THE DERIVE - NEWSLETTER #69

ISSN 1990-7079

THE BULLETIN OF THE

DERIVE®

USER GROUP

+ CAS-TI

C o n t e n t s:

March 2008

[1] *Rainer Wonisch*, Parabeln und Co. erforschen mit TI-Nspire CAS (Software)
bk teachware SR-58, ISBN 978-3-901769-79-5

[2] *Rainer Wonisch*, Parabeln und Co. erforschen mit TI-Nspire CAS (Handheld)
bk teachware SR-59, ISBN 978-3-901769-80-1

(*Inhalt von* [1] *und* [2]*: Geraden im Koordinatensystem, Parabeln im Koordinatensystem, Nullstellen von Parabeln, Anwendungen*)

[3] *Josef Böhm*, Programmieren mit dem TI-Nspire CAS
bk teachware SR-60, ISBN 978-3-901769-81-8

(*Inhalt: Anstelle einer Einführung: die Finanzen, Über eine Sutra zur ersten Bibliothek, Das Chaos-Spiel, Eine Bibliothek für Dreiecke, Wir haben ein CAS: ein Programm für Extremwertaufgaben, Trainingsprogramme für Grundfertigkeiten, Wir überwinden die Regression 4. Grades, Mit TI-Nspire zum Lottohaupttreffer*)

[4] *Pat und Carl Leinbach/Josef Böhm*, Forensische Mathematik für den Unterricht
bk teachware SR-57, ISBN 978-3-901769-78-8

(*Inhalt: Die Formel von Sherlock Holmes, Kriminologie mit dem Wetterhäuschen, Eine Buchstabensuppe als letzte Nachricht, Tödliche Geschwindigkeit, Die Botschaft der Bremsspuren, Blutspritzer an der Wand, Mit der Matrix zu den Tatverdächtigen, Knochenreste vom Bruderkrieg, Stumme Zeugen: Fingerabdrücke, Ein GPS - nicht nur für das Auto, Die Aufgaben eines Coroners*)

If you know about CAS-related publications then please inform me. Books dealing with the use of technology in math education in general are also interesting. Josef

TIME 2008 – Call for Papers

22. September– 26. September 2008

http://time.tut.ac.za

- 30 April 2008: Submission of abstracts on or before this date
- 31 May 2008: Acceptance of abstract notification by this date

Abstracts can be submitted as e-mail attachments to Josef Böhm (nojo.boehm@pgv.at) or Michel Beaudin (Michel.Beaudin@etsmtl.ca) for the Derive/Nspire part of the conference and for the ACDCA part to Bernhard Kutzler (b@kutzler.com) or Vlasta Kokol-Voljc (vlasta.kokol@uni-mb.si).

The deadline for submission is 30 April 2008 with acceptance of notification by 31 May 2008.

Liebe DUG-Mitglieder,

Mit dieser 1. Ausgabe des DNL im Jahr 2008 möchte ich Sie nochmals recht herzlich zur Teilnahme an der TIME 2008 einladen. Die Einreichfrist für einen Beitrag endet mit 30. April 2008. Vielleicht können Sie diese Konferenz auch mit einem weiteren Aufenthalt in Südafrika verbinden.

Nun zu diesem DNL#69: er ist wieder eine Mischung aus unterschiedlichen Einsatzgebieten wie Statistik (ANOVA), 3D-Geometrie (Tetraeder), Schulmathematik und Einstieg in die Hochschulmathematik (Devoir #1). Diesen letzten Artikel (von Michel Beaudin) habe ich gerne in seiner Originalfassung belassen, zumal auch die Lösungen beigefügt sind.

Ich wollte eigentlich auch noch einen Beitrag zu den Fußpunktkurven und –flächen liefern, aber – wie so oft – muss ich diesen für den nächsten DNL aufheben. Als kleinen Vorgeschmack können Sie sich auf den Seiten 14 und 36 holen.

Wenn Sie sich auch für schöne und interessante 3D-Grafiken interessieren, habe ich einen Hinweis für Sie: das ausgezeichnete Programm *Surfer* kann frei bezogen werden. Sehen Sie bitte die entsprechende URL im User Forum. (Einige damit erzeugte Grafiken finden Sie in diesem DNL.)

Von unserer Homepage können Sie neben den DNLs den aktualisierten Index für DNL#1 – #16 und nun neu auch den Index für DNL#53 – #68 finden. Daneben habe ich von allen downloadbaren DNLs die Titelseiten zusammengestellt. Damit kann auch so nach Inhalten geblättert werden.

In den letzten Märztagen gab es wieder ein Treffen mit Damen und Herren aus dem TI-Nspire-Entwicklungsteam. Es werden große Anstrengungen unternommen, diese Software auch den Wünschen und Erwartungen der DERIVE- und Voyage 200 (TI-89/92) User anzupassen. Es wurde auch die Gelegenheit wahrgenommen, insbesondere Anregungen, Forderungen und Empfehlungen hinsichtlich eines sinnvollen didaktischen Einsatzes eines CAS weiterzugeben. Viele Probleme in Bezug auf Benützerfreundlichkeit konnten offen diskutiert werden. Die DUG ist nach wie vor ein wichtiger Gesprächspartner für TI.

Bei dieser Gelegenheit darf ich die Nspire-Anwender (Software und Handheld) wieder um Beiträge, Hinweise, Tipps und Tricks bitten. Natürlich sind auch weiterin alle anderen Beiträge sehr willkommen. Auf der Info-Seite können Sie einige Neuerscheinungen zum TI-Nspire finden. Auch auf der ACDCA-Homepage (www.acdca.ac.at) gibt es Beiträge zum Programmieren mit dem TI-Nspire.

Mit einem nochmaligen Hinweis auf die TIME 2008 verbleibe ich Euer

Dear Dug-Members,

At the beginning of this 1st issue 2008 I'd like to invite you once more participating at TIME 2008. Deadline for submitting an abstract is 30 April. Maybe that you can combine this conference with a holiday in South Africa.

Some comments on this DNL#69: it is again a mixture of various applications as statistics (ANOVA), 3D-Geometry (Tetrahedron), secondary school math (From a Spreadsheet to a CAS) and first steps in higher mathematics (Devoir #1). I left the last article in its original version (French), the more so as it contains the solutions.

I intended to deliver a contribution on pedalcurves and –surfaces but – as so often – I had to postpone it for the next DNL because of lack of space. On Pages 16 and 36 are shown mouthwatering examples .

If you are interested in beautiful and interesting 3D-graphics then I have a tipp for you: download for free the great program *Surfer*. You can find the respective URL(s) in the User Forum. (Some *Surfer* produced graphs are presented in this DNL.)

You are invited to download from our homepage the updated index for DNLs#1 – #16 and new also the index for DNLs#53 – #68. Besides this I collected all frontpages of the said DNLs in two papers. So you can easily browse the contents of all downloadable DNLs.

In the last days of March we had a 3-days meeting with people of the TI-development team. There are big efforts to accommodate the CAS software to the wishes and expectations of the DERIVE- and the TI-89/92/V200 users. We took also the opportunity to pass a bundle of ideas, demands and recommendations with respect to a meaningful didactical use of a CAS. Many user friendliness problems were discussed very open. The DUG proved again to be an important interlocutor for TI.

At this occasion I´d like to ask the Nspire-users (software and handheld) to submit contributions, comments, tipps and tricks. Of course, all other contributions are very welcome as ever. On our info page you can find some new publications for the TI-Nspire. You can also download contributions (in German) from the ACDCA-homepage (www.acdca.ac.at).

Repeating my reminder on TIME 2008 I remain your

The *DERIVE-NEWSLETTER* is the Bulletin of the *DERIVE* & CAS-*TI User Group*. It is published at least four times a year with a contents of 40 pages minimum. The goals of the *DNL* are to enable the exchange of experiences made with *DERIVE*, *TI*-CAS and other CAS as well to create a group to discuss the possibilities of new methodical and didactical manners in teaching mathematics.

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Contributions:
Please send all contributions to the Editor. Non-English speakers are encouraged to write their contributions in English to reinforce the international touch of the *DNL*. It must be said, though, that non-English articles will be warmly welcomed nonetheless. Your contributions will be edited but not assessed. By submitting articles the author gives his consent for reprinting it in the *DNL*. The more contributions you will send, the more lively and richer in contents the *DERIVE* & CAS-*TI Newsletter* will be.

Next issue: June 2008
Deadline 15 May 2008

Preview: Contributions waiting to be published

Some simulations of Random Experiments, J. Böhm, AUT, Lorenz Kopp, GER
Wonderful World of Pedal Curves, J. Böhm
Tools for 3D-Problems, P. Lüke-Rosendahl, GER
Financial Mathematics 4, M. R. Phillips
Hill-Encription, J. Böhm
Farey Sequences on the *TI,* M. Lesmes-Acosta, COL
Simulating a Graphing Calculator in *DERIVE*, J. Böhm
Henon & Co, J. Böhm
Do you know this? Cabri & CAS on PC and Handheld, W. Wegscheider, AUT
An Interesting Problem with a Triangle, Steiner Point, P. Lüke-Rosendahl, GER
Overcoming Branch & Bound by Simulation, J. Böhm, AUT
Diophantine Polynomials, D. E. McDougall, Canada
Graphics World, Currency Change, P. Charland, CAN
Precise Recurring Decimal Notation, P. Schofield, UK
Cubics, Quartics – interesting features, T. Koller & J. Böhm
Logos of Companies as an Inspiration for Math Teaching
Exciting Surfaces in the FAZ / Pierre Charland´s Graphics Gallery
BooleanPlots.mth, P. Schofield, UK
Old traditional examples for a CAS – what´s new? J. Böhm, AUT
What is hiding in Dr. Pest? B. Grabinger, GER
Truth Tables on the TI, M. R. Phillips
Advanced Regression Routines for the TIs, M. R. Phillips
Where oh Where is IT? (GPS with CAS), C. & P. Leinbach, USA
Embroidery Patterns, H. Ludwig, GER
Mandelbrot and Newton with *DERIVE*, Roman Hašek, CZ
Snail-shells, Piotr Trebisz, GER
A Conics-Explorer, J. Böhm, AUT

and others

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Website for downloading the "*Surfer*" and other interesting information:
`www.imaginary2008.de`

Website with more information about a "*Surfer*"-competition:
`www.spektrum.de/mathekunst`

Website with results of a former competition (600 graphics):
`www.zeit.de/matheskulptur`

Website for additional information:
`www.algebraicsurface.net`

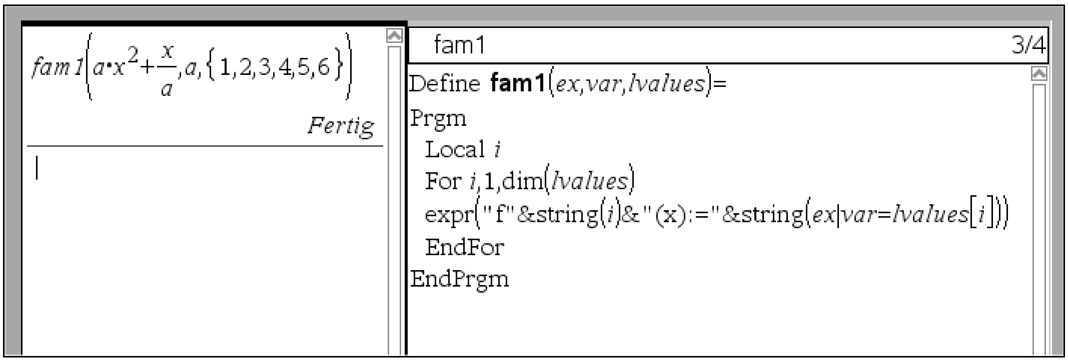
Interruption of calculation for TI-Nspire CAS software (PC)

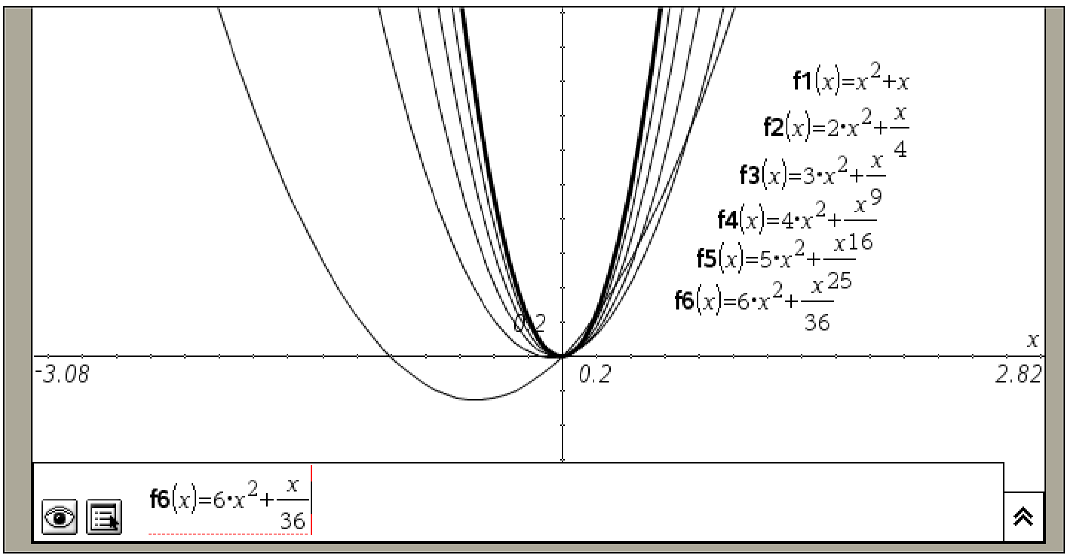
There can be problems cancelling a calculation or interrupting a program with a non-English keyboard. You should press the Break (=Pause-) key for some seconds. But it can happen that even this does not work. This is a bug and should be resolved with the next release.

Wolfgang Pröpper

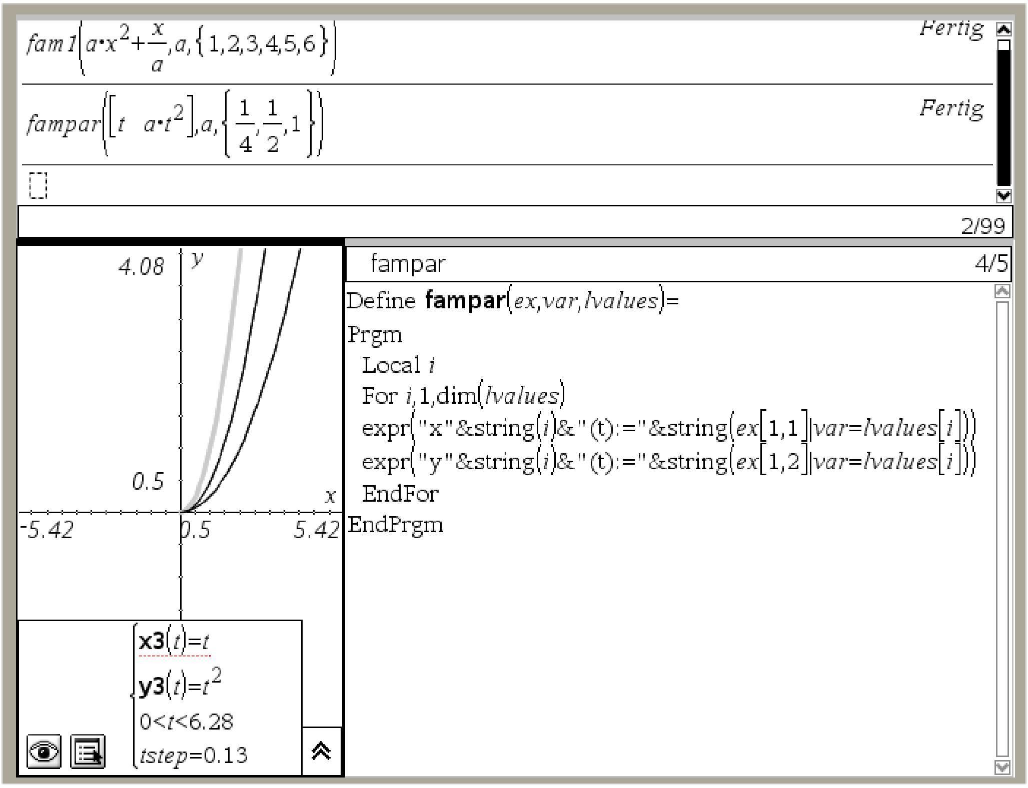
… I have tried to generate a family of curves for the G&G-application. It did not work. Can you help me? …

I tried (using many tricks including the indirection command) but I failed as Wolfgang. Then I wrote to Philippe Fortin and he knew the solution:





This worked now, thanks to Philippe. And what´s about a family of curves given in parameter form. Let me try to adapt Philippe's program:



The domain for the parameter must be adjusted in the G&G-application.

Hubert Langlotz

… Thanks for the rich information in DNL#68. Here is a small program which I used in class…

wurf(1000)	
	.02
	Fertig
wurf2(1000)	.014
wurf2(10000)	.0153
$\frac{6!}{(6.)^6}$	.015432

4/99

Simulation of a problem from Probability Theory

What is the probability to receive numbers from 1 to 6 rollig 6 dies?

(Bild der Wissenschaft Januar 2008 = Scientific American)

wurf2(1000)=**wurf2**(1000)

(Highlight righthand side and evaluate. See examples:

wurf2(1000)=.02, **wurf2**(1000)=.02, **wurf2**(5000)=.0136,

In the notes one can only call a function, not a program. **wurf2** is the function, **wurf** is the respective program!

