THE DERIVE-NEWSLETTER #71/72

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THE BULLETIN OF THE

DERIVE®

USER GROUP

+ CAS-TI

Contents:

December 2008

Bookstore:

[1] Bernhard Kutzler, Technology and the Yin & Yang of Teaching and Learning Mathematics, BK-01, ISBN 978-3-901769-83-2

[2] Proceedings/Tagungsband TIME 2008 (Buffelspoort, South Africa) bkt SR-61, ISBN 978-3-901769-82-5

Please notice the information about all bk-teachware publications on the opposite page

We can recommend some very valuable websites:

Nils Hahnfeld´s program packages for the TI-handheld devices (see also page 47):

http://www.ti89.com/apps

Visit **Eberhard Lehmann´s** website: http://www.snafu.de/~mirza

DUG Member **Lorenz Kopp** updated his programs for DERIVE 6 including applications of the slider bar in many presentations:

http://www.lzkopp.de/index.htm

Wonderful geometry based on background pictures - study the mathematics in famous paintings - and on conics:

http://jmora7.com/Arte/arte.htm
http://viajarnamatematica.ese.ipp.pt/moodle/file.php/1/Geometria_Dinamica/arte/vasarely/arte_conicas.html

If you like Dynamic Geometry then it is a must visiting **Michael de Villiers'** website:

http://mysite.mweb.co.za/residents/profmd/

Interesting websites full of information for TI-handheld users are (thanks to **Nils Hahnfeld**):

http://www.ibiblio.org/technicalc/buglist/bugs.pdf
http://technicalc.org/tiplist/en/html/index.htm
http://technicalc.org/tiplist/en/html/index.htm

Datenbank des **Kompetenzzentrums für Mathematik** an der Uni Klagenfurt:

http://www.uni-klu.ac.at/iff-idm/technologie-db/

Andras Ringler from Szeged, Hungary published the ultimative form of "*Let´s think Greek*":

http://www.mozaik.info.hu/Homepage/Mozaportal/matematika.php

Download all *DNL-DERIVE*- and TI-files from
http://www.austromath.at/dug/

Dear DUG Members,

I am really late with this DNL and I am very grateful for your patience. I promised to upload a double issue and indeed as you can see we have a very contentful DNL#71/72.

We had a great TIME 2008 in South Africa. Once again many thanks to Steve Joubert and his colleagues for organizing this meeting. It was really a surprise for us to learn about the very intensive use of DERIVE in technical applications at the Tshwane Technical University. We enjoyed four exciting keynote lectures (three presenters are members of the DUG-family). One of them – Stephen Arnold from Australia – gave a lecture on the use of CAS (TI-Nspire) and he also provided a contribution for this DNL. All the TIME 2008 lectures are collected in the Conference Proceedings (see the information on the opposite page).

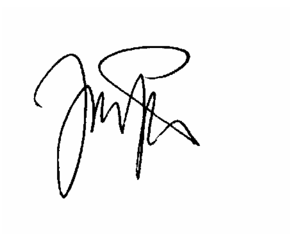
Browsing this issue you will see that we have some papers dealing with NspireCAS. When you will compare the list of future submissions you will find that it became longer again. I am very grateful for you not ending cooperation.

As you can read below bk-teachware – the most eager publisher of DERIVE and TI-related books – is ending its business at the end of this year. Many thanks to Bernhard for so many years of wonderful cooperation. We wish you the very best for your future. Bernhard has been member #1 of the DUG and he promised to remain in tight contact to technology supported teaching and learning. He shares his long experience with us in his novel booklet "Yin & Yang".

I´d like to invite you to visit some of the recommended websites.

Finally the greatest news: Have a look on page 14 – great news about TIME 2010!! (See page

My wife Noor and I wishyou and your family a wonderful Christmas and a healthy and peaceful New Year 2009 which hopefully will recover from the recent economic problems



Dear customers of bk teachware!

This is to let you know that we will shut down "bk teachware" at the end of 2008. It has been a pleasure to serve the math education community for more than thirteen years. Thank you for your loyalty.

I will continue to work as a freelance author and teacher. I will continue to publish my own books, you can order them via my personal website

http://b.kutzler.com.

Books from the bk teachware series will continue to be available through www.school-scout.de (look for publisher "bk teachware").

Sincerely,
bernhard kutzler

The *DERIVE-NEWSLETTER* is the Bulletin of the *DERIVE* & CAS-*TI User Group*. It is published at least four times a year with a contents of 40 pages minimum. The goals of the *DNL* are to enable the exchange of experiences made with *DERIVE*, *TI*-CAS and other CAS as well to create a group to discuss the possibilities of new methodical and didactical manners in teaching mathematics.

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Contributions:
Please send all contributions to the Editor. Non-English speakers are encouraged to write their contributions in English to reinforce the international touch of the *DNL*. It must be said, though, that non-English articles will be warmly welcomed nonetheless. Your contributions will be edited but not assessed. By submitting articles the author gives his consent for reprinting it in the *DNL*. The more contributions you will send, the more lively and richer in contents the *DERIVE* & CAS-*TI Newsletter* will be.

Next issue: March 2009
Deadline 15 February 2009

Preview: Contributions waiting to be published

Some simulations of Random Experiments, J. Böhm, AUT, Lorenz Kopp, GER
Wonderful World of Pedal Curves, J. Böhm, AUT
Tools for 3D-Problems, P. Lüke-Rosendahl, GER
Financial Mathematics 4, M. R. Phillips, USA
Hill-Encription; Henon & Co, J. Böhm AUT
Simulating a Graphing Calculator in *DERIVE*, J. Böhm, AUT
Do you know this? Cabri & CAS on PC and Handheld, W. Wegscheider, AUT
An Interesting Problem with a Triangle, Steiner Point, P. Lüke-Rosendahl, GER
Overcoming Branch & Bound by Simulation, J. Böhm, AUT
Diophantine Polynomials, D. E. McDougall, Canada
Graphics World, Currency Change, P. Charland, CAN
Cubics, Quartics – interesting features, T. Koller & J. Böhm
Logos of Companies as an Inspiration for Math Teaching
Exciting Surfaces in the FAZ / Pierre Charland´s Graphics Gallery
BooleanPlots.mth, P. Schofield, UK
Old traditional examples for a CAS – what´s new? J. Böhm, AUT
Truth Tables on the TI, M. R. Phillips, USA
Advanced Regression Routines for the TIs, M. R. Phillips, USA
Where oh Where is IT? (GPS with CAS), C. & P. Leinbach, USA
Embroidery Patterns, H. Ludwig, GER
Mandelbrot and Newton with *DERIVE*, Roman Hašek, CZ
Snail-shells, Piotr Trebisz, GER
A Conics-Explorer; Some tools for exercising basic manipulation skills, J. Böhm, AUT
Stationary Points of Functions of 2 Variables, G. Mann, ISR
Working with DiffEqu Made Easy and Statistics Made Easy
The Role and Function of Proof with SketchPad?, de Villiers, SA
Stocks and Medicine in Math End Examinations, Hofbauer & Metzger Schuhäcker, AUT
Using Science as a Tool for Learning Mathematics, H. Urban-Woldron, AUT
The Horror Octahedron, W. Alvermann, GER
Contributions from J. L. Galan, Spain
Study Cards for the TIs, G. Scheu, GER
and others

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Prof. Paditz sent an interesting CAS question and we exchanged some emails. It began with a request to the TI Cares Customer Support Team which answered not to have direct contact to the Software developers. So Prof. Paditz forwarded his question to me:

Prof. Ludwig Paditz, Germany paditz@informatik.htw-dresden.de

I have a question for the team (Bug in CAS):

Please check the following problem in Real Mode (TI-NspireCAS):

Integral(1/sqrt(x^2-a^2),x) for x<-a and a>0.

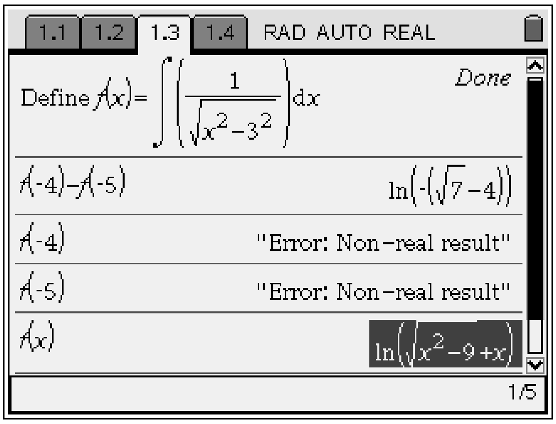
The result ln(sqrt(x^2-a^2)+x) is not correct because the argument of ln(…) is or can be negative. The absolute value in the logarithm is missing: ln(|sqrt(x^2-a^2)+x|)

Here is an example performed with TI-NspireCAS:

$$f(-4)-f(-5)=\ln(\sqrt{(-4)^2-3^2}-4)-\ln(\sqrt{(-5)^2-3^2}-5)$$

is not real and the correct result is

$$\ln\left(\left|\sqrt{(-4)^2-3^2}-4\right|\right)-\ln\left(\left|\sqrt{(-5)^2-3^2}-5\right|\right).$$



What is the opinion of my colleagues?

DNL:

Dear Prof. Paditz,
Good to hear from you.

The Nspire-output does not surprise – because of two reasons:

Firstly because the Nspire uses the same CAS-machine as the Voyage 200 and this CAS calculator behaves completely the same,

and secondly because this CAS was developed by David Stoutemyer who has a special view of ln(x) or ln|x|. He explained his point of view very detailed in a DERIVE Newsletter (#26) several years ago. I attach the respective paper.

With my best regards

Josef Böhm

Prof. Ludwig Paditz, Germany paditz@informatik.htw-dresden.de

Dear Mr Böhm,
many thanks for your quick response concerning the problem of integrating a positive function over an interval of the negative x-axis.

David has good arguments from the view of function theory (analytic functions in the Gaussian plane, main- and side branches of the log-function) and CAS.

But he did not have in mind the pupils and students which are in the learning process and want to use the calculator as a tool in real analysis.

Didactics of mathematics goes short.

Pupils learn in real analysis that the antiderivative F(x) can be given as function of the area with a variable upper boundary x with a given integrand f(t) and we have: $F'(x) = f(x)$.

Hence in real analysis the two formulations ***F*(*x*) = Integral(*f*(*t*),*t*,*a*,*x*)** and ***F*'(*x*) = *f*(*t*)** are identical.

The arguments of David are not interesting at this position and does not support a basic course in "Real Analysis" from the didactical point of view.

The pupil(student) expects for example to recieve for $f(t) = \frac{1}{\sqrt{t^2 - 9}}$ with $t < -3$ a real area function

$$F(x) = \int_{-5}^{x} f(t)\,dt \text{ for } x < -3.$$

Nspire cannot meet this expectation at the moment. So we have a situation as follows:

Nspire is only intended and suitable for a restricted target group i.e. those people who can follow David´s argumentation and have some knowledge in function theory.

And these are not pupils and even not students in basic courses. Even electrical engineering students hear about function theory in higher semesters.

That´s the reason that I prefer to recommend the ClassPad330 (OS 3.03 including a free of charge university licence for the PC emulator) because this tool is more meeting the expectations of the students (and mine, too) from the **didactical point of view**.

What is your opinion?

With my nest regards

Ludwig Paditz

<u>DNL:</u>

Dear Prof. Paditz,

I tried your integral with several popular CASes (not the worst ones!) and found that all of them are behaving more or less the same as the Nspire and Voyage 200 (see attachment).

But both TI-systems offer the possibility to force real valued results by setting complex Format (in the Mode Menu on V200 resp in the document settings on Nspire) to "Real". So all possibilities are open the school – mathematical and the higher view.

I don´t find it bad to show the pupils/students that school mathematics sometimes "simplifies" the matter and that there is another more general treatise of the issues.

With my best regards and many thanks for the interesting discussion

Yours Josef Boehm

Josef Böhm

(The attachments mentioned above are on the next page.)

Mathematica

In[1]:= `f[x_] := Integrate[1/Sqrt[x^2 - 9], x]`

In[2]:= `f[x]`

Out[2]= $\log\left(x+\sqrt{x^2-9}\right)$

In[5]:= `g[x_] :=` $\log\left(x+\sqrt{x^2-9}\right)$

In[6]:= `g[3]`

Out[6]= $\log(3)$

In[7]:= `g[-4] - g[-5]`

Out[7]= $\log\left(4-\sqrt{7}\right)$

In[8]:= `g[-4]`

Out[8]= $i\pi+\log\left(4-\sqrt{7}\right)$

In[9]:= `g[-5]`

Out[9]= $i\pi$

MuPad

`f(x):=int(1/sqrt(x^2-9), x)`

$\ln\left(x+\sqrt{x^2-9}\right)$

`g(x):=ln(x + (x^2 - 9)^(1/2))`

$\ln\left(x+\sqrt{x^2-9}\right)$

`evalAt(g(x),x = -4)`

$\ln\left(\sqrt{7}-4\right)$

`int(1/sqrt(x^2-9), x = -5..-4)`

$\ln\left(4-\sqrt{7}\right)$

`float(int(1/sqrt(x^2-9), x = -5..-4))`

0.3032468274

`evalAt(g(x),x = -4)-evalAt(g(x),x = -5)`

$\ln\left(\sqrt{7}-4\right)-\pi\cdot i$

Maxsyma

(%i1) `f(x):=integrate(1/sqrt(x^2-9),x);`

(%o1) $f(x):=\mathrm{integrate}\left(\frac{1}{\sqrt{x^2-9}},x\right)$

(%i2) `f(x);`

(%o2) $\log\left(2\sqrt{x^2-9}+2x\right)$

(%i4) `g(x):=log(2*sqrt(x^2-9)+2*x);`

(%o4) $g(x):=\log\left(2\sqrt{x^2-9}+2x\right)$

(%i5) `g(-5)-g(-4);`

(%o5) $\log(-2)-\log(2\sqrt{7}-8)$

(%i7) `integrate(1/sqrt(x^2-9),x);`

(%o7) $\log\left(2\sqrt{x^2-9}+2x\right)$

(%i9) `integrate(1/sqrt(x^2-9),x,-5,-4);`

Is $\sqrt{x^2-10x+16}+x-5$ *positive or negative?* neg;

Is $\sqrt{x^2-8x+7}+x-4$ *positive or negative?* neg;

(%o9) $\log(8-2\sqrt{7})-\log(2)$

(%i10) `g(-5);`

(%o10) $\log(-2)$

(%i11) `g(-4);`

(%o11) $\log(2\sqrt{7}-8)$

(%i16) `rectform(g(-4));`

(%o16) $\log(8-2\sqrt{7})+\%i\,\%pi$

Derive

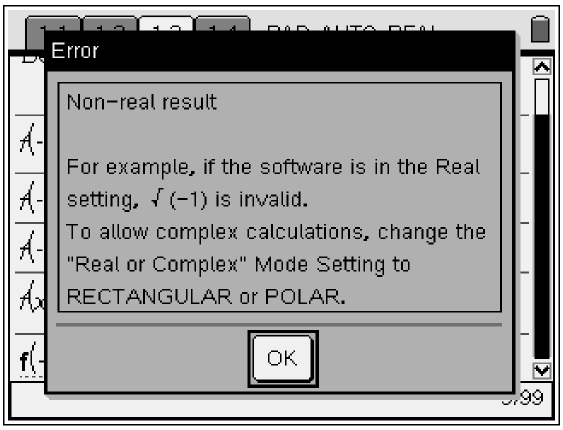
#1: $f(x) := \int \frac{1}{\sqrt{x^2-9}}\,dx$

#2: $f(-4) - f(-5) = \mathrm{LN}(4 - \sqrt{7})$

#3: $[f(-4), f(-5)] = [\mathrm{LN}(4 - \sqrt{7}) + \pi\cdot i, \pi\cdot i]$

#4: $f(x) = \mathrm{LN}(\sqrt{x^2 - 9} + x)$

The Nspire presents a message about the complex results when calculating *f*(-4) and *f*(-5):



Prof. Ludwig Paditz, Germany paditz@informatik.htw-dresden.de

Dear Mr Böhm,
many thanks for your explanations and I don´t mind to agree with them.

Complex analysis eplains many things simpler and does not need the absolute value for the log-function.

Maybe that complex numbers will get more importance in school mathematics in the future. Then several issues will be easier to explain to the learning pupils/students.

Friendly regards from Dresden
Yours
Ludwig Paditz

By the way, do you know that DES-TIME 2006 was awarded the Dresden Congress Award 2007?

I did not know? This is the picture from the ceremony. Congratulations to Rainer Heinrich (2nd from right) and his team.

Giuseppe Ornaghi

Clicking approximate on the improper integral $\int_0^\infty x e^x dx$ Derive 6 gives 17.27826738 as a result, but this is wrong because the integral is divergent. Can I obtain the correct result? Has anyone an explanation?
Tanks in advance.
Giuseppe Ornaghi

Valeriu Anisiu

As you can see in the status bar, Derive displays "Dubious accuracy" when you approximate the obviously divergent integral.
If you increase the precision you will obtain a larger number (but never INF).

The fact is that Derive is rather weak in computing numerically integrals (even proper ones!); but it has too many other qualities which force us to love it :-)
V.A.

first simplified in Exact Mode then in Approximate Mode:

$$\#1: \quad \int_0^\infty x \cdot e^x \, dx = \infty$$

$$\#2: \quad \int_0^\infty x \cdot e^x \, dx = 4.760539886 \cdot 10^{1731}$$

Wolfgang Pröpper, Nürnberg, Germany w.proepper@franken-online.de

Hello Peter,

I read your very nice contribution on Recurring Decimals in DNL# 70. But may I point out, that the `dtoq` function in paragraph 3 has a little bug:

A problem arises, unless your TI is in exact mode:

Line 15 must read `c:=exact(expr(left(x,c-1)))` because otherwise c is treated as a decimal number and not a fraction.

Best regards,
Wolfgang
Peter Schofield []

Peter Schofield p.schofield@leedstrinity.ac.uk

Hello Wolfgang,

Thank you for your email.
I'm glad that you liked my article in DNL#70 on recurring decimals.
From his response, Josef appears to have enjoyed developing this idea as well.

I accept your point about the TI must be in exact mode to display the fractions as fractions.
Perhaps Josef could mention your correction comment in DNL#71?

I'm sure that both my Derive and TI-89 coding can be improved.
The programs main virtue is that they bring this issue on recurring decimals to the attention of interested people like yourself and Josef.

All the best,

Peter

Peter Schofield

Dear Josef,

Thank you for featuring my paper in DNL #70. I'm also very impressed with your procedures "dn_to_fr" and "fr_to_dn". They provide alternative methods to my procedures "DtoQ" and "QtoD". I too was basing my methods on algorithms I learned in my schooldays. In "The Good Old Days" we could send off an idea like this to Albert Rich and, if he liked it and found it feasible, he would update Derive to include it in Derive's basic number notations.
If I remember correctly, I sent my original paper in response to a plea from you for articles involving TI-calculator programs. My paper also considers how to convert "DtoQ" and "QtoD" into TI-89 routines (and also, how these algorithms for non-decimal number bases can be programmed in Derive). This might be something for further research?
Your program "dn_to_fn" is clearly more efficient (and works faster that) "DtoQ", however you might include a couple more lines to trap the cases when the decimal point (or quotes) is not included in the number string.
Finally, I've just encountered another nail in the coffin of poor old Derive (boy am I bitter!). I was trying out your procedure "dn_to_fn" using Derive6 on my latest laptop - which has Windows-Vista. Although the basic program works OK, it is not possible to access Derive's HelpFile in this recent version of Windows. Do you know of any way round this?

Yours,

Peter

DNL:

Dear Peter,

thanks for your mail and your positive reaction on my memories from schooldays. I am very grateful for providing the TI-89 routines (which were published, too in DNL#70). Fact is that mainly the people from the "old DERIVE crew" send papers which deal at least partially with the TI-handheld. Finding articles for the TI-NSpire is not so easy. Thanks for pointing to some possible improvements for my routines.

Concerning DERIVE & VISTA I often have received your complaint, but in DNL#65 (March 2007) is an URL for downloading a patch provided by Guenter Schoedl. VISTA does not support the old html-format of the original DERIVE help file.

This is the respective text from DNL#65, information page:

http://www.microsoft.com/downloads/details.aspx?displaylang=de&FamilyID=6ebcfad9-d3f5-4365-8070-334cd175d4bb

X64 for 64 Bit platforms
X86 for 32 Bit platforms

Just launch the file in Vista. Much luck, Günter.

I understand that in times when one does not work with VISTA he/she does not show any interest in these "tricks". Until now the patch always worked.
Much luck and have a nice summer.

Did you ever try working with Maxima? It is open source and can be downloaded for free. It is a very powerful CAS with a syntax similar to DERIVE.

Best regards
Josef

Integration is always an issue!

