}{6\cdot n}\cdot\left(\mathrm{SUBST}\left(u, v, a+\frac{(i-1)\cdot(b-a)}{n}\right)+4\cdot\mathrm{SUBST}\left(u, v, a+\frac{(i-1)\cdot(b-a)}{n}+\frac{b-a}{2\cdot n}\right)+\mathrm{SUBST}\left(u, v, a+\frac{i\cdot(b-a)}{n}\right)\right)$$

#27: trapsum(trd(t), t, 0, 24, 60) = 11689.47

#28: midptsum(trd(t), t, 0, 24, 60) = 11689.368

#29: simps(trd(t), t, 0, 24, 60) = 11689.561

All estimated results are ≈ 11690 vehicles.

h) Calling finally the defined integral should give a very similar result:

$$\#30:\quad \int_0^{24}\mathrm{trd}(t)\,dt = 1601.0860\cdot\sqrt{\pi}\cdot\mathrm{ERF}(4.3831537)+1601.0860\cdot\sqrt{\pi}\cdot\mathrm{ERF}(2.4234259)+1099.285\cdot\sqrt{\pi}\cdot\mathrm{ERF}(5.5120661)$$

$$+\,1099.285\cdot\sqrt{\pi}\cdot\mathrm{ERF}(2.4218181)+2119.92$$

$$\#31:\quad \int_0^{24}\mathrm{trd}(t)\,dt = 11689.557$$

Appearance of the error function in #30 might lead to some discussion.

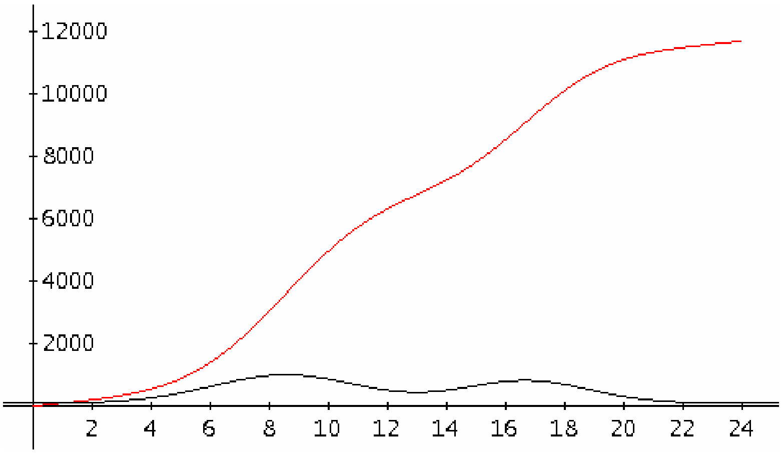
i) Now we meet the undefined integral.

```
          veh_sum(t) :=
            If 0 ≤ t ≤ 24
#32:           ∫(trd(x), x, 0, t)
               ?
```

j) The average number of vehicles…

$$\#33:\quad \frac{11689.557}{24} = 487.06487$$

… passing per hour is 487.



k) I hope that the students will find some examples (consumption of water, electricity during a day, touristic seasons, ….)

l) The last point is really a challenge and a great occasion to demonstrate the power of a CAS. This point is always a final and closing highlight when working through this example with teacher students and with experienced teachers on seminars.

We want to extend this function over a week – let´s say without weekend, because these two days will follow other rules. So we have a periodic function and periodic functions provoke the idea of involving trigonometric functions.

The moment will come when somebody whispers "Fourier??". That is it, but how to perform a Fourier analyse? Most of us – except teachers on technical schools – did this the last time during their university studies.

Easy done let´s ask DERIVE. We call the Online Help, type in *Fourier* into the Index search and, here it is:

FOURIER(y, t, t1, t2, n) simplifies to a Fourier series approximation of the expression y(t) as the variable t varies from t1 to t2, truncated to the nth harmonic. The Fourier series has the form b0 plus the sum of

ak·sin(2·p·k·t/(t2-t1)) + bk·cos(2·p·k·t/(t2-t1))

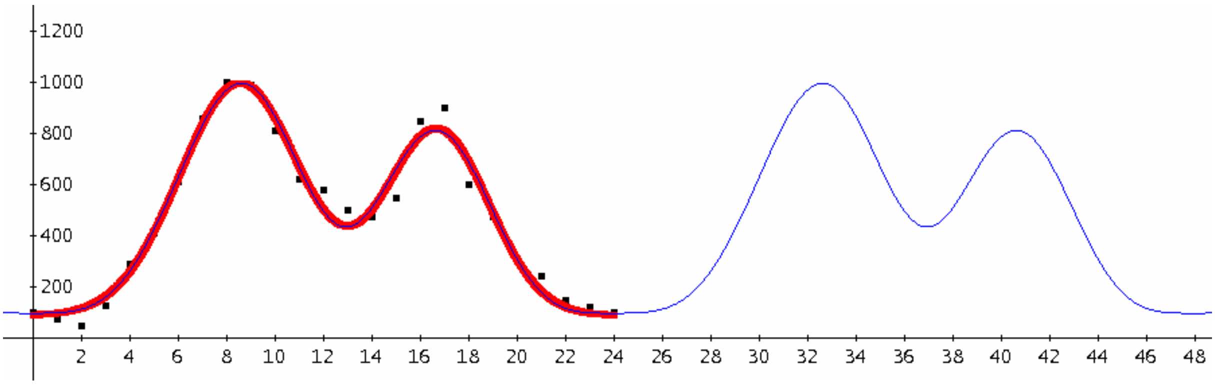
for k=1 to n, where b0 through bn and a1 through an are expressions independent of t. If simplifying an expression involving the FOURIER function does not provide a closed form, you can try approximating it instead. For example, to see a Fourier series approximation for a square wave truncated to the 12th harmonic, simplify and plot the expression

```
FOURIER(SQUARE_WAVE(t), t, 0, 2, 12)
```

So we use this provided tool:

The exact result amazes the students, the approximated form look a bit "friendlier".

```
#34:  FOURIER(trd(t), t, 0, 24, 4)

#35:  - 9.2944483·COS(1.0471975·t) - 18.785045·SIN(1.0471975·t) + 131.41071·COS(0.78539816·t) +
        68.874177·SIN(0.78539816·t) - 181.12044·COS(0.52359877·t) - 85.037440·SIN(0.52359877·t) -
        330.92730·COS(0.26179938·t) + 39.435113·SIN(0.26179938·t) + 487.06490
```



But the plot is much more amazing. It is a perfect fit of the daily function and repeats periodically.

One of my colleagues was so astonished that he said: "This is really wonderful, I am feeling like an apprentice of a wizard" ("Das ist ja wunderbar, ich fühl´ mich wie ein Zauberlehrling"). Maybe that students might feel like Harry Potter.

I will close thanking Thomas Himmelbauer for designing this wonderful example which offers so many occasions to work on calculus problems, which forces the students to interpret the results and which gives the opportunity to use various tools up to change the technology. He is the author of many other examples which can be found an the homepage of the ACDCA.

www.acdca.ac.at

In the last DNL Peter Schofield and I showed some plots of contour- or level curves and as a useful side product ways to program implicit 2D-plots. Families of 3D-level curves give a nice presentation of surfaces and solids. Peter did a lot more – he wrote tools for presenting 3D implicitly given surfaces. I present some results to show the use of his functions. Many thanks, Peter.