Program and function are among the DNL#69 – files.

École de technologie supérieure

Mat 801-01 : Compléments de mathématiques (profil Génie Mécanique) **Session A-07**
Michel Beaudin

Devoir #1
Corrigé sur 100, pondération 15%

Donné le mercredi 19 septembre 2007
À remettre le lundi 1er octobre au début du T.P.
Ce travail peut être fait en équipe de 2 personnes

Résolutions symbolique et numérique d'équations/systèmes d'équations non linéaires.
L'importance de pouvoir augmenter la précision.

1- **[25 points]** Considérons l'équation algébrique $x+\sqrt[3]{x+1}=8$. Un logiciel de calcul symbolique a donné la solution suivante :

$$x=-\frac{\alpha}{6}+\frac{\beta}{6}+8 \quad \text{où} \quad \alpha=\sqrt[3]{12\sqrt{6573}+972} \quad \text{et}\ \beta=\sqrt[3]{12\sqrt{6573}-972}\,.$$

Ce qui vaut approximativement 6.07982.

a) Comment cette réponse a-t-elle pu être obtenue?

b) Montrez que la fonction $g(x)=8-\sqrt[3]{x+1}$ satisfait les hypothèses du théorème du point fixe en prenant, par exemple, l'intervalle I = [6, 6.2]. Reconfirmez par la suite la valeur 6.07982.

Aide : pour a), transformez l'équation en une équation polynomiale de degré 3, et utilisez la formule de Cardan donnée à la section 0.2 du résumé 1.

2- **[25 points]** Considérons l'équation $z^{26}=1.01^{-z}+3$.

a) Combien de solutions réelles possède cette équation? Justifiez votre réponse.

b) Trouvez (par la méthode de votre choix) la plus petite solution réelle de cette équation. La « méthode de votre choix » inclut aussi le solveur numérique de votre système de calcul.

c) Trouvez aussi une solution complexe, en utilisant la méthode de Newton à 2 variables, après avoir considéré les parties réelle et imaginaire de l'équation (une fois la substitution $z\to x+iy$ effectuée).

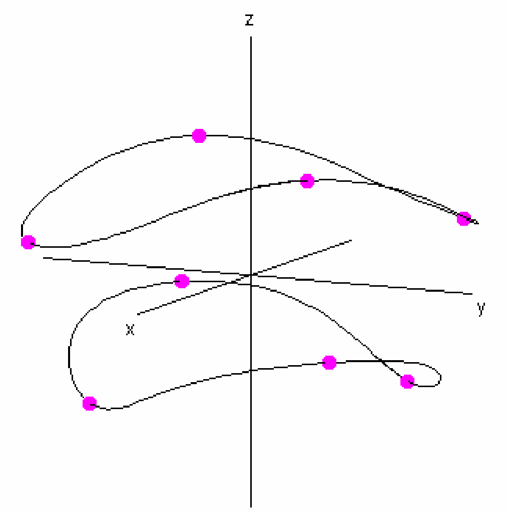
3- **[25 points]** L'importance de pouvoir augmenter la précision.

a) Trouvez les 10 solutions réelles de l'équation $\prod_{k=1}^{20}(x-k)-\frac{x^{19}}{10^7}=0$. Attention à la précision par défaut de votre système…

b) Soit la fonction *f*, définie par $f(x)=x\sqrt{x}\left(\sqrt{x+1}+\sqrt{x-1}-2\sqrt{x}\right)$. Approximez $f\left(10^6\right)$ en utilisant la précision par défaut de votre système. Que trouvez-vous? Augmentez maintenant la précision et recommencez. Que trouvez-vous? Finalement, montrez que $\lim_{x\to\infty} f(x)=-\frac{1}{4}$.

4- **[25 points]** Soit la courbe d'intersection des 2 surfaces $x^2+xyz+y^2=3,\ x^2+y^2+2z^2=4$. Trouvez les points sur cette courbe qui sont le plus proche et le plus loin de l'origine.

Aide : notez que « la » courbe d'intersection des 2 surfaces précédentes est en fait constituée de 2 courbes fermées (voir figure plus bas). Optimisez le carré de la distance $f=x^2+y^2+z^2$ en appliquant la méthode des multiplicateurs de Lagrange (voir rappel plus loin). Vous aurez un système polynomial et, en se restreignant aux solutions réelles, vous devriez trouver 8 solutions (vérifiable en tenant compte de la base de Gröbner associée).



Quelques rappels en calcul à plusieurs variables

Rappelons que si $w = w(x, y, z)$ est un champ scalaire à 3 variables, alors son gradient est le vecteur $\nabla w=\left[\frac{\partial w}{\partial x},\frac{\partial w}{\partial y},\frac{\partial w}{\partial z}\right]$. En général, l'équation $f(x,y,z)=0$ représente une surface dans l'espace et le vecteur ∇f est perpendiculaire à cette surface. De plus, un vecteur qui est perpendiculaire à la fois aux vecteurs v_1 et v_2 est le vecteur $v_1\times v_2$, le produit vectoriel de ces 2 vecteurs.

Lorsqu'on cherche les valeurs extrêmes de la fonction $f(x, y, z)$ sous les contraintes $g(x, y, z) = 0$ et $h(x, y, z) = 0$, on procède habituellement par multiplicateurs de Lagrange, méthode qui dit qu'une condition nécessaire est donnée par les solutions du système de 5 équations à 5 inconnues suivant :

$$(*)\qquad \nabla f=\lambda\nabla g+\mu\nabla h,\ \ g=0,\ \ h=0.$$

Solutions du devoir #1, Mat 801-01, session A-07

Problème #1: La réponse a été obtenue en utilisant la formule de Cardan. En effet:

#1: $x + (x + 1)^{1/3} = 8$

#2: $\text{SOLVE}(x + (x + 1)^{1/3} = 8, x, \text{Real})$

#3: $$x = \frac{(12\cdot\sqrt{6573} - 972)^{1/3}}{6} - \frac{(12\cdot\sqrt{6573} + 972)^{1/3}}{6} + 8$$

#4: $x = 6.079824878$

Voyons d'où cela est sorti...

#5: $(x + (x + 1)^{1/3} = 8) - x$

#6: $(x + 1)^{1/3} = 8 - x$

#7: $((x + 1)^{1/3} = 8 - x)^3$

#8: $x + 1 = (8 - x)^3$

#9: $x = - x^3 + 24\cdot x^2 - 192\cdot x + 511$

#10: $(x = - x^3 + 24\cdot x^2 - 192\cdot x + 511) + x^3 - 24\cdot x^2 + 192\cdot x - 511$

#11: $x^3 - 24\cdot x^2 + 193\cdot x - 511 = 0$

Effectuons le changement de variable $x = y - -24/3$:

#12: $$\left(y + \frac{24}{3}\right)^3 - 24\cdot\left(y + \frac{24}{3}\right)^2 + 193\cdot\left(y + \frac{24}{3}\right) - 511 = 0$$

#13: $y^3 + y + 9 = 0$

#14: $\text{SOLVE}(y^3 + y + 9 = 0, y, \text{Real})$

#15: $$y = \frac{(12\cdot\sqrt{6573} - 972)^{1/3}}{6} - \frac{(12\cdot\sqrt{6573} + 972)^{1/3}}{6}$$

Et, puisque $x = y + 8$, on obtient bien ce que la commande solve du début a donné.

#16: $g := 8 - (x + 1)^{1/3}$

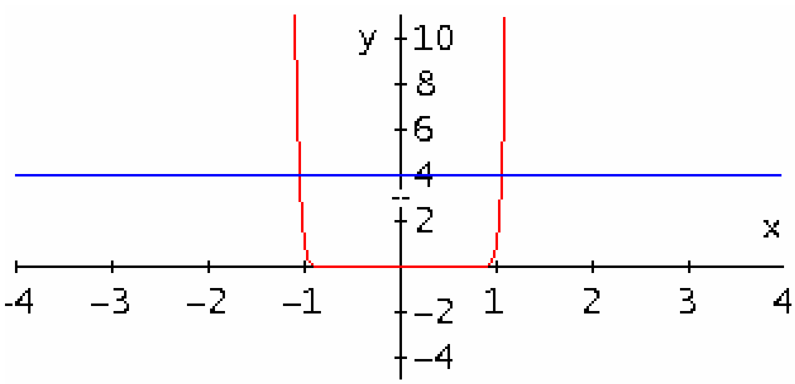
Des graphiques permettent de vérifier que g envoie bien l'intervalle [6, 6.2] dans lui-même et que, dans ce même intervalle, la valeur absolue de la dérivée de g est majorée par une constante K strictement inférieure à 1. Le théorème du point fixe s'applique donc. Prenons l'extrémité gauche de l'intervalle:

#17: FIXED_POINT(g, x, 6)

#18: [6, 6.087068817, 6.079170206, 6.079884066, 6.079819527, 6.079825362, 6.079824834, 6.079824882, 6.079824878, 6.079824878, 6.079824878, 6.079824878]

Problème #2: la présence du terme "+3" fait en sorte que la fonction LambertW de Maple ne peut être utilisée... Une fenêtre standard montre 2 intersections (donc 2 solutions) mais puisque l'exponentielle finira par dominer la puissance (pour des valeurs très très négatives), une TROISIÈME solution réelle existe.

#19: $z^{26} = 1.01^{-z} + 3$



La fonction "nsolve" de Derive peut être guidée avec –∞ comme borne inférieure!

#20: NSOLVE($z^{26} = 1.01^{-z} + 3$, z, -∞, -2)

#21: $z = - 2.662524454 \cdot 10^{4}$

Une table de valeurs nous aurait permis de constater qu'une troisième solution existe, située entre –26000 et –27000:

#22: TABLE($\left[z^{26}, 1.01^{-z} + 3\right]$, z, -15000, -30000, -1000)

#23:

$$\left[\begin{array}{ccc}
-1.5\cdot10^{4} & 3.787675244\cdot10^{108} & 6.616171278\cdot10^{64} \\
-1.6\cdot10^{4} & 2.02824096\cdot10^{109} & 1.386693635\cdot10^{69} \\
-1.7\cdot10^{4} & 9.8100666\cdot10^{109} & 2.906392772\cdot10^{73} \\
-1.8\cdot10^{4} & 4.335958657\cdot10^{110} & 6.091553847\cdot10^{77} \\
-1.9\cdot10^{4} & 1.768453418\cdot10^{111} & 1.276738251\cdot10^{82} \\
-2\cdot10^{4} & 6.7108864\cdot10^{111} & 2.675935572\cdot10^{86} \\
-2.1\cdot10^{4} & 2.386171548\cdot10^{112} & 5.608535013\cdot10^{90} \\
-2.2\cdot10^{4} & 7.998152883\cdot10^{112} & 1.175501582\cdot10^{95} \\
-2.3\cdot10^{4} & 2.540526541\cdot10^{113} & 2.463752062\cdot10^{99} \\
-2.4\cdot10^{4} & 7.682318074\cdot10^{113} & 5.163816292\cdot10^{103} \\
-2.5\cdot10^{4} & 2.220446049\cdot10^{114} & 1.082292293\cdot10^{108} \\
-2.6\cdot10^{4} & 6.15611958\cdot10^{114} & 2.268393262\cdot10^{112} \\
-2.7\cdot10^{4} & 1.642320326\cdot10^{115} & 4.754360743\cdot10^{116} \\
-2.8\cdot10^{4} & 4.227745295\cdot10^{115} & 9.964738678\cdot10^{120} \\
-2.9\cdot10^{4} & 1.052805015\cdot10^{116} & 2.088525088\cdot10^{125} \\
-3\cdot10^{4} & 2.541865828\cdot10^{116} & 4.377372238\cdot10^{129}
\end{array}\right]$$

#24: $f2(z) :\equiv z^{26} - 1.01^{-z} - 3$

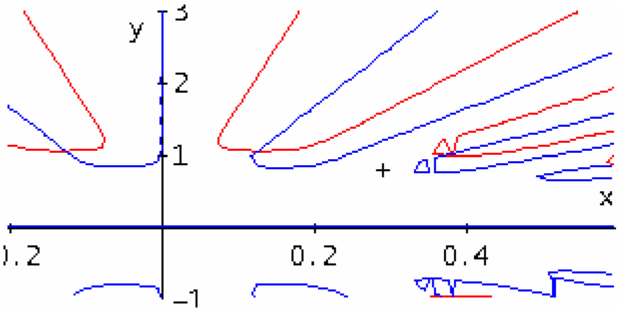
#25: $RE(f2(x + i\cdot y)) = 0$

#26: $$-\left(\frac{101}{100}\right)^{-x}\cdot COS\left(y\cdot LN\left(\frac{101}{100}\right)\right) + x^{26} - 325\cdot x^{24}\cdot y^{2} + 14950\cdot x^{22}\cdot y^{4} - 230230\cdot x^{20}\cdot y^{6} + 1562275\cdot x^{18}\cdot y^{8} - 5311735\cdot x^{16}\cdot y^{10} + 9657700\cdot x^{14}\cdot y^{12} - 9657700\cdot x^{12}\cdot y^{14} + 5311735\cdot x^{10}\cdot y^{16} - 1562275\cdot x^{8}\cdot y^{18} + 230230\cdot x^{6}\cdot y^{20} - 14950\cdot x^{4}\cdot y^{22} + 325\cdot x^{2}\cdot y^{24} - y^{26} - 3 = 0$$

#27: $IM(f2(x + i\cdot y)) = 0$

#28: $\left(\frac{101}{100}\right)^{-x}\cdot \mathrm{SIN}\left(y\cdot \mathrm{LN}\left(\frac{101}{100}\right)\right) + 2\cdot x\cdot y\cdot(13\cdot x^{24} - 1300\cdot x^{22}\cdot y^{2} + 32890\cdot x^{20}\cdot y^{4} - 328900\cdot x^{18}\cdot y^{6} + 1562275\cdot x^{16}\cdot y^{8} - 3863080\cdot x^{14}\cdot y^{10} + 5200300\cdot x^{12}\cdot y^{12} - 3863080\cdot x^{10}\cdot y^{14} + 1562275\cdot x^{8}\cdot y^{16} - 328900\cdot x^{6}\cdot y^{18} + 32890\cdot x^{4}\cdot y^{20} - 1300\cdot x^{2}\cdot y^{22} + 13\cdot y^{24}) = 0$

En traçant les 2 courbes (#26 et #28) dans une même fenêtre, une infinité d'intersections (solutions complexes de l'équation $f2(z) = 0$) se présentent. Nous allons choisir le couple (0.12, 1)



#29: NEWTONS([RE(f2(x + ι·y)), IM(f2(x + ι·y))], [x, y], [0.12, 1], 10)

#30:

0.12	1
0.1356956448	1.088548039
0.1311422562	1.061873117
0.1281462587	1.049333933
0.1272824115	1.04710648
0.1272412487	1.047049788
0.127241196	1.047049764
0.127241196	1.047049764
0.127241196	1.047049764
0.127241196	1.047049764
0.127241196	1.047049764

On aurait pu utiliser Newton à une seule variable complexe:

#31: NEWTON(f2(z), z, 0.12 + ι, 10)

#32: [0.12 + ι, 0.1356956449 + 1.08854804·ι, 0.1311422563 + 1.061873118·ι, 0.1281462588 + 1.049333934·ι, 0.1272824116 + 1.04710648·ι, 0.1272412488 + 1.047049788·ι, 0.127241196 + 1.047049764·ι, 0.127241196 + 1.047049764·ι, 0.127241196 + 1.047049764·ι, 0.127241196 + 1.047049764·ι, 0.127241196 + 1.047049764·ι]

Problème #3: pour a), "nsolve" ou "nsolutions" est rapide mais des solutions entières sont évidemment impossible...

#33: $\text{exp1} := \left(\prod_{k=1}^{20} (x - k)\right) - \frac{x^{19}}{10^{7}}$

#34: NSOLUTIONS(exp1, x, Real)

#35: [1, 2, 3, 4, 5, 20.78880561, 6.999745978, 6.000005824, 8.006075438, 8.928803402]

#36: APPROX(NSOLUTIONS(exp1, x, Real), 20)

#37: [1, 2, 3, 20.78880561, 8.928803402, 8.006075438, 6.999745978, 4.000000000, 6.000005824, 4.999999939]

#38: APPROX(NSOLUTIONS(exp1, x, Real), 30)

#39: [1, 2, 20.78880561, 8.928803402, 4.999999939, 8.006075438, 6.000005824, 2.999999999, 6.999745978, 4.000000000]

#40: APPROX(NSOLUTIONS(exp1, x, Real), 40)

#41: [1, 8.928803402, 20.78880561, 2.000000000, 2.999999999, 8.006075438, 4.000000000, 6.000005824, 6.999745978, 4.999999939]

#42: APPROX(NSOLUTIONS(exp1, x, Real), 50)

#43: [6.000005824, 8.006075438, 6.999745978, 4.999999939, 20.78880561, 2.999999999, 0.9999999999, 8.928803402, 2.000000000, 4.000000000]

#44: APPROX(NSOLUTIONS(exp1, x, Real), 60)

#45: [4.000000000, 6.999745978, 2.999999999, 0.9999999999, 8.006075438, 6.000005824, 20.78880561, 8.928803402, 2.000000000, 4.999999939]

#46: exp2(x) := x·√x·(√(x + 1) + √(x − 1) − 2·√x)

Pour calculer la limite de exp2(x) lorsque x tend vers l'infini, on peut, par exemple, remplacer x par $1/h$, simplifier et laisser h tendre vers 0 par la droite. La règle de l'Hospital s'applique alors et, en 2 coups, on trouve la valeur $-1/4$ (voir #54 plus bas). Derive trouve cette valeur sans nous indiquer comment...