Paulie paulienator@GMAIL.COM

Dear Derive users,

yesterday I was trying to calculate a primitive of $\frac{1}{\cos(3x)+5}$, I found what I thought was a solution and used DERIVE to verify it. After writing the expression and pressing Ctrl+B it displayed an expression which was a bit different from mine:

My solution:	$\frac{1}{3\sqrt{6}}\text{atan}\left(\sqrt{\frac{2}{3}}\tan\frac{3x}{2}\right)$
DERIVE´s solution	$\int \frac{1}{\text{COS}(3\cdot x)+5}\,dx = \frac{\sqrt{6}\cdot x}{12} - \frac{\sqrt{6}\cdot \text{ATAN}\left(\frac{\text{SIN}(3\cdot x)}{\text{COS}(3\cdot x)+2\cdot\sqrt{6}+5}\right)}{18}$

I plotted them and realised that the derivatives of both were the same (1/(cos(3x)+5)) but the difference of the functions was not a constant but rather a step function, this was verified by asking DERIVE to derive them.
I went a bit further and wanted to know how DERIVE got to his expression so I pressed Ctrl+D (step by step simplifying) but by doing it this way I got the same expression I calculated in my notebook.
Conclusion: simplifying step by step and directly do not give the same solution to the problem!
I'd like anyone to tell me how is it that DERIVE gets to the expression which is different from mine.
(I apologize for my bad English)
Thanks in advance!

Maxima

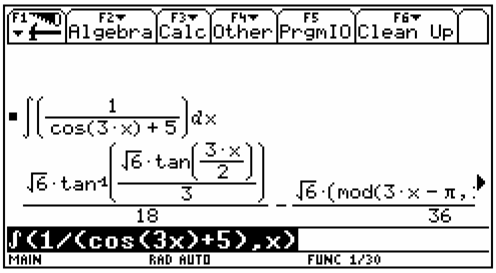
(%i2) integrate(1/(cos(3*x)+5), x);

$$\text{(\%o2)}\quad \frac{\operatorname{atan}\left(\frac{2\sin(3x)}{\sqrt{6}\,(\cos(3x)+1)}\right)}{3\sqrt{6}}$$

WIRIS

$$\int 1/(\cos(3\cdot x)+5)\,dx \rightarrow \frac{\sqrt{6}\cdot \operatorname{atan}\left(\frac{\sqrt{6}\cdot\tan\left(\frac{3\cdot x}{2}\right)}{3}\right)}{18}$$

All TI-CAS devices:



Johann Wiesenbauer, Vienna j.wiesenbauer@tuwien.ac.a

Hi,

Frankly, I don't think that this is problem of Derive, but to me it looks that it has rather to do with your basic understanding of an anterderivative. Let's take a simpler example than yours, namely the antiderivates of 1/x. What is it? Many students would say ln(x) +c for any constant c, which is non-sense, of course. Some of the more clever ones, would say ln(abs(x)) +c, and obviously this is the answer you have in mind too. What is the correct answer then? Think a while about it, then scroll down to see it.

Cheers,
Johann

Paulie paulienator@GMAIL.COM

I understand what you say, I know both approaches are OK if you think of anterderivatives as functions whose derivatives give the original function. In my case, I was thinking of anterderivatives as functions expressing the area under a curve and, looking it that way, the function derive calculated was more accurate than mine (just plot them).

Either way, I just wanted somebody to tell me how is it that derive calculates his anterderivative, because it could be useful for me in the future.

Thanks for your answers!

Johann Wiesenbauer, Vienna

Hi,

Sorry, if I possibly backed the wrong horse in my first answer. I thought your problem is that the two antiderivates differ by a step function rather than a constant. On the other hand I had a problem to read your expressions as they arrived here in a very unreadable form. (Simply copying expressions without editing them thereafter doesn't work in general!)

I succeeded now in reconstructing those expressions and in my opinion your problem looks like this. First of all, you considered the function

```
f(x) := 1/(COS(3x) + 5)
```

and Derive's antiderivative looked like

```
F(x) := SQRT(6)x/12 - SQRT(6)ATAN(SIN(3x)/(COS(3x) + 2SQRT(6) + 5))/18
```

I agree with you that this expression is very satisfactory, as this is a continuous function on the region where it is defined, which comes as a big surprise in view of those infinitely many singularities of f(x). Ok, your own solution looked like this

```
G(x) := 1/(3SQRT(6))ATAN(SQRT(2/3)TAN(3x/2))
```

and indeed,

```
F'(x) - G'(x) = 0
```

Surprisingly enough, by using Derive's step-by-step option one gets a third solution, namely

```
H(x) :=SQRT(6)ATAN(SQRT(6)SIN(3x)/(3(COS(3x) + 1)))/18
```

which coincides with yours, as a simplification of G(x)–H(x) with Derive shows. Obviously, Derive "thinks" that the solution H(x) is much nicer as it is more concise than F(x), and frankly, most humans would be tricked into thinking the same. Strictly speaking, the fact that F(x) and H(x) don't coincide is a bug in Derive though, even if both results are correct.

Unfortunately, as to your original question, how the solution F(x) can be computed I cannot make any contribution being not very skilled at trigonometric transformations, but maybe some of the readers here might get his teeth into it, now that the problem has been posed in a more readable form.

Cheers,
Johann

It is a pleasure for me to have one of the ver very few occasions to support Johann. The manipulation of the trig functions is not too difficult to derive one result of the antiderivative from the other one:

I use two identities: $\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}$ and $\cos x = \cos^2\frac{x}{2} - \sin^2\frac{x}{2}$. Josef

$$\frac{\sqrt{6}}{18}\,\mathrm{atan}\frac{\sqrt{6}\sin(3x)}{3(\cos(3x)+1)} = \frac{1}{3\sqrt{6}}\,\mathrm{atan}\left(\sqrt{\frac{2}{3}}\frac{\sin(3x)}{\cos(3x)+1}\right) =$$

Then

$$= \frac{1}{3\sqrt{6}}\,\mathrm{atan}\left(\sqrt{\frac{2}{3}}\frac{2\sin\frac{3x}{2}\cos\frac{3x}{2}}{\cos^2\frac{3x}{2}-\sin^2\frac{3x}{2}+\cos^2\frac{3x}{2}+\sin^2\frac{3x}{2}}\right) = \frac{1}{3\sqrt{6}}\,\mathrm{atan}\left(\sqrt{\frac{2}{3}}\frac{2\sin\frac{3x}{2}\cos\frac{3x}{2}}{2\cos^2\frac{3x}{2}}\right) =$$

$$= \frac{1}{3\sqrt{6}}\,\mathrm{atan}\left(\sqrt{\frac{2}{3}}\tan\frac{3x}{2}\right)$$

Paulie paulienator@GMAIL.COM

Hi,
thanks again for your answers. I'm really clueless as to what trigonometric transformation could get a SQRT(6)x/12 out of ATAN(f(x)). Anyway, I was in fact trying to solve the differential equation y'=cos(3y)+5 and after undoing some change of variables I got a satisfactory solution.

Ralph Freese ralph@MATH.HAWAII.EDU

> "surprise in view of those infinitely many singularities of f(x)".

I can't see any. It looks to me that f(x) is defined (and continuous) on the whole real line. In fact f(x) > 0 everywhere.

F(x) is a continuous antiderivative while H(x) has infinitely many discontinuities but elsewhere its derivative is f(x). This means the definite integral of f(x) from a to b is F(b) - F(a), while H(b) - H(a) can give the wrong answer.

Al Rich is the one who deserves credit for getting this right. It is interesting that circa 1990, Maple and Mathematica would get these wrong while Derive would get them right. While some might argue that H(x) is a valid antiderivative (it is as far as the Risch algorithm is concerned, for example), no one could argue that it was ok to give 3 (say) as the answer of a definite integral that is supposed to be 5.

(see the and of this discussion on page 61)

Making Algebra Meaningful With Technology

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ABSTRACT

We have been using technology for classroom teaching and learning now for around 30 years. Clearly there have been enormous advances in the nature and form of this technology over that time, but there have also been great developments in our understanding of how children learn. This paper addresses key questions drawn from research and from classroom practice: what have we learned over the past thirty years about good teaching and learning, about the teaching and learning of mathematics in general, and of algebra in particular? And what is the role of technology in this process?

1. Introduction

I begin by observing that this is my thirtieth year as a teacher of mathematics. Soon after I began my teaching career, personal computers also made their introduction to the classroom. I remember being responsible for the purchase of an Apple IIe computer back in 1980, intended to bring my school rapidly into the new “computer age”. It is interesting to look back over that time and, in particular, to ponder what we have learned from both classroom research and the wisdom of practice concerning the use of technology as an aid to learning.

From my perspective, as classroom teacher, researcher and academic, it is possible to make some fairly well-supported and sensible statements at this point in time concerning good teaching and learning, the teaching and learning of mathematics, and of algebra in particular. It is then possible to relate these to the appropriate and effective use of technology for the learning of algebra in a meaningful way.

1. Students learn best when they are actively engaged in constructing meaning about content that is relevant, worthwhile, integrated and connected to their world.
2. Students learn **mathematics** best when
   - They are active participants in their learning, not passive spectators;
   - They learn mathematics as integrated and meaningful, not disjoint and arbitrary;
   - They learn mathematics within the context of challenging and interesting applications.
3. Students learn **algebra** best when
   - It is not presented as meaningless symbols following arbitrary rules;
   - The understanding of algebra is based upon concrete foundations, with opportunities for manipulation and visualisation;
   - Algebra is presented as a vital tool for modeling real-world applications.

And the role of technology in the process?

Technology in mathematics and science learning plays two major roles:

- As a tool for REPRESENTATION, and
- As a tool for MANIPULATION

Good technology supports students in building skills and concepts by offering multiple pathways for viewing and for approaching worthwhile tasks, and scaffolds them appropriately throughout the learning process.

Bearing these principles in mind, it is timely to look now at ways in which they may be integrated using appropriate tools. My vehicle of choice for this exploration is the new TI-Nspire platform from Texas Instruments which, in both handheld and computer software forms, offers a very complete mathematical toolkit, with dynamically linked multiple representations. Such a tool represents the current end-point of thirty years of research and classroom experience in the teaching and learning of mathematics in general, and of algebra in particular.

If algebra is to be taught in an effective and meaningful way, then it must be taught differently than has been the case in general to this point. High school algebra is probably the clearest example of the malaise which affects almost all of school mathematics. We can scarcely claim to be successful in the teaching and learning of a subject in which the vast majority of students, after studying the subject for at least 11 years, leave school not only being unable to apply much of what they have "learned" in any practical or realistic way to their lives, but with an active and often virulent dislike of the subject. Even many of our "success stories" may be very capable "technicians" but can scarcely claim to have any deep mastery or understanding of this discipline. They can make the moves and perform the manipulations, but do they really understand what they are doing?

By most reasonable measures, it is fair to claim that the teaching of mathematics in schools generally has been less than successful. Some might say spectacularly unsuccessful!

We can identify two significant factors which have contributed to this current state:

1. Much of school Mathematics is taught in a decontextualised, fragmented way, with little connection to the lives of students or to the world beyond the classroom and examination.
2. Much of school mathematics is taught in a socially and intellectually isolated way, as a series of routines to be learned rather than processes to be understood. It is algorithmic rather than meaningful, for what is algorithm but a suspension of meaning, designed to break learning down to a memorized series of steps. Efficient? Perhaps. Meaningful? No.

So what might be done?

First, look for opportunities to teach school mathematics within contexts that are rich in meaning and significance for students, engaging them and encouraging them to interact both with the mathematics and with their peers in the learning of that mathematics.

Second, reward informal as well as formal approaches to mathematical thinking. Encourage multiple representations and multiple approaches to problems and to solutions. While algorithmic approaches may be considered efficient in reaching a specified solution, the cost of that efficiency has been high, since it robs students of the opportunities to play with the mathematics they are seeking to learn, to make mistakes (and to learn from those mistakes), and to explore individually and with others in a co-operative learning environment.

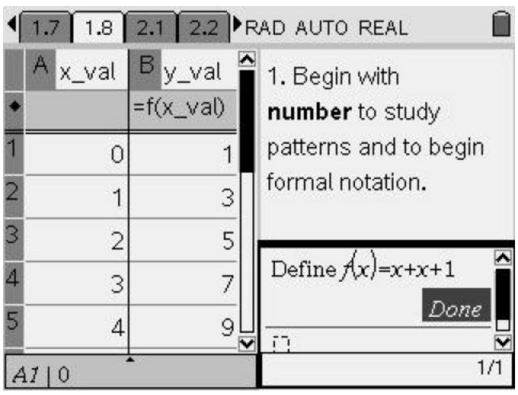
There is a clear and highly significant role for good technology in this review of school practice. We may consider the example of the learning of algebra in seeing how such an approach may begin in our classrooms.

Research over the past thirty years points to some clear steps in the process of learning algebra effectively, and the possibilities of new technologies point to some new steps with great potential to assist in bringing meaning to the learning.

2. **Begin with Number**

Just as algebra is, most purely, a generalization of the rules by which we operate with numbers, the path to algebra logically grows from students' knowledge and understanding of numbers and their operations. Number patterns, in particular, offer a perfect "jumping off" point by which students may be actively engaged in studying these rules and operations, and tables of values provide a powerful tool for exploring and conjecturing. The simple "guess my rule" games which teachers have used for many years may go well beyond just building simple patterns. They may also be used to introduce the symbolic notation of algebra in a practical and meaningful way.

From simple linear functions such as **y = 2x + 1** students can be challenged to find the rules for variations on the same theme (what about **y = x + x + 1**? **y = 3x + 2 – x – 1**?) – Yes, that rule is correct but it is not what I have – how else could the rule be written?



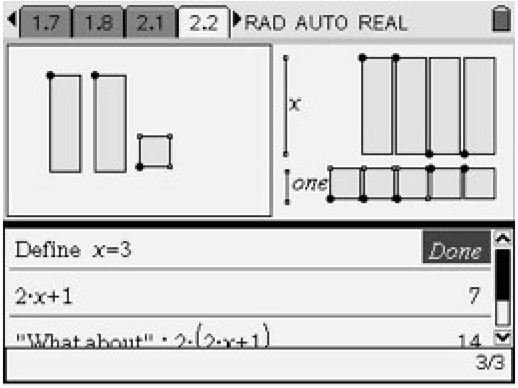
Then on to factors, such as **2(2x + 1)** – stressing the careful use of appropriate language: multiplication is always "lots of" –3 x 4 is 3 "lots of" 4 and **2(2x + 1)** is **2 lots of 2x + 1**!

We do have much to learn from primary school: subtraction is "how far from?" So 5 – 3 is really "how far from 3 to 5"? Up two steps. Simple?

Then what about -3 – 4? How far from 4 to -3? Clearly, "down 7 steps" if we use a ladder metaphor.

Careful use of correct language is a huge step towards students making their mathematics meaningful, initially with work on number and later, inevitably, with their algebra.

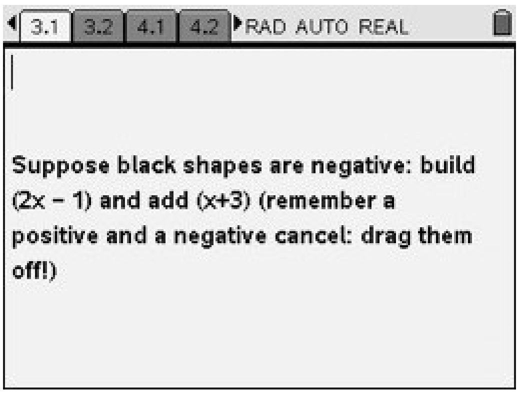
3. Build firm concrete foundations

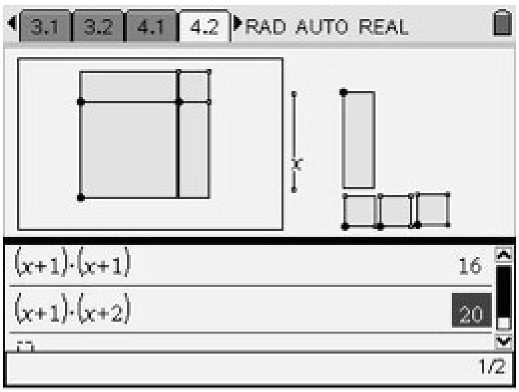


The second "golden rule" from my own teaching experience and also well-grounded in classroom research concerns the appropriate use of concrete materials to provide a firm foundation for the symbolic forms and procedures of high school algebra. "Area models" provide a powerful and robust means for students to interact with symbolic forms in ways both tactile, meaningful and transferable.

Two major limitations may be identified with the use of such concrete materials in this context: there is no direct link between the concrete model and the symbolic form, other than that drawn by the teacher – students working with cardboard squares and rectangles must be reminded regularly what these represent.

Of even greater concern, these concrete models promote a static rather than dynamic understanding of the variable concept. Both these limitations may be countered by the use of appropriate technology to scaffold and support the tactile forms of these models.

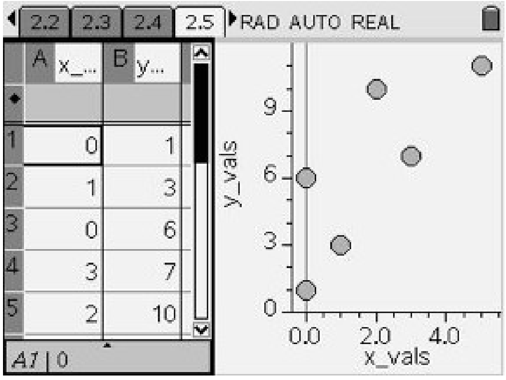


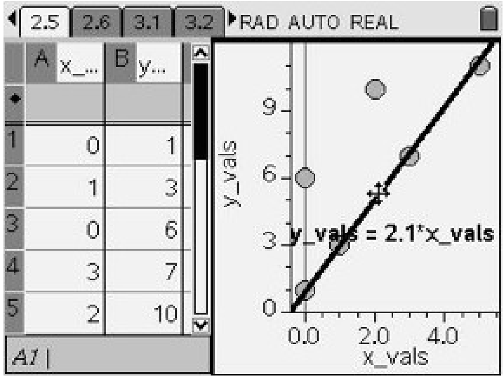


These basic shapes may be readily extended to model negative values (color some of the shapes differently and then these "cancel" out their counterparts) and even to quadratics, using x^2 shapes! After even a brief exposure, students will never again confuse $2x$ with x^2 since they are clearly different shapes.

4. Move carefully into graphs

The introduction of the graphical representation is too often rushed and much is assumed on the part of the students. Like the rest of algebra, the origins of graphs should lie firmly in number.



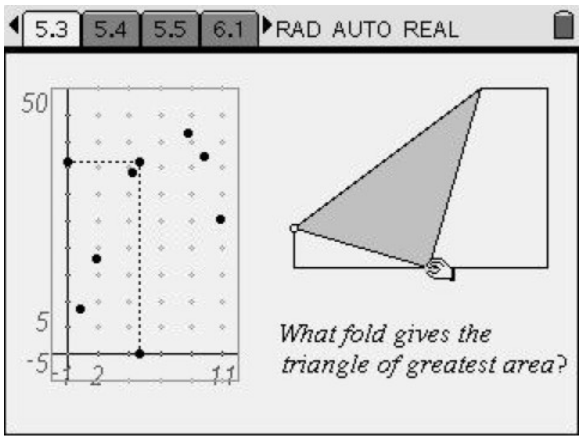


The use of scatter plots of number patterns and numerical data should precede the more usual continuous line graphs, which we use to represent functions. Such conceptual "objects" have little meaning for students, in the same way that symbolic "objects" (such as "2x + 1") need to be conceptually expanded to include more diverse ways of thinking.

We now have tools which make it easy for students to manipulate scatter plots and so further build understanding of the relationship between table of values and graphical representation. Only then should we encourage the use of the more formal "straight line" representation.

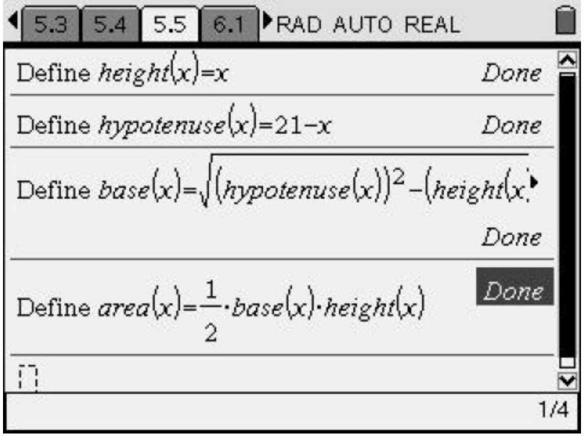
5. Bring it all together with modeling

Once we have built firm numerical foundations for symbol and graph, our students are ready to begin to *use* algebra – perhaps a novel idea in current classrooms! The real power of algebra lies in its use as a tool for modeling the real world (and, in fact, all possible worlds!) research is clear that students in the middle years of schooling (which is when we introduce algebra) most strongly need their mathematics to be relevant and significant to their lives. Teaching algebra from a modeling perspective most clearly exemplifies that approach, and serves to bring together the symbols, numbers and graphs that they have begun to use.



Opportunities for algebraic modeling abound, especially around such topics as Pythagoras' Theorem. The simple paper folding activity shown - in which the top left corner of a sheet of A4 paper is folded down to meet the opposite side, forming a triangle in the bottom left corner – is a great example of a task which begins with measurement, involves some data collection and leads to the building of an algebraic model. Students measure the base and height of their triangles, use these to calculate the area of the triangle, and then put their data into lists, which can then be plotted.