Implicit Plots in 3D

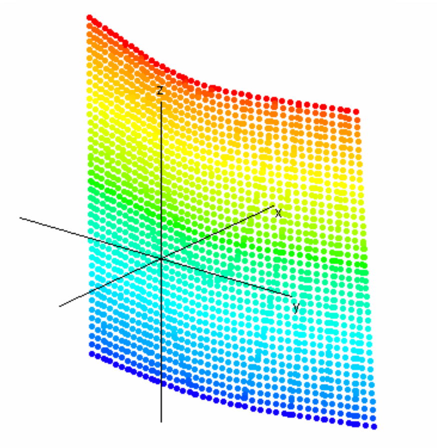
Peter Schofield

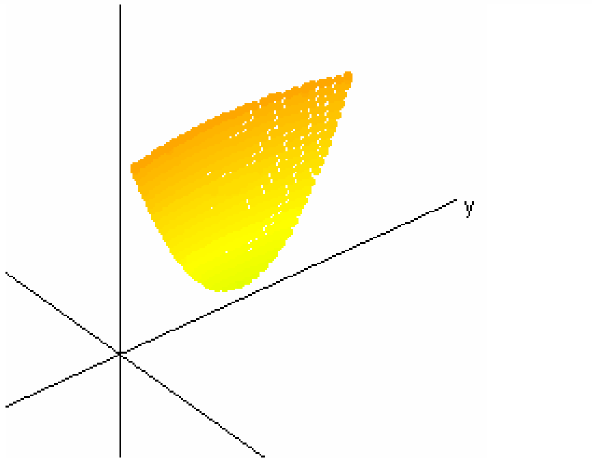
The following three functions do all the work:

```
ImplicitPts(f, st := –5, end := 5, step := 0.25)

ImplicitDots(f, st := –5, end := 5, step := 0.25)

InterPts(f, g, st := –5, end := 5, step := 0.1, rad := 0.3)
```

If you are satisfied with the standard settings ($-5 \le x, y, z \le +5$ and a scanning step of 0.25) then you only need to enter the function(s) in implicit form(s). For changing the settings you can enter vectors for `st`, `end` and `step`. See a demonstration of applying standard settings and "individual settings":

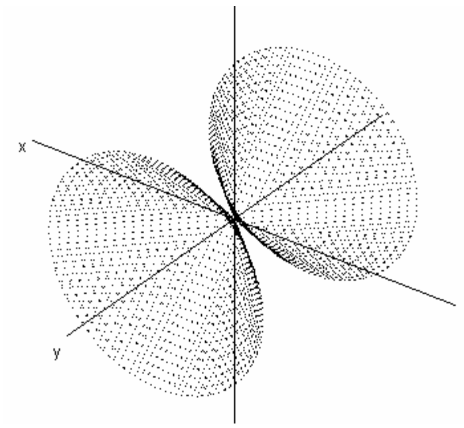
$\text{ImplicitPts}(x^3 - y^2 + z = 10)$ $\text{ImplicitPts}(x^3 - y^2 + z = 10, [-2, -3, -4], [2, 3, 4], 0.1)$

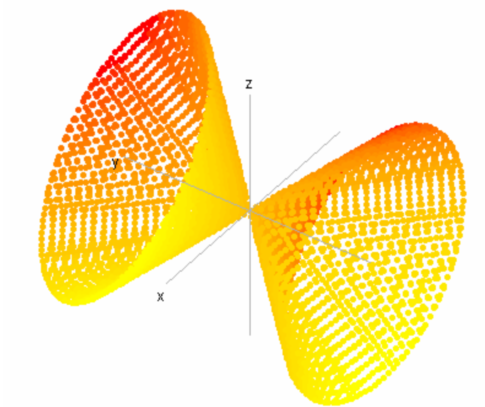




Lets start with a simple cone $x^2 - y^2 + z^2 = 0$ appearing in "Dots" and in "Points" (Size Medium)

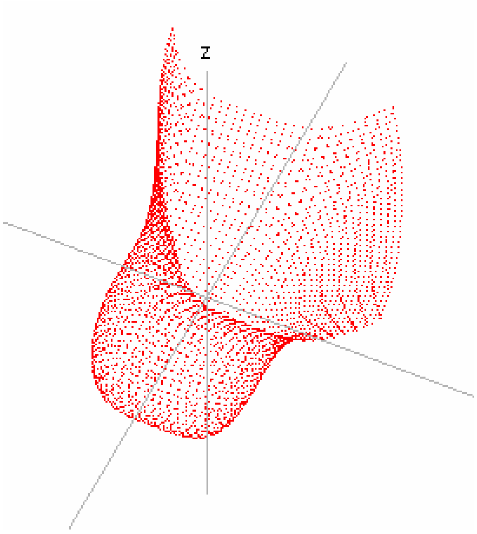
$\text{ImplicitDots}(x^2 - y^2 + z^2)$ $\text{ImplicitPts}(x^2 - y^2 + z^2)$

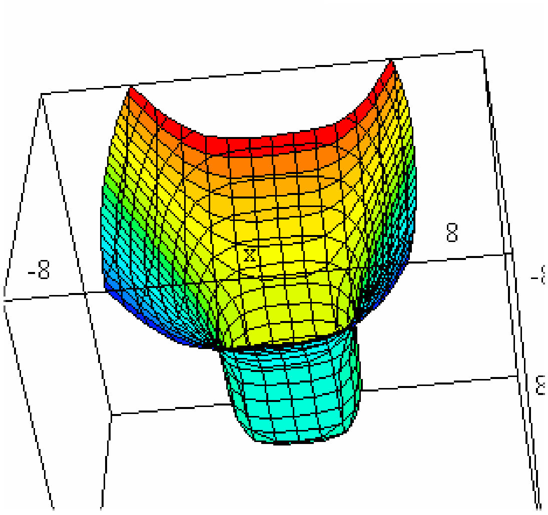




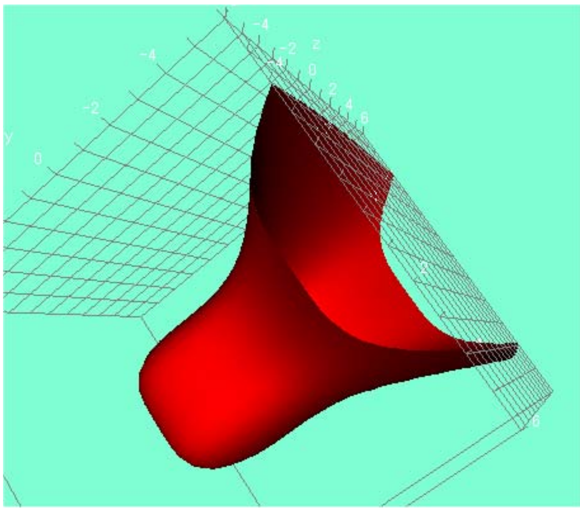
The next surface is a product of my fantasy. I compare the 3D-plots created by *DERIVE,* DPGraph and Autograph:

$$\text{ImplicitDots}(x^4 + (y - 1)^3 + (2\cdot z)^2 = 32)$$





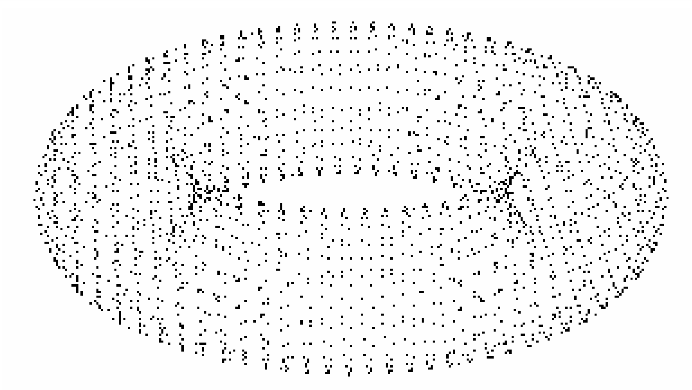
DERIVE DPGraph



Autograph

Now we have a torus with special values for scanning ranges in *x*-, *y*- and *z*-direction.