#47: exp2(x)

#48: $\lim_{x \to \infty} \text{exp2}(x)$

#49: $-\frac{1}{4}$

#50: $\text{exp2}\left(\frac{1}{h}\right)$

#51: $$\frac{\left(\sqrt{\left(\frac{1 - h}{h}\right)} + \sqrt{\left(\frac{h + 1}{h}\right)}\right)\cdot \text{SIGN}(h)}{h^{3/2}} - \frac{2}{h^2}$$

#52: h_ :∈ Real (0, ∞)

#53: $$\frac{\left(\sqrt{\left(\frac{1 - h\_}{h\_}\right)} + \sqrt{\left(\frac{h\_ + 1}{h\_}\right)}\right)\cdot \text{SIGN}(h\_)}{h\_^{3/2}} - \frac{2}{h\_^2}$$

#54: $$\frac{\sqrt{(1 - h\_)} + \sqrt{(h\_ + 1)} - 2}{h\_^2}$$

#55: $\text{exp2}(10^6)$

#56: $$0$$

#57: $\text{APPROX}(\text{exp2}(10^6), 20)$

#58: $$-0.2500000003$$

#59: $\text{APPROX}(\text{exp2}(10^6), 50)$

#60: $$-0.2500000000$$

Problème #4: notez que le graphique a été produit en utilisant RK pour le système d'É.D. provenant du produit vectoriel des 2 gradients...

#61: $\left[\text{dist2} := x^2 + y^2 + z^2,\ \text{cont1} := x^2 + x\cdot y\cdot z + y^2 - 3,\ \text{cont2} := x^2 + y^2 + 2\cdot z^2 - 4\right]$

Les 5 équations provenant de la méthode des multiplicateurs de Lagrange sont rapidement obtenues par Derive:

#62: APPEND_COLUMNS(GRAD(dist2) − λ·GRAD(cont1) − μ·GRAD(cont2), [cont1], [cont2])

#63: $\left[-x\cdot(2\cdot\lambda + 2\cdot\mu - 2) - \lambda\cdot y\cdot z,\ -\lambda\cdot x\cdot z - y\cdot(2\cdot\lambda + 2\cdot\mu - 2),\ z\cdot(2 - 4\cdot\mu) - \lambda\cdot x\cdot y,\ x^2 + x\cdot y\cdot z + y^2 - 3,\ x^2 + y^2 + 2\cdot z^2 - 4\right]$

#64: $mat := \text{SOLUTIONS}\left(\left[-x\cdot(2\cdot\lambda + 2\cdot\mu - 2) - \lambda\cdot y\cdot z,\ -\lambda\cdot x\cdot z - y\cdot(2\cdot\lambda + 2\cdot\mu - 2),\ z\cdot(2 - 4\cdot\mu) - \lambda\cdot x\cdot y,\ x^2 + x\cdot y\cdot z + y^2 - 3,\ x^2 + y^2 + 2\cdot z^2 - 4\right], [x, y, z, \lambda, \mu], \text{Real}\right)$

#65: mat COL [1, ..., 3]

#66:
$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \\ \frac{\sqrt{(6\cdot\sqrt{5} - 6)}}{2} & \frac{\sqrt{(6\cdot\sqrt{5} - 6)}}{2} & \frac{\sqrt{5}}{2} - \frac{3}{2} \\ \frac{\sqrt{(6\cdot\sqrt{5} - 6)}}{2} & -\frac{\sqrt{(6\cdot\sqrt{5} - 6)}}{2} & \frac{3}{2} - \frac{\sqrt{5}}{2} \\ -\frac{\sqrt{(6\cdot\sqrt{5} - 6)}}{2} & \frac{\sqrt{(6\cdot\sqrt{5} - 6)}}{2} & \frac{3}{2} - \frac{\sqrt{5}}{2} \\ -\frac{\sqrt{(6\cdot\sqrt{5} - 6)}}{2} & -\frac{\sqrt{(6\cdot\sqrt{5} - 6)}}{2} & \frac{\sqrt{5}}{2} - \frac{3}{2} \end{bmatrix}$$

#67:
$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \\ 1.361654128 & 1.361654128 & -0.3819660112 \\ 1.361654128 & -1.361654128 & 0.3819660112 \\ -1.361654128 & 1.361654128 & 0.3819660112 \\ -1.361654128 & -1.361654128 & -0.3819660112 \end{bmatrix}$$

#68:
$$\text{VECTOR}\left(\left|\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \\ 1.361654128 & 1.361654128 & -0.3819660112 \\ 1.361654128 & -1.361654128 & 0.3819660112 \\ -1.361654128 & 1.361654128 & 0.3819660112 \\ -1.361654128 & -1.361654128 & -0.3819660112 \end{bmatrix}_i\right|, i, 1, 8\right)$$

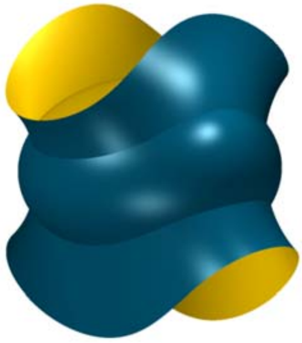
#69: [1.732050807, 1.732050807, 1.732050807, 1.732050807, 1.963186685, 1.963186685, 1.963186685, 1.963186685, 1.963186685]

Donc, les points (1, 1, 1), (1, −1, −1), (−1, 1, −1) et (−1, −1, 1) sont tous à une distance de √3 = 1.732 unité de l'origine (et sont donc les plus proches) tandis que les 4 autres points sont tous à une distance de 1.96 unité de l'origine et sont les plus éloignés.

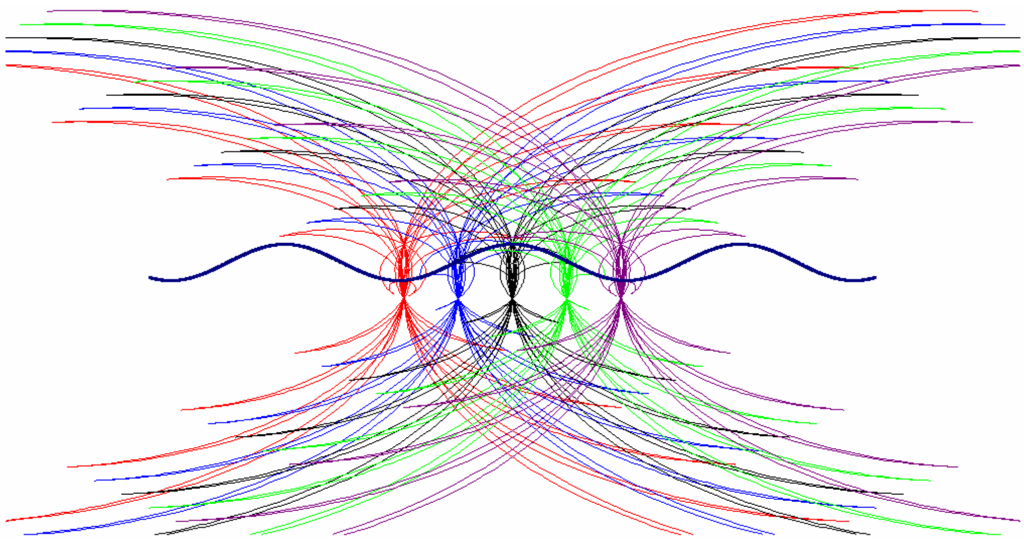
#70: $\left\|\left[\frac{\sqrt{(6\cdot\sqrt{5} - 6)}}{2}, \frac{\sqrt{(6\cdot\sqrt{5} - 6)}}{2}, \frac{\sqrt{5}}{2} - \frac{3}{2}\right]\right\|$

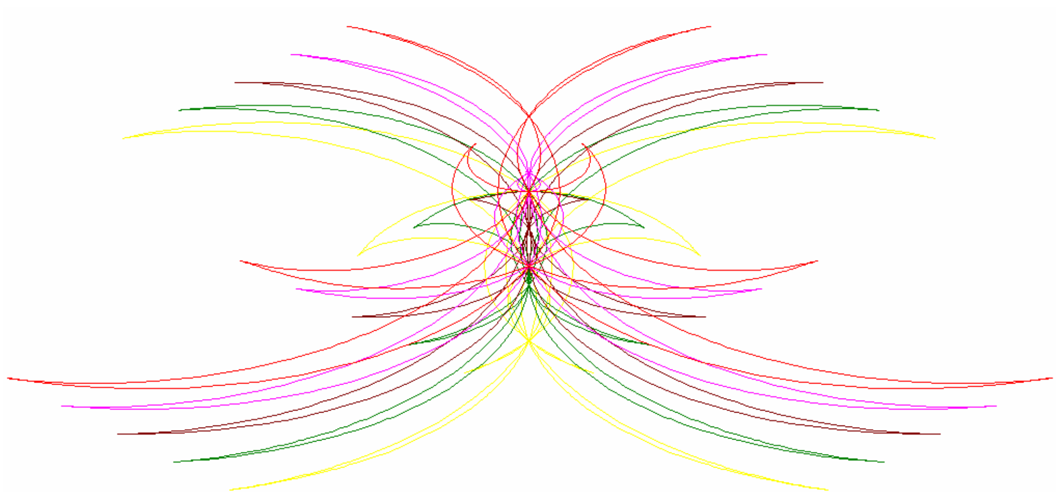
#71: $\frac{\sqrt{(6\cdot\sqrt{5} + 2)}}{2}$

#72: 1.963186686



Two examples of pedalcurves





An Optimization Problem: From a Spreadsheet to a CAS

Josef Böhm, Würmla, Austria

In the MNU Journal 60/6 from 2007 (MNU = Mathematisch-Naturwissenschaftlicher Unterricht = *Education in Maths and Science*) Dr. Helmut Brunner presented an optimization problem and its solution using the Solver from MS Excel recommending the spreadsheet as the appropriate tool for solving this and similar problems.

I read his contribution with interest – one reason is, that I have known Dr. Brunner since many years very well from my times as teacher. He was headmaster of a College of Business Administration in Mödling, Lower Austria, for many years, speaker in numerous seminars, textbook author and well known in the math community. I appreciate him as an excellent mathematician. He started very early using the programmable pocket computers for math teaching and he was very successful as head of the teachers working group in our federal state propagating the importance of probability theory and statistics in math education.

But I cannot agree with Dr. Brunner (Helmut) praising a spreadsheet as “the“ appropriate tool for tackling this problem.

> The problem: Find the pairs of points P(x_1,y_1) and Q(x_2,y_2) having minimal or maximal distance from each other with P being a point on the ellipse $x^2 + 4y^2 = 4$ and Q being a point on the circle $(x-2)^2 + (y-2)^2 = 1$.

This extremal value problem with four variables and two side conditions will be solved by using *Lagrange* multipliers: the extended main condition must be derived partially wrt to the four variables x_1, y_1, x_2 und y_2. These derivatives will be equated 0 and form together with the side conditions a nonlinear system of equations with 6 unknowns.

Main condition: $d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2} = \max, \min$

Side condition 1 $\quad x_1{}^2+4y_1{}^2-4=0 \qquad$ (eq 1)

Side condition 2 $\quad (x_2-2)^2+(y_2-2)^2-1=0 \qquad$ (eq 2)

Helmut takes the square of the distance which gives the following main condition extended by the multipliers:

$$mcl=(x_2-x_1)^2+(y_2-y_1)^2+\lambda\cdot\left(x_1{}^2+4y_1{}^2-4\right)+\mu\cdot\left((x_2-2)^2+(y_2-2)^2-1\right)$$

The partial derivatives give equations (3) to (6):

$$\frac{\partial\, mcl}{\partial x_1}=-2(x_2-x_1)+2\lambda x_1=0 \qquad \text{(eq 3)} \qquad \frac{\partial\, mcl}{\partial y_1}=-2(y_2-y_1)+8\lambda y_1=0 \qquad \text{(eq 4)}$$

$$\frac{\partial\, mcl}{\partial x_2}=2(x_2-x_1)+2\mu(x_2-2)=0 \qquad \text{(eq 5)} \qquad \frac{\partial\, mcl}{\partial y_2}=2(y_2-y_1)+2\mu(y_2-2)=0 \qquad \text{(eq 6)}$$

Helmut eliminates (by hand) the multipliers λ and μ. (Could be a nice task for students to repeat basic skills in manipulating expressions!) The remaining system of equations is complicated enough to resist all efforts to solve it by hand.

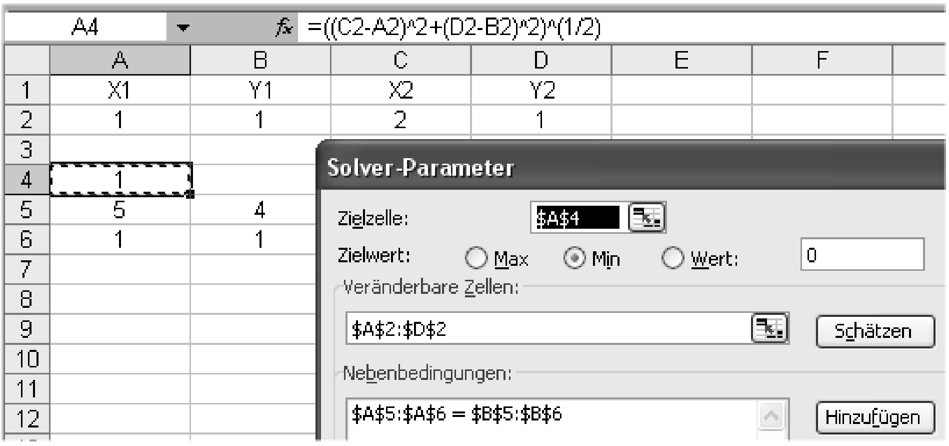
$$x_1^2 + 4y_1^2 = 4 \qquad \text{(eq 1)}$$

$$(x_2 - 2)^2 + (y_2 - 2)^2 = 1 \qquad \text{(eq 2)}$$

$$4y_1(x_2 - x_1) = x_1(y_2 - y_1) \qquad \text{(eq 3)}$$

$$(y_2 - 2)(x_2 - x_1) = (x_2 - 2)(y_2 - y_1) \qquad \text{(eq 4)}$$

Helmut does not even try to solve this system but he uses Excel and its Solver and finally obtains after setting appropriate initial guesses (which ones??) for the – hidden in the dark – iteration algorithm in a straight forward way two solutions. I show the mimimum solution:



One can read off the coordinates of the points P and Q and the minimum distance. For finding the maximum one has to change from Min to Max and change also the initial values.

A4: =((C2-A2)^2+(D2-B2)^2)^(1/2)

	A	B	C	D
1	X1	Y1	X2	Y2
2	1,38562112	0,72111969	1,56698698	1,09861245
3				
4	0,41880109			
5	4,00000035	4		
6	0,99999979	1		

As you can see, he does not solve the system, but tries to find the minimum/maximum under the given side conditions. You can also try to solve the complete – reduced - system using the solver:

A	B	C	D
X1	Y1	X2	Y2
1,38564120926219	0,721110075022646	1,56698736779241	1,09861204348036
=((C2-A2)^2+(D2-B2)^2)^(1/2)			
=A2^2+4*B2^2	4		
=(C2-2)^2+(D2-2)^2	1		
=4*B2*(C2-A2)-A2*(D2-B2)	0		
=(D2-2)*(C2-A2)-(C2-2)*(D2-B2)	0		

(The Excel files are among the downloadable files.)

Doubtless the Solver is an excellent numerical tool and I have used it very often – when working with Excel. But nowadays in many schools other – mathematical – tools are at disposal. First of all I have CAS in mind – what else? Most of the Computeralgebra Systems have built in the Groebner Bases (Learn more about Groebner Bases in DNL#68!) and are able to solve polynomial equation systems. Application of these systems set the main activity of the students into the mathematical definition and description of the problem and then into the interpretation of the solution (if there is one). Of course, the students need a certain familiarity with "their" CAS. But when students have to solve problems like this they will have acquired some experience.

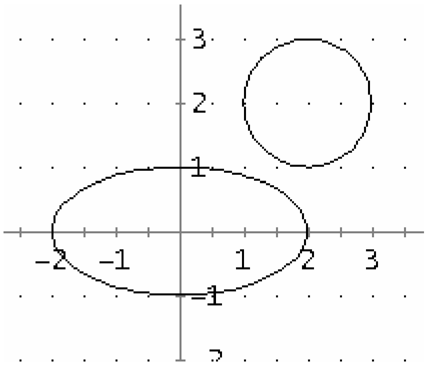
In the following I´ll show the complete procedure using *DERIVE* (followed by the solution process performed on the handheld and finally with TI-*Nspire*).

DERIVE allows parallel presentation in form of „abstract formulae“ and „visible geometric objects“. This is an enormous advantage for teacher and student.

#1: InputMode := Word

#2: $x^2 + 4\cdot y^2 = 4$

#3: $(x - 2)^2 + (y - 2)^2 = 1$



We set up the main condition (taking the square of the distance) and the two side conditions (the points are lying on the two respective curves) and introduce the Lagrange multipliers. Having done this, the mathematical problem is solved. We can leave the solution of the system of equations to our tool, a CAS. I try to keep the problem as general as possible and introduce parameters *m*, *n* and *r* for the coordinates of the center of the circle and its radius. (This proves to be helpful for further investigations.)