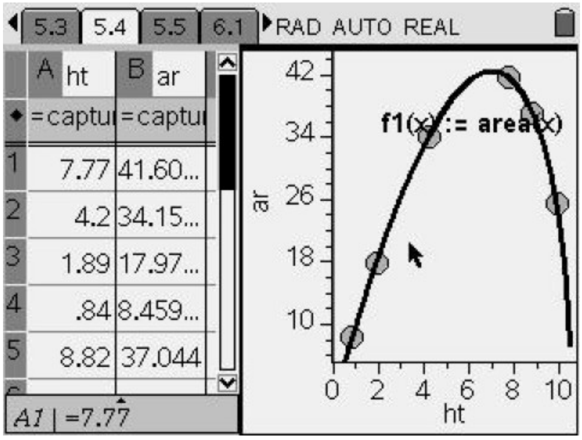
They may then begin to build their algebraic model, but using appropriate technology, may use real language to scaffold this process and develop a meaningful algebraic structure, as shown. In this problem, if we call the height of the triangle "x" (define a function as shown!) then the hypotenuse will be 21 – x, since the width of the sheet of paper is exactly 21 cm. Then apply a little Pythagoras to obtain the function for the base (also dependent on the height – see how the key understandings of variable and function are developed?), and the area follows.



Returning to the graphical representation, students may now plot the graph of their function, *area(x)*, and see how it goes through each of their measured data points – convincing proof that their model is correct – and usually a dramatic classroom moment!

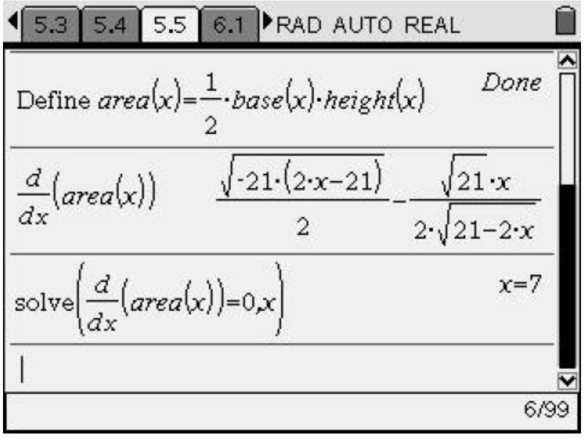
6. Build algebraic structure using real language

This is powerful, meaningful use of algebraic symbolism. The building of purposeful algebraic structures using real language supports students in making sense of what they are doing, and validates the algebraic expressions which they can then go on to produce. Able students should still be expected to compute the algebraic forms required and perhaps validate them using a variety of means.



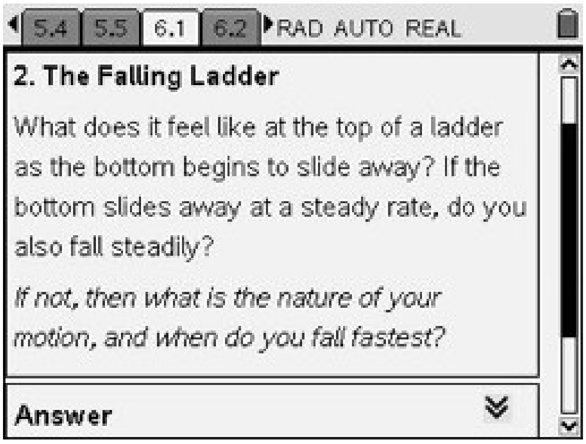
This use of real language for the definition of functions and variables has previously only existed on CAS (computer algebra software) and even there only rarely used. The new TI-Nspire is a numeric platform (non-CAS) and so allowable in all exams supporting graphic calculators, but it supports this use of real language.

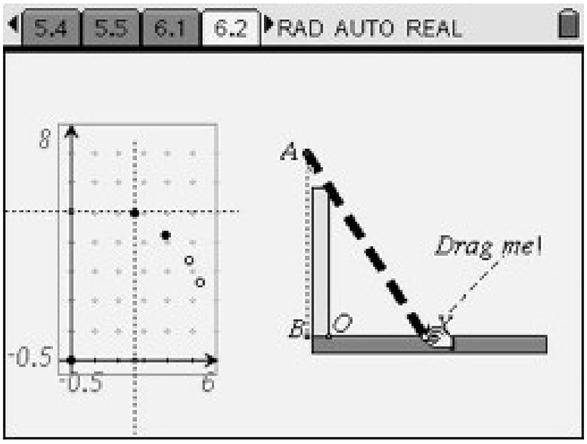
Of course, it is wonderful to have CAS facilities when they are needed. Using CAS we can actually display the function in its symbolic form, and then compute derivative and exact solution, arriving at the theoretical solution to this problem. The best fold occurs when the height of the fold is 7 cm, exactly one third of the width of the page.

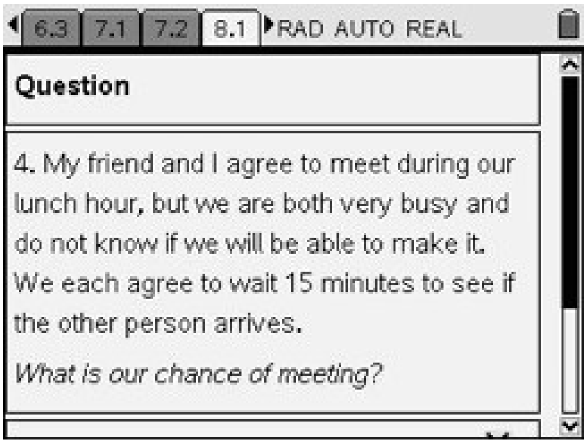


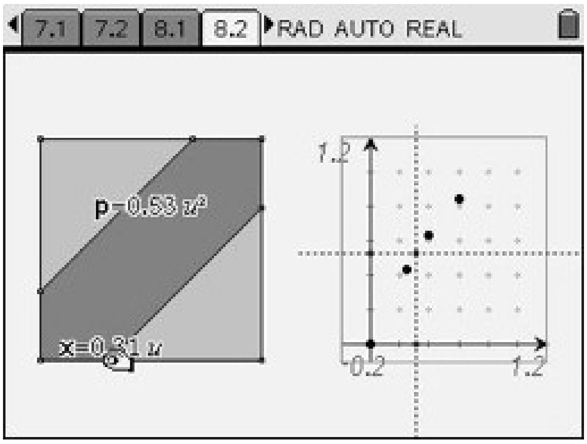
Using non-CAS tools, this same result may be found using the numerical function maximum command, or by using numeric derivative and numeric solve commands.

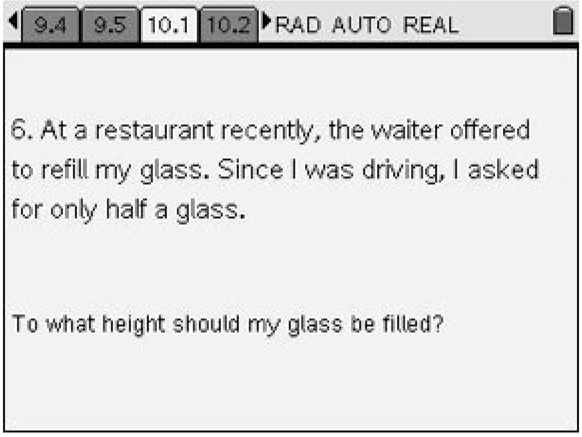
Once we begin looking for such problems, we find that they abound!

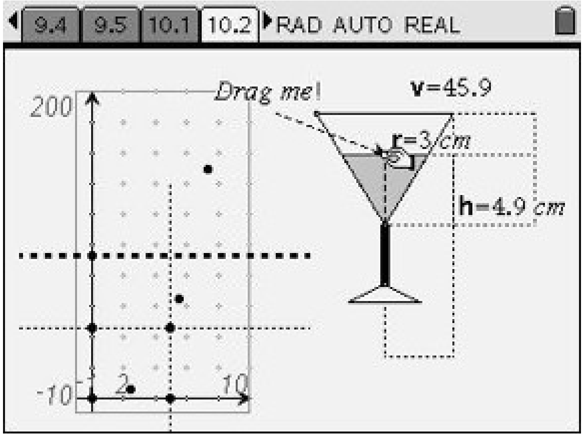






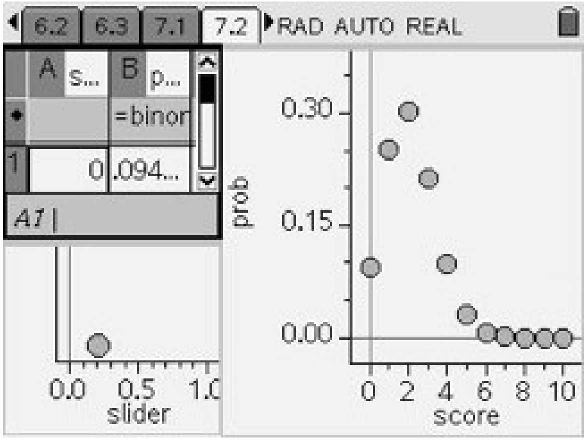


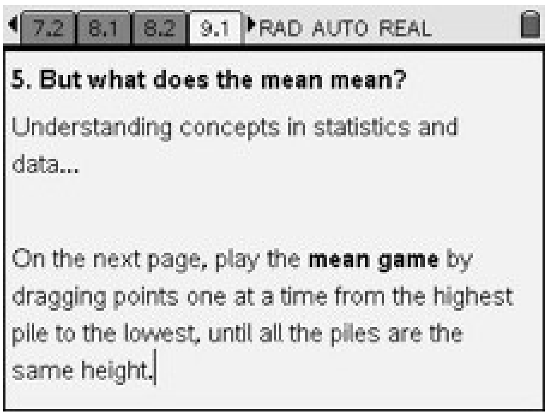


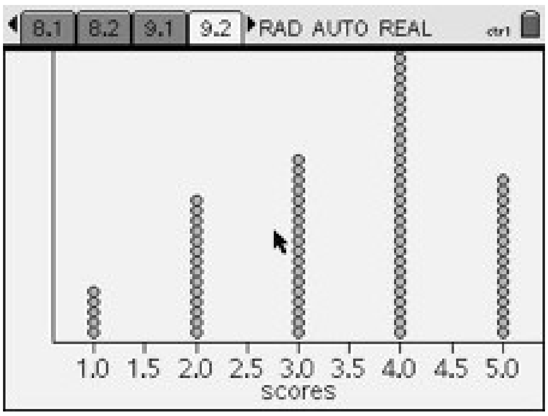


Statistics provides a ready source of good material, and often overlaps with our study of algebra. Consider binomial distribution: if my chance of scoring a bulls-eye at darts is not good (say, 20%) then the distribution of probabilities of scoring between 0 and 10 bulls-eyes will look as shown.

Using appropriate technology, we may vary that chance of a bulls-eye and students may investigate the effect this has upon the possible distributions – in this case, using a slider!

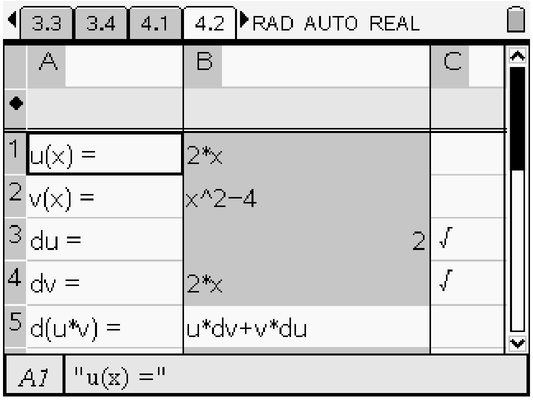


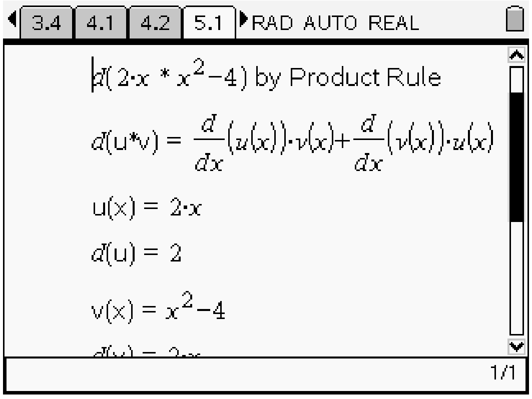




Look for Scaffolding Opportunities

Scaffolding is an important aspect of meaningful algebra learning, and computer algebra offers some powerful opportunities for such support. The real challenge in using CAS for teaching and learning, however, lies in finding ways to NOT let the tool do all the work!





Innovative use of CAS may include taking advantage of the algebraic capabilities of the *Lists & Spreadsheet* application, or writing programs which offer model solutions – but which stop short of giving the final result. Certainly these tools may readily provide automated solutions to extended algebraic processes, but there seems to me to be greater value in having the students do some or all of the work, and having the tool check and verify this work.

Such applications of these powerful tools remain yet to be explored.

8. Conclusion

Why do I like to use technology in my Mathematics teaching?

It helps my students to be better learners:

- It scaffolds their learning, allowing them to see more and to reach further than would be possible unassisted
- Good technology extends and enhances their mathematical abilities, potentially offering a more level playing field for all
- It is inherently motivating, giving them more control over both their mathematics and the ways that they may learn it
- Good technology encourages them to ask more questions about their mathematics, and offers insight into the true nature and potential of mathematical thinking and knowledge

Good technology also helps me to be a better teacher:

- It offers better ways of teaching, new roads to greater understanding than was previously possible
- It encourages me to talk less and to listen more: Students and teacher tend to become co-learners
- It makes my students' thinking public, helping me to better understand their strengths and weaknesses, and to better evaluate the quality of my own teaching and of their learning
- It frequently renews my own wonder of Mathematics, helping me to think less like a mathematics teacher and more like a mathematician

Why do I love using technology in my mathematics classroom?

Because, like life, mathematics was never meant to be a spectator sport.

TIME 2010

I have been asked several times about TIME 2010. I am very happy to announce that Jose Luis Galan and his colleagues from the University Malaga, Spain, will host the next TIME 2010 in the famous region of Andalusia. The date will be begin of July in 2010.

So TIME is back in Europe again after an exciting stay in South Africa. North Europe, West Europe and Central Europe were venues of our conferences but we missed South Europe. Many thanks to Jose Luis and his team.
Let´s meet in Spain in 2010 for Mathematics, Paella and Flamenco!!
You will receive more details as soon as possible

Stochastische Simulationen mit dem TI-Nspire

Stochastic Simulations with TI-Nspire

Benno Grabinger & Josef Böhm, Neustadt, Germany / Würmla, Austria

„Man sieht jeden Tag, dass die gelehrtesten Leute auf Grund von bloßen Analogien Schlüsse ziehen; da wo sie sich einbilden, in die Dinge klare Einsicht zu haben betrachten sie das als höchst evident, was es gar nicht ist. Und daher kommt es, dass nur diejenigen, deren Verstand durch mathematische Studien geschärft ist, fähig sind den lrrtum zu entdecken.“

Jakob Bernoulli (1654-1705)

Die Stochastik ist ein Gebiet der Mathematik in dem intuitive Vorannahmen und Vorurteile dem Lernenden Schwierigkeiten bereiten. Man denke z.B. an das Ziegenproblem das auch dann noch zu heißen Diskussionen Anlass gibt, wenn die Lösung bekannt ist. Freudenthal meinte zu derartig gelagerten Problemen, „dass die meisten immer noch nicht glauben, was die Logik ihnen verordnet“. Man muss den Schülerinnen und Schülern deshalb Zeit lassen, bewusste Erfahrungen mit dem Zufall zu machen. Nur damit sind die emotional geprägten und tief verwurzelten Voreinstellungen zum Zufall zu revidieren. Eine Möglichkeit solche Erfahrungen zu machen, bieten Simulationen von Zufallsexperimenten. Die Qualität von Softwarewerkzeugen hängt deshalb auch davon ab, wie leicht und anschaulich sich Simulationen mit dem jeweiligen System realisieren lassen.

An drei einfachen Beispielen sollen Vorzüge und Schwierigkeiten mit dem Nspire-System beim Erstellen von Simulationen betrachtet werden.

Stochastics is a field of mathematics where intuitive assumptions and prejudices make problems for the learning. Take for example the well known “Goat Problem” which leads to hot discussions even then when the solution is known. Freudenthal´s opinion to such problems was that “most people don´t believe what is prescribed for them by logic“. One must leave time to the students to make conscious experiences with chance. This is the only way to revise the emotional based and deep rooted views to chance. Simulations of random experiments offer one possibility to collect such experiences. Quality of software tools depends – among others – on the way how easy and clear simulations can be realized using the respective tool.

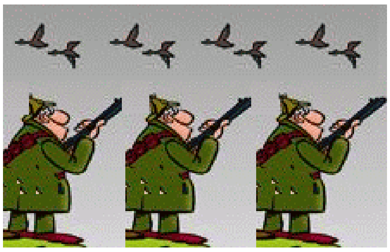
Presentating three simple examples we will observe benefits and disadvantages of TI-Nspire in preparing simulations.

1. Ein Besetzungsproblem (An Occupancy Problem)

Bei dem klassischen Besetzungsproblem werden m Kugeln zufällig auf n Zellen verteilt. Situationen dieser Art spielen z.B. in der Thermodynamik eine Rolle. Typische Fragestellungen sind:

Wie viele Kugeln sind erforderlich, um alle Zellen zu füllen (Sammlerproblem, Coupon Collector Problem)? Wie groß ist die maximale Anzahl der Kugeln in einer beliebigen Zelle? Welche Verteilung der Kugelanzahlen liegt vor? Das folgende Problem ist eine „lustige“ Einkleidung der Frage nach den leer gebliebenen Zellen:

m balls are distributed randomly in n cells. How many balls are needed to fill all cells (Coupon Collector Problem)? What is the maximum number in any cell? Which is the distribution of the balls? The next problem is a “funny” version of the question, how many cells will remain empty?



Zehn Jäger, lauter perfekte Schützen (d. h. jeder Jäger trifft das Ziel, auf das er schießt), schießen auf 10 Enten. Die Jäger können nur einmal schießen und sie können sich nicht ab-sprechen, wer auf welche Ente schießt. Sie schießen gleichzeitig und wählen ihr Opfer zufällig aus. Wie viele Enten überleben im Durchschnitt, wenn dieses Experiment oft wiederholt wird?

Ten hunters - all of them are never failing perfect huntsmen – shoot on ten ducks. They have only one shoot each and cannot agree about their target ducks. So they shoot all at the same time and choose their target randomly. What is the average number of surviving ducks?

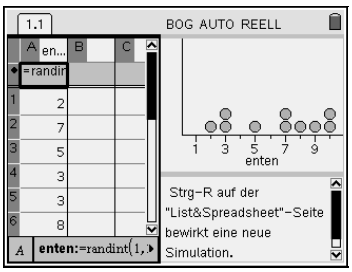
Mit minimalem Aufwand lässt sich dieses Problem in ansprechender Weise mit dem Nspire-System nachbilden. Mit **randint(1,10)** wird eine Zufallszahl zwischen 1 und 10 erzeugt. Sie stellt die Nummer der Ente dar, auf die – erfolgreich – geschossen wird.

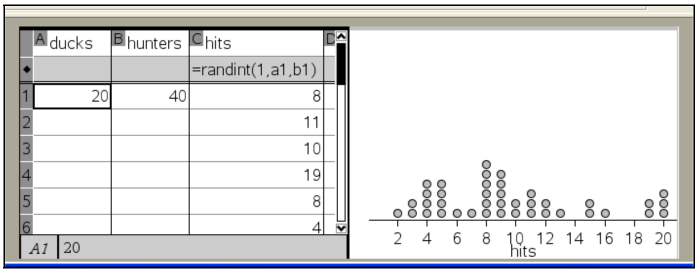
Trägt man in die Kopfzeile einer **List & Spreadsheet** Seite den Text **enten:=randint(1,10,10)** ein, dann wird dort eine Liste mit 10 Zufallszahlen erzeugt. Fügt man jetzt eine **Daten & Statistik** Seite ein, dann kann mit einem Klick die Häufigkeit der Treffer pro Ente angezeigt werden. (Man muss am Fuß der Grafik das Variablenfeld anklicken und die Variable **enten** angeben. Dann erst ordnet sich der erst ungeordnete Haufen von Kugeln.) Das linke Bild zeigt den screenshot vom Taschenrechner.

Das rechte Bild zeigt die PC-Version mit einer Verallgemeinerung: Hier wird in den Zellen *A1* und *B1* die Anzahl der Enten und Jäger geschrieben und dementsprechend die Anweisung zur Gewinnung der Zufallstreffer in Spalte C verallgemeinert.

Mit Strg+R im Spreadsheet wird eine neue Simulation vorgenommen, die sich auf Grund der Verlinkung der Variablen unmittelbar auf die Grafik auswirkt.

(Ersetzt man **randint(1,10,10)** durch **randint(1,365,25)** und erweitert die *x*-Skala der **Data-**Seite bis 365, so liefert dasselbe Dokument die Simulation des Geburtstagsproblems für 25 Personen.)