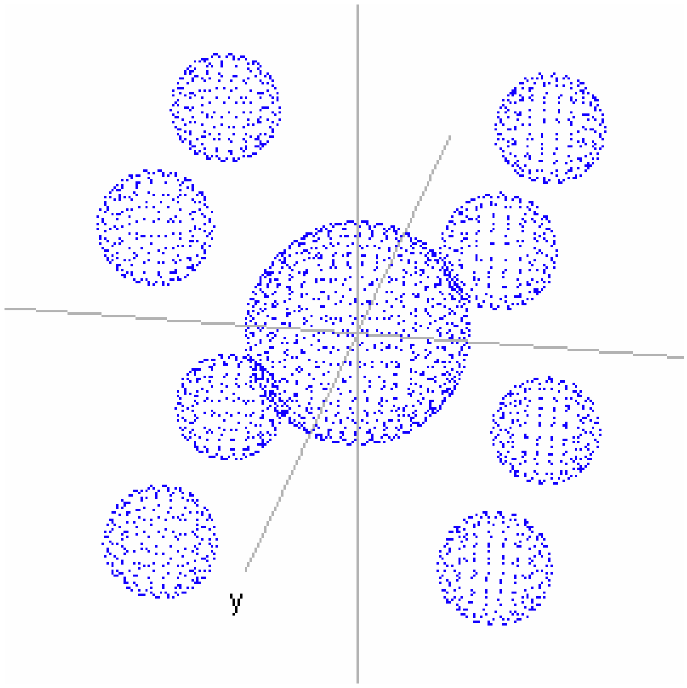
$$\text{ImplicitDots}(36\cdot z^2 + (x^2 + y^2 + z^2 - 10)^2 = 36, [-4, -4, -1], [4, 4, 1])$$



Peter produced one large sphere surrounded by 8 smaller spheres. For convenience define the SPHERE-function and then put them all together:

$$\text{sphere}(a, b, c, r) := (x - a)^2 + (y - b)^2 + (z - c)^2 - r^2$$

```
[         ImplicitDots(sphere(3, 3, 3, 1), 2, 4)                  ]
[  ImplicitDots(sphere(3, 3, -3, 1), [2, 2, -4], [4, 4, -2])      ]
[  ImplicitDots(sphere(3, -3, 3, 1), [2, -4, 2], [4, -2, 4])      ]
[ ImplicitDots(sphere(3, -3, -3, 1), [2, -4, -4], [4, -2, -2])    ]
[  ImplicitDots(sphere(-3, 3, 3, 1), [-4, 2, 2], [-2, 4, 4])      ]
[ ImplicitDots(sphere(-3, 3, -3, 1), [-4, 2, -4], [-2, 4, -2])    ]
[ ImplicitDots(sphere(-3, -3, 3, 1), [-4, -4, 2], [-2, -2, 4])    ]
[       ImplicitDots(sphere(-3, -3, -3, 1), -4, -2)               ]
[         ImplicitDots(sphere(0, 0, 0, 2), -2, 2)                 ]
```



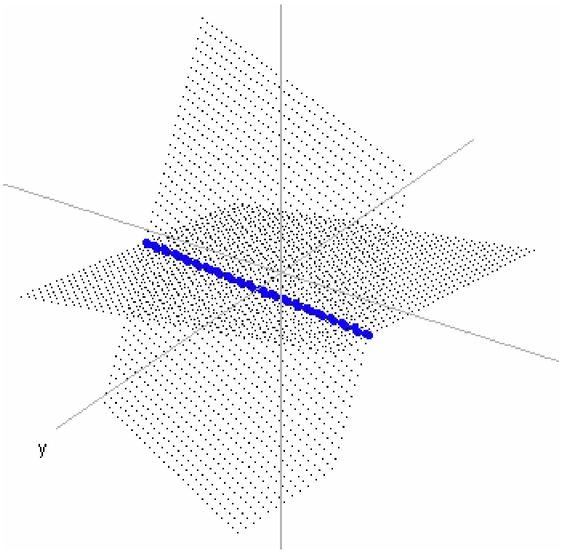
InterPts(surface1, surface 2, [settings for domain, step and rad]) shows the intersection of implicitly given surfaces. This is Peter´s advice:

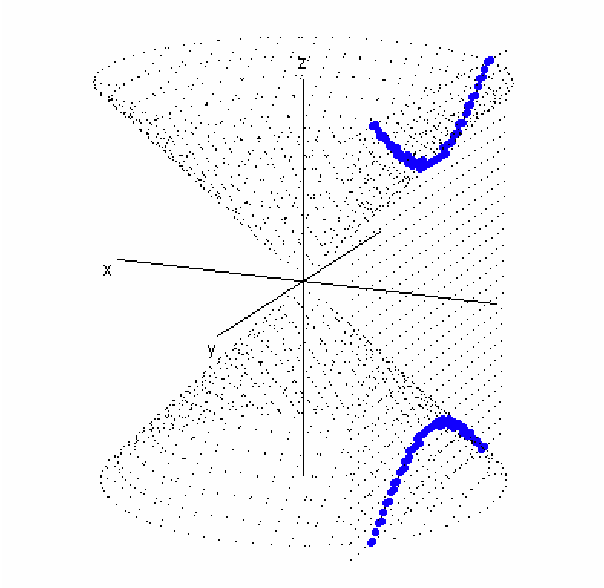
```
Intersections of surfaces

For each of #36-#39 3D-plot use: Insert > Plot > Medium > Custom

Select Point Colour using Top (Min,Max) and Dot Colour using Grids.

Set Apply parameters to rest of plot list (ON) and then Finish (be patient):
```

We produce the intersection of two planes and the intersection of a cone and a plane resulting in a hyperbola:

#36: $\left[\begin{array}{c} \text{InterPts}(x + 2\cdot y + z = 3,\ x - y + 7\cdot z = 0) \\ \text{ImplicitDots}(x + 2\cdot y + z = 3) \\ \text{ImplicitDots}(x - y + 7\cdot z = 0) \end{array}\right]$

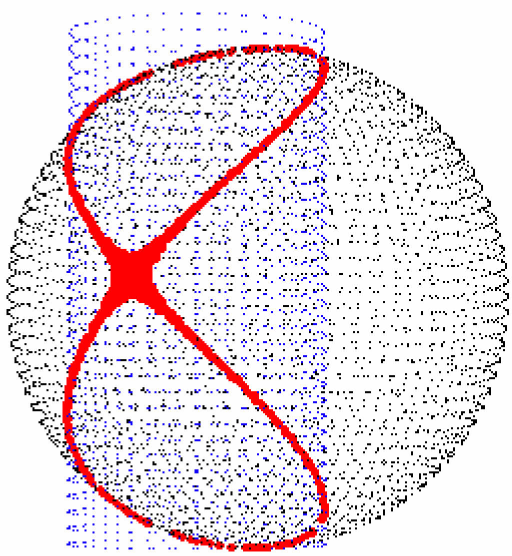
#37: $\left[\begin{array}{c} \text{InterPts}(x + 2,\ x^2 + y^2 - z^2) \\ \text{ImplicitDots}(x + 2) \\ \text{ImplicitDots}(x^2 + y^2 = z^2) \end{array}\right]$



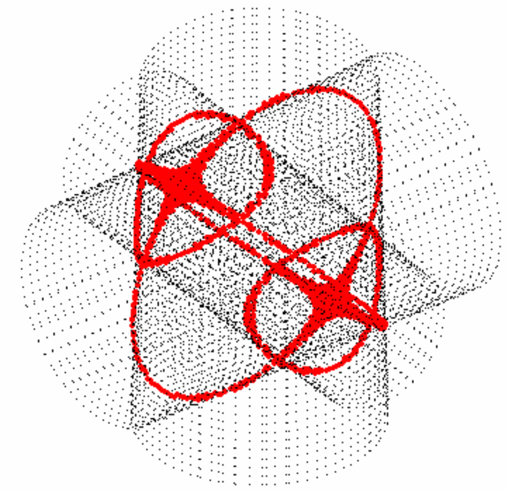


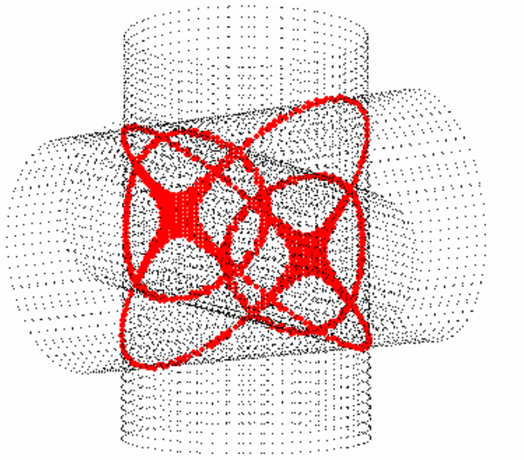
Sphere and cylinder give in a special configuration the *Window of Viviani*:

$$\left[\begin{array}{c} \text{ImplicitDots}(x^2 + y^2 + z^2 = 16,\ -4,\ 4) \\ \text{ImplicitDots}((x - 2)^2 + y^2 = 4,\ -4,\ 4) \\ \text{InterPts}(x^2 + y^2 + z^2 = 16,\ (x - 2)^2 + y^2 = 4,\ -4,\ 4,\ 0.05,\ 0.05) \end{array}\right]$$



Intersection of cylinders:

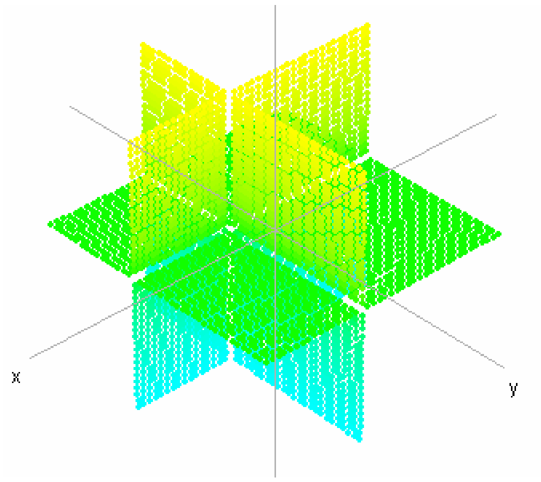




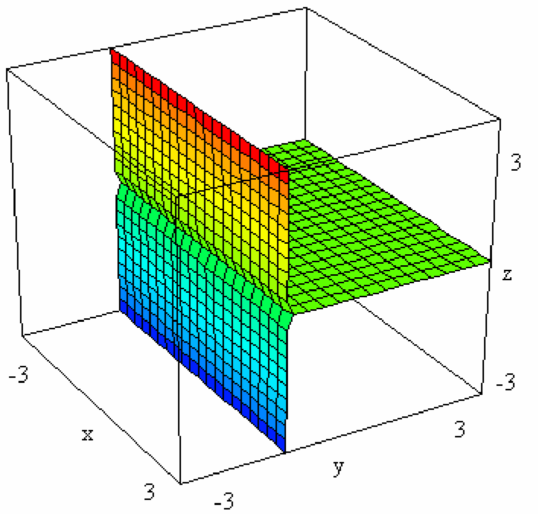
3D_PLOT (notice all three solution planes are plotted):

ImplicitPts($(x - 1)^2 \cdot (y + 1) \cdot z^2 = 0$)

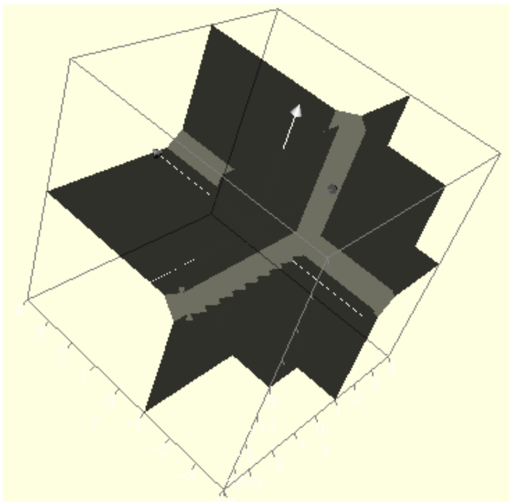
(If you try this equation with 'DPgraph' you don't get the correct plot?)



DERIVE



DPGraph



Autograph

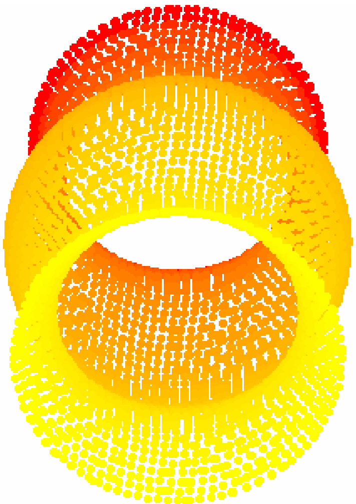
ContourPts_XY, ContourPts_XZ, ContourPts_YZ and Contour Dots_XY, Contour Dots_XZ, Contour Dots_YZ were introduced in DNL#63

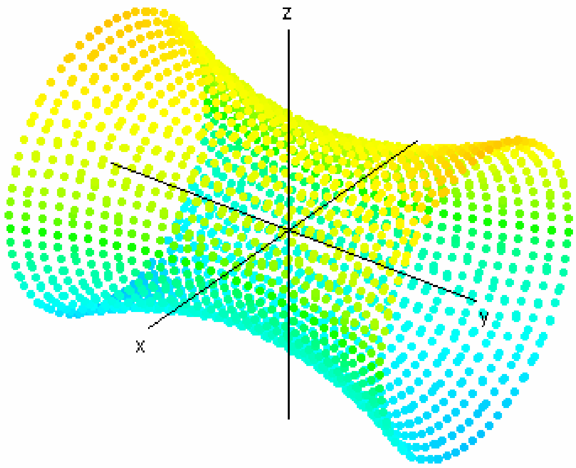
Surfaces of Revolution, etc. – try these with Insert>Plot> Large Points > Rainbow!

Set Apply parameters to rest of plot list (ON) and then Finish:

$$\text{VECTOR}(\text{ContourPts\_XY}(x^2 + y^2 = 9 + 3\cdot\text{COS}(z),\ z\_,\ -4,\ 4,\ 0.2),\ z\_,\ -5,\ 5,\ 0.2)$$

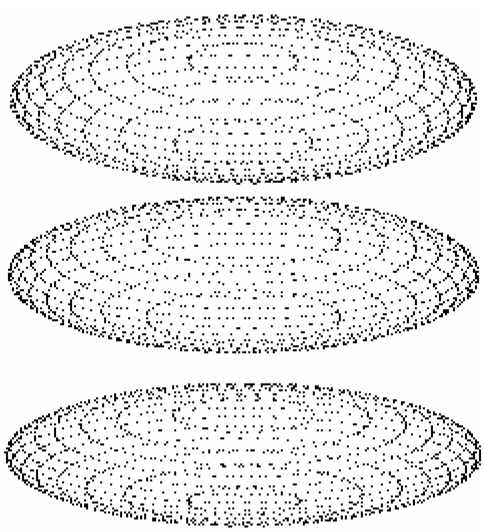
$$\text{VECTOR}\left(\text{ContourPts\_XZ}\left(x^2 + z^2 = 2\cdot\text{COSH}\left(\frac{y}{2}\right),\ y\_,\ -4,\ 4,\ 0.2\right),\ y\_,\ -5,\ 5,\ 0.2\right)$$

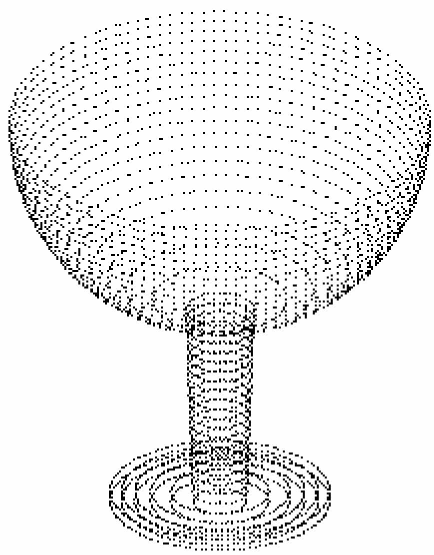




$$\text{VECTOR}(\text{ContourDots\_XY}(x^2 + y^2 - 16\cdot\text{COS}(2\cdot z),\ z\_),\ z\_,\ -5,\ 5,\ 0.1)$$