#4: $mc := (x2 - x1)^2 + (y2 - y1)^2$

#5: $sc1 := x1^2 + 4\cdot y1^2 - 4 = 0$

#6: $sc2(m, n, r) := (x2 - m)^2 + (y2 - n)^2 - r^2 = 0$

#7: $mcl(m, n, r) := mc + \lambda\cdot LHS(sc1) + \mu\cdot LHS(sc2(m, n, r))$

The MNU-Problem: m = n = 2, r = 1.
#8 is for the plot and #9 gives the solutions.

#8: [sc1, sc2(2, 2, 1)]

#9: $\left(\text{SOLUTIONS}\left(\frac{d}{d\ x1}\ mcl(2, 2, 1) \wedge \frac{d}{d\ x2}\ mcl(2, 2, 1) \wedge \frac{d}{d\ y1}\ mcl(2, 2, 1) \wedge \frac{d}{d\ y2}\ mcl(2, 2, 1) \wedge sc1 \wedge sc2(2, 2, 1), [x1, y1, x2, y2, \lambda, \mu]\right)\right)\ \text{COL}\ [1, 2, 3, 4]$

Expression #9 contains the complete system. As we are not interested in the values for λ and μ output of the first four columns will be sufficient:

#10:

1.38564	0.72111	2.43301	2.90138
-1.92051	-0.279113	2.86453	2.50257
1.38564	0.72111	1.56698	1.09861
-1.92051	-0.279113	1.13546	1.49742
3.85358 + 4.06302·i	-2.16447 + 1.80843·i	3.85358 + 4.06302·i	-2.16447 + 1.80843·i
3.85358 - 4.06302·i	-2.16447 - 1.80843·i	3.85358 - 4.06302·i	-2.16447 - 1.80843·i
2.9341 + 1.44216·i	-0.887665 + 1.19173·i	2.16497 + 0.624219·i	0.829455 + 0.0879741·i
2.9341 - 1.44216·i	-0.887665 - 1.19173·i	2.16497 - 0.624219·i	0.829455 - 0.0879741·i
2.9341 + 1.44216·i	-0.887665 + 1.19173·i	1.83502 - 0.624219·i	3.17054 - 0.0879741·i
2.9341 - 1.44216·i	-0.887665 - 1.19173·i	1.83502 + 0.624219·i	3.17054 + 0.0879741·i
1.47974 + 0.729097·i	0.83114 - 0.324517·i	1.47974 + 0.729097·i	0.83114 - 0.324517·i
1.47974 - 0.729097·i	0.83114 + 0.324517·i	1.47974 - 0.729097·i	0.83114 + 0.324517·i

We don´t need the complex solution, so we include the restriction on real values only. As I am planning to perform this procedure not only once, I define a function to deliver the real solutions for various parameters *m*, *n* and *r*. Then I apply this function for the MNU-paper parameters:

$$\#11:\quad sols(m, n, r) := \left(SOLUTIONS\left(\frac{d}{d\ x1} mc1(m, n, r) \wedge \frac{d}{d\ x2} mc1(m, n, r) \wedge \frac{d}{d\ y1} mc1(m, n, r) \wedge \frac{d}{d\ y2} mc1(m, n, r) \wedge sc1 \wedge sc2(m, n, r), [x1, y1, x2, y2, \lambda, \mu], Real\right)\right)\ COL\ [1, 2, 3, 4]$$

#12: sols(2, 2, 1)

#13:

1.38564	0.72111	2.43301	2.90138
-1.92051	-0.279113	2.86453	2.50257
1.38564	0.72111	1.56698	1.09861
-1.92051	-0.279113	1.13546	1.49742

Here we find the second important difference to the Excel-Solver-solution. (The first one is – in my opinion – the representation of the equations in *DERIVE* which is very close to the hand written procedure. I really appreciate the concept of the relations between cells in a spreadsheet and all the other valuable benefits of a spreadsheet, but this is a better job for a CAS.) Four solutions are presented in one single step and we need not enter any guesses. The criteria for determining the type of the extremal values are so difficult, that they are even rarely given explicit in textbooks. It is very easy to check the validity of the solutions by visualisation and inspection. We plot the solution points together with their connecting segments and we (let) calculate the respective distances. This can be done without any problems.

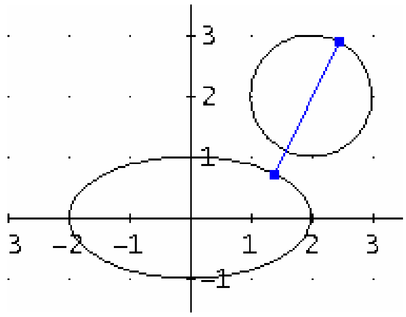
The third – important difference – is the easiness of visualisation of the solutions which can inspire for further investigations.

1. Solution

#14: $\begin{bmatrix} 1.38564 & 0.72111 \\ 2.433 & 2.90138 \end{bmatrix}$

#15: $\sqrt{(\mathrm{SUBST}(mc, [x1, y1, x2, y2], (sols(2, 2, 1))_1))}$

#16: 2.418801041

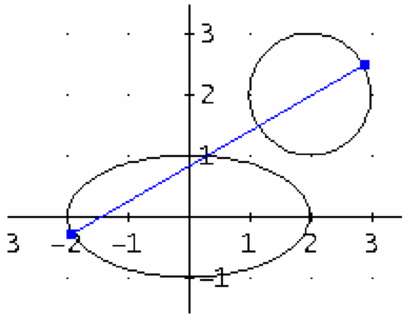


2. Solution

#17: $\begin{bmatrix} -1.92227 & -0.27915 \\ 2.86491 & 2.50258 \end{bmatrix}$

#18: $\sqrt{(\mathrm{SUBST}(mc, [x1, y1, x2, y2], (sols(2, 2, 1))_2))}$

#19: 5.53484358



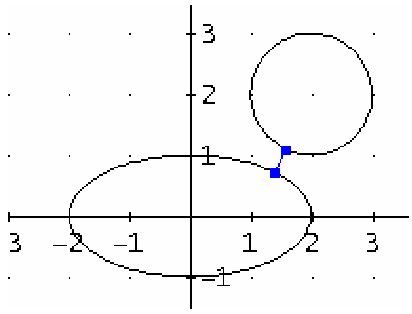
The Maximum

3. Solution

#20: $\begin{bmatrix} 1.38564 & 0.721109 \\ 1.56698 & 1.09861 \end{bmatrix}$

#21: $\sqrt{(\mathrm{SUBST}(mc, [x1, y1, x2, y2], (sols(2, 2, 1))_3))}$

#22: 0.4188010415



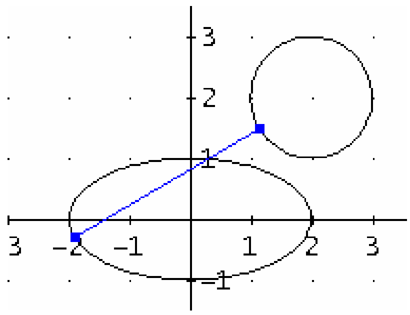
The Minimum

4. Solution

#23: $\begin{bmatrix} -1.92227 & -0.27915 \\ 1.13547 & 1.49742 \end{bmatrix}$

#24: $\sqrt{(\mathrm{SUBST}(mc, [x1, y1, x2, y2], (sols(2, 2, 1))_4))}$

#25: 3.534843535



An experienced *DERIVE* user will find the values for all distances belonging to the pairs of solution points in one single step:

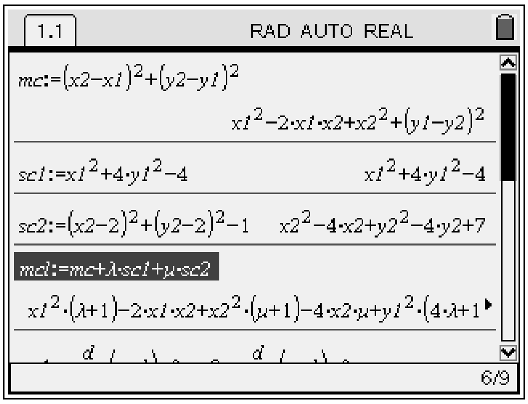
#26: $\mathrm{VECTOR}\left(\sqrt{\left((v_3 - v_1)^2 + (v_4 - v_2)^2\right)}, v, sols(2, 2, 1)\right)$

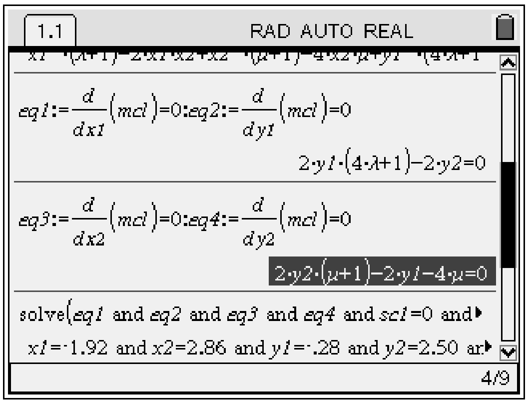
#27: [2.418801041, 5.53484358, 0.4188010415, 3.534843535]

The 2nd pair leads to the maximum and the 3rd pair to the minimum.

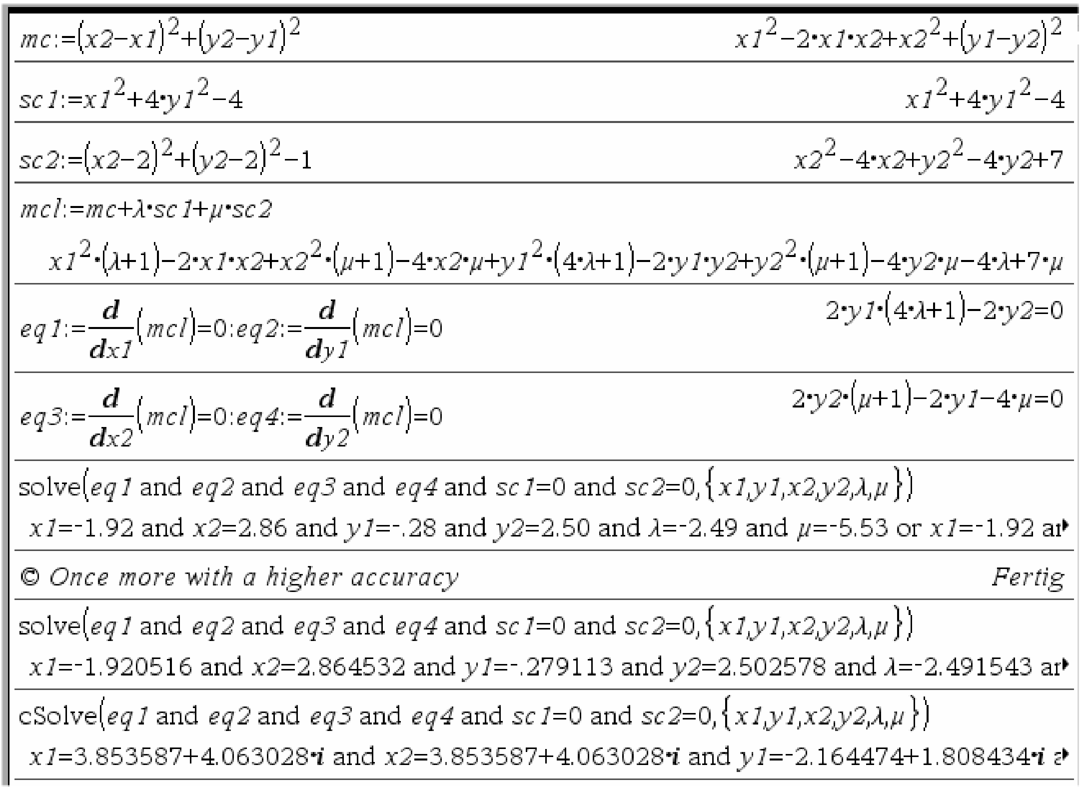
The CAS-solution offers an interesting question for the students: Where do the other solutions come from? Do they have any interpretation? (I will come back to this question later in this paper.)

As you might know the algorithm for solving systems of polynomial equations (Groebner bases) is also implemented in symbolic pocket calculators as TI-92+, Voyage 200 and TI-*Nspire*CAS. The next figures show the procedure performed using TI-*Nspire*CAS. Capacity of the devices is restricted by their smaller memory, but often one can help himself.

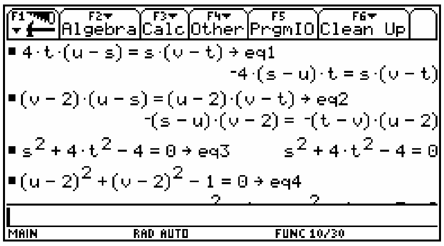


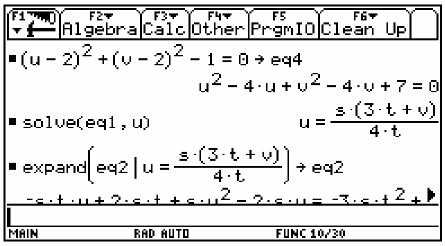


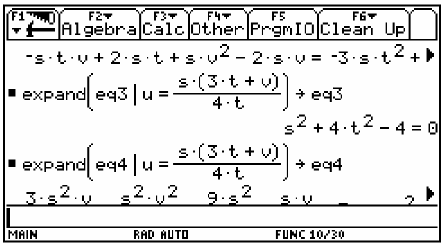
The above screens are from the *Nspire*-handheld and the screen below is from the PC-version.

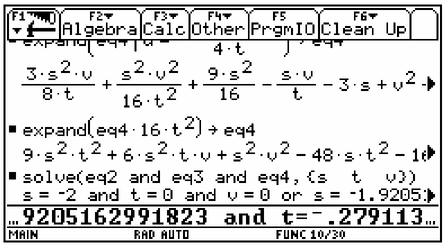


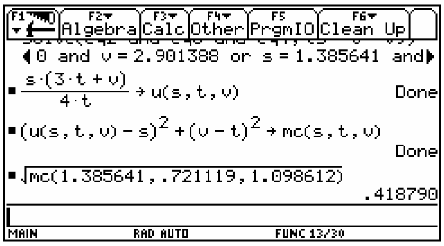
Even the reduced system from MNU6/60 proves to be too complex for the TI-92+/Voyage 200 and you will receive the message about too little memory. With little support, which addresses the students´ competence in handling the device one can "persuade" even the TI-92+ to release the solution what shall be demonstrated in the following:











I have to use variables *s*, *t*, *u* and *v*, because *y*1 and *y*2 are system variables on the TI-92 and Voyage 200. The system consisting of the four equations from above can be reduced once more leading to a – nonlinear - system with three unknowns. I substitute for *u* and this system can now be solved completely. The device gives 6 triples of solutions:

$s = 2$, $t = v = 0$
$s = -2$, $t = v = 0$
$s = 1{,}385641$, $t = 0{,}721119$, $v = 1{,}098612$
$s = 1{,}385641$, $t = 0{,}721119$, $v = 2{,}901388$
$s = -1{,}920516$, $t = -0{,}279113$, $v = 2{,}502578$
$s = -1{,}920516$, $t = -0{,}279113$, $v = 1{,}497422$

The first two solutions don´t make sense because of $t = 0$, the other four solutions are the same as in the *DERIVE* calculation. The missing value for *u* can be found by substitution. Then it is easy to calculate the distances. Maximum and minimum are found by inspection. Plotting is not so easy as it is using *DERIVE*.

In addition to the discussion about the "other" solutions hopefully the interest to inform about *Gröbner Bases* and the *Buchberger* algorithm will be roused. So a "Black Box" can be coloured – not bright white - but at least grey.

Dr. Brunner´s method is justified when students only have access to and experience with Excel. But it is my strong opinion that if a CAS can be used this should be preferred.