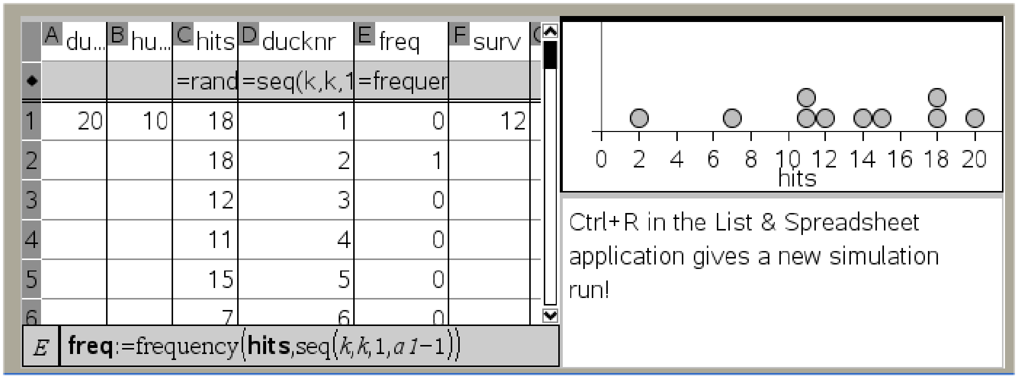
It only requires minimal effort to reproduce this problem with the TI-Nspire in an attractive way. **randint(1,10)** creates an integer random number $1 \le r \le 10$ wich represents the number of the duck which is hit by a hunter.

Entering **enten:=randint(1,10,10)** into the headline of a **List & Spreadsheet** page results in a list of 10 randomn numbers. We split the screen and insert a **Data & Statistics** page. After switching to this application you are faced with a lot of balls. Click on the variable button on the bottom of the screen and choose **enten**, then the balls are sorted and you find a nice diagram corresponding with the experiment on the left side of the screen. The left graph from aboveshows the screen of the hand held device.

The right figure shows the PC-version – with English variable names – which is more general. We can enter the number of hunters and ducks in cells *A1* and *B1* and generalize the command in column *C* to show the numbers of the hit ducks accordingly.

When the **List & Spreadsheet** is active pressing Ctrl+R calls a new simulation, which leads immediately to its updated representation because of successful linkage of the applications.

(Replacing **randint(1,10,10)** by **randint(1,365,25)** and adjusting the *x*-scale in the **Data** page you can simulate the well known birthday problem for 25 persons using the same document.)



Wir können drei weitere Spalten hinzufügen, um die Trefferhäufigkeit zu zeigen. Hier haben wir 20 Enten und 10 Jäger. Spalte D zeigt die Enten (von #1 bis #*n*). (Die Eingabe in die graue Kopfzelle lautet: **ducknr:=seq(k,k,1,a1)**.) In Spalte E wird gezählt, wie viele Treffer auf die entsprechenden Enten fallen. Hier nützen wir eine wertvolle Funktion, die auf dem TI-92 und V200 nicht verfügbar ist: **frequency(list,binslist)**. Dabei ist **list** die Liste der Elemente und **binslist** die Liste der rechten Grenzen der Bereiche. (Automatisch wird dann noch ∞ als letzte Grenze hinzugefügt.) Zelle *F1* enthält schließlich die Anzahl der „überlebenden" Enten, die wir mit Hilfe einer weiteren neuen Funktion erhalten: **countif(freq,0)**. Ich denke, dass die Syntax selbst erklärend ist.

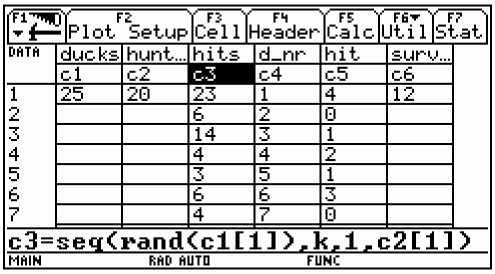
Schön und einfach geht das hier, weil die Grundidee dieser Simulation sich perfekt in das Nspire-System einbauen lässt.

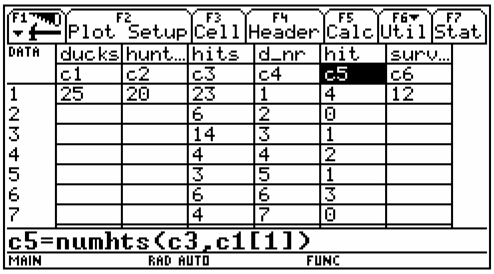
Wie geht das nun für den TI-92 bzw. Voyage 200, für den diese beiden Funktionen nicht verfügbar sind?

We can add three more columns to show the frequency of the hits. Here we have 20 ducks and 10 hunters. Column D shows the ducks (from #1 to #n). (Enter in the grey header **ducknr:=seq(k,k,1,a1).**) Column E counts how many hunters have hit the respective ducks. Here we use a function which is not available on the TI-92 and V200: **frequency(list, binslist)** with **list** being the list of elements and **binslist** being the list of the right boundaries (<=) of the buckets (= ranges). (∞ is added automatically as last boundary.) Finally cell *F1* contains the number of the surviving ducks which is obtained using another new function, **countif**. Write into the cell **= countif(freq,0)**.I believe that the syntax is self explanatory.

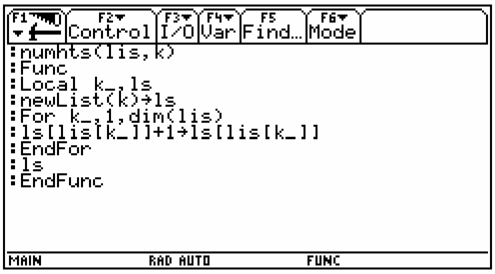
This works all very pretty and not too difficult because the basic idea of this simulation can be realised perfectly using the concept of the Nspire system of linking all applications.

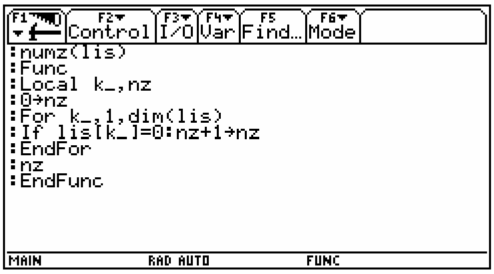
And how to do on the TI-92 / Voyage 200 (without **frequency** and **countif**)?





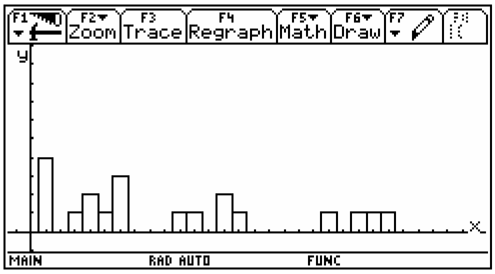
The headers of c4 and c6 are **seq(k,k,1,c1[1])** and **numz(c5)**. **numhts** and **numz** are functions which are substituting frequency and countif(list,0).





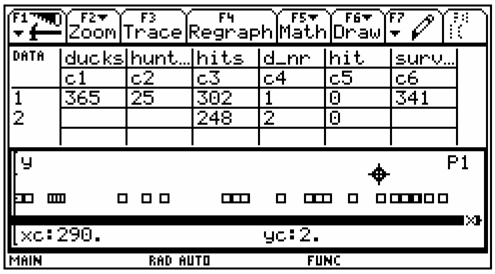
Graphic representation works, but not so immediate as with Nspire because adjusting to results of a new simulation does not work automatically. One has to switch between the two applications forth and back.

Hier geht es nicht so unmittelbar wie mit TI-NspireCAS, da man zwischen den Applikationen hin und her schalten muss.

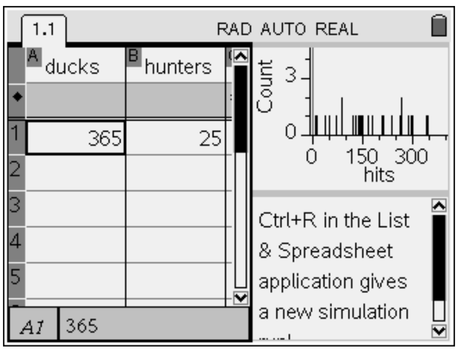


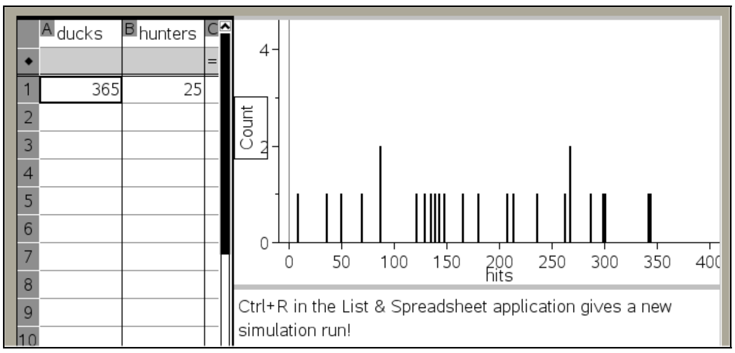
This is the realisation of the birthday problem (25 persons). It is not possible to present the data as a histogram. A Plot Setup error message appears. It seems to be that we have to draw too many buckets.

Das Geburtstagsproblem mit 25 Personen. Ein Histogramm ist nicht machbar, da offensichtlich zu viele Säulen gezeichnet werden müssten. Wir erhalten immer eine Fehlermeldung



Let´s try with TI-Nspire:



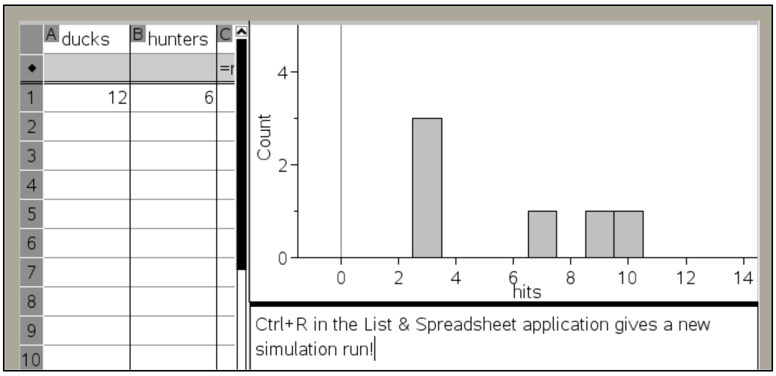


Wie man sieht, es macht die Realisierung mit TI-Nspire kein Problem: wenn mindestens eine der Säulen (hier Stäbe) länger als 1 ist, haben mindestens 2 Personen am gleichen Tag Geburtstag.

Fragen an die Schüler: Welche Fragestellung wird in der nächsten Simulation behandelt? Kannst Du andere Problemstellungen erfinden, die sich mit diesem Modell simulieren lassen?

As you can see, it is no problem to produce a histogram with TI-Nspire (left: handheld, right: software). If at least one bar is higher than 1 then at least 2 persons celebrate their birthday on the same day of the year.

Questions for students: Which problem is treated in the simulation below? Can you find other problems which can be simulated using this model?



Das Geburtstagsproblem ist ein "Dauerbrenner" in der Wahrscheinlichkeitstheorie und kommt sicherlich in allen Einführungskursen vor. Das Ergebnis ist – für alle, die die Lösung nicht kennen – mit großer Sicherheit überraschend. Wir geben noch eine Möglichkeit an, die Häufigkeit des Eintreffens von zusammenfallenden Geburtstagen von *n* Personen zuerst experimentell zu approximieren und dann theoretisch zu berechnen:

The Birthday Problem is surely a fixpoint in introductory courses for probability theory. The result is – for all who don´t know the solution – a surprise. We show a way to calculate the frequency that among *n* persons at least two of them have the same birthday using a simulation first and comparing the result with the theoretical probability.

sim_birth(25,100)	0.49
sim_birth(25,500)	0.568
theo_birth(25)	0.5687
sim_birth(30,1000)	0.676
sim_birth(30,2000)	0.721
theo_birth(30)	0.706316

6/99

sim_birth 1/2

```
Define sim_birth(n,k)=
Func
© k simulations with n persons
```

$$\text{approx}\left(\frac{\sum_{k\_=1}^{k}(success(n))}{k}\right)$$

```
EndFunc
```

success 3/7

```
Define success(n)=
Func
© n persons
Local liste,i
liste:=newList(365)
For i,1,n
liste[randInt(1,365)]:=1
EndFor
when(sum(liste)<n,1,0)
EndFunc
```

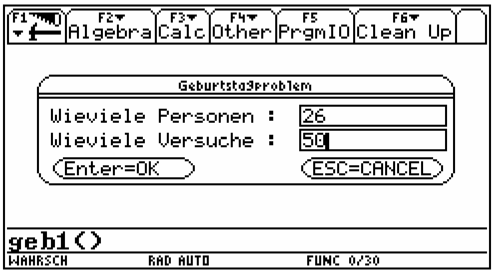
"theo_birth" stored successfully

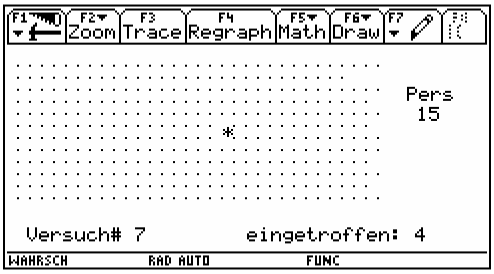
```
Define theo_birth(n)=
Func
```

$$\text{approx}\left(1-\prod_{i=1}^{n}\left(\frac{365-i+1}{365}\right)\right)$$

```
EndFunc
```

This is a simulation of the birthday problem visialised on the TI-92 /Voyage 200





What is the probability that among 26 students of a class at least two of them have the same birthday? The dots represent the days of the year. We see that after 6 experiments (of the intended 50 ones) the event occurred four times. Experiment 7 is in progress where student #15 just found a “birthday partner“). Finally we had 22 positive results in this simulation run.

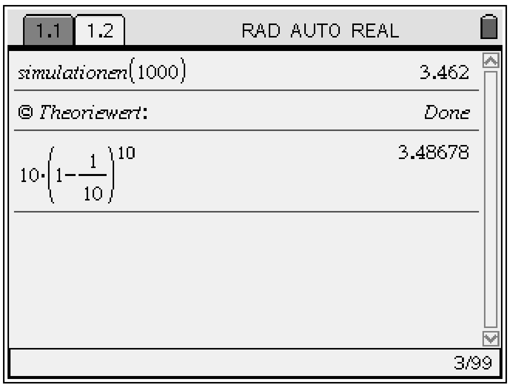
Die TI-92/V200-Visualisierung finden Sie unter den Dateien zu diesem DNL. Hier ist eben der 7. Lauf der Simulation im Gange, in dem Person #15 einen "Geburtstagspartner" gefunden hat. Insgesamt wurden in 22 von 50 Versuchen zusammen fallende Geburtstage gefunden.

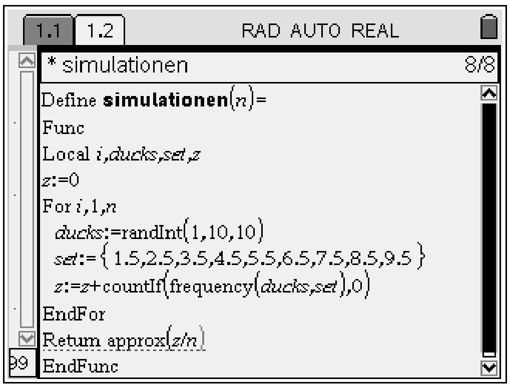
Zurück zu unseren Enten:

Mit der CAS-Funktionalität des Nspire-Systems kann auch die durchschnittliche Anzahl der nicht getroffenen Enten in einer großen Anzahl von Simulationen ermittelt werden. Die linke Abbildung zeigt einen Vergleich des Mittelwerts aus 1000 Simulationen mit dem Theoriewert. Die rechte Abbildung zeigt die dabei verwendete Funktion **simulationen(n).**

Back to our ducks:

Based on the CAS functionality of the Nspire system we can find the average number of the surviving ducks after a large number of simulations. The left figure below shows the comparison of the mean derived from 1000 runs of the simulation and the theoretical mean value. The right figure shows the function used to do *n* runs, **simulationen(n)**.



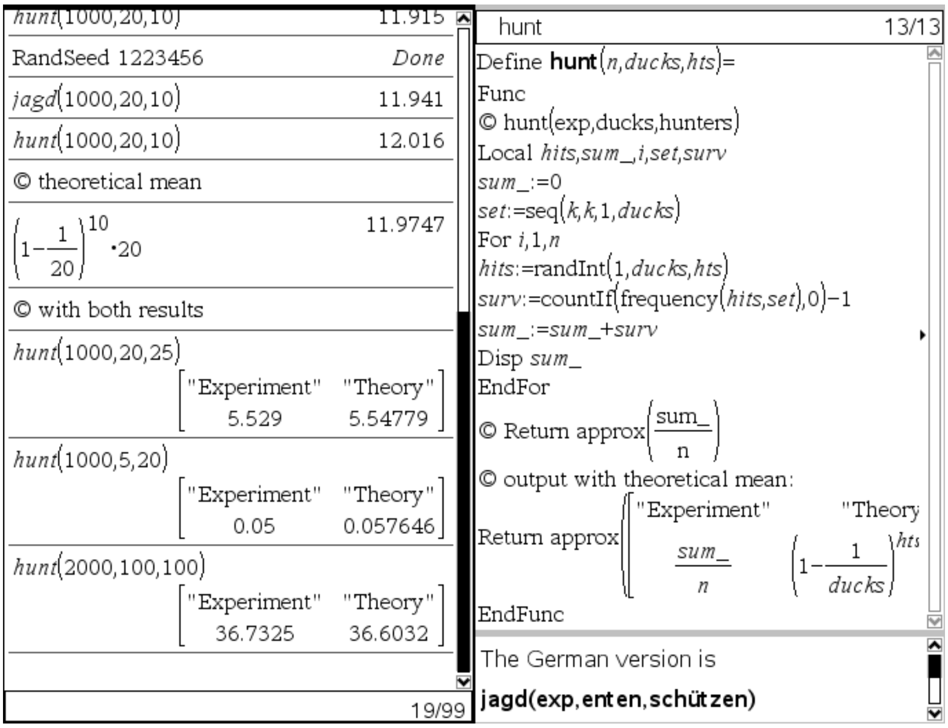


An dieser Stelle wird deutlich, dass hier eine Vertrautheit mit dem Nspire-System erforderlich ist, die über das übliche Maß eines durchschnittlichen Schülers weit hinausgeht. Vielleicht gibt es aber auch einen einfacheren Weg als den vom Autor vorgeschlagenen.

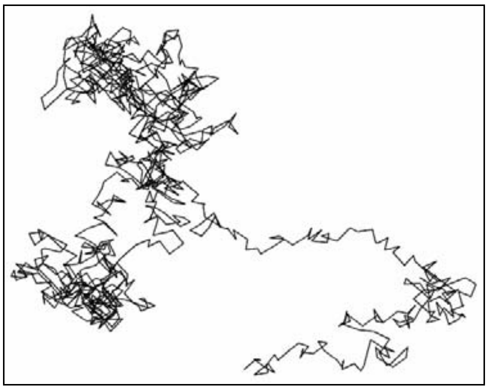
Latest now it becomes clear that you need some familiarity with the Nspire-system which lies beyond the usual level of an average student. But perhaps there is an easier way to do this simulation than the one proposed here by the author(s).

This is the generalized version of the simulation program (function) from above. Enter the number of simulations, ducks and hunters. You can perform the output without or together with the theoretical mean value – just change the position of comment character (©).

Das ist eine allgemeinere Form des Simulationprogramms (-funktion) von oben. Die Argumente der Funktion sind die Anzahl der durchzuführenden Simulationsläufe, die Anzahlen der Enten und der Jäger. Die Ausgabe kann mit oder ohne dem theoretischen Mittelwert erfolgen. Das Kommentarzeichen (©) muss nur dementsprechend versetzt werden.



2. Ein Irrfahrtproblem (Random Walk)



Ein „*random walk*“ ist eine Zufallsbewegung, bei der jeder Schritt unabhängig vom vorherigen ist. Das bekannteste Beispiel, das mit einem random walk beschrieben werden kann, ist die Brownsche Molekularbewegung. Wird ein kleines Teilchen in eine Flüssigkeit gebracht, dann beobachtet man unter dem Mikroskop, dass das Teilchen eine Zickzackbewegung ausführt. Diese ist eine Folge der unregelmäßigen Stößen der sich ständig bewegenden Atome und Moleküle.

Mit dieser Erklärung hat Albert Einstein 1905 die Brownsche Molekularbewegung beschrieben. Modellhaft soll hier zunächst angenommen werden, dass sich das Teilchen längs einer Geraden bewegt, d.h. einen eindimensionalen *random walk* (= eindimensionale Irrfahrt) ausführt. Bei jedem Schritt kann sich das Teilchen auf dem Zahlenstrahl um eine Einheit nach links oder nach rechts bewegen, jeweils mit der Wahrscheinlichkeit 0,5.
Dabei interessieren Fragen wie: Mit welcher Wahrscheinlichkeit befindet sich das Teilchen nach n Schritten an einer bestimmten Stelle? Wie weit entfernt sich das Teilchen dabei maximal vom Startpunkt?