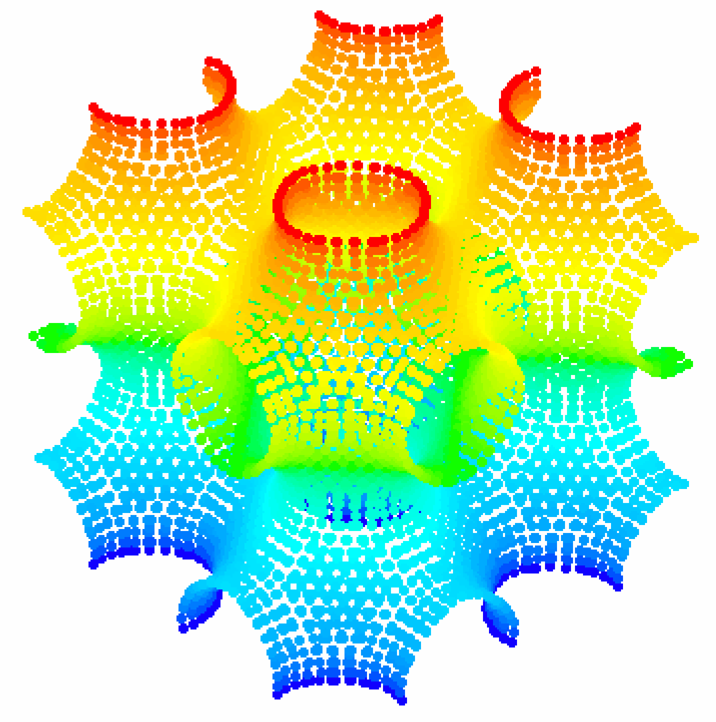
$$\left[\begin{array}{c}\text{VECTOR}(\text{ContourDots\_XY}(x^2 + y^2 + (z - 4)^2 = 16,\ z\_,\ -4,\ 4,\ 0.2),\ z\_,\ 0,\ 4,\ 0.2)\\ \text{VECTOR}(\text{ContourDots\_XY}(x^2 + y^2 = 0.4\cdot(1 + 0.1\cdot z),\ z\_,\ -1,\ 1,\ 0.1),\ z\_,\ -4,\ 0,\ 0.2)\\ \text{VECTOR}(\text{ContourDots\_XY}(x^2 + y^2 = z,\ [z\_,\ -4],\ -2,\ 2,\ 0.1),\ z\_,\ 0,\ 4,\ 0.8)\\ \text{VECTOR}(\text{ContourDots\_XY}(x^2 + y^2 = z,\ [z\_,\ -3.8],\ -2,\ 2,\ 0.1),\ z\_,\ 0,\ 4,\ 0.8)\end{array}\right]$$





With Peter´s special recommendations (one of his favourite graphs):

$$\mathrm{ImplicitPts}(\mathrm{SIN}(x) + \mathrm{SIN}(y) + \mathrm{SIN}(z) = 0)$$

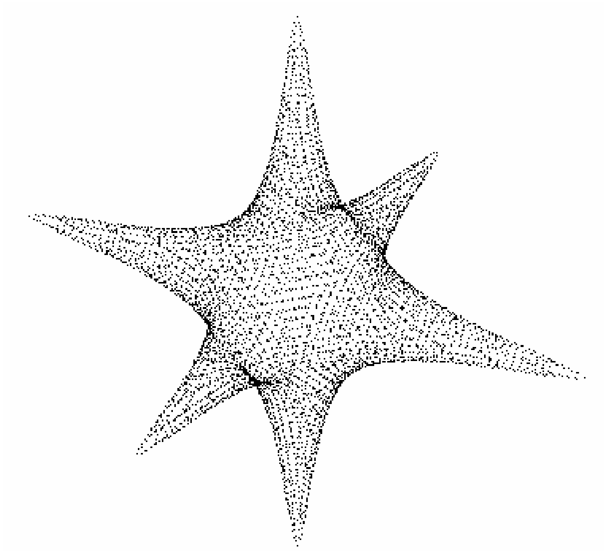


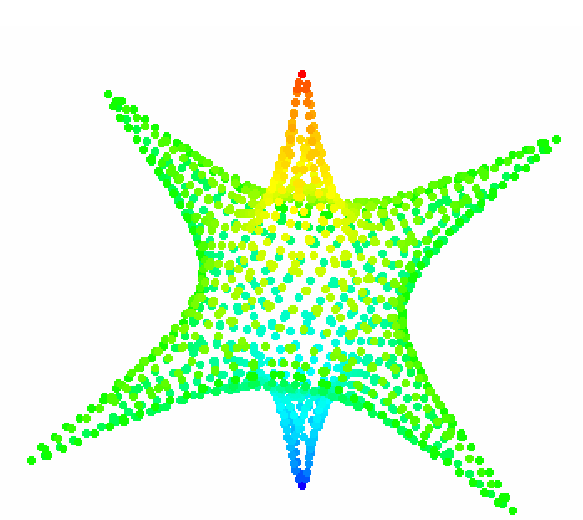
I found the information in the German edition of the *Scientific American* (= *Spektrum der Wissenschaft*) from January 2007 that the Association of German Mathematicians (DMV = Deutsche Mathematiker-Vereinigung) named a special and spectacular mathematical object after a German writer and poet, *Hans Magnus Enzensberger* ("Der Zahlenteufel", "The Number Devil"). This is the DERIVE-image of this impressive star with six vertices:

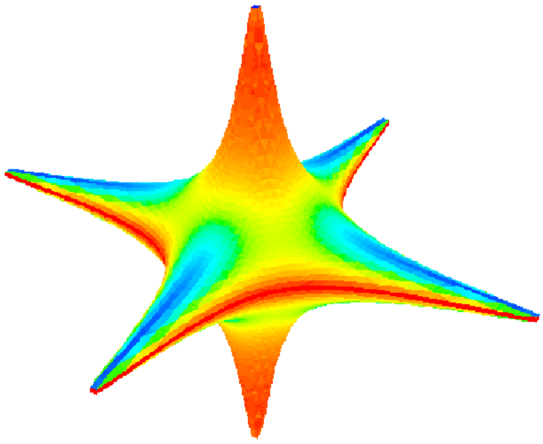
Magnus Enzensberger Star

$$\mathrm{ImplicitDots}(400\cdot(x^2\cdot y^2 + y^2\cdot z^2 + x^2\cdot z^2) = 1 - x^2 - y^2 - z^2, -1, 1, 0.025, 0.025)$$

$$\mathrm{ImplicitPts}(400\cdot(x^2\cdot y^2 + y^2\cdot z^2 + x^2\cdot z^2) = 1 - x^2 - y^2 - z^2, -1, 1, 0.05, 0.05)$$







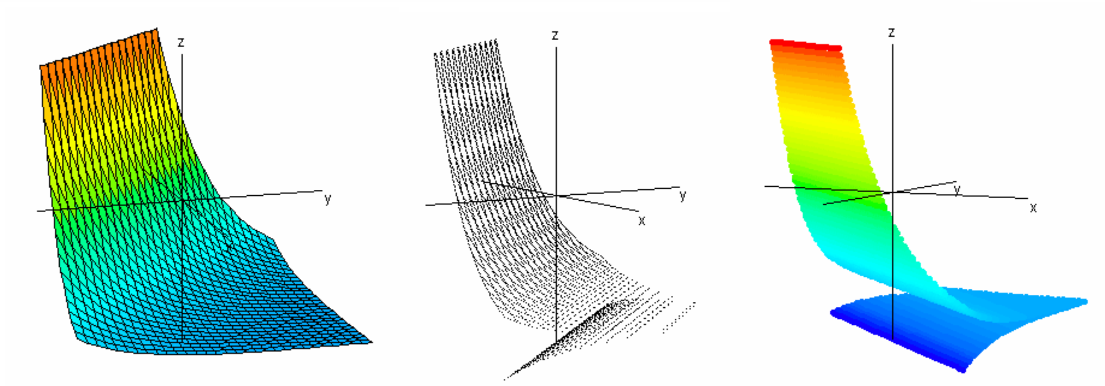
Magnus-Enzensberger-Star created by DPGraph (setting Color by Steepness)