This is what – except the remarks on further investigations and some smaller changes – I wrote in my comments to Dr. Brunner´s article which appeared – without showing the TI-92 & TI-*Nspire* treatments – in MNU 61/1. As I am here not restricted in space in the DNL I can proceed …

My first intention was to solve the problem without using the multipliers because they are not part of the curriculum in secondary schools. Students might be inspired by the plots to try the following approach:

Find pairs of points on the two curves with parallel tangents and with the connecting segment perpendicular to the segments. In this case we need implicit differentiation of the curves.

$$x^2+4y^2=4 \qquad 2x+8yy'=0 \qquad \to y'=-\frac{x}{4y}$$

$$(x-2)^2+(y-2)^2=1 \qquad 2(x-2)+2(y-2)y'=0 \to y'=-\frac{x-2}{y-2}$$

Points P(x_1,y_1) and Q(x_2,y_2) are lying on the ellipse and circle, the slopes in P and Q must be equal and the slope of segment PQ is the negative reciprocal value of the slope of the tangents. This leads to four equations (without needing the multipliers because it is no extremal value problem any longer!):

$$x_1^2+4y_1^2-4=0 \qquad \text{(eq 1)}$$

$$(x_2-2)^2+(y_2-2)^2-1=0 \qquad \text{(eq 2)}$$

$$\frac{x_2-2}{y_2-2}=\frac{x_1}{4y_1} \to 4y_1(x_2-2)=x_1(y_2-2) \qquad \text{(eq 3)}$$

$$\frac{y_2-y_1}{x_2-x_1}=\frac{y_2-2}{x_2-2} \to (x_2-2)(y_2-y_1)=(y_2-2)(x_2-x_1) \qquad \text{(eq 4)}$$

Without looking back in the article we leave the solution again to the CAS:

```
#28: [eq1 := sc1, eq2 := sc2(2, 2, 1)]

#29: eq3 := 4·y1·(x2 - 2) = x1·(y2 - 2)

#30: eq4 := (x2 - 2)·(y2 - y1) = (y2 - 2)·(x2 - x1)

#31: SOLUTIONS(eq1 ∧ eq2 ∧ eq3 ∧ eq4, [x1, y1, x2, y2], Real)
```

#32:

$$\begin{bmatrix} -1.92051 & -0.279113 & 1.13546 & 1.49742 \\ 1.38564 & 0.721110 & 1.56698 & 1.09861 \\ -1.92051 & -0.279113 & 2.86453 & 2.50257 \\ 1.38564 & 0.721110 & 2.43301 & 2.90138 \end{bmatrix}$$

Compared with the solution from above we don´t find any difference -
- except the order of the solutions:

#33:

$$\begin{bmatrix} 1.38564 & 0.72111 & 2.43301 & 2.90138 \\ -1.92051 & -0.279113 & 2.86453 & 2.50257 \\ 1.38564 & 0.72111 & 1.56698 & 1.09861 \\ -1.92051 & -0.279113 & 1.13546 & 1.49742 \end{bmatrix}$$

Comparing the system with the reduced system from page 16 we find that both systems are nearly the same. Only in eq 3 the variables x_1 and y_1 are both substituted by 2. Any idea why?

Before talking about this we will use our generalized functions to investigate other cases. What will happen if the circle lies completely within the ellipse, what if both curves will intersect, …

We choose $m = -3$, $n = 2$ and $r = 7$:

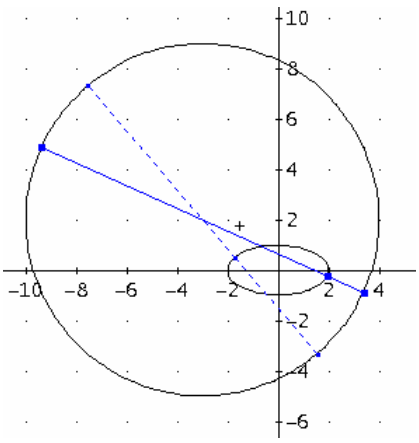
```
#39:  NotationDigits := 5

#40:  [sc1, sc2(-3, 2, 7)]

#41:  sols(-3, 2, 7)

       ⎡  1.9516   -0.2186    3.388   -0.86222 ⎤
       ⎢ -1.7254    0.5057   -7.5427   7.3257  ⎥
#42:   ⎢ -1.7254    0.5057    1.5427  -3.3257  ⎥
       ⎣  1.9516   -0.2186   -9.388    4.8622  ⎦

#43:  [[ 1.9516 -0.2186 ], [ -1.7254  0.5057 ], [ -1.7254  0.5057 ], [ 1.9516 -0.2186 ]]
      [[ 3.388 -0.86222 ]  [ -7.5427  7.3257 ]  [ 1.5427  -3.3257 ]  [ -9.388  4.8622 ]]

#44:  VECTOR(√((v₃ - v₁)² + (v₄ - v₂)²), v, sols(-3, 2, 7))

#45:  [1.574, 8.964, 5.0359, 12.425]

#46:  [[ 1.9516 -0.2186 ], [ 1.9516 -0.2186 ]]
      [[ 3.388 -0.86222 ]  [ -9.388  4.8622 ]]
```

There are again 4 solutions.

Plot of #43 with medium sized points and dotted connection lines show all solutions, #46 with large sized points and solid connections give the presentation of maximum and minimum.

Any idea to reduce the system of equations once more?

In the next experiment we have intersecting curves:



m = -3, n = -2, r = 4

```
#47:  [sc1, sc2(-3, -2, 4)]

#48:  sols(-3, -2, 4)

       ⎡ -1.7254   -0.5057   -5.5958   -5.0432  ⎤
       ⎢ -0.33802   0.98561  -0.33802   0.98561 ⎥
       ⎢  1.9516    0.2186   -6.6503   -3.6355  ⎥
#49:   ⎢ -1.7254   -0.5057   -0.40416   1.0432  ⎥
       ⎢  1.9516    0.2186    0.65033  -0.36444 ⎥
       ⎣  0.84742  -0.90579   0.84742  -0.90579 ⎦
```

#51: $\text{VECTOR}\left(\sqrt{\left((v_3 - v_1)^2 + (v_4 - v_2)^2\right)}, v, \text{sols}(-3, -2, 4)\right)$

#52: [5.964, 0, 9.4259, 2.0359, 1.4259, 0]

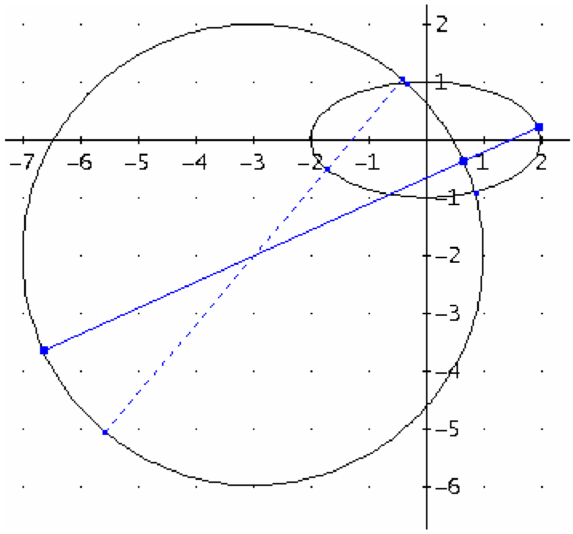
#53: $\left[\begin{bmatrix} 1.9516 & 0.2186 \\ -6.6503 & -3.6355 \end{bmatrix}, \begin{bmatrix} 1.9516 & 0.2186 \\ 0.65033 & -0.36444 \end{bmatrix}\right]$

This problem leads to six solutions of the system.

Two of them show the final function value 0. What is the interpretation of them?

Any idea to reduce the system of equations once more?

What if the circle lies completely inside of the ellipse?



m = n = 1/4, r = 1/2

#54: $\left[\text{sc1}, \text{sc2}\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}\right)\right]$

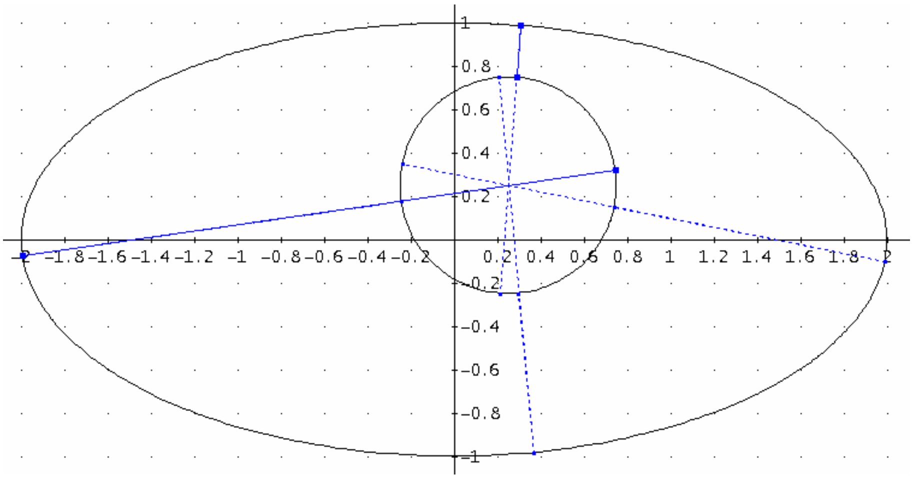
#55: $\text{sols}\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}\right)$

#56: $\begin{bmatrix} 1.9899 & -0.1001 & 0.74017 & 0.15137 \\ 0.36419 & -0.98328 & 0.20389 & 0.74787 \\ -1.9948 & -0.071402 & 0.74495 & 0.32086 \\ -1.9948 & -0.071402 & -0.24495 & 0.17913 \\ 1.9899 & -0.1001 & -0.24017 & 0.34862 \\ 0.3074 & 0.98811 & 0.21122 & -0.24849 \\ 0.36419 & -0.98328 & 0.2961 & -0.24787 \\ 0.3074 & 0.98811 & 0.28877 & 0.74849 \end{bmatrix}$

#58: $\text{VECTOR}\left(\sqrt{\left((v_3 - v_1)^2 + (v_4 - v_2)^2\right)}, v, \text{sols}\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}\right)\right)$

#59: [1.2748, 1.7385, 2.7677, 1.7677, 2.2748, 1.2403, 0.73855, 0.24034]

#60: $\left[\begin{bmatrix} 0.3074 & 0.98811 \\ 0.28877 & 0.74849 \end{bmatrix}, \begin{bmatrix} -1.9948 & -0.071402 \\ 0.74495 & 0.32086 \end{bmatrix}\right]$



This gives us eight solutions. Is it possible to have more than eight solutions? The plot demonstrates very clear what we sould have known for a while:

The connecting lines of all pairs P and Q are passing the center of the circle (which is a consequence of the fact that the segment is perpendicular to the parallel tangents). This is presented algebraically by substitution of x_2 and y_2 by 2 in the above equations on pages 16 and 22.

We can again redesign the solution process (which leads to a reduction the system of the equations):

Find the point(s) P on the ellipse where the tangent is perpendicular to the line PM (M = center of the circle).

$$x_1^2 + 4y_1^2 - 4 = 0 \qquad \text{(eq 1)}$$

$$\frac{y_1 - n}{x_1 - m} = \frac{4y_1}{x_1} \rightarrow x_1(y_1 - n) = 4y_1(x_1 - m) \qquad \text{(eq 5)}$$

```
#61:  eq5(m, n) := x1·(y1 - n) = 4·y1·(x1 - m)
#62:  sols2(m, n) := SOLUTIONS(eq1 ∧ eq5(m, n), [x1, y1], Real)
#63:  sols2(2, 2)
```

$$\#64: \begin{bmatrix} -1.92051 & -0.279113 \\ 1.38564 & 0.721110 \end{bmatrix}$$

$$\#65: \text{sols2}\left(\frac{1}{4}, \frac{1}{4}\right)$$

$$\#66: \begin{bmatrix} 1.98995 & -0.100101 \\ -1.99489 & -0.0714024 \\ 0.307407 & 0.988116 \\ 0.364199 & -0.983280 \end{bmatrix}$$

We have to find the intersection points of all lines PM with the ellipse in order to find the complete solutions and then select the points giving maximum and minimum distance.

I hope that you enjoyed my CAS-treatment and the extensions of the given Excel-Solver-application and that you share my preference of using CAS – symbolic calculation **and** plotting simultaneously. Using only the spreadsheet we miss a lot of opportunities to do mathematics.

Analysis of Variance - ANOVA

MacDonald R. Phillips
don.phillips@gmail.com

#1: Notation := Decimal

#2: Precision := Approximate

In general, the statistical technique analysis of variance, ANOVA, is used to test the hypothesis that the means of three or more populations are equal. The routines below can be used for one-way analysis of variance, ANOVA, two-way analysis of variance, TwoWayANOVA, and two-factor analysis of variance with interaction, TwoFactorANOVA.

1. One-Way Analysis of Variance (ANOVA)

ANOVA takes a matrix of three or more rows, each row representing a sample from a different population. The samples do not have to be of equal size. ANOVA tests whether the means of the samples are equal.

Example 1: Quality-awareness examinations were given to 6 employees at three different manufacturing plants. Based on the scores below, are the average scores the same in each plant?

#3: data :=

$$\begin{bmatrix} 85 & 75 & 82 & 76 & 71 & 85 \\ 71 & 75 & 73 & 74 & 69 & 82 \\ 59 & 64 & 62 & 69 & 75 & 67 \end{bmatrix}$$

#4: ANOVA(data)

#5:

One-Way Analysis of Variance

Ho:	Means Equal
Ha:	Means Not Equal
α:	0.05
F(α):	3.682320331

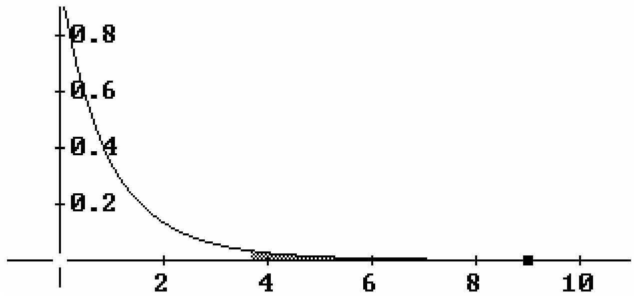
Source	DF	SS	MS	F	P(F)
Between	2	516	258	9	0.002702899474
Within	15	430	28.66666666		
Total	17	946			

Conclusion: Reject Ho

The ANOVA output shows Ho, Ha, α (the level of confidence) and the F value of α. Next it shows the ANOVA table with DF (the degrees for freedom), SS (sum-of-squares), MS (mean sum of squares), the F value for the test statistic, and P(F) (the probability of F. Since P(F) is less than α, the conclusion is to reject Ho and accept the Ha that not all the means are equal. This means that some of the means may be equal, but not all of them are. Other tests are needed to determine if all the means are not equal, or if some are, which ones they are.

The test may also be plotted showing the curve of the F-distribution, the rejection region, and the value of the test statistic.

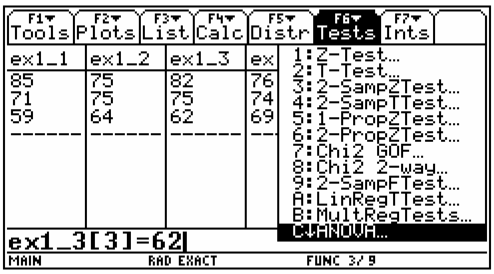
```
#6:    ANOVAPlot(data)
```

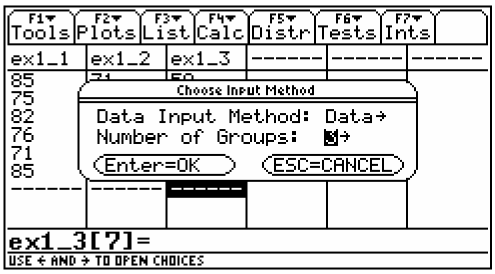


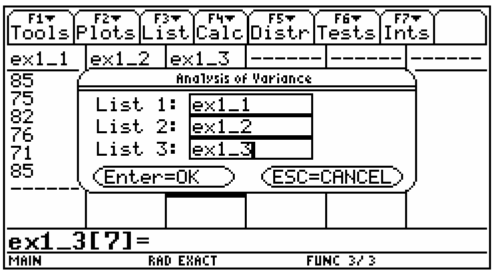
How to do it without Derive?

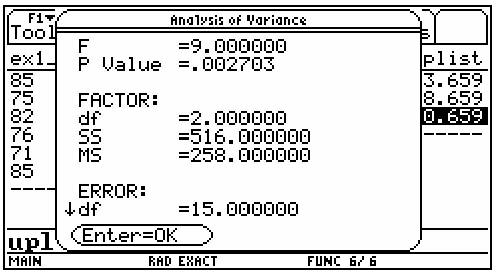
The TI-Voyage and Nspire offer a bundle of statistics tools with ANOVA among them. You can follow how to treat the first example using the Stats/List Editor-Application of the Voyage 200/ TI-89/92.

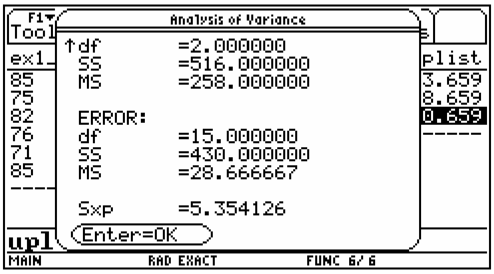
Please compare the results. Plotting the F-distribution and the rejection area is not so easy, but it can be done using the built-in fpdf-function together with the shade-option in the Graph-Window:

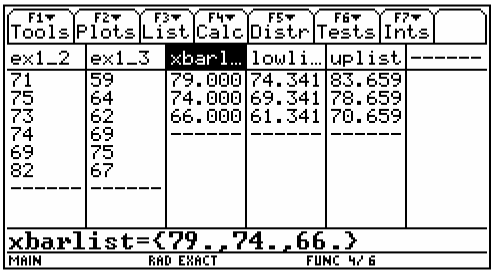




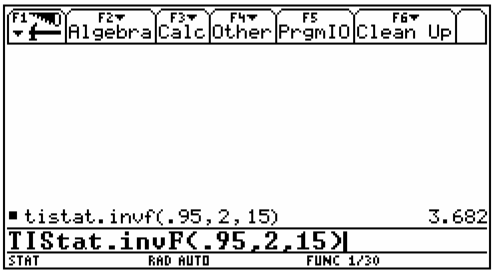


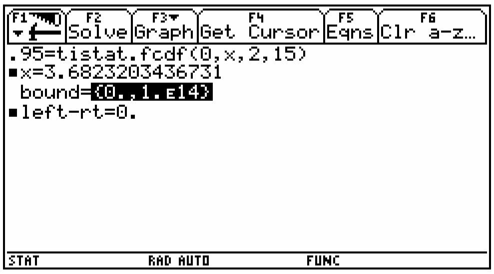


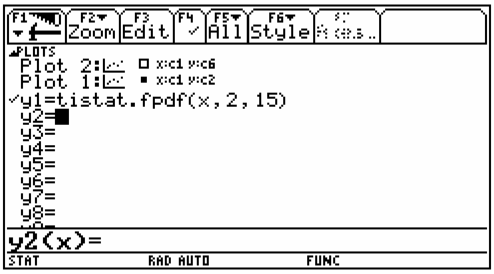


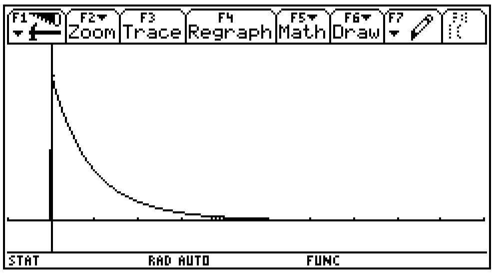


The Numeric Solver provides the bound of the rejection area. Then we can plot the distribution function and shade the rejection area.