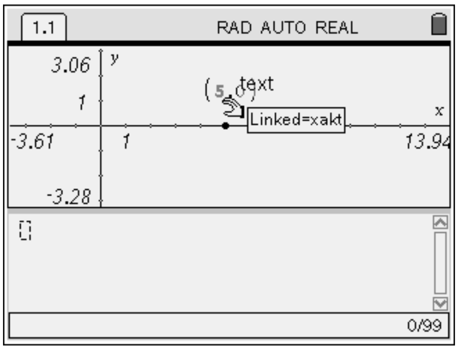
A *random walk* is a random motion where each step is independent of the previous one. The best known example which can be described by a random walk is the Brownian motion: *Brownian motion refers to the erratic movements of small particles of solid matter suspended in a liquid. These movements can only be seen under a microscope*[1]. This is a consequence of irregular collisions of the moving atoms and molecules. Albert Einstein gave this explanation of the Brownian motion in 1905. In the first model we will assume that the particle is moving along a straight line, i.e. it performs a linear random walk. The particle moves with each step either to the left or to the right with a probability of 0.5.

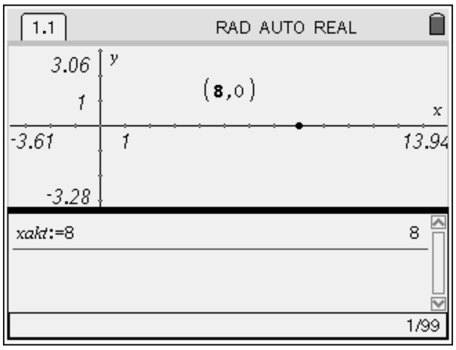
Interesting questions are: What is the probability that the particle reaches a certain position after *n* steps? Which is the maximum distance to the starting point during its walk to this position?

[1] Definition from *Fractals for the Classroom, Peitgen a.o.*

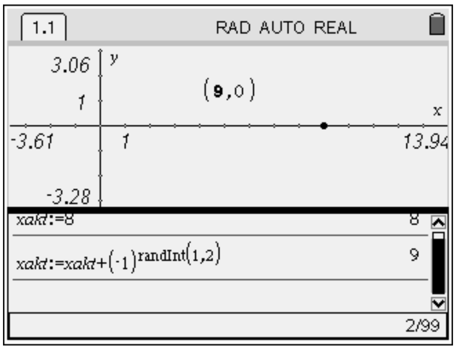
Um diesen Zufallsprozess mit dem Nspire-System nachzubilden, wird ein Punkt an die Markierungen der *x*-Achse des Koordinatensystems gebunden. Bewegt man danach den Punkt, dann kann er nur mit der Schrittweite der Gittereinheit verschoben werden. Dem Punkt werden dann seine Koordinaten zugewiesen. Die *x*-Koordinate wird in der Variablen **xakt** gespeichert.

For modeling this random process with the Nspire we fix a point to the grid marks on the *x*-axis. The point can only move to the left or to the right by steps of one grid unit. The *x*-coordinate of this point is stored as variable **xaxt**.





Der untere Teil der abgebildeten Seite ist eine **Calculator**-Applikation. Verändert man dort den Wert der Variablen **xakt**, dann bewegt sich der Punkt im Grafik-Fenster entsprechend. Um diese Bewegung zufallsgesteuert durchführen zu können, verwendet man den Befehl **xakt := xakt + (–1)$^{\textbf{randint(1,2)}}$**. Da **randint(1,2)** entweder 1 oder 2 als Ergebnis liefert, wird damit der Wert von **xakt** um 1 erhöht oder erniedrigt, d.h. der Punkt bewegt sich bei jedem Drücken der Enter-Taste „zufallsgesteuert" auf der *x*-Achse.



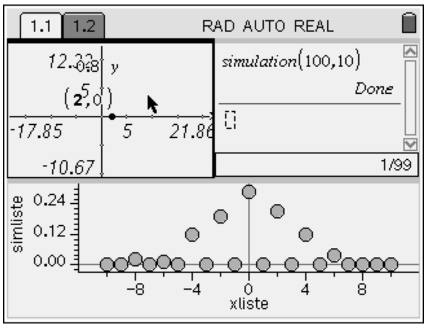
The bottom half of the screen is a **Calculator** page. Changing the value of **xakt** leads to the respective movement of the point. In order to perform the random movement we use the assignment: **xakt := xakt + (–1)$^{\textbf{randint(1,2)}}$**. As **randint(1,2)** gives either 1 or 2, the value of **xakt** is increased or decreased by 1, i.e. the point moves to the right or to the left at every call of this command.

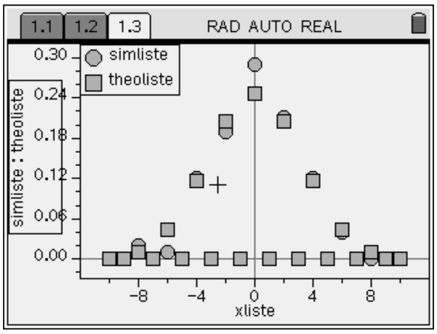
Während im ersten Beispiel das Zusammenspiel von **List & Spreadsheet**-Seite und der **Daten & Statistik**-Seite benutzt wurde, ist in diesem Beispiel zu sehen, wie sich Änderungen auf einer **Calculator**-Seite auf die **Graphs & Geometry**-Seite auswirken. In beiden Beispielen erweist sich das Nspire-System als geeignetes und einfach zu bedienendes Instrument, um die Simulation anschaulich durchzuführen.

In the first example we used the interplay of the applications **List & Spreadsheet** with **Data & Statistics**. The second one demonstrates how changes on the **calculator** page have an effect on the **Graphs & Geometry** App. TI-Nspire proves in both cases to be an appropriate and easy to handle tool to run simulations very clear.

Ziel einer Simulation ist nicht nur die graphische Repräsentation des Geschehens, vielmehr ist auch eine Auswertung gewünscht, um einen Vergleich mit der Theorie zu ermöglichen. Der folgende linke Bildschirm zeigt das Ergebnis einer Auswertung von 100 Irrfahrten, wobei jede Irrfahrt im Ursprung begann und 10 Schritte dauerte. Auf der *y*-Achse ist die relative Häufigkeit aufgetragen, mit der die Irrfahrt im zugehörigen *x*-Wert endete. Die Beobachtung, dass jeder zweite Wert gleich Null ist, ergibt sich daraus, dass eine Irrfahrt, die in 0 beginnt, und eine gerade Zahl von Schritten dauert, auch nur in einem Punkt mit einem geradzahligen *x*-Wert enden kann (entsprechend endet eine Irrfahrt nach einer ungeraden Schrittzahl in einem Punkt mit einer ungeraden *x*-Koordinate).

The aim of a simulation is not only the graphic representation of the problem, but we would like to have an evaluation to enable a comparison with the theory behind. The left screen shows the evaluation of 100 random 10-steps-walks each of them starting in the origin. The *y*-values are the relative frequencies of the walks ending in the point (*x*,0). (≈ 30% of the walks ended in the origin, 2% in (–8,0), …). Every second value is 0, because an *n*-step walk starting from (0,0) with *n* = even will end in a point with an even *x*-value (if *n* = odd, then the final point will have an odd *x*-value, accordingly).





In der rechten Abbildung sind die simulierten relativen Häufigkeiten (Rechtecke) und die Wahrscheinlichkeiten (Kreise) in einem Bild eingezeichnet. Es zeigt sich vom Augenschein her eine gute Übereinstimmung zwischen Theorie und Simulation.

The right figure shows the results of the simulation (boxes) and the theoretical probabilities (balls). The correspondence between theory and experiment is convincing, isn´t it?

Wie man sich leicht klar machen kann, ist die Wahrscheinlichkeit, dass eine in 0 beginnende Irrfahrt nach *n* Schritten im Punkt (*k*,0) steht, gegeben durch

$$p(k)=\begin{cases}\dbinom{n}{\frac{n+k}{2}}\cdot\left(\dfrac{1}{2}\right)^n & k\in\{-n,-n+2,\ldots,n-2,n\}\\ 0 & \text{sonst}\end{cases}$$

Zwei nette Aufgaben für Schüler sind:

- *Leite die Formel für diese Wahrscheinlichkeit her.*
- *Wie ändert sich die Formel, wenn man mit der Wahrscheinlichkeit p den Schritt nach rechts macht, und natürlich mit q = 1–p den Schritt nach links?*

Diese Auswertung lässt sich mit dem Nspire-System zwar durchführen, allerdings ist dazu ein nicht geringes Maß an Erfahrung im Umgang mit Variablen und Listen erforderlich. Es ist zu bezweifeln, dass Schüler (und auch die meisten Lehrer) in der Lage sind, das Nspire-System in dieser Weise zu nutzen.
Die hier verwendeten Programme und Funktionen werden unten ohne nähere Erläuterung angegeben.

It is not difficult to find out that the probability that a random walk starting in the origin will end in point (k,0) is given by

$$p(k)=\begin{cases}\dbinom{n}{\frac{n+k}{2}}\cdot\left(\dfrac{1}{2}\right)^n & k\in\{-n,-n+2,\ldots,n-2,n\}\\ 0 & \text{else}\end{cases}$$

These might be nice problems for the students:

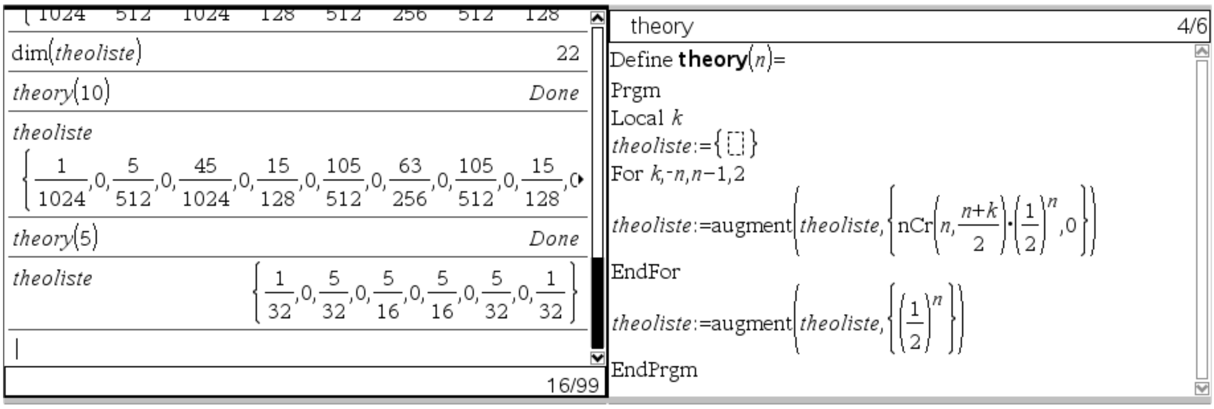
- *Derive the formula for this probabilitiy.*
- *How changes the formula if a step to the right is done with probability p – and then with probability q = 1–p the step to the left?*

One can perform this evaluation using the features of the Nspire system but it is necessary to have an not too small experience in handling variables and lists. It is doubtful if students (and even many of the teachers) are able to use the Nspire system in a such extended way.

The used programs are given below without more explications.

```
walk                                   0/4
Define walk(n)=
Prgm
  Local i
  For i,1,n
    xakt:=xakt+(-1)^randInt(1,2)
  EndFor
EndPrgm
```

```
simulation                             1/13
Define simulation(z,n)=
Prgm
© Führt z Irrfahrten mit je n Schritten durch
Local i
simliste:=newList(2·n+1)
For i,1,z
  xakt:=0
  walk(n)
  simliste[xakt+n+1]:=simliste[xakt+n+1]+1
EndFor
simliste:=simliste/z
xliste:={[]}
For i,-n,n
  xliste:=augment(xliste,{i})
EndFor
EndPrgm
```

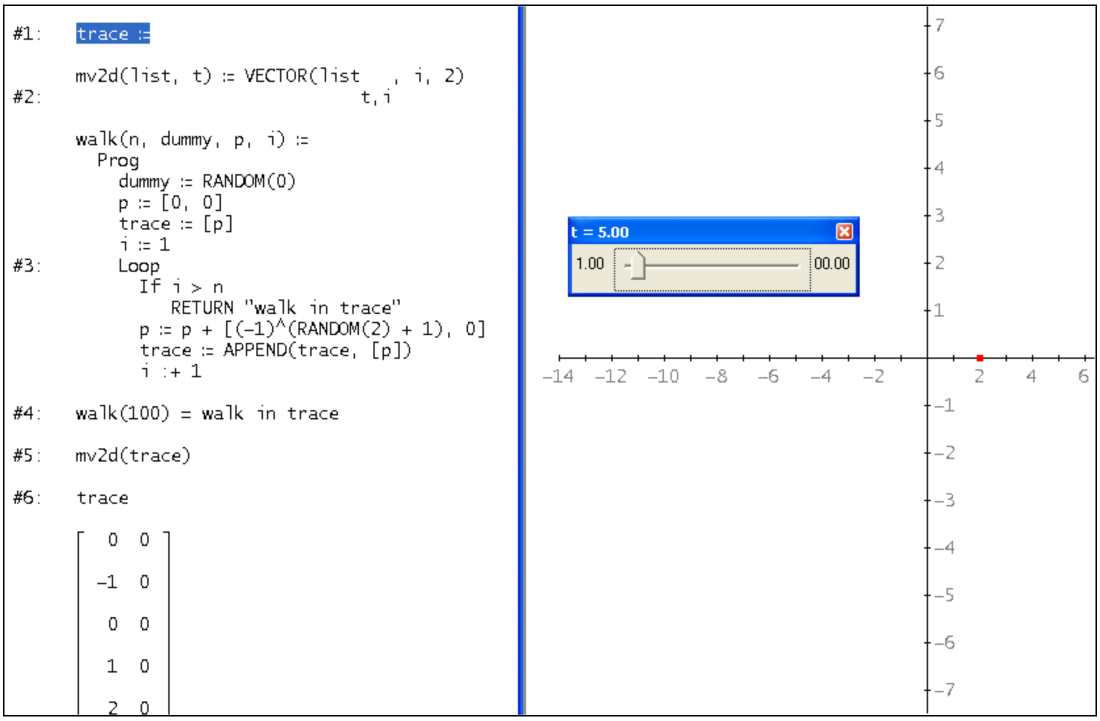


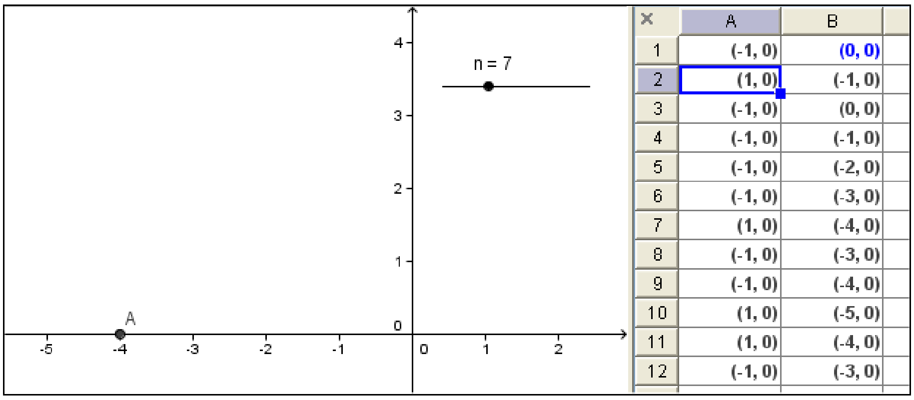
walk(n) performs a random walk of *n* steps. **simulation(z,n)** evaluates z of these *n*-step random walks and collects the results in **simliste** (which is needed for the graph). **theory(n)** gives the list of the theoretical values of the probabilities.

The concept of the linked variables together with the slider bar which is now implemented in the Nspire gives us the idea to model the random walk step by step or even automatically. This can be done in DERIVE and in GeoGebra (using its future spreadsheet feature):

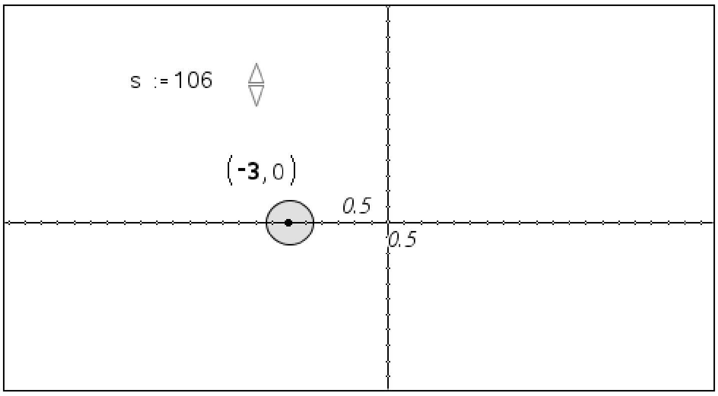
walk(n) erzeugt eine Irrfahrt über *n* Schritte. **simulation(z,n)** wertet *z* dieser *n*-Schritt Irrfahrten aus und sammelt die Ergebnisse in der Liste **simliste** (sie wird für die Erstellung des Graphen benötigt). **theory(n)** berechnet die Liste der theoretischen Werte der Wahrscheinlichkeiten.

Die vorliegende Möglichkeit der verlinkten Variablen zusammen mit den nun auch mit Nspire möglichen Schiebereglern bringt uns auf die Idee, die Irrfahrten Schritt für Schritt oder sogar automatisch ablaufen zu lassen. Wir können das mit DERIVE (nur schrittweise) und GeoGebra (indem wir das in einer Pre-Release-Version implementierte Spreadsheet nützen).





And finally on the Nspire:



GeoGebra and TI-Nspire offer the feature to animate the slider. So you can lean back watching the point jumping from one position to the other and finally reaching its destination after n steps.

The random walk offers some other questions:
One of them is to find out the mean distance travelled, another one to find out the mean displacement, which must be distinguished.

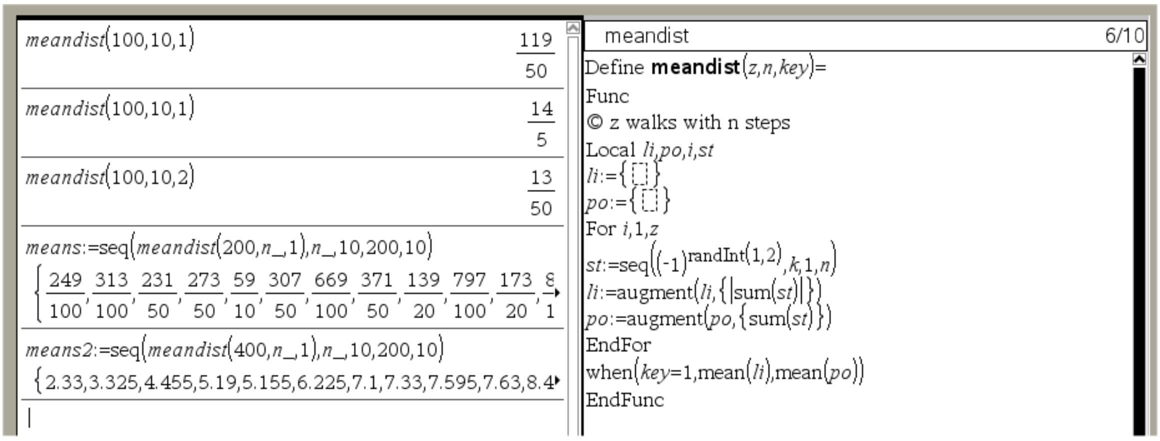
We have a short function (see below) which provides the mean distance (key = 1) and the mean displacement (key ≠ 1) for *z* experiments each of them walking *n* random steps.

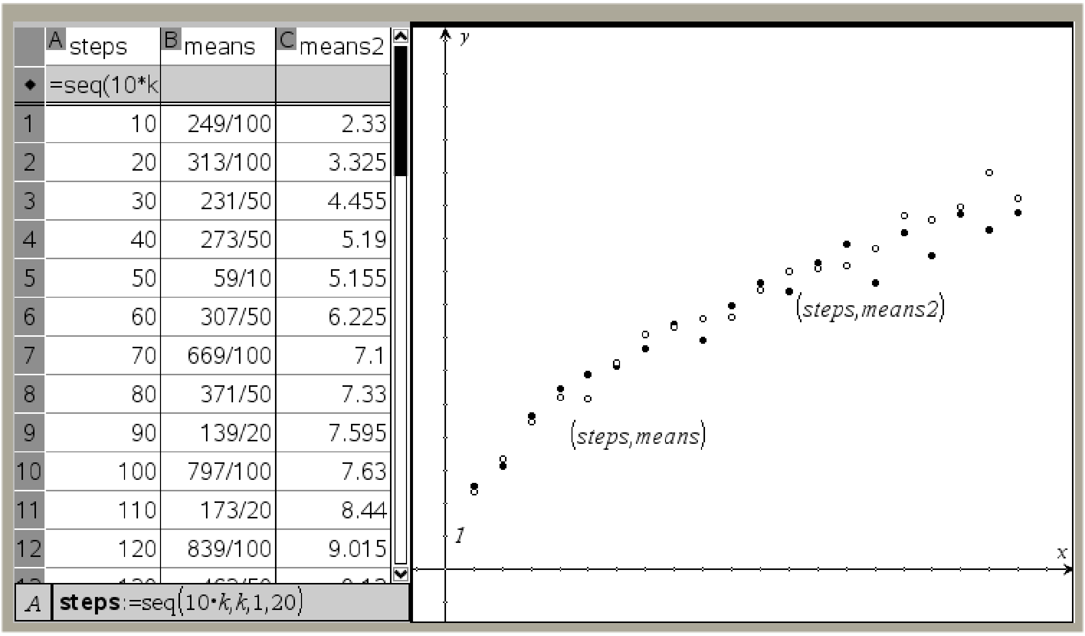
means and **means2** are two sequences for the mean distances after 10, 20, …, 190 and 200 steps for a series of 200 and 400 random walks. We transfer the two lists to a Lists & Spreadsheet page and plot the means versus the number of steps.