Compare 3D-plots of:

$$z = \sqrt{e}^{\,-(x + y)} - 4$$

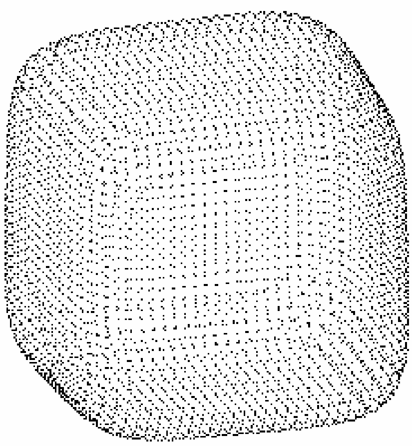
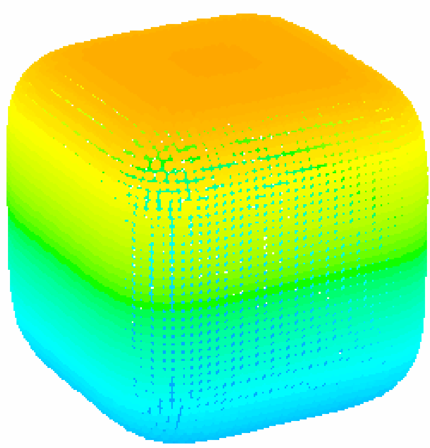
$$\text{VECTOR}(\text{ContourDots\_XY}(\text{LN}((z + 4)^2) + x + y,\ z\_,\ -5,\ 5,\ 0.5),\ z\_,\ -3.9,\ 5,\ 0.05)$$

$$\text{ImplicitPts}(\text{LN}((z + 4)^2) + x + y)$$



$$\text{ImplicitDots}(|x|^5 + |y|^5 + |z|^5 - 2^5,\ -2,\ 2,\ 0.125)$$

$$\text{ImplicitPts}(|x|^5 + |y|^5 + |z|^5 - 2^5,\ -2,\ 2,\ 0.1)$$

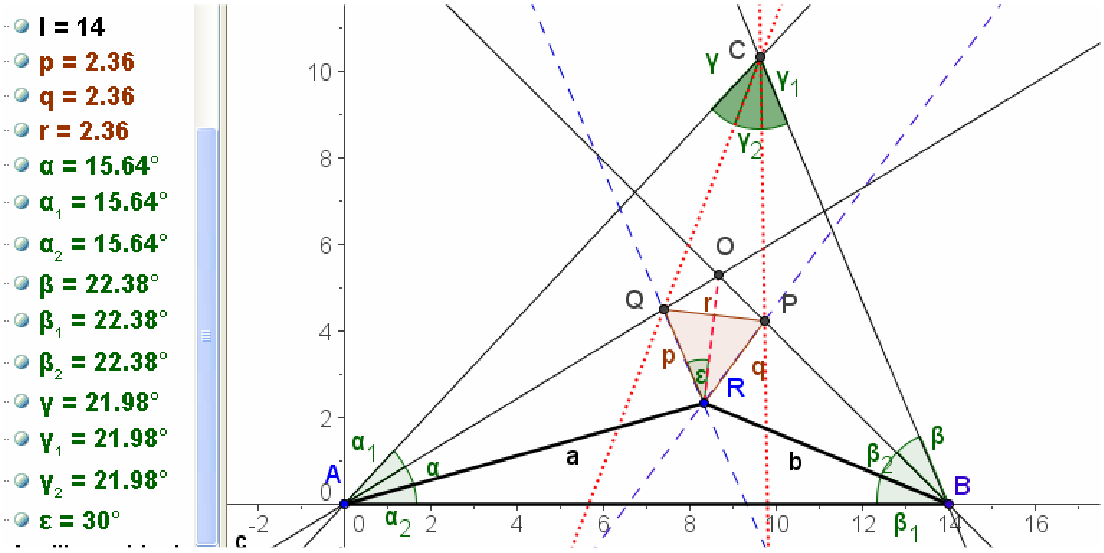



Morley in the Mirror

Peter Lüke-Rosendahl, Germany

Given is a triangle ABR. Intersection of reflection of AB wrt AR and reflection of AB wrt BR gives intersection point O. Draw two lines which form an angle of 30° with OR and intersect them with AO and BO giving points Q and P.
(1) Show that the **Morley Triangle** PQR is equilateral.
Further reflections of AR and BR with respect to AQ and BP intersect in point C. It is obvious that angles CAB and CBA are divided into three equal parts by the straight lines, but
(2) Show that lines CQ and CP also divide angle ACB into three equal parts.

Before starting with Derive and the computer algebra realisation and proof I´d like to make clear the problem using the free dynamic geometry program GeoGebra. You see the plot and the last part of the protocol which is verifying both statements from above.



Moreover we can ask for an accurate report on the construction process so we can reproduce it point for point and line for line.

You can see how to find point C together with all coordinates and equations of lines (or circles or conics if there were some to appear).

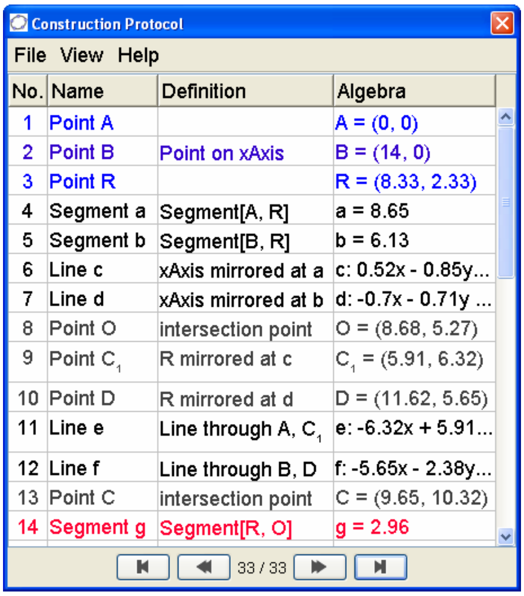
I find this as a good starting point to make the next step: the formal proof.

This will be done by Peter´s Derive paper.

Peter worked with fixed points A, B and R and did the proof finally with generalized coordinates.

I follow his tracks but include slider bars for the coordinates and have one fixed triple A_, B_, R_ for performing parallel numerical checks.

The screen shots show various basic triangles (various settings in the slider bars).


Construction Protocol
File View Help

No.	Name	Definition	Algebra
1	Point A		A = (0, 0)
2	Point B	Point on xAxis	B = (14, 0)
3	Point R		R = (8.33, 2.33)
4	Segment a	Segment[A, R]	a = 8.65
5	Segment b	Segment[B, R]	b = 6.13
6	Line c	xAxis mirrored at a	c: 0.52x - 0.85y...
7	Line d	xAxis mirrored at b	d: -0.7x - 0.71y ...
8	Point O	intersection point	O = (8.68, 5.27)
9	Point C_1	R mirrored at c	C_1 = (5.91, 6.32)
10	Point D	R mirrored at d	D = (11.62, 5.65)
11	Line e	Line through A, C_1	e: -6.32x + 5.91...
12	Line f	Line through B, D	f: -5.65x - 2.38y...
13	Point C	intersection point	C = (9.65, 10.32)
14	Segment g	Segment[R, O]	g = 2.96

33 / 33

#1: CaseMode := Sensitive

#2: [bx :=, rx :=, ry :=]

#3: [A := [0, 0], B := [bx, 0], R := [rx, ry]]

#4: [A_ := [0, 0], B_ := [18, 0], R_ := [12, 3]]

We start by providing a toolbox for further use.
Line (= **G**erade) defined by points U,V with parameter s

#5: g(U, V, s) := U + s·(V – U)

Pedalpoint (= **L**ot**f**uß**p**unkt) of P0 with respect to line(U,V):

#6: $$\mathrm{LFP}(P0,\ U,\ V) := U + \frac{(P0 - U)\cdot(V - U)}{(V - U)^2}\cdot(V - U)$$

Result of reflecting P0 with respect to line(U,V) (=**B**ild**p**unkt):

#7: BP(P0, U, V) := P0 + 2·(LFP(P0, U, V) – P0)

Intersection point (= **S**chnitt**p**unkt) of two lines g1 and g2:

#8: SP(g1, g2, p, q) := SUBST(g1, p, FIRST(FIRST(SOLUTIONS(g1 = g2, [p, q]))))

Rotation about α°:

#9: $$\mathrm{rotation}(\alpha) := \begin{bmatrix} \mathrm{COS}(\alpha) & -\mathrm{SIN}(\alpha) \\ \mathrm{SIN}(\alpha) & \mathrm{COS}(\alpha) \end{bmatrix}$$

#10: TRIANGLE(U, V, W) := [U, V, W, U]

#11: TRIANGLE(A, B, R)

Figure 1

#12: [BP(B, A, R), g(A, BP(B, A, R)), BP(A, B, R), g(B, BP(A, B, R))]