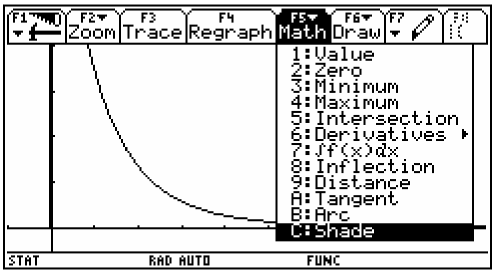


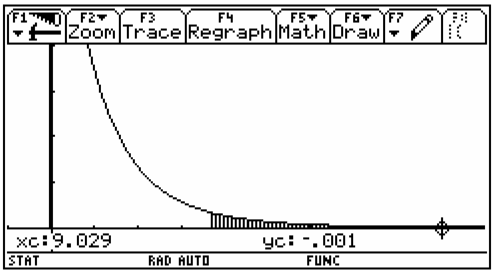






I adapt the Window-settings to obtain a better visualisation of the rejection area. The shading needs the lower bound 3.68.





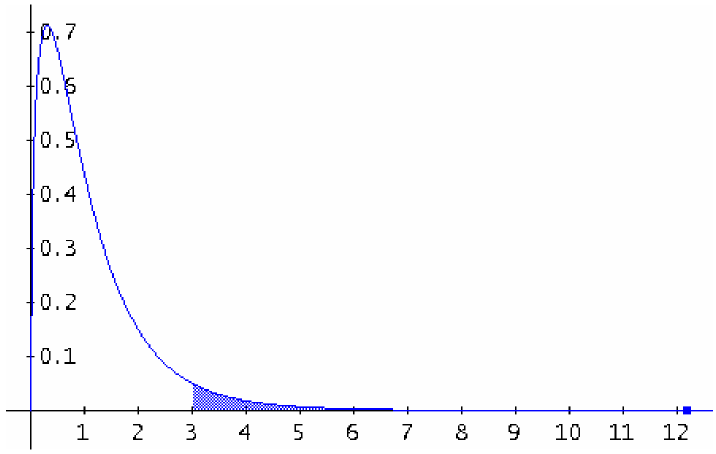
There is a second example illustrating the One-Way-Analysis followed by the TI-Nspire solution.

Example 2:

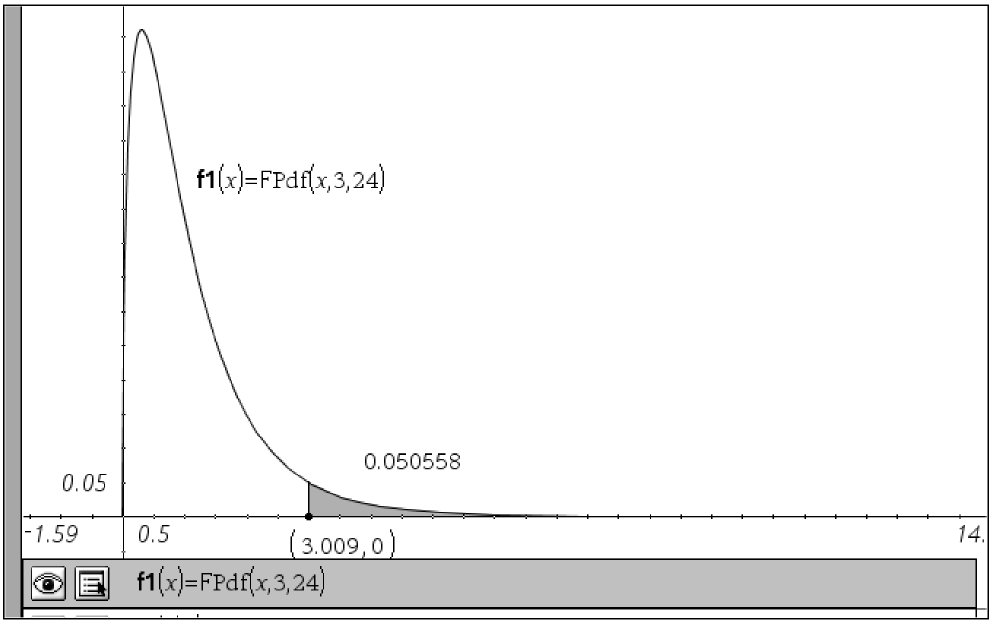
#17: example :=

$$\begin{bmatrix} 9 & 11 & 7 & 8 & 9 & 9 & 10 \\ 11 & 12 & 8 & 9 & 10 & 11 & 10 \\ 12 & 13 & 11 & 9 & 10 & 12 & 12 \\ 15 & 17 & 15 & 10 & 16 & 14 & 12 \end{bmatrix}$$

```
#18:   ANOVA(example)

                              One-Way Analysis of Variance

                                  Ho:      Means Equal
                                  Ha:    Means Not Equal
                                  α:           0.05
                                 F(α):     3.008786556

#19:    Source    DF       SS              MS              F              P(F)
                                                                                     -5
        Between    3   102.2857142   34.09523809   12.18723404   4.813001956·10
        Within    24   67.14285714   2.797619047
         Total    27   169.4285714

                               Conclusion:   Reject Ho

#20:   ANOVAPlot(example)
```



l1:={9,11,7,8,9,9,10}	{9,11,7,8,9,9,10}
l2:={11,12,8,9,10,11,10}	{11,12,8,9,10,11,10}
l3:={12,13,11,9,10,12,12}	{12,13,11,9,10,12,12}
l4:={15,17,15,10,16,14,12}	{15,17,15,10,16,14,12}
ANOVA *l1,l2,l3,l4*,0	*Fertig*
stat.results	"Titel" "ANOVA" "F" 12.1872 "PVal" .000048 "df" 3. "SS" 102.286 "MS" 34.0952 "dfError" 24. "SSError" 67.1429 "MSError" 2.79762 "sp" 1.67261 "CLowerList" "{...}" "CUpperList" "{...}" "x̄List" "{...}"
invF(.95,3,24)	3.00879



Compare again the results. Working with TI-Nspire is very similar with working on the TI-92 or TI-Voyage 200. The CAS-machine which does the work is the same one.

The additional advantage of Don´s tool is that the answer if the hypothesis should be rejected or not is given in the result.

2. Two-Way Analysis of Variance (TwoWayANOVA)

Unlike one-way ANOVA, two-way Analysis of Variance separates the column effects from the row effects. Or, in some textbooks, it separates the Treatment effects from the Block effects.

Example 3: Six air traffic controllers were assigned to test 3 proposals for the modification and redesign of their workstations to determine to what extent the 3 alternatives differ in terms of ther effects on controller stress. The results for Systems A, B, and C are in columns 1, 2, and 3 of the data matrix atc. The result for the six controllers are the rows of the data matrix. The Treatments are Systems A, B, and C, the 3 workstation redesigns. The Blocks are the six controllers. Since there is expected to be a large variance due to individual controller differences, two-way analysis of variance is used to control for this variance.

#7: $\text{atc} := \begin{bmatrix} 15 & 15 & 18 \\ 14 & 14 & 14 \\ 10 & 11 & 15 \\ 13 & 12 & 17 \\ 16 & 13 & 16 \\ 13 & 13 & 13 \end{bmatrix}$

#8: TwoWayANOVA(atc)

#9:

Two–Way Analysis of Variance

Source	DF	SS	MS	F	P(F)
Column	2	21	10.5	5.526315789	0.02418065429
Row	5	30	6	3.157894736	0.0573991616
Error	10	19	1.9		
Total	17	70			

The two-way ANOVA shows that the column effect, with an F-value of 5.526 and a probability of 0.02418, is significant at the 5 percent level of confidence. This means that the treatment means are different, controlling for individual controller differences. Taking the column averages, it appears that System B, with a mean value of 13 may be the least stress inducing workstation. However, it would be a good idea to test whether there are any significant differences between System A, with a mean value of 13.5, and System B.

#10: AVERAGE(atc')

#11: [13.5, 13, 15.5]

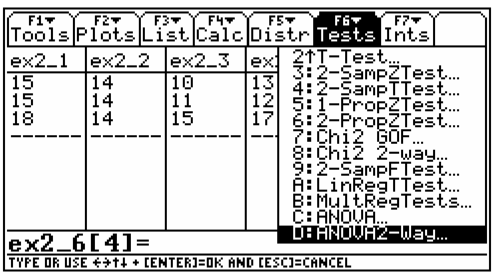
The Treatments and Blocks may be reversed in the data matrix. That is, the Treatments may be the rows and the blocks the columns. The ANOVA table has the same output, but with the column and row effects switched.

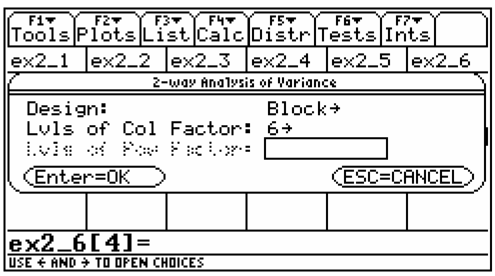
#12: TwoWayANOVA(atc')

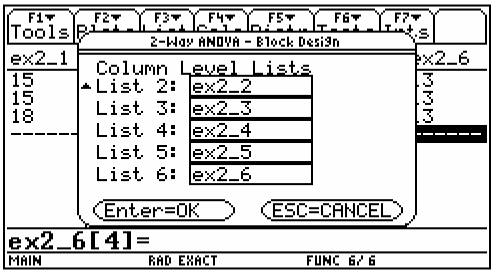
#13:

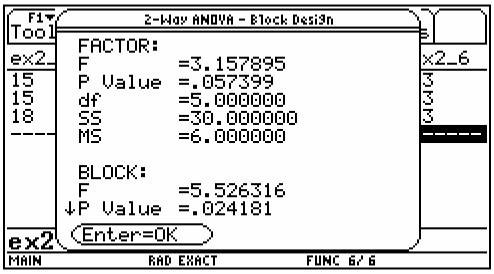
Two–Way Analysis of Variance

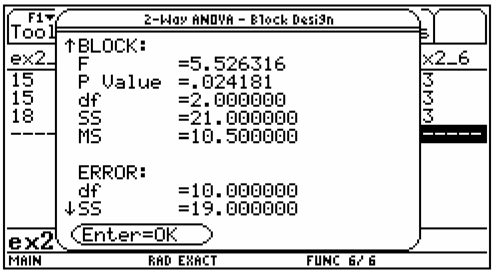
Source	DF	SS	MS	F	P(F)
Column	5	30	6	3.157894736	0.0573991616
Row	2	21	10.5	5.526315789	0.02418065429
Error	10	19	1.9		
Total	17	70			

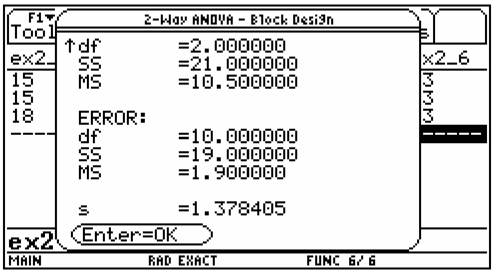












I´d like to add another example[1] to illustrate the application - Josef:

An extra example: Twelve calves are based on their weight at begin of the experiment grouped into 4 categories C1, C2, C3 and C4 with 3 calves in each group. They are fed with three different mixtures of feed, F1, F2 and F3. At the end of the experiment the increase of weight is measured (see table below). Test the hypothesis that the increase in weigth does neither depend on the weight at the begin of the experiment nor on the feed.

	F1	F2	F3
C1	7.0	14.0	8.5
C2	16.0	15.5	16.5
C3	10.5	15.0	9.5
C4	13.5	21.0	13.5

#21: animals :=

$$\begin{bmatrix} 7 & 14 & 8.5 \\ 16 & 15.5 & 16.5 \\ 10.5 & 15 & 9.5 \\ 13.5 & 21 & 13.5 \end{bmatrix}$$

#22: TwoWayANOVA(animals)

#23:

Two–Way Analysis of Variance

Source	DF	SS	MS	F	P(F)
Column	2	54.125	27.0625	5.756277695	0.0402165846
Row	3	87.72916666	29.24305555	6.220088626	0.02847615714
Error	6	28.20833333	4.701388888		
Total	11	170.0625			

#24: TwoWayANOVA(animals')

#25:

Two-Way Analysis of Variance

Source	DF	SS	MS	F	P(F)
Column	3	87.72916666	29.24305555	6.220088626	0.02847615714
Row	2	54.125	27.0625	5.756277695	0.0402165846
Error	6	28.20833333	4.701388888		
Total	11	170.0625			

The result shows that hypothesis 1 – the starting weight does not influence the increase in weight – can be rejected and that hypothesis 2 – the type of feed does not influence the increase in weight – can be rejected, too.

3. Two-Factor Analysis of Variance with Interaction (TwoFactorANOVA)

In some experiments it is desired to draw conclusions about more than one factor or variable. In "factorial experiments" all possible combination of factors are involved in the analysis. In a two-factor experiment, all combinations include the two factors and their interaction, i.e., factor One, factor Two, and One∗Two. If there are 'a' levels of factor One, and 'b' levels of factor Two, the factorial experiment is considered to have 9 treatment combinations. There must be 2 or more samples selected for each treatment combination.

Example 4: A university wants to determine how three different preparation programs affect students' GMAT scores before selecting one to adpot. They include a 3-hour review, a 1-day course, and a 10-week course. It is also thought that the college a student is in may affect the scores. The university looks at students from the college of business, engineering, and arts and sciences.
There are thus 3 treatments in each of the two factors for a total of 9 treatment cominations. The unversity decides to select 2 students for each treatment combination. In experimental design terminology, the sample size of 2 for each treatment combination indicates that there are 2 replications.

I´d like to start with the TI-.procedure, because for applying Don´s Derive tool it will be necessary to arrange the data in a special way. (I must admit that I had problems to transfer the Derive-arranged data into an appropriate form to tackle the problem with the TI-implemented function. I wrote to Don and on the next morning I found his answer in my mailbox:

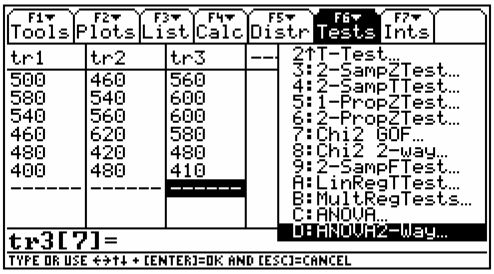
Josef,

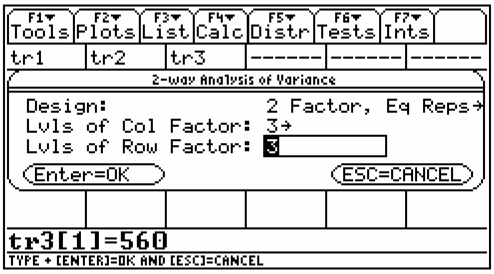
To use the TI-89 2-Way Analysis of Variance function the data has to be arranged a certain way. Factor One becomes the list1, list2, list3, etc columns in the Stats List/Editor. Factor Two becomes the rows of the lists. But the replications, which must be 2 or more, are entered in sequential rows. In the following matrix, Factor Two is listed in the rows and Factor One in the columns.

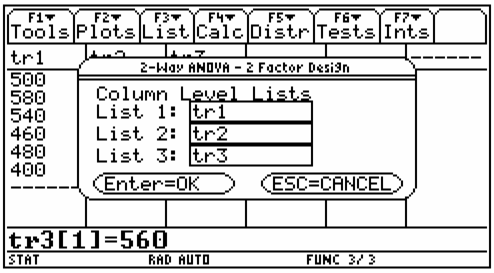
Factor Two/Factor One	1	2	3
1	500	460	560
1	580	540	600
2	540	560	600
2	460	620	580
3	480	420	480
3	400	480	410

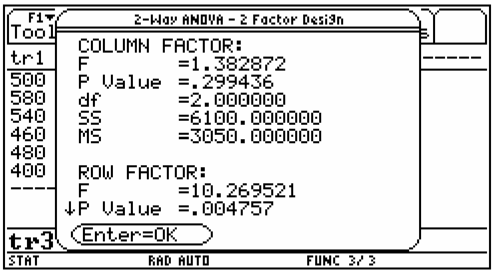
I am proud that I nearly had done it. Unfortunately I did not arrange the rows in the correct order, I took 1, 2, 3, 1, 2, 3 and did not consider the "replications" in the right way.