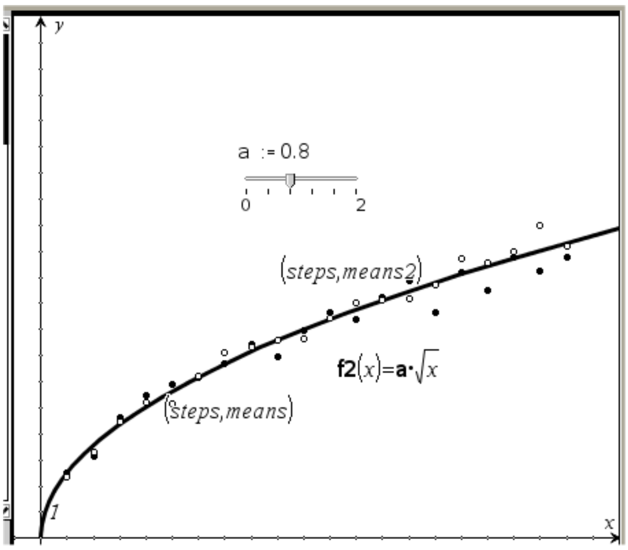
Im Zusammenhang mit den Irrwegen ergeben sich weitere Fragen:
Eine davon wäre die Frage nach der durchschnittlichen Entfernung des letzten Punkts vom Urpsprung, eine zweite die Frage nach der durchschnittlichen Position des letzten Punkts der Wanderung. Die beiden Mittelwerte sind sehr wohl zu unterscheiden.

Die kleine – unten angegebene – Funktion berechnet die durchschnittliche Entfernung (key = 1) und die durchschnittliche letzte Position (key ≠ 1) für *z* Simulationen mit je *n* Zufallsschritten.

means und **means2** sind zwei Folgen für die mittleren Entfernungen vom Ursprung nach 10, 20, …, 190 and 200 Schritten für 200 und 400 Irrwege. Wir übertragen die Listen in eine Lists & Spreadsheet Seite und zeichnen das Streudiagramm Schrittanzahl – mittlere Entfernung vom Ursprung.



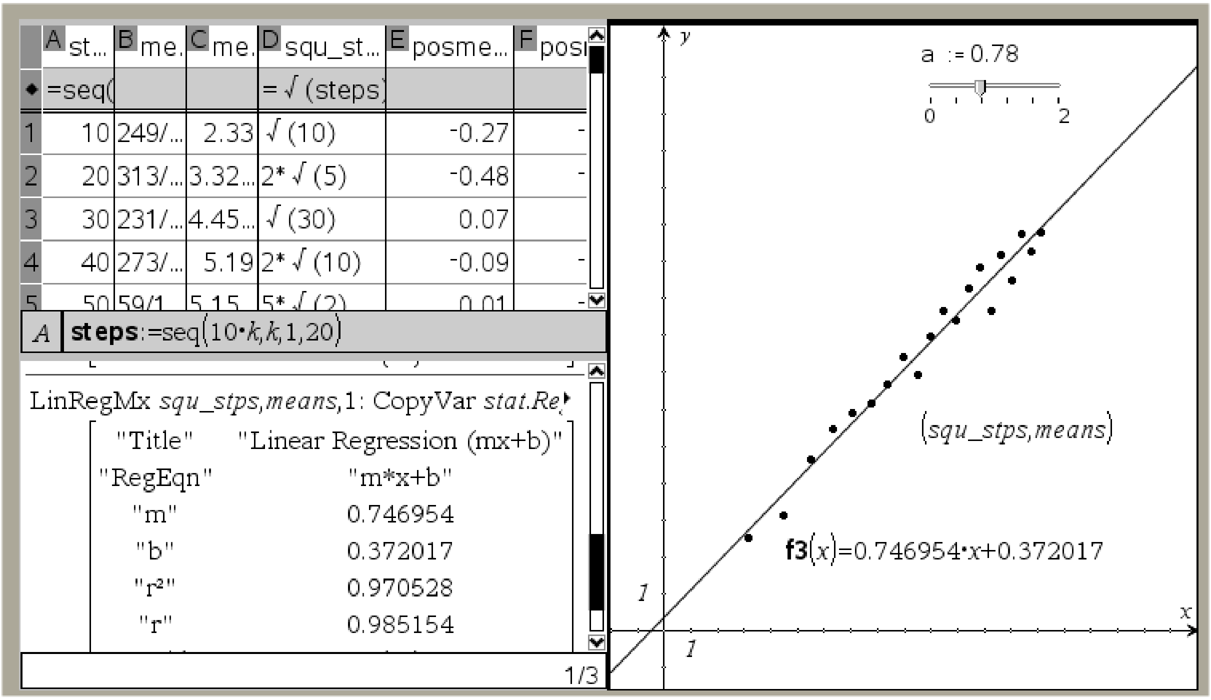




Das Streudiagramm lässt uns an eine Parabel der Form $y = a \cdot \sqrt{x}$ denken. Wir führen einen Schieberegler für den Koeffizienten *a* ein und tasten uns an $a \approx 0{,}8$ heran.

The scatter plot reminds us on a parabola of the form $y = a \cdot \sqrt{x}$. We insert a slider and find an appropriate estimation for *a* with $a \approx 0.8$.

Another idea is to plot the square root of the number of steps against the mean values, which leads to a linear regression line with the slope $m \approx 0.75$.



Theory says that the expectation value for *n* jumps of size *s* in one dimension is $s \cdot \sqrt{\frac{2n}{\pi}}$ which is for *a* = 1 ≈ 0.799*n*. Hence, simulation gives a reasonable result.

The mean displacement is relatively small value because the numbers of left and right steps will be approximately equal for large *n*.

Nach der Theorie ist der Erwartungswert für die Entfernung vom Ursprung nach *n* Schritten der Schrittlänge *s* bei einer eindimensionalen Irrfahrt $s \cdot \sqrt{\frac{2n}{\pi}}$. Für *s* = 1 ist dies 0,799 *n*. So können wir mit dem Ergebnis der Simulation recht zufrieden sein.

3. Das Galton-Brett (The Galton-Board)

Several years ago – it was 1997 – we had a simulation of the Galton Board on the TI-92 which was programmed by Wolfgang Pröpper. See a copy from DNL#28. (The program is available on request.)

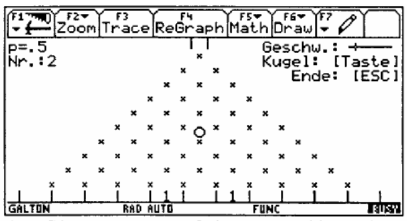


Fig. G1 Galton´s Board in mode 1

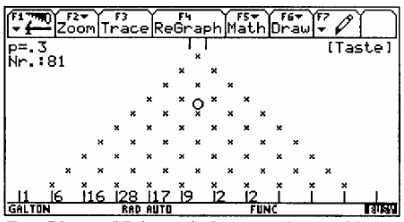


Fig. G2 GB in mode 2, stopped at marble 81

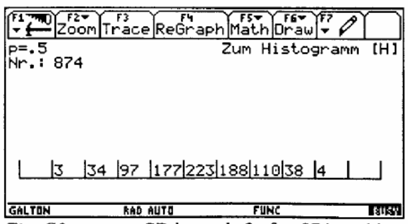


Fig. G3 GB in mode 3 after 874 marbles

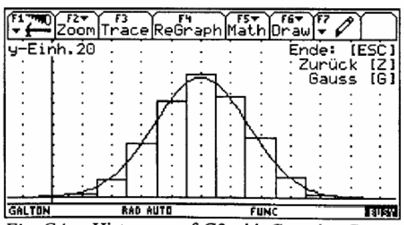


Fig. G4 Histogram of *G3* with Gaussian Curve

In the following you can see this famous simulation of the binomial distribution realised with TI-NspireCAS. The next contribution will show Lorenz Kopp´s visualisation with DERIVE.

Benno Grabinger Nov. 08

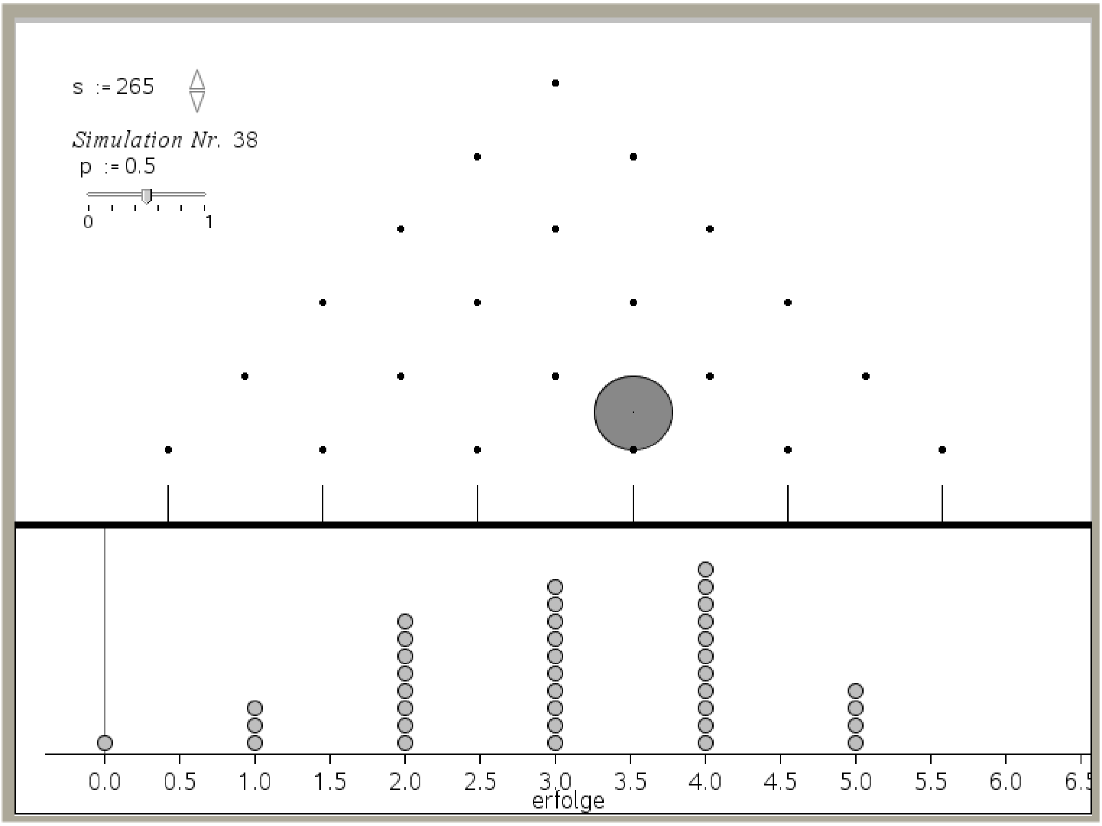
Galton-Board Vers. 1.2

1. Set counter s in the resp. slider bar by editing the textbox to 1 (don´t forget ENTER)
2. Set the probability (for choosing the right side) p on the slider bar.
3. Start the simulation by animating the slider s.
4. The maximum number of simulations is 200. **Before reaching simulation #200 the animation must be stopped using the slider bar, otherwise it would restart from the beginning.**
5. The Data & Statistics page shows the comparison between simulation and theory.
6. Before running a new simulation set slider **s** back to **1**!

Note: Repeating steps 1 – 6 the simulation gives always the same results. If you would like to have a simulation with other data, then create a new data set in the Lists % Spreadsheet page by pressing **Ctrl+R**.

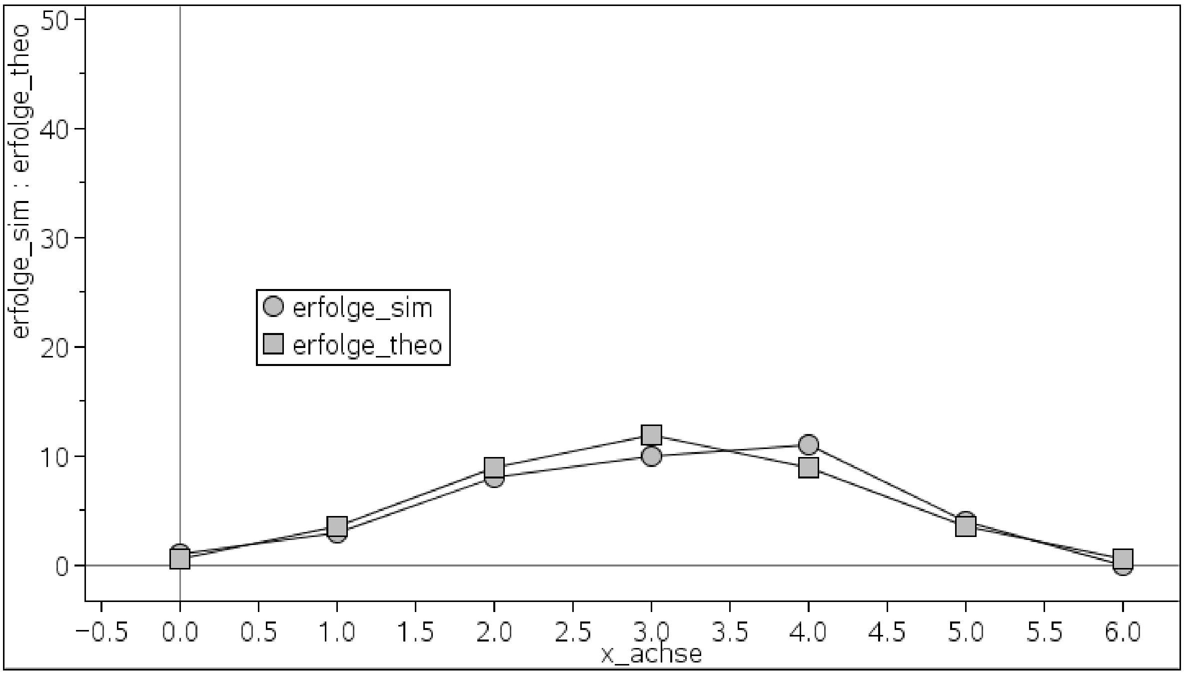
The Calaculator page contains the used functions. (*German version on page above*)

Es gibt auch eine deutschsprachige Anweisung im Dokument!



Wir halten nach 38 Kugeln und vergleichen Theorie und Experiment. Dann lassen wir die Kugeln weiter fallen (über den Schieberegler).

We stop after 38 balls and compare theory and experiment. Then we continue the simulation (using the slider for s).

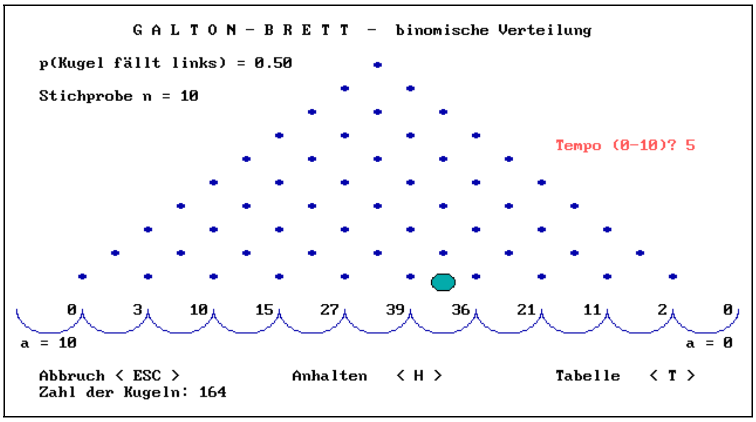


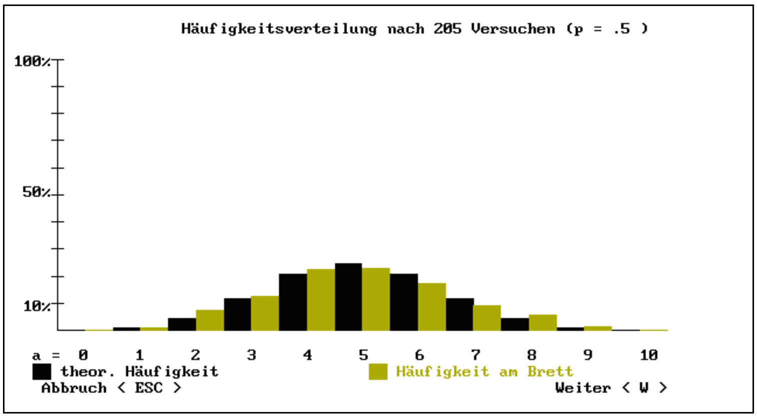
Fazit:
Mit dem Konzept der Verlinkung von Seiten lassen sich mit dem Nspire-System optisch ansprechende Simulationen durchführen. Voraussetzung ist dabei (wie bei allen anderen Software-Werkzeugen), dass beim Anwender eine Idee vorhanden ist, wie sich das Problem mit der Funktionalität der Software darstellen lässt. Dazu ist viel Erfahrung erforderlich, die auch nicht durch die angepriesene intuitive Bedienung eines Nspire ersetzt werden kann. Der durchschnittliche Schüler wird nicht in der Lage sein, Simulationen wie die hier vorgestellten selbst zu entwickeln. Als Ausweg bieten sich fertige Dokumente an, die die Schüler selbst verändern und mit diesen dann auch experimentieren können. Auf diese Weise lassen sich dann gezielt Erfahrungen mit dem Zufall sammeln, was uns im Alltag nicht immer möglich ist.

Summing it up:
The concept of linking the applications makes possible to perform pretty looking simulations. The prerequisite is (like with all other software tools) that the user has the idea how the problem can be presented using the functionality of the special software. This requires much experience which cannot be substituted by the highly praised intuitive manipulation of the Nspire. The average student will not be able to develop simulations like the ones which are presented in this article. An alternative are ready made documents, which can be varied and/or adapted by the students. In this way they can collect experiences with chance what is not always possible in daily life.

*In Pre-CAS times I produced a BASIC-program to simulate the GALTON board. I reanimated my Professional BASIC system for this DNL and created an English version od this program, too. In its compiled form it runs under my Windows-XP. Both versions are among the downloadable files. But I had to rename both programs with the extension *.txt. So the files are named GALTON_D.txt and GALTON_E.txt. Otherwise it wouldn´t be able to send them by email – the mail program denies to transfer *.EXE files. You only have to rename its extension to EXE. Double click on the programs in the Windows Explorer and it should run in a DOS-window. (Like in good old times!) We tried this, it works. Much luck. Josef*

*In den Zeiten vor dem CAS habe ich ein BASIC-Programm zur Simulation des GALTON-Bretts geschrieben. Ich habe mein BASIC Professional Development System aus dem Jahre Schnee wieder installiert und habe eine englische Version erzeugt. Außerdem mussten Warteschleifen eingebaut werden, da die Kugeln sonst zu rasch über die Nagelreihen getanzt wären. Die Programme wurden neu kompiliert und laufen unterWindows XP. Aber ich musste beide Programme umbenennen. Sie erhielten die Dateierweiterung *.txt. Sie heißen nun GALTON_D.txt und GALTON_E.txt. Als EXE-Datei lassen sie sich nicht per e-mail verschicken. Sie müssen den Programmen nur wieder die Originalextension EXE verpassen. Mit einem Doppelklick auf den Programmnamen im Explorer sollten sie im DOS-Fenster ablaufen. (Wie in der guten alten Zeit!) Ich wünsche viel Glück dazu. Josef*





[1] W. Pröpper, *The TI-92 as a Medium in Math Classes*, DNL#28, 1997
[2] P. Schofield, D. Sjöstrand, *Moving the Particles, Tracing their Path*, DNL#62, 2006
[3] A. Engel, *Wahrscheinlichkeitsrechnung und Statistik* 1 und 2, Klett,
[4] W. Feller, *An Introduction to Probability Theory and Its Applications*, Wiley, 1968
[4] B.H. Kaye, *A Randolm Walk Through Fractal Dimensions*, VCH, 1989
[5] H.-O. Peitgens a.o., *Fractals for the Classroom*, Springer, 1992
[6] R. Beare, *Mathematics in Action*, Chartwell-Bratt, 1997

Graphic Simulation of the Galton Board

including a histogram for the absolute frequency (substituting the pile of balls)

Lorenz Kopp, Germany

elli(c_,r_,i_,f_): ellipse for plotting the circles with f_ = ratio ymax/xmax
in the 2D Plot Window

ran(p): result is 1 for probability p_
p_<0.5, p_>0.5: the board is tipped to the left / to the right

```
#1:   [Notation :≡ Decimal, NotationDigits :≡ 4]

#2:   bs :≡

#3:   elli(c_, r_, i_, f_) :≡ VECTOR([r_·COS(φ), f_·r_·SIN(φ)] + c_, φ, 0, 2·π, π/i_)

#4:   ran(p_) :≡ FLOOR(RANDOM(1) + p_)
```

b_sim(n_,p_,k_,nb,sr,z_): simulation of a Bernoulli-chain with length n_,
parameter p_ to have exact k_ hits

nb: list of numbers 0 to k_ hits

bs: last value of nb, remains constant until next call of **b_sim**.