Figure 2

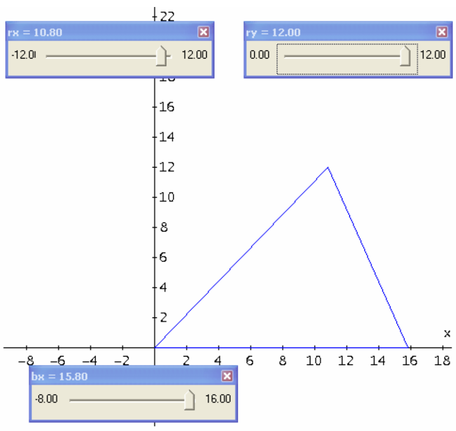


figure 1

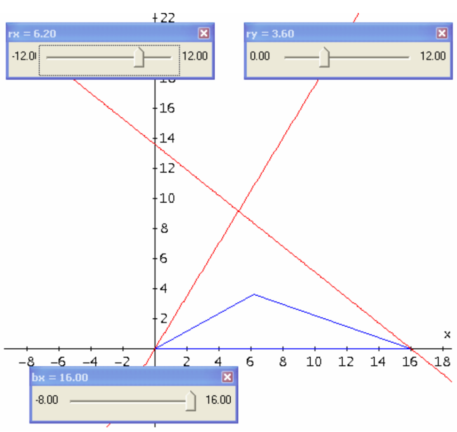


figure 2

#13: O(U, V, W) := SP(g(U, BP(V, U, W), p), g(V, BP(U, V, W), q))

#14: [O(A, B, R), R]

#15: gQ(A, B, R, p) := R + p·rotation(30°)·(O(A, B, R) – R)

#16: gP(A, B, R, p) := R + q·rotation(– 30°)·(O(A, B, R) – R)

#17: [gQ(A, B, R, p), gP(A, B, R, p)]

Figure 3

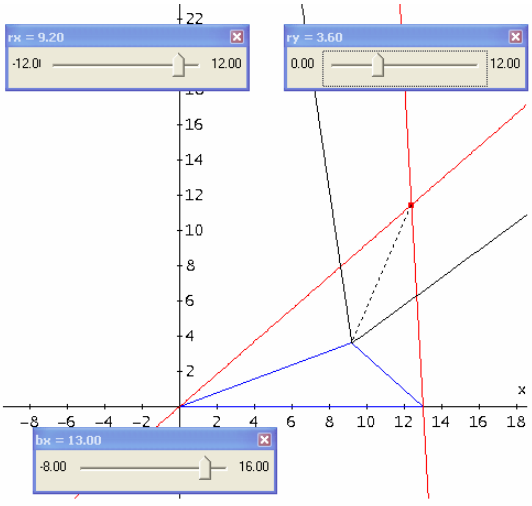


figure 3

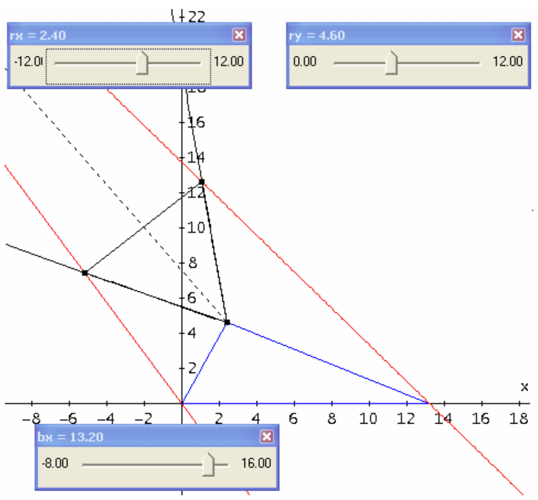


figure 4

Finding the Morley Triangle points Q and P:

#18: P(U, V, W) := SP(gP(U, V, W, q), g(V, O(U, V, W), p), q, p)

#19: Q(U, V, W) := SP(gQ(U, V, W, p), g(U, O(U, V, W), q), p, q)

#20: MorleyTriangle(U, V, W) := TRIANGLE(P(U, V, W), Q(U, V, W), W)

#21: MorleyTriangle(A, B, R)

Figure 4

We have to show that all angles of ΔPQR are 60° (#22-#24) and/or that sides PQ, PR and QR are of equal length (#25, #26). We measure the angles in positive direction:

#22:
$$W\alpha(U,\ V,\ W) := \frac{\mathrm{ACOS}\left(\dfrac{(V - U)\cdot(W - U)}{|V - U|\cdot|W - U|}\right)}{1°}$$

#23: Wα(R, Q(A, B, R), P(A, B, R)) = 60

#24: Wα(Q(A, B, R), P(A, B, R), R) = 60

#25: |Q(A, B, R) − R| − |P(A, B, R) − R| = 0

#26: |Q(A, B, R) − P(A, B, R)| − |P(A, B, R) − R| = 0

Finding point C:
Reflection of AR and BR wrt to AQ and AP gives two lines, their intersection point is C.
We calculate and plot lines CQ and CP.

#27: gAC(U, V, W, p) := g(U, BP(W, U, O(U, V, W)), p)

#28: gBC(U, V, W, q) := g(V, BP(W, V, O(U, V, W)), q)

#29: C(U, V, W) := SP(gAC(U, V, W, p), gBC(U, V, W, q), p, q)

#30: [gAC(A, B, R), gBC(A, B, R), C(A, B, R)]

#31: [g(C(A, B, R), Q(A, B, R)), g(C(A, B, R), P(A, B, R))]

Figure 5

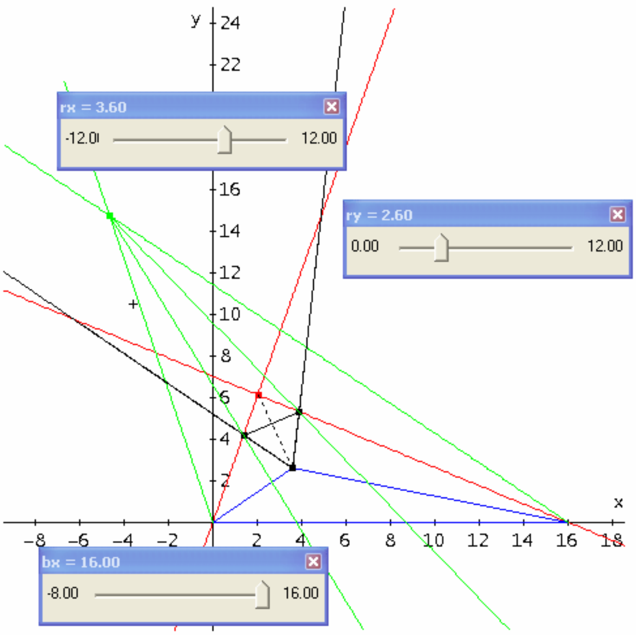


figure 5

Finally we want to show that gCQ and gCP form a third of angle ACB with sides AC and BC. This is no problem using basic triangle A_B_R_ (#33, #34), but does not work in the general form (expression #32 simplifies to a huge expression without any useful meaning).

```
#32:  IF(3·Wα(A, C(A, B, R), Q(A, B, R)) = Wα(A, C(A, B, R), B), OK, false)

#33:  IF(3·Wα(A_, C(A_, B_, R_), Q(A_, B_, R_)) = Wα(A_, C(A_, B_, R_), B), OK, false)

#34:  IF(bx > 0, OK, false)
```

So we try the other way round: Q1 is mirror image of Q with respect to line PC. Our conjecture is true if Q1 is element of line BC.

```
#35:  Q1(A, B, R) := BP(Q(A, B, R), P(A, B, R), C(A, B, R))

#36:  SOLVE(B = g(C(A_, B_, R_), Q1(A_, B_, R_), s))
```

#37: $$bx = 18 \wedge s = \frac{235\cdot\sqrt{3}}{187} - \frac{235}{374}$$

```
#38:  SOLVE(B = g(C(A, B, R), Q1(A, B, R)), s)
```

#39: $$(s = \pm\infty \wedge ry = 0) \vee s = \frac{(bx^2 - 2\cdot bx\cdot rx + rx^2 + ry^2)\cdot(3\cdot rx^2 - ry^2)}{2\cdot ry\cdot(rx^2 + ry^2)\cdot(\sqrt{3}\cdot bx - \sqrt{3}\cdot rx + ry)}$$

Expression #37 demonstrates that Q1 is on BC for the special case and #39 does the same for the general case. The same procedure can be done for point P.

```
#40:  P1(A, B, R) := BP(P(A, B, R), Q(A, B, R), C(A, B, R))

#41:  SOLVE(A = g(C(A, B, R), P1(A, B, R)), s)
```