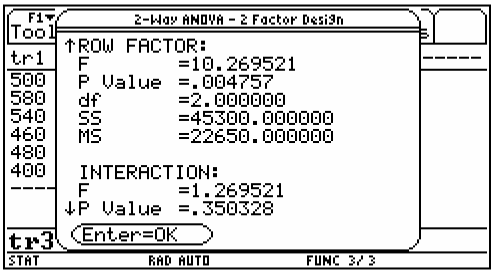
I´d like to start with the TI-.procedure, because for applying Don´s Derive tool it will be necessary to arrange the data in a special way.

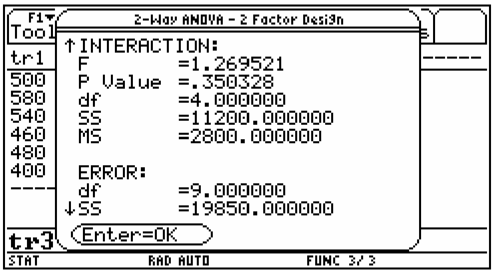


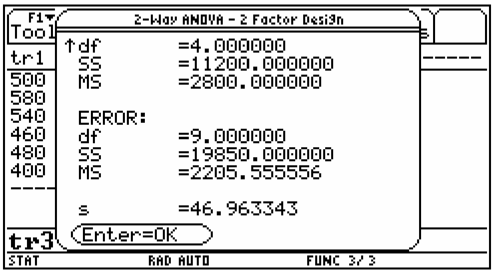












Interpretation will be given after performing Don´s Derive procedure, which is following.

The data matrix for the two-factor analysis of variance is entered in a different manner. The first column indicates the level of Factor One, the second column indicates the level of Factor Two, and the third column is the value of the treatment combination. The program requires that the levels of each factor be numbered sequentially from 1 to n, n being the number of levels for that factor. Since 2 students are selected for each of the 9 treatment combinations, there are 18 data points to enter.

#14: gmat :=

1	1	500
1	2	540
1	3	480
1	1	580
1	2	460
1	3	400
2	1	460
2	2	560
2	3	420
2	1	540
2	2	620
2	3	480
3	1	560
3	2	600
3	3	480
3	1	600
3	2	580
3	3	410

#15: TwoFactorANOVA(gmat)

#16:

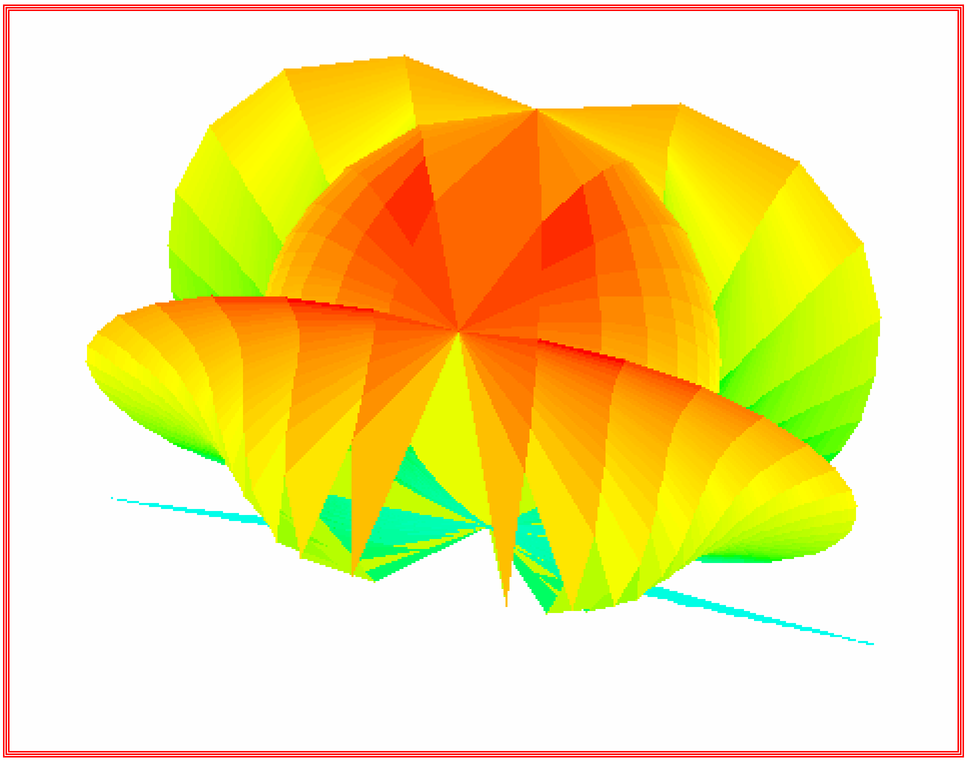
Two-Factor Analysis of Variance with 2 Replicates

Source	DF	SS	MS	F	P(F)
Factor One	2	6100	3050	1.382871536	0.2994361085
Factor Two	2	$4.53\cdot10^{4}$	$2.265\cdot10^{4}$	10.26952141	0.004756718049
Interaction	4	$1.12\cdot10^{4}$	2800	1.26952141	0.3503277693
Error	9	$1.985\cdot10^{4}$	2205.555555		
Total	17	$8.245\cdot10^{4}$			

The results allow us to conclude three things, assuming an α of 5%. First, Factor One, with a F-value of 1.38 and a p-value of .299, does not allow us to reject the Null Hypothesis so we conclude that there is no difference in the three test preparation programs on GMAT scores. However, Factor Two, with a F-value of 10.269 and a p-value of .00476, allows us to reject the Null Hypothesis and conclude that there is a difference in the colleges' ability to prepare students for the GMAT. And, F-value and p-value for the interaction effect gives us no reason to believe that the 3 preparation programs differ in their ability to prepare students from different colleges for the GMAT.

Don wrote:
….. Also, for the Two Factor ANOVA I could have programed the function to accept the data the same as the TI-89 does. But I chose the way I did because the data does not have to be entered in any particular order; the data points could have been entered in any random order.

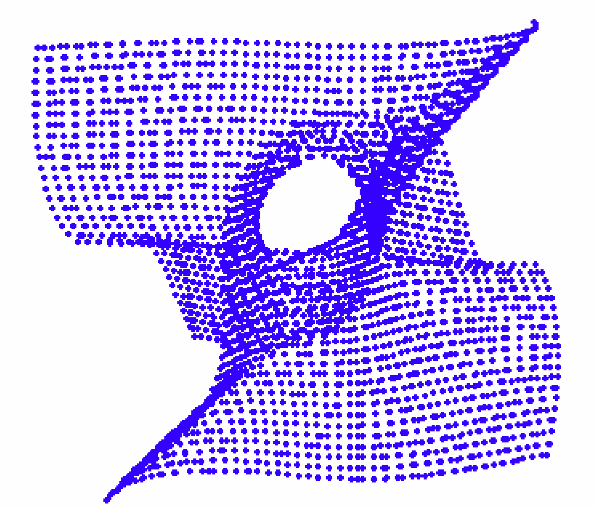
The 3D-plot of a nice Pedal Surface

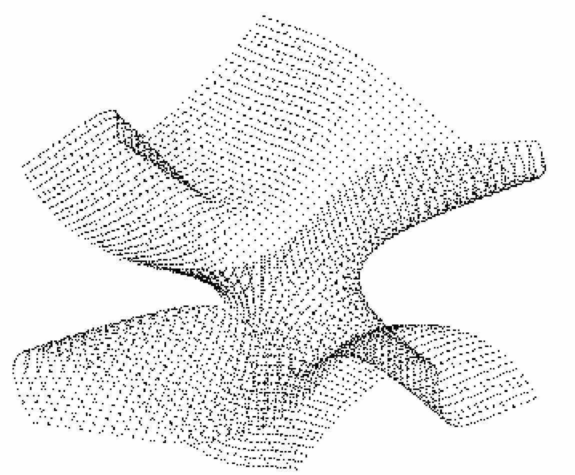


$$\text{pedsurf}\left(\left[u^2 - 2\cdot v^2,\ u + v,\ \frac{u^2 + v^2}{2}\right],\ u,\ v,\ [0,\ 0,\ -5]\right)$$

(see in the next DNL)

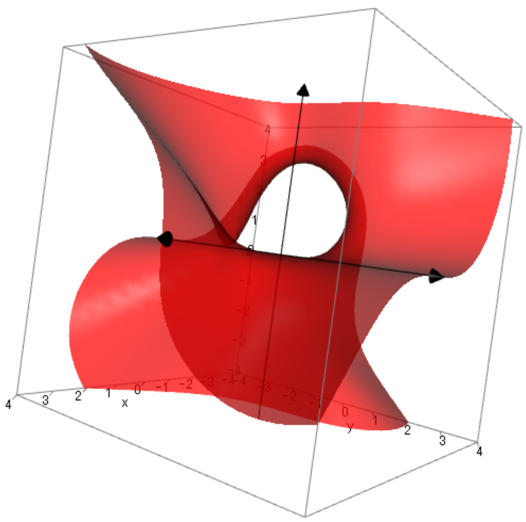
Surface #4: $x^3y + z^3x + y^3z + z^3 + 5z = 0$

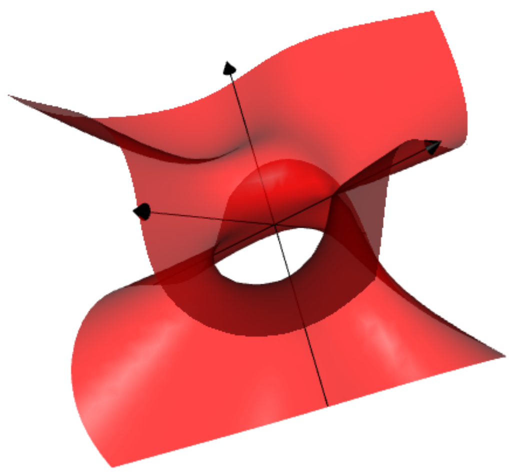




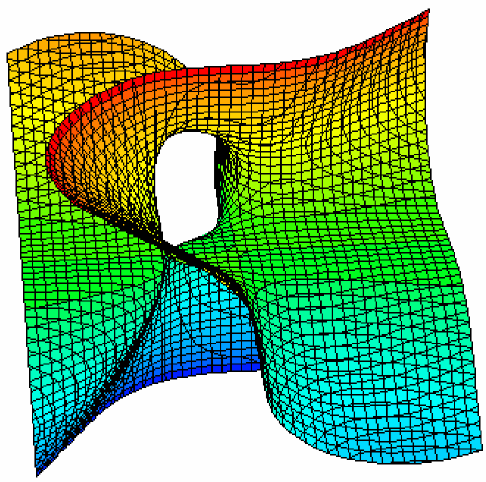
```
VECTOR(ContourPts_XY(x^3·y + z^3·x + y^3·z + z^3 + 5·z = 0, z_, -5, 5, -5, 5, 0.25, 0.25), z_, -5, 5, 0.25)
```

```
ImplicitDots(x^3·y + z^3·x + y^3·z + z^3 + 5·z = 0, [-5, -5, -5], [5, 5, 5], 0.2)
```

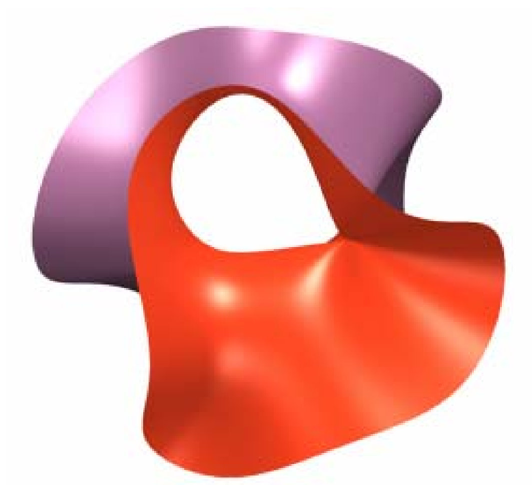




Autograph plots

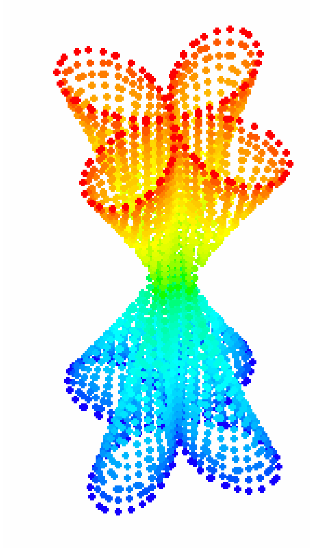


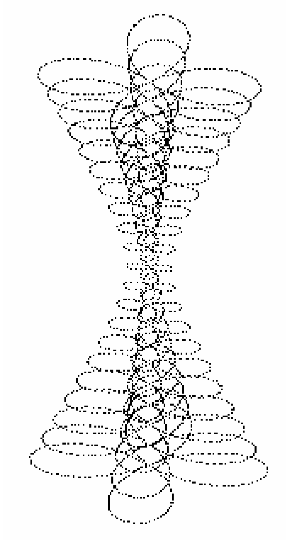
DPGraph



Surfer

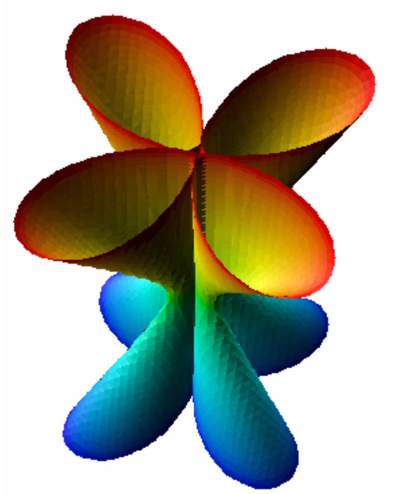
Surface #5: $\left(x^2+y^2\right)^3 = x^2y^2\left(z^2+1\right)$

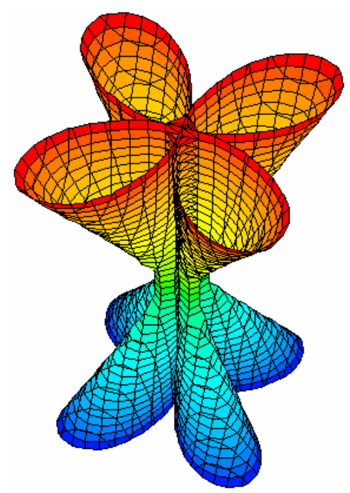




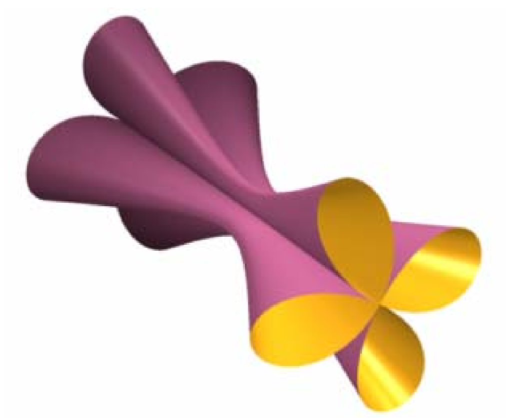
```
VECTOR(ContourPts_XY((x^2 + y^2)^3 = x^2·y^2·(z^2 + 1), z_, -5, 5, -5, 5, 0.25, 0.25), z_, -5, 5, 0.25)
```

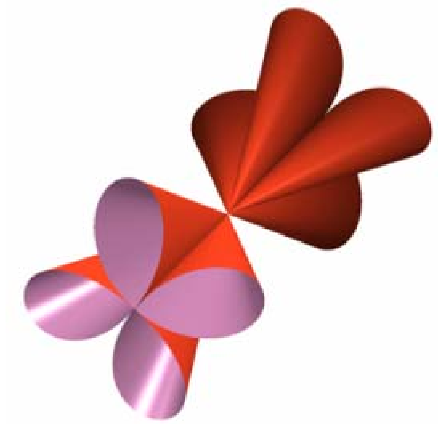
```
VECTOR(ContourDots_XY((x^2 + y^2)^3 - x^2·y^2·(z^2 + 1), 1), 1, -5, 5, 0.5)
```





DPGraph Plots





Surfer Plots (right one with a variation = instead of z^2+1 simply z^2)

Incenter and Excenters of a Tetrahedron

David Sjöstrand, Onsala, Sweden

In this paper we often identify a point *P* with the vector $\overrightarrow{OP}$.

Some definitions

$line(A,B)$ is the line passing the points *A* and *B*.

Triangle (*A,B,C*) is the triangle with vertices *A*, *B* and *C*.

Area(*A,B,C*) is the area of *Triangle* (*A,B,C*)

$d = Area(B,C,A)$

$a = Area(C,D,B)$

$b = Area(A,C,D)$

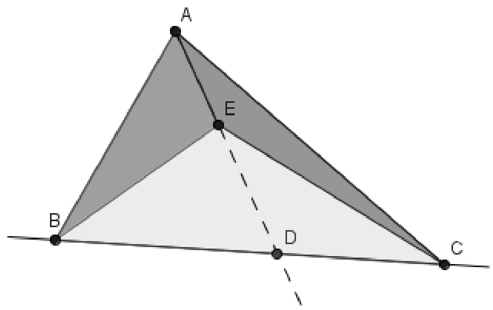
$c = Area(B,A,D)$

$Tetrahedron(A,B,C,D)$ is the tetrahedron with vertices *A*, *B, C* and *D.*

$Volume(A,B,C,D)$ is the volume of $Tetrahedron(A,B,C,D)$.