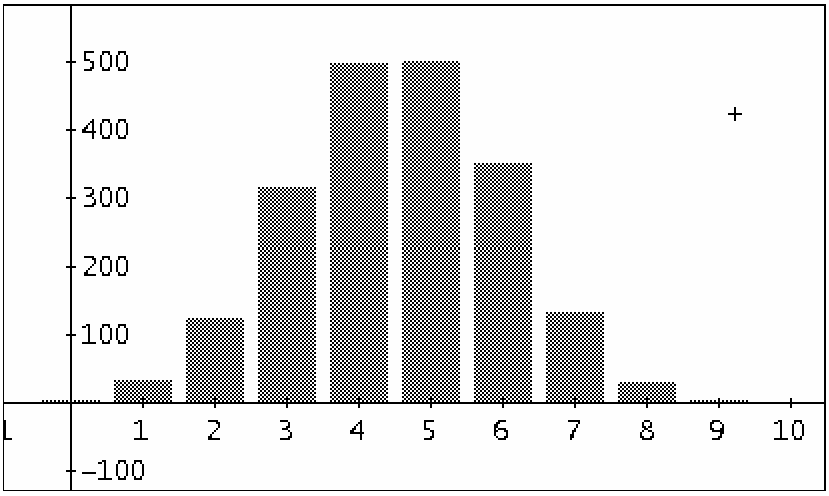
```
      b_sim(n_, p_, k_, nb, sr, z_, dummy) :≡
        Prog
          dummy :≡ RANDOM(0)
          nb :≡ VECTOR(0, j_, 1, k_ + 1)
          z_ :≡ 0
          Loop
#5:         If z_ = n_
               Prog
                 bs :≡ nb
                 RETURN bs
            sr :≡ ∑(ran(p_), j_, 1, k_)
            nb↓(sr + 1) :+ 1
            z_ :+ 1

#6:   b_sim(2000, 0.5, 9)

#7:   [3, 34, 124, 317, 499, 501, 352, 135, 32, 3]
```

histo(k_): Building the histogram belonging to **b_sim** from the bottom to the top and from the left to the right. (needs the global variable **bs** from **b_sim**)

```
      histo(k_) := VECTOR(VECTOR([j_ - 0.4 < x < j_ + 0.4 ∧ 0 < y < i_·bs      ],
#8:                                                                     j_ + 1

         j_, 0, k_), i_, 0, 1, 0.05)
```

```
#9:   histo(9)
```

b_histo(k_): histogram for simulating the Bernoulli chain




```
      b_histo(n_, p_, k_, nb, sr, z_, dummy) :=
        Prog
          dummy := RANDOM(0)
          nb := VECTOR(0, j_, 1, k_ + 1)
          z_ := 0
#10:      Loop
            If z_ = n_
               RETURN VECTOR([j_ - 0.4 < x < j_ + 0.4 ∧ 0 < y < nb↓(j_ + 1)],
            sr := ∑(ran(p_), j_, 1, k_)
            nb↓(sr + 1) :+ 1
            z_ :+ 1






        j_, 0, k_)
```

Parameters for plotting the GALTON Board

galton(k_): creates the board with **k_** rows of pins, **xmax** and **ymax** are
the maximum values on the axes

ho: top of the board

dx and **dy**: shift in direction of the axes; **rx** half horizontal axis of the ellipse

fx: stretch factor for the ellipse in vertical direction, set n_ < 2*ymax, k_ < 11

#11: [xmax := 11, ymax := 1100]

#12: $\left[ho := 0.925 \cdot ymax,\ dx := 0.5,\ dy := \frac{0.5 \cdot ymax}{xmax}\right]$

#13: $\left[rx := dx \cdot 0.5,\ fy := \frac{ymax}{xmax}\right]$

You can find the extended expression #14 galton(k_):= ... in the file

Recommended domain for plotting: $-7 \le x \le 17$, $-100 \le y \le 1100$

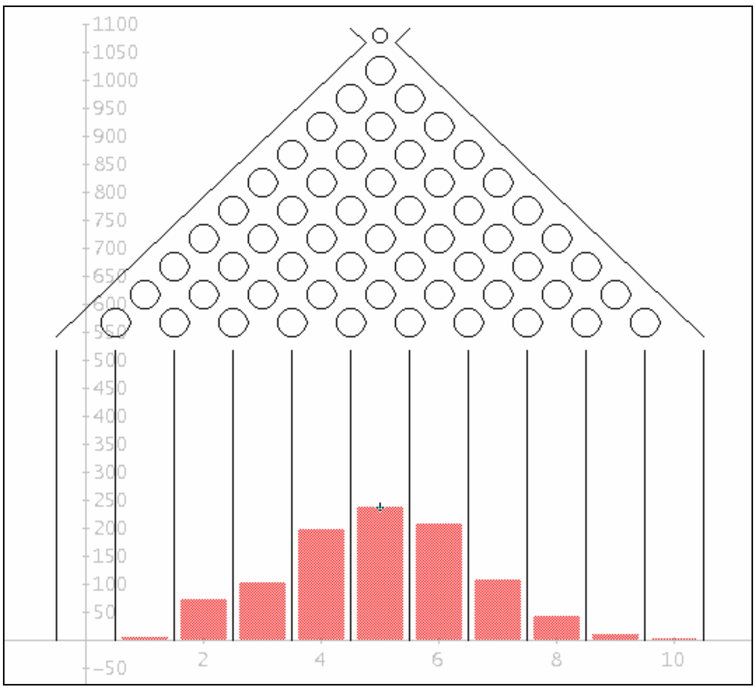
First application:

simplify **b_sim(n_,p_,k_)** (calculates the result bs),
then enter **[galton(k_), histo(k_)]**, highlight first **galton(k_)** and plot,
then highlight **histo(k_)** and plot.

Lets have 1000 balls, probability = 0.5 and 10 rows of pins:

#15: b_sim(1000, 0.5, 10) = [1, 6, 75, 105, 198, 238, 210, 110, 43, 11, 3]

#16: [galton(10), histo(10)]

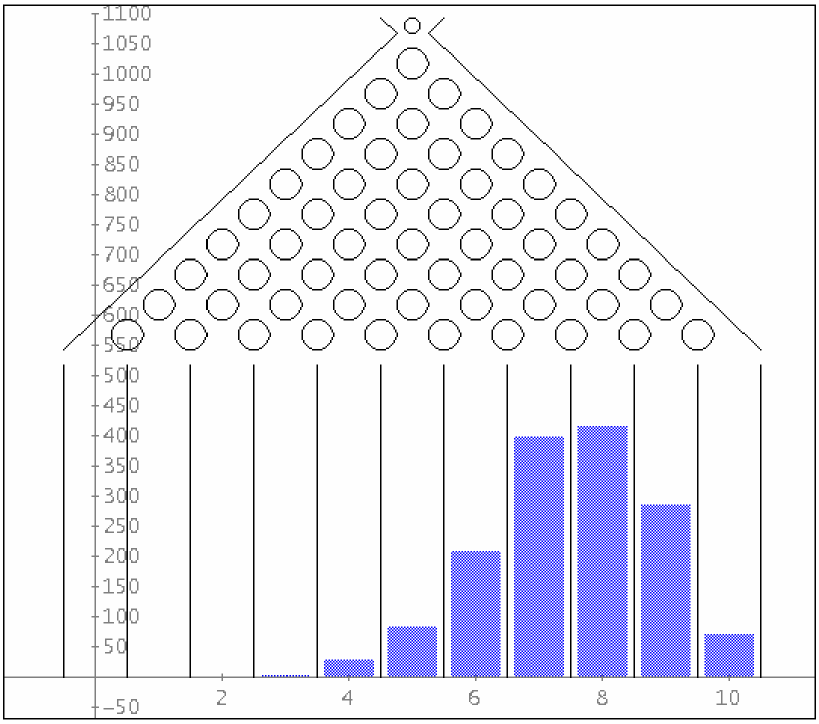


Second application:

Enter `[galton(k_), b_histo(k_)]`, highlight first `galton(k_)` and plot, then highlight `b_histo(k_)` and plot.

Lets have 1500 balls, probability = 0.75 and 10 rows of pins:

```
#17:  [galton(10), b_histo(1500, 0.75, 10)]
```




```
      b_sim_rel(n_, p_, k_, nb, pb, sr, z_, dummy) :=
        Prog
          dummy := RANDOM(0)
          nb := VECTOR(0, j_, 1, k_ + 1)
          pb := VECTOR(0, j_, 1, k_ + 1)
          z_ := 0
          Loop
#18:        If z_ = n_
               Prog
                 bs := [nb, pb]
                 RETURN [nb, pb]
            sr := ∑(ran(p_), j_, 1, k_)
            nb↓(sr + 1) :+ 1
            pb↓(sr + 1) := nb↓(sr + 1)/n_
            z_ :+ 1

#19:  b_sim_rel(1000, 0.5, 9)
```

#20:

2	10	77	172	240	234	177	70	17	1
0.002	0.01	0.077	0.172	0.24	0.234	0.177	0.07	0.017	0.001

```
      histo_rel(k_) ≔ VECTOR(VECTOR([j_ - 0.45 < x < j_ + 0.45 ∧ 0 < y <
#21:
        i_·bs          ], j_, 0, k_), i_, 0, 1, 0.1)
             2,j_ + 1

      b_histo_rel(n_, p_, k_, nb, pb, sr, z_) ≔
        Prog
          nb ≔ VECTOR(0, j_, 1, k_ + 1)
          pb ≔ VECTOR(0, j_, 1, k_ + 1)
          z_ ≔ 0
          Loop
#22:        If z_ = n_
               RETURN VECTOR([j_ - 0.45 < x < j_ + 0.45 ∧ 0 < y < pb↓(j_ +
            sumrd(k_) ≔ sr ≔ ∑(ran(p_), j_, 1, k_)
            nb↓(sr + 1) ≔ nb↓(sr + 1) + 1
            pb↓(sr + 1) ≔ nb↓(sr + 1)/n_
            z_ ≔ z_ + 1

        1)], j_, 0, k_)
```

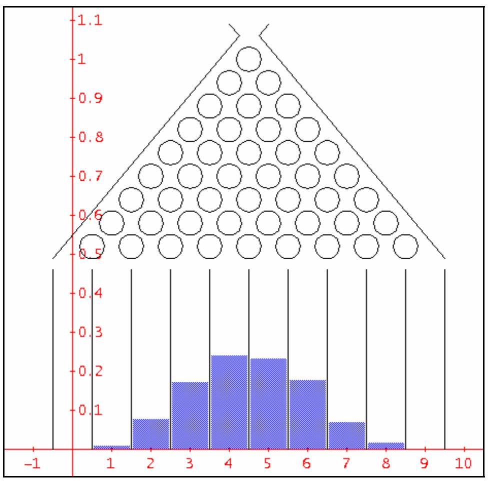
The Pin-Board for the relative frequency:

```
#23:  [ho_ ≔ 1, dx_ ≔ 0.5, dy_ ≔ 0.06]
```

#24: $\left[rg\_ := \frac{dx\_}{1.6},\ f\_ := 0.1\right]$

#25: `galton_rel(k_) ≔ ...` is again a huge construction.

```
#26:  galton_rel(9)

#27:  histo_rel(9)
```

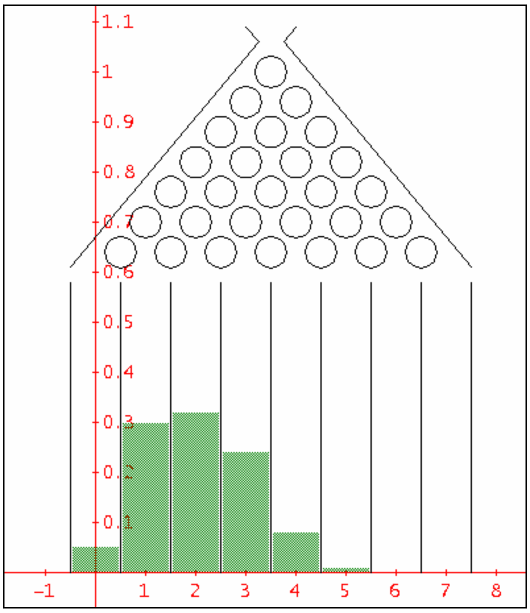


#28: b_sim_rel(100, 0.3, 7)

#29:

5	30	32	24	8	1	0	0
0.05	0.3	0.32	0.24	0.08	0.01	0	0

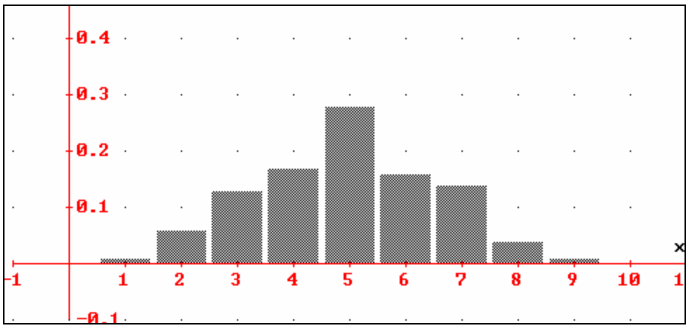
#30: [galton_rel(7), histo_rel(7)]



Examples for plotting histograms for simulating Bernoulli chains (binomial distribution)

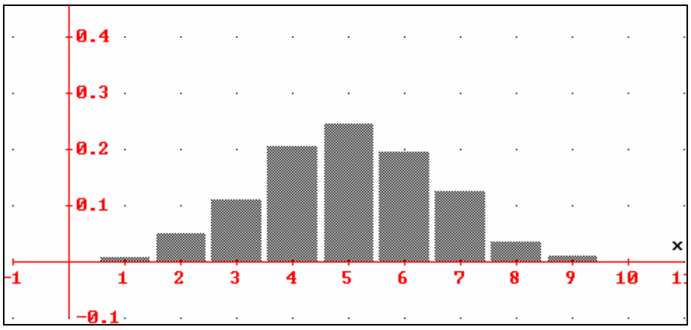
(n small, eg n = 100)

#31: b_histo(100, 0.5, 10)



(n large, eg n = 2000)

#32: b_histo(2000, 0.5, 10)



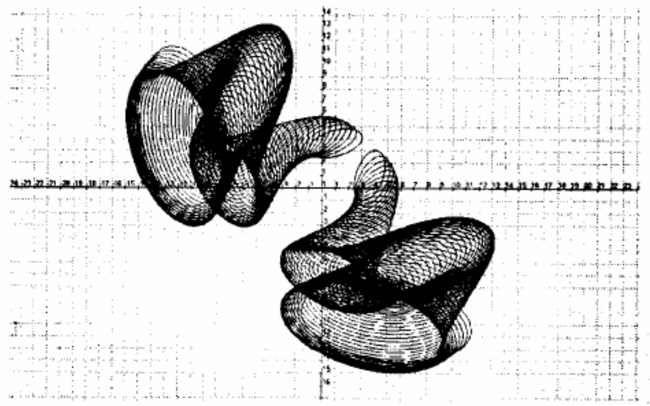
Zu Eberhard Lehmanns "Innovative Materialien zur Analytischen Geometrie"

Auf über 170 Seiten werden 5 Kapitel behandelt:

- Andere Wege in die Analytische Geometrie
- Abbildungsmatrizen, Schrägbilder und Transformationen (zB "Vom Kreis zur Banane")
- Mehrfach abbilden – Abbildungsfolgen
- Projekte mit Objekten der Analytischen Geometrie
- CAS-Hilfen für die Analytische Geomtrie der Kerncurricula

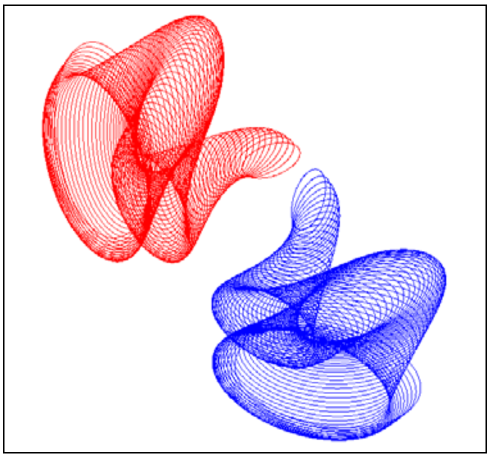
Eberhard shows a rich collection of inspiring math projects. He uses background pictures and animations. Although most of his examples are based on working with Eberhard´s program ANIMATO (which can be purchased) it is not too difficult to transfer his ideas to other CAS if you want to. But Eberhard treats his problems also with DERIVE and the TI-92/Voyage 200. There is a nice graph of "trumpet fish". See how Eberhard produces the graph with ANIMATO and then the translation to "DERIVIAN:

Trompetenfische, siehe f1 und f4

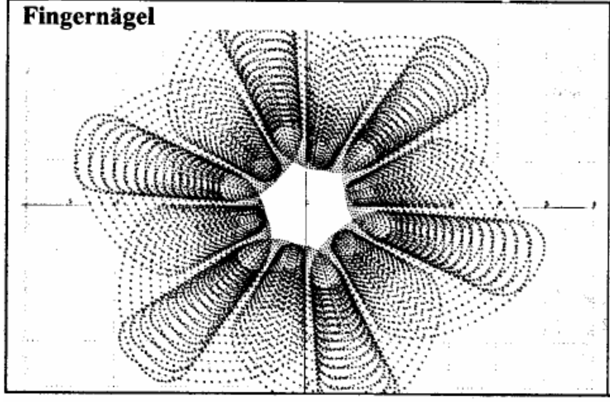


f1: 1
f2: pi/30
f3: f1^n*cos(n*f2)
f4: f1^n*sqrt(n*f2)
f5: 2cos(t)+2
f6: sin(t)+3
f8: f3(u)*f5-f4(u)*f6,f4(u)*f5+f3(u)*f6
f10: f4(u)*f5+f3(u)*f6,f3(u)*f5-f4(u)*f6

```
#1:  [f2 := π/30, f1 := 1]
#2:  [f3(n) := f1^n·COS(n·f2), f4(n) := f1^n·√(n·f2)]
#3:  [f5 := 2·COS(t) + 2, f6 := SIN(t) + 3]
#4:  [f3(u)·f5 - f4(u)·f6, f4(u)·f5 + f3(u)·f6]
#5:  VECTOR([f3(u)·f5 - f4(u)·f6, f4(u)·f5 + f3(u)·f6], u, 1, 100)
#6:  VECTOR([f4(u)·f5 + f3(u)·f6, f3(u)·f5 - f4(u)·f6], u, 1, 100)
```

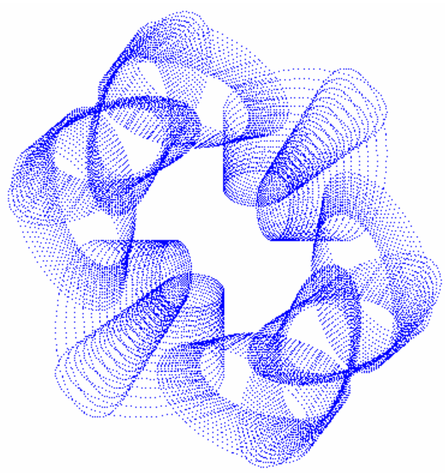


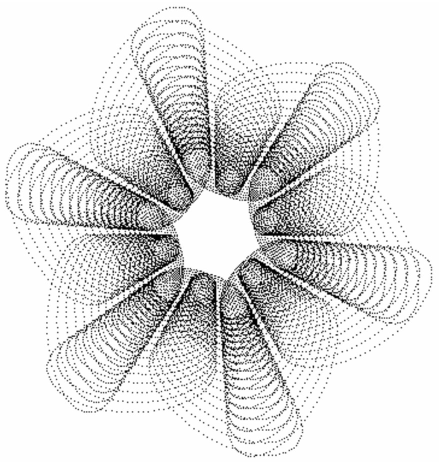
Or see the "Fingernails":



f1: 0.99
f2: pi/3
f3: f1^n*cos(n*f2)
f4: f1^n*sin(n*f2)
f5: 2cos(t)+2
f6: sin(t)+3
f8: f3(u)*f5-f4(u)*f6,f4(u)*f5+f3(u)*f6
f10: f4(u)*f5+f3(u)*f6,f3(u)*f5-f4(u)*f6

```
#7:   [f1 := 0.99, f2 := π/3]

#8:   [f3(n) := f1^n·COS(n·f2), f4(n) := SIN(n·f2)]

#9:   VECTOR([f3(u)·f5 - f4(u)·f6, f4(u)·f5 + f3(u)·f6], u, 1, 100)

#10:  VECTOR([f4(u)·f5 + f3(u)·f6, f3(u)·f5 - f4(u)·f6], u, 1, 100)

#11:  [f3(n) := f1^n·COS(n·f2), f4(n) := f1^n·SIN(n·f2)]

#12:  VECTOR([f3(u)·f5 - f4(u)·f6, f4(u)·f5 + f3(u)·f6], u, 1, 100)

#13:  VECTOR([f4(u)·f5 + f3(u)·f6, f3(u)·f5 - f4(u)·f6], u, 1, 100)
```