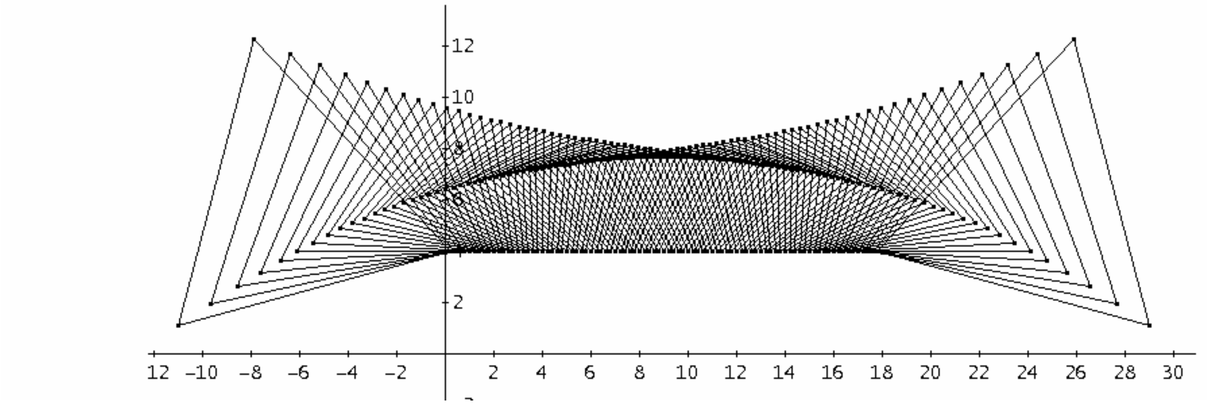
#42: $$s = \frac{(rx^2 + ry^2)\cdot(3\cdot bx^2 - 6\cdot bx\cdot rx + 3\cdot rx^2 - ry^2)}{2\cdot ry\cdot(bx^2 - 2\cdot bx\cdot rx + rx^2 + ry^2)\cdot(\sqrt{3}\cdot rx + ry)}$$

There are some restrictions for *s* dependent on the coordinates, because of $3\alpha + 3\beta < 180°$ follows $\alpha + \beta < 60°$.

The intersection points of the angle trisecting lines form an equilateral triangle

What happens if R is moving on a line parallel to the *x*-axis?

```
#43:  MorleyFam(t) := VECTOR(MorleyTriangle(A_, B_, [t, 4]), t, 0.2, 17.8, 0.2)
```



What is the locus of points P and Q? The power of CAS helps to give an answer:

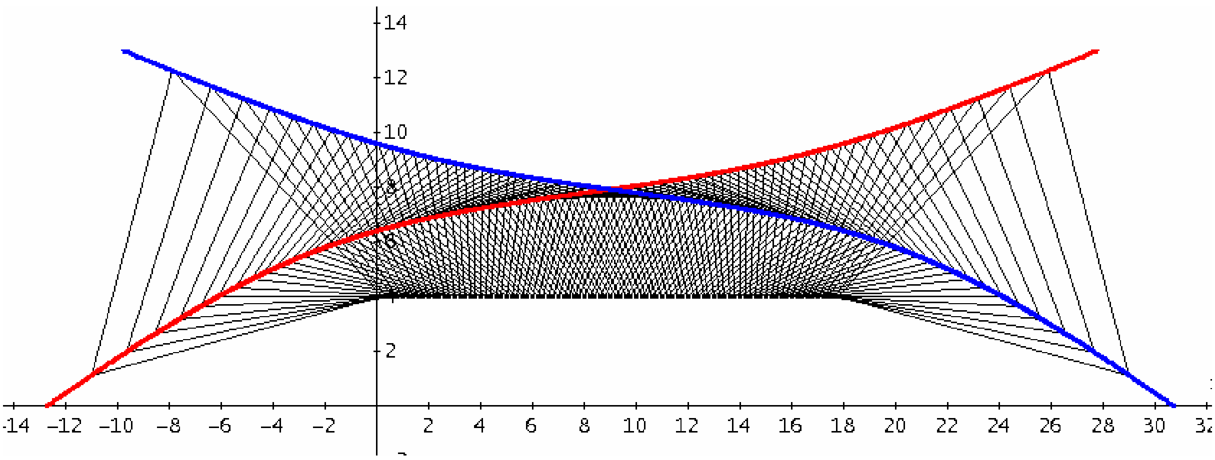
```
#44:  Q(A_, B_, [t, 4])
```

#45: $$\left[-\frac{32\cdot\sqrt{3}\cdot(\sqrt{3}\cdot t - 6\cdot\sqrt{3} - 4)}{3\cdot(t^2 - 18\cdot t - 24\cdot\sqrt{3} + 16)} + t - \frac{4\cdot\sqrt{3}}{3},\ \frac{8\cdot\sqrt{3}\cdot t\cdot(\sqrt{3}\cdot t - 18\cdot\sqrt{3} - 4)}{3\cdot(t^2 - 18\cdot t - 24\cdot\sqrt{3} + 16)}\right]$$

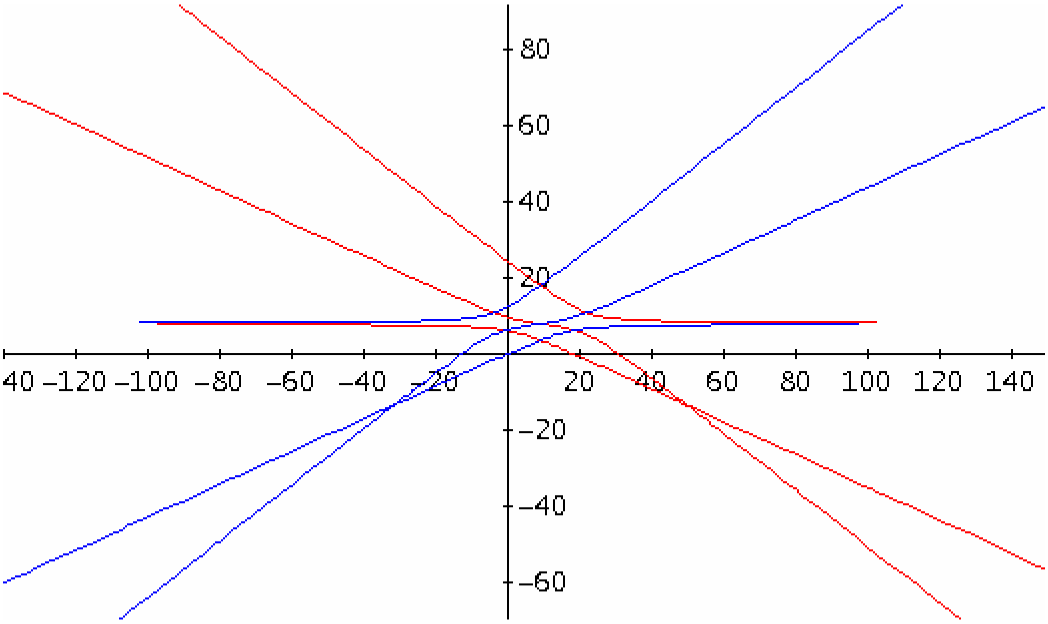
```
#46:  P(A_, B_, [t, 4])
```

#47: $$\left[-\frac{32\cdot\sqrt{3}\cdot(\sqrt{3}\cdot t - 12\cdot\sqrt{3} + 4)}{3\cdot(t^2 - 18\cdot t - 8\cdot(3\cdot\sqrt{3} - 2))} + t + \frac{4\cdot\sqrt{3}}{3},\ \frac{32\cdot\sqrt{3}\cdot(t - 4\cdot\sqrt{3})}{3\cdot(t^2 - 18\cdot t - 8\cdot(3\cdot\sqrt{3} - 2))} + 8\right]$$

```
#48:  VECTOR(Q(A_, B_, [t, 4]), t, 0, 18, 0.01)

#49:  VECTOR(P(A_, B_, [t, 4]), t, 0, 18, 0.01)
```



The locus for both points with $-100 \le t \le 100$ has an interesting form:



A nice Morley family picture at the end

$$\text{MorleyFam(t) := VECTOR}\left(\text{MorleyTriangle(A\_, B\_, [t, 2·SIN(t)]), t, } -2\cdot\pi,\ 2\cdot\pi,\ \frac{\pi}{40}\right)$$