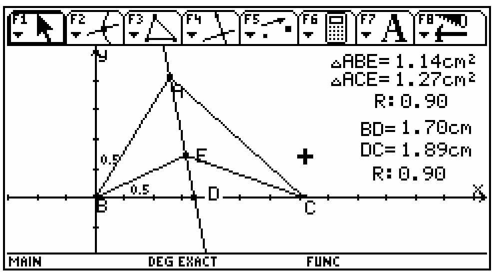
Lemma 1

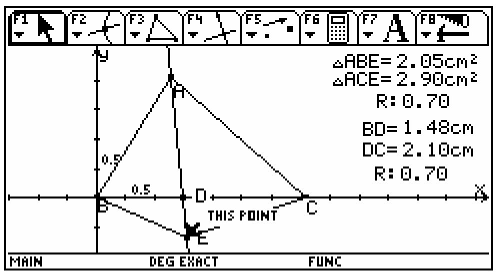
D divides the line segment *BC* in the ratio $Area(AEC) : Area(AEB)$ counted from *C*.

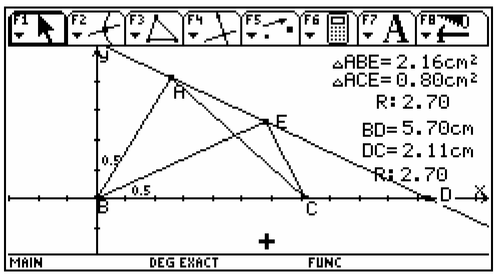


We can immediately verify this lemma using any dynamic geometry software.

Below can see the screens of a Voyage 200 with the Cabri-Application:







Proof:

$$\frac{Area(A,B,D)}{Area(A,D,C)} = \frac{BD}{DC} = \frac{Area(E,B,D)}{Area(E,D,C)}.$$

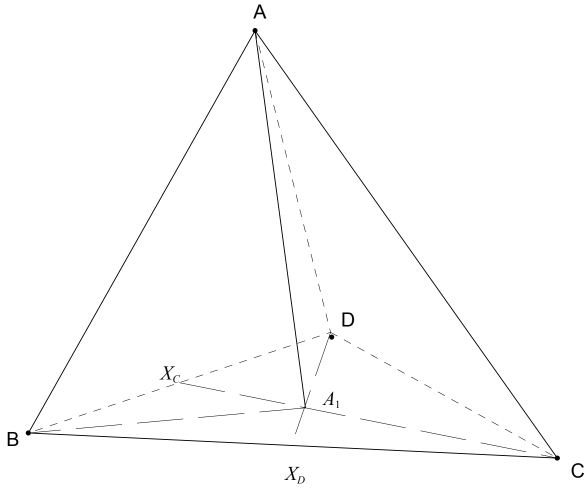
Therefore $\dfrac{Area(A,E,B)}{Area(A,E,C)} = \dfrac{BD}{DC}$.

Q.E.D.

Theorem 1

The incenter of $Tetrahedron(A,B,C,D)$ is

$$\frac{aA+bB+cC+dD}{a+b+c+d}.$$



Proof:

$A_1 \in Triangle(B,C,D)$ and $line(A,A_1)$ is the line having equal distances to the planes $plane(A,B,C)$, $plane(A,B,D)$ and $plane(A,C,D)$.

We define points $B_1 \in Triangle(A,C,D)$, $C_1 \in Triangle(B,A,D)$ and $D_1 \in Triangle(B,C,A)$ and lines $line(B,B_1)$, $line(C,C_1)$ and $line(D,D_1)$ in a corresponding manner. The incenter of $Tetrahedron(A,B,C,D)$ is the intersection point of these four lines.

Let *e* be the common distance between A_1 and $plane(A,C,D)$ and between A_1 and $plane(A,B,D)$.

Now $Volume(A,A_1,C,D)=\frac{be}{3}$ and $Volume(A,A_1,B,D)=\frac{ce}{3}$.

Let h_a be the altidude of $Tetrahedron(A,B,C,D)$ from *A* to $plane(B,C,D)$. We receive

$Volume(A,A_1,C,D)=\frac{h_a area(A_1,C,D)}{3}$ and $Volume(A,A_1,B,D)=\frac{h_a area(A_1,B,D)}{3}$.

This means that $$\frac{b}{c}=\frac{\frac{be}{3}}{\frac{ce}{3}}=\frac{Volume(A,A_1,C,D)}{Volume(A,A_1,B,D)}=\frac{\frac{h_a area(A_1,C,D)}{3}}{\frac{h_a area(A_1,B,D)}{3}}=\frac{area(A_1,C,D)}{area(A_1,B,D)}$$

By Lemma 1 we now conclude that X_D divides the line segment *BC* in the ratio *b:c* counted from *C* to *B*. This means that $X_D=\frac{bB+cC}{b+c}$.

By symmetry we have that $X_C=\frac{bB+dD}{b+d}$.

The point $\frac{(b+c)X_D+dD}{b+c+d}=\frac{bB+cC+dD}{b+c+d}$ is on $line(X_D,D)$

The point $\frac{(b+d)X_C+cC}{(b+d)+c}=\frac{bB+dD+cC}{b+d+c}=\frac{bB+cC+dD}{b+c+d}$ is on $line(X_C,D)$.

Therefore $\frac{bB+cC+dD}{b+c+d}$ is the intersection point between $line(X_d,D)$ and thus

$$A_1=\frac{bB+cC+dD}{b+c+d}.$$

By symmetry $B_1 = \dfrac{aA + cC + dD}{a + c + d}$.

As $\dfrac{(b+c+d)A_1 + aA}{(b+c+d)+a} = \dfrac{aA + bB + cC + dD}{a+b+c+d}$ is on line $line(A, A_1)$

and $\dfrac{(a+c+d)B_1 + bB}{(a+c+d)+b} = \dfrac{aA + bB + cC + dD}{a+b+c+d}$ is on line $line(B, B_1)$, we see that

$\dfrac{aA + bB + cC + dD}{a+b+c+d}$ is the intersection point between $line(A, A_1)$ and $line(B, B_1)$.

Thus the incenter of $Tetrahedron(A, B, C, D)$ is $\dfrac{aA + bB + cC + dD}{a+b+c+d}$.

Q.E.D.

Theorem 2

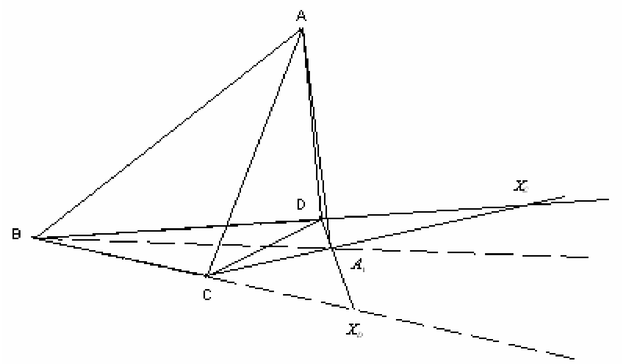
The excenters of $Tetrahedron(A, B, C, D)$ are given by

$\dfrac{-aA + bB + cC + dD}{-a+b+c+d}$, $\dfrac{aA - bB + cC + dD}{a-b+c+d}$, $\dfrac{aA + bB - cC + dD}{a+b-c+d}$ and $\dfrac{aA + bB + cC - dD}{a+b+c-d}$.

We will first prove that the excenter of $Tetrahedron(A, B, C, D)$ lying opposite to the plane defined by *A*, *D* and *C* counted from *B* is given by $\dfrac{aA - bB + cC + dD}{a-b+c+d}$

Proof: Everything we did in the proof of Theorem 1 can be repeated word by word with the only exceptions that we have to replace *b* by –*b* in the expressions for X_C and X_D and in all the subsequent expressions. It is straightforward to justify this.

If the line having equal distances to the planes $plane(A, B, C)$, $plane(A, B, D)$ and $plane(A, C, D)$ is parallel to $plane(B, C, D)$ we argue as follows:



Since $\dfrac{aA - bB + cC + dD}{a-b+c+d}$ and excenter of $Tetrahedron(A, B, C, D)$ lying opposite to the plane defined by *A*, *D* and *C* counted from *B* both are continuous functions of the vertices of the tetrahedron it follows that the expression, $\dfrac{aA - bB + cC + dD}{a-b+c+d}$ for the excenter, is valid even in this case.

By symmetry the three other excenters of $Tetrahedron(A, B, C, D)$ are given by

$\dfrac{-aA + bB + cC + dD}{-a+b+c+d}$, $\dfrac{aA + bB - cC + dD}{a+b-c+d}$ and $\dfrac{aA + bB + cC - dD}{a+b+c-d}$.

Q.E.D.

This is the *DERIVE*-application of the results from above together with a graphic representation of the tetrahedron together with the spheres:

Inspheres and Exspheres of a Tetrahedron

David Sjöstrand

#1: CaseMode := Sensitive

#2: InputMode := Word

Area of a triangle with vertices A, B and C:

#3: $\text{area}(A, B, C) := \frac{1}{2}\cdot|(B - A) \times (C - A)|$

Area of the faces of a tetrahedron with vertices A, B, C and D:

#4: [a := area(B, C, D), b := area(A, C, D), c := area(B, A, D), d := area(A, B, C)]

Volume of a tetrahedron with vertices A, B, C and D.
Radius of the inscribed circle of this tetraedron.

#5: $\left[V := \frac{1}{6}\cdot|(D - A) \bullet (B - A) \times (C - A)|,\ \text{radius}(a, b, c, d) := \frac{3\cdot V}{a + b + c + d}\right]$

Sphere with radius r and center M:

#6: $\text{Sphere}(r, s, t, M) := \left[M_1 + r\cdot\text{COS}(s)\cdot\text{SIN}(t),\ M_2 + r\cdot\text{SIN}(s)\cdot\text{SIN}(t),\ M_3 + r\cdot\text{COS}(t)\right]$

Incenter of a tetrahedron with vertices A, B, C and D:

#7: $\text{incenter}(a, b, c, d) := \frac{1}{a + b + c + d}\cdot(a\cdot A + b\cdot B + c\cdot C + d\cdot D)$

#8: plane(X, Y, Z) := DET([X − Y, X − Z, [x, y, z] − X]) = 0

#9: [plane(A, B, C), plane(A, B, D), plane(A, D, C), plane(D, B, C)]

The insphere:

#10: Sphere(radius(a, b, c, d), s, t, incenter(a, b, c, d))

The four exspheres:

#11: Sphere(radius(−a, b, c, d), s, t, incenter(−a, b, c, d))

#12: Sphere(radius(a, −b, c, d), s, t, incenter(a, −b, c, d))

#13: Sphere(radius(a, b, −c, d), s, t, incenter(a, b, −c, d))

#14: Sphere(radius(a, b, c, −d), s, t, incenter(a, b, c, −d))

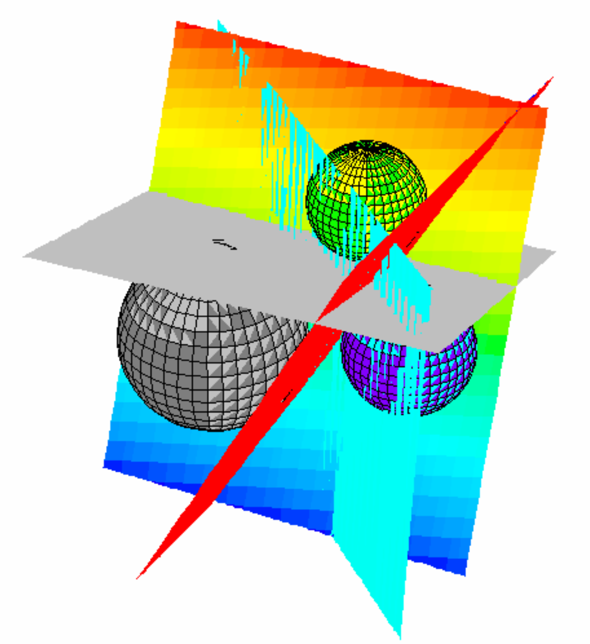
#15: [A := A, B := B, C := C, D := D]

#16: [A := [0, 0, 0], B := [1, 0, 0], C := [0, 1, 0], D := [0, 0, 1]]

#17: [A := [−3, −4, −5], B := [1, 2, 0], C := [0, 1, 3], D := [0, −2, 1]]

#18: [A := [0, 0, −5], B := [5, 4, 3], C := [−4, 5, 3], D := [5, −5, 6]]

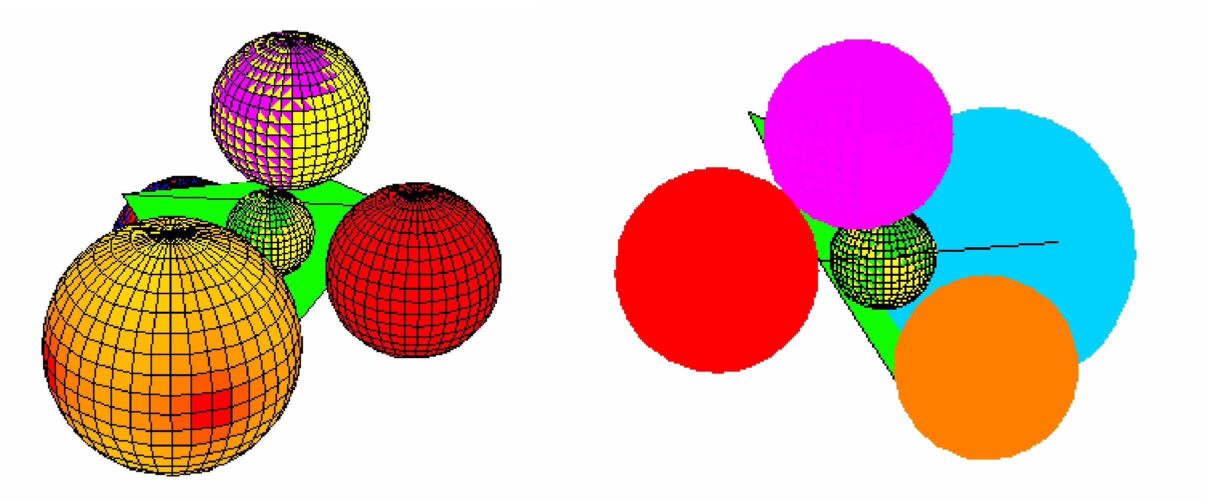
Here are the four planes together with the spheres. The insphere is invisible.



The next plots show the tetrahedron with its edges together with the four spheres. In the second plot I changed the colour and didn´t show the mesh lines.

#19: [A, B, C, D, B, D, A, C]

#20: $\left[\begin{bmatrix} A & B \\ A & C \end{bmatrix}, \begin{bmatrix} A & D \\ A & B \end{bmatrix}, \begin{bmatrix} B & D \\ B & C \end{bmatrix}, \begin{bmatrix} A & C \\ A & D \end{bmatrix}\right]$



Expression #19 plots the wire grid of the tetrahedron.

Expression #20 plots the sides of the tetrahedron.

David asked, if I could find a proof Lemma 1 for an arbitrary position of point E (even outside of the triangle). I answered that in my opinion his proof is also valid for these positions but I felt inspired to do another proof which is also valid for all positions of point E.

Appendix: Another proof of Lemma 1:

#1: [InputMode := Word, CaseMode := Sensitive]

#2: $y - a2 = \frac{e2 - a2}{e1 - a1}\cdot(x - a1)$

#3: $0 - a2 = \frac{e2 - a2}{e1 - a1}\cdot(d1 - a1)$

#4: $\text{SOLVE}\left(0 - a2 = \frac{e2 - a2}{e1 - a1}\cdot(d1 - a1),\ d1\right)$

#5: $d1 = \frac{a1\cdot e2 - a2\cdot e1}{e2 - a2}$

#6: $BD := \frac{a1\cdot e2 - a2\cdot e1}{e2 - a2}$

#7: DC := BD − c1

#8: $\text{area}(X, Y, Z) := \frac{1}{2}\cdot\text{DET}\begin{bmatrix} X_1 & X_2 & 1 \\ Y_1 & Y_2 & 1 \\ Z_1 & Z_2 & 1 \end{bmatrix}$

#9: [A := [a1, a2], B := [0, 0], C := [c1, 0], E := [e1, e2]]

#10: $\frac{\frac{\text{area}(A, E, B)}{\text{area}(A, E, C)}}{\frac{BD}{DC}} = 1$

Q. E. D. Now in detail:

#11: $\text{area}(A, E, B) = \frac{a1\cdot e2 - a2\cdot e1}{2}$

#12: $\text{area}(A, E, C) = \frac{a1\cdot e2 + a2\cdot(c1 - e1) - c1\cdot e2}{2}$

#13: $\frac{\text{area}(A, E, B)}{\text{area}(A, E, C)} = \frac{a1\cdot e2 - a2\cdot e1}{a1\cdot e2 + a2\cdot(c1 - e1) - c1\cdot e2}$

#14: $\frac{BD}{DC} = \frac{a1\cdot e2 - a2\cdot e1}{a1\cdot e2 + a2\cdot(c1 - e1) - c1\cdot e2}$

David´s comment: Your dfw-file is interesting. As I understand it, the four subexpressions of area(A, E, B)/area(A, E, C)/(BD/DC) can be negative or positive. which seems to be nice.