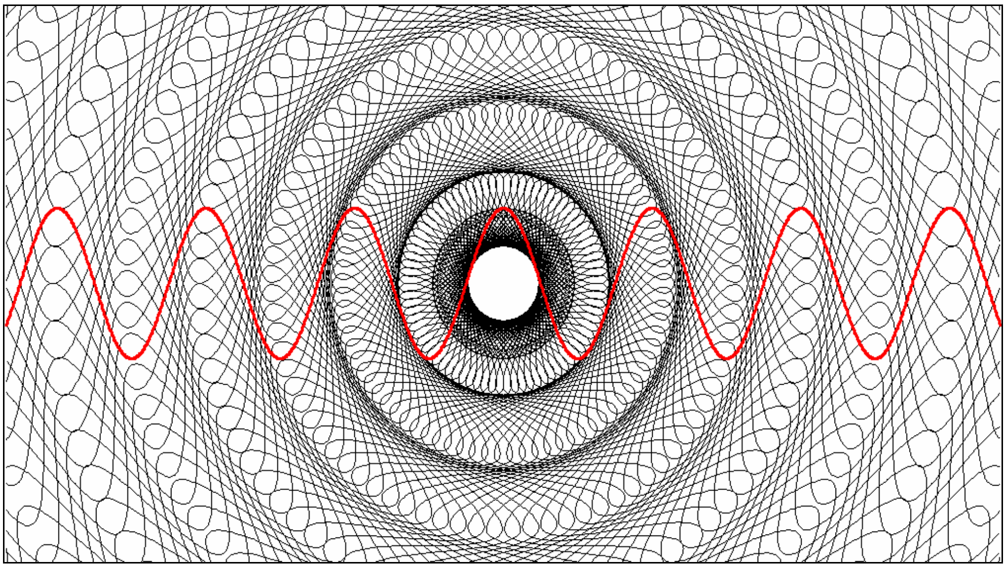


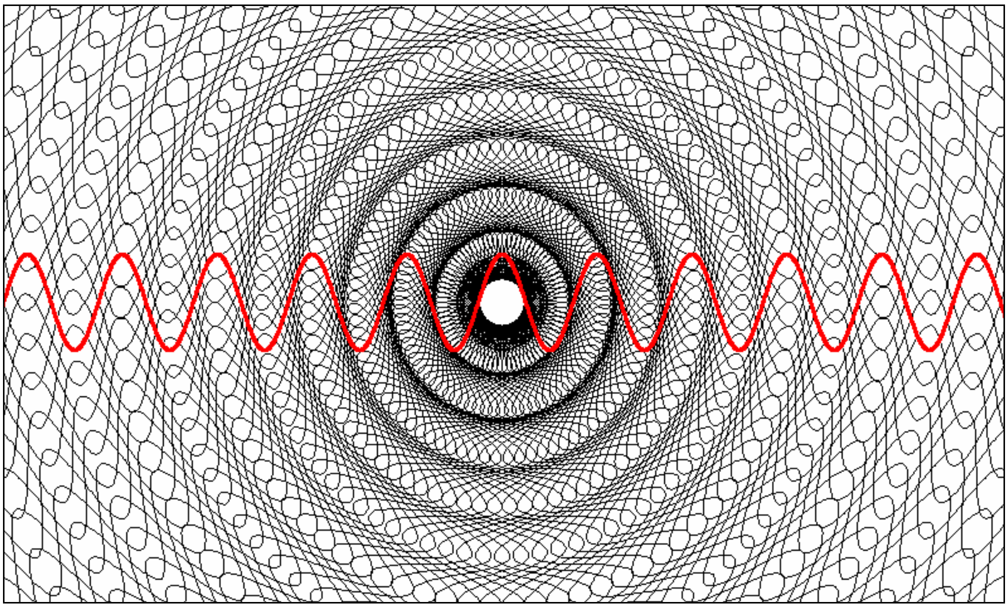


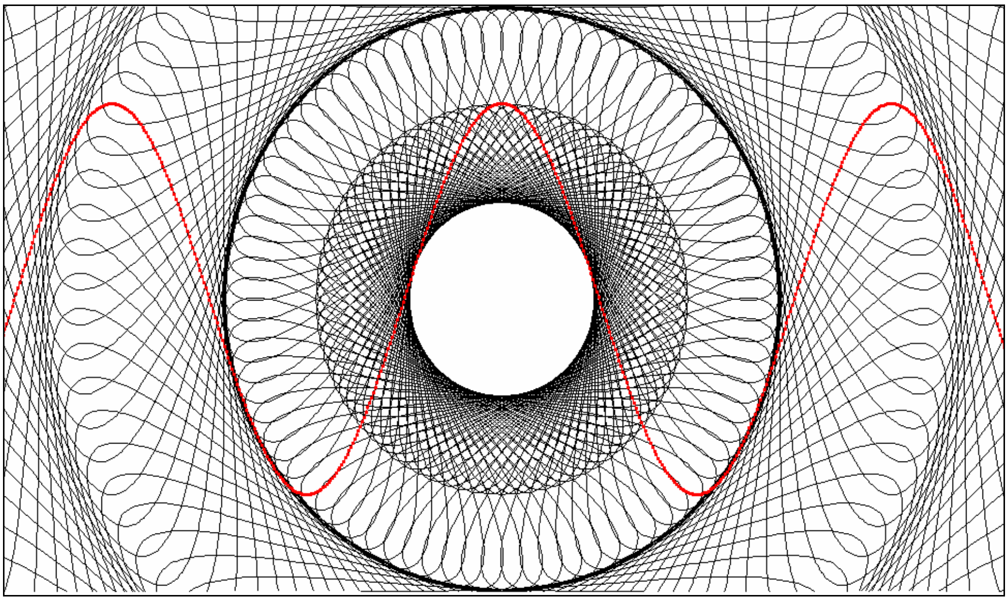
Among many others Eberhard gives a short introduction into POVRAY. I enjoyed this inexpensive book and can really recommend it. (www.snafu.de/~mirza)

Finally a nice example for applying the rotation matrix (in its DERIVE realisation)

```
#14:  [rot := [COS(u)  - SIN(u)
               SIN(u)    COS(u)], curve := [t, COS(3·t)]]

#15:  VECTOR(rot·curve, u, 0, 2·π, π/40)

#16:  TABLE(COS(3·t), t, - 6·π, 6·π, 0.01)
```

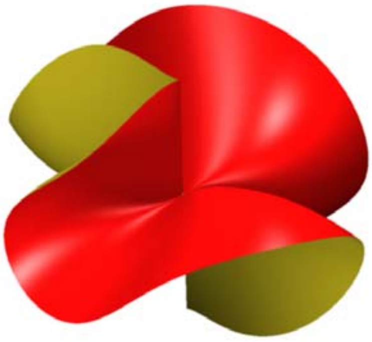


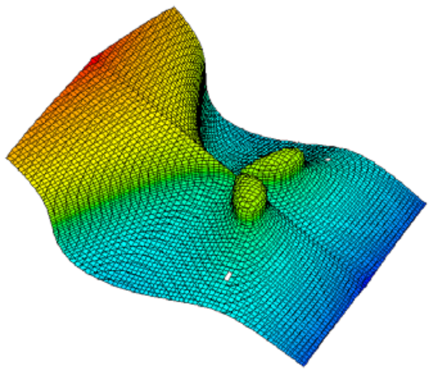


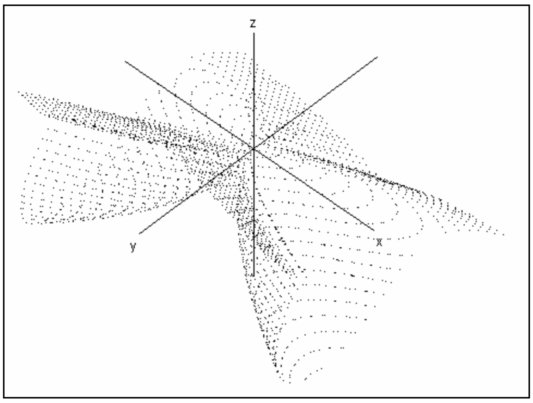


More information at: http://www.snafu.de/~mirza

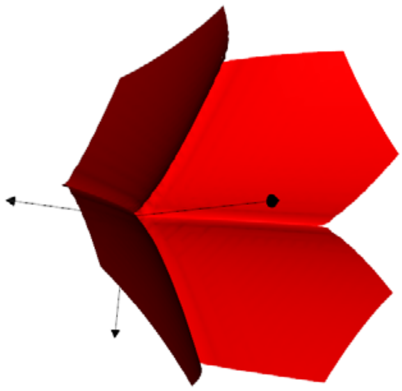
Surface #8: $(y^3 - x^2 - z^3)^3 = 27x^2y^3$

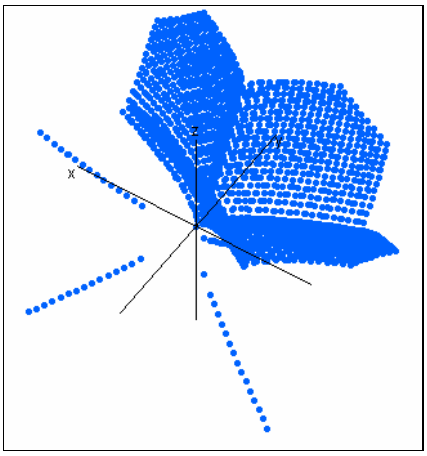




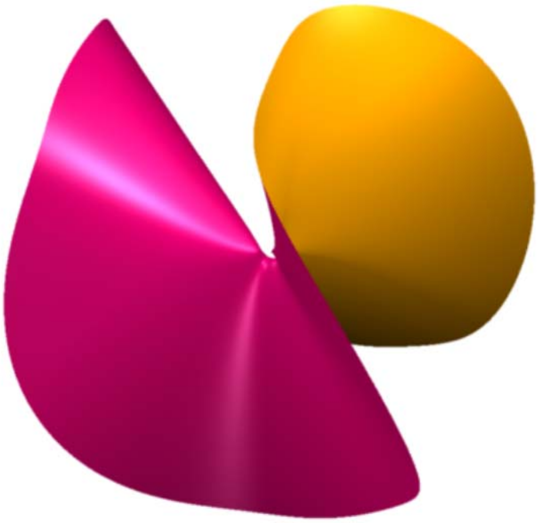


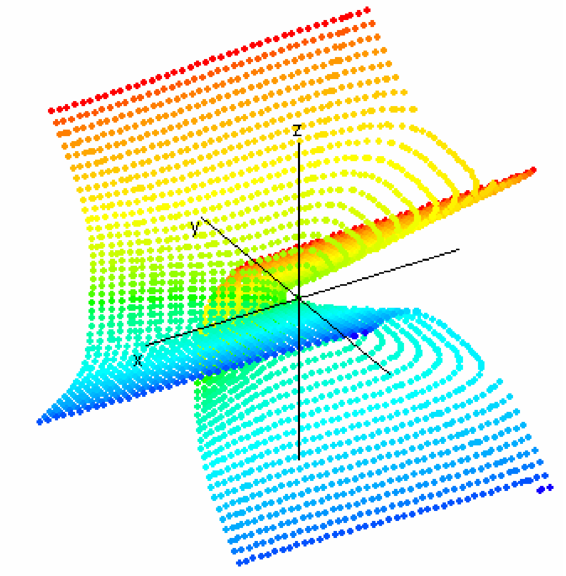
Surface #8a: $(y^3 - x^2 - z^2)^3 = 27x^2y^3z^2$



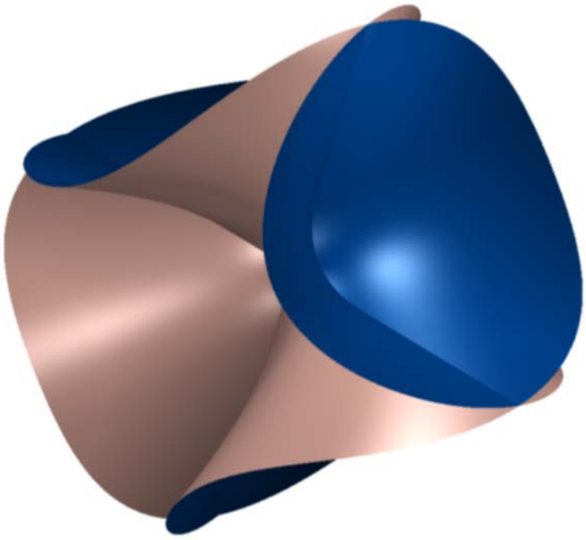


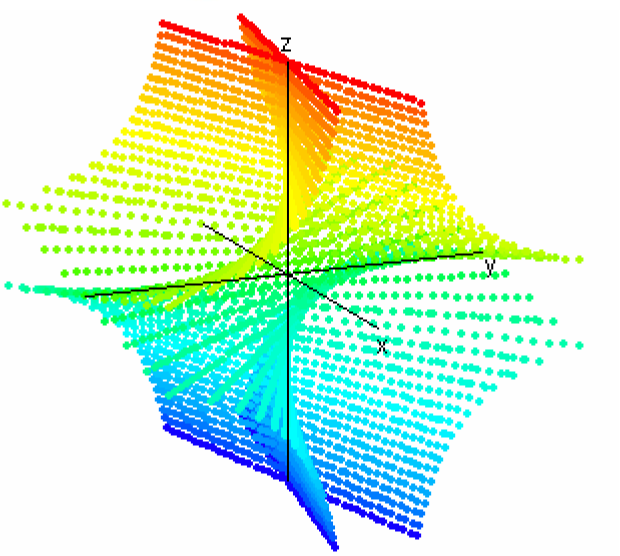
Surface #9: $x^2 - x^3 + y^2 + y^4 + z^3 = z^4$





Surface #10: $x^2 = y^2 z^2$





Long Division – Step by Step

Josef Böhm, Würmla, Austria

In my lecture at TIME 2008 I presented some CAS-tools which could serve as training tools for manipulating skills. I had already a lot of functions and programs but at the occasion of TIME 2008 I realized some new ideas using several software tools. This is "Long Division":

```
#1:   [DisplayFormat:=Compressed,TimesOperator:=Implicit]

#2:   [rd(n_):=RANDOM(n_)+1,num:=,res:=,quot:=0,task:=,v1,v2]

#3:   rs:=2 rd(2)-3

#4:   hk(u):=POLY_COEFF(u,x,POLY_DEGREE(u,x))

#5:   [l1:=[a,b,c,d,e,f,g,h,i,j,k,l,m],l2:=[n,o,p,q,r,s,t,u,v,w,x,y,z]]

      start_(dummy):=
        Prog
          dummy:=RANDOM(0)
          DISPLAY("Mit div wird eine Polynomdividision aufgerufen.")
          DISPLAY("Fortlaufende Ausführung von step liefert die schrittweise")
          DISPLAY("Durchführung der Division.")
          DISPLAY("")
          DISPLAY("div2 liefert Polynomdivisionen (ohne Rest) mit 2 Variablen.")
          DISPLAY("Fortlaufende Ausführung von step2 zeigt die schrittweise")
          DISPLAY("Ausführung dieser Aufgabe.")
#6:       DISPLAY("")
          DISPLAY("Der =-Button in der Eingabezeile ist sehr nützlich!")
          DISPLAY("")
          DISPLAY("div offers a task for long division of polynomials.")
          DISPLAY("Simplifying step gives stepwise execution of the division.")
          DISPLAY("")
          DISPLAY("div2 and step2 do the same with two variables (no remainder).")
          DISPLAY("")
          DISPLAY("Use the =-button in the Entry line!")
          DISPLAY("")

#7:   start:=start_()

      division(hdn,hdq,hdr,rest):=
        Prog
          hdn:=rd(2)+2
          hdq:=IF(hdn=3,2,rd(hdn-2))
          hdq:=IF(hdq=1,hdq+1,hdq)
          hdr:=IF(hdq=1vhdq=2,1,rd(hdq 1)+1)
          num:=rs rd(10) x^hdn+∑((5-rd(10)) x^k,k,0,hdn-1)
#8:       res:=rs rd(10) x^hdq+∑((5-rd(10)) x^k,k,0,hdq-1)
          rest:=rs rd(10) x^hdr+∑((5-rd(10)) x^k,k,0,hdr-1)
          rest:=[rest,0]↓rd(2)
          num:=EXPAND(num res+rest)
          task:=["","Dividiere/Divide","","","","";"(",num,") : (",res,")"," = "]
          "ausg:=task"
          quot:=0
          task

#9:   div:=division()
```

```
    step_(q_,prod):=
      Prog
        q_:=hk(num) x^POLY_DEGREE(num)/(hk(res) x^POLY_DEGREE(res))
        If POLY_DEGREE(q_)<0
           Prog
#10:         task:=APPEND(task,[["","Rest/Remainder:",num,"","Quotient:",quot]])
             RETURN task
        num:=num-q_ res
        quot:=quot+q_
        task:=APPEND(task,["",EXPAND(-q_ res),"","",quot,"";"",EXPAND(num),"","","",""])
        task

#11:  step:=step_()
```

Load `division.mth` as a Utility file and simplify `start`. Then you are shown how the program works and the random number generator is automatically initialized. I decided not to write the instru-ctions in a text box because then it would not be able to store the file as an mth-file which can be loaded as a utility file. I make use of the `DISPLAY`-command.

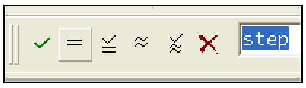
```
#17:  start

Mit div wird eine Polynomdividision aufgerufen.
Fortlaufende Ausführung von step liefert die schrittweise
Durchführung der Division.

div2 liefert Polynomdivisionen (ohne Rest) mit 2 Variablen.
Fortlaufende Ausführung von step2 zeigt die schrittweise
Ausführung dieser Aufgabe.

Der =-Button in der Eingabezeile ist sehr nützlich!

div offers a task for long division of polynomials.
Simplifying step gives stepwise execution of the division.

div2 and step2 do the same with two variables (no remainder)

Use the =-button in the Entry line!
```

The command `dummy:=random(0)` makes sure that at every start new problems are generated by the program `division()`, which can simply be called by simplifying `div`.

In the following you can see a sample session:

#19: $\text{div}=\begin{bmatrix} \text{Dividiere/Divide} \\ (\ -8x^5+25x^4-4x^3+3x^2\ ) : (\ -8x^2+x-1\ ) \quad = \end{bmatrix}$

Write `step` into the entry line and click on the =-button to execute the function. You will see the first step performed.



Consecutive clicking on this button the long division algorithm is presented "step by step".

#20: Dividiere/Divide

$$(-8x^5+25x^4-4x^3+3x^2) : (-8x^2+x-1) =$$
$$8x^5-x^4+x^3 \qquad x^3$$
$$24x^4-3x^3+3x^2$$

#21: Dividiere/Divide

$$(-8x^5+25x^4-4x^3+3x^2) : (-8x^2+x-1) =$$
$$8x^5-x^4+x^3 \qquad x^3$$
$$24x^4-3x^3+3x^2$$
$$-24x^4+3x^3-3x^2 \qquad x^3-3x^2$$
$$0$$

#22: Dividiere/Divide

$$(-8x^5+25x^4-4x^3+3x^2) : (-8x^2+x-1) =$$
$$8x^5-x^4+x^3 \qquad x^3$$
$$24x^4-3x^3+3x^2$$
$$-24x^4+3x^3-3x^2 \qquad x^3-3x^2$$
$$0$$

Rest/Remainder: 0 Quotient: x^3-3x^2

This division had no remainder. Let´s try the next one:

#23: Dividiere/Divide

$$(81x^5+18x^4-35x^3-6x^2-5x) : (9x^2+4x-1) =$$

After several steps we come to an end and see a remainder is left:

#27:

Dividiere/Divide

$$(81x^5+18x^4-35x^3-6x^2-5x):(9x^2+4x-1) =$$

$$-81x^5-36x^4+9x^3 \qquad\qquad 9x^3$$

$$-18x^4-26x^3-6x^2-5x$$

$$18x^4+8x^3-2x^2 \qquad\qquad 9x^3-2x^2$$

$$-18x^3-8x^2-5x$$

$$18x^3+8x^2-2x \qquad\qquad 9x^3-2x^2-2x$$

$$-7x$$

Rest/Remainder: $-7x$ Quotient: $9x^3-2x^2-2x$

I don´t show here the code for `div2` but only start and end of one run of this exercise.

#28: div2

#29: Dividiere/Divide

$$(15j^6s^3-15j^5s^2-3j^4s^3-20j^3s^4+3j^3s^2-20j^2s^3+4js^4+4s^3):(5j^2s-s) =$$

This is the final table showing the complete division:

#33:

Dividiere/Divide

$$(15j^6s^3-15j^5s^2-3j^4s^3-20j^3s^4+3j^3s^2-20j^2s^3+4js^4+4s^3):(5j^2s-s) =$$

$$3j^4s^3-15j^6s^3 \qquad\qquad 3j^4s^2$$

$$-15j^5s^2-20j^3s^4+3j^3s^2-20j^2s^3+4js^4+4s^3$$

$$15j^5s^2-3j^3s^2 \qquad\qquad 3j^4s^2-3j^3s$$

$$-20j^3s^4-20j^2s^3+4js^4+4s^3$$

$$20j^3s^4-4js^4 \qquad\qquad 3j^4s^2-3j^3s-4js^3$$

$$4s^3-20j^2s^3$$

$$20j^2s^3-4s^3 \qquad\qquad 3j^4s^2-3j^3s-4js^3-4s^2$$

$$0$$

Solve Statistics or Probability problems instantly...Step by Step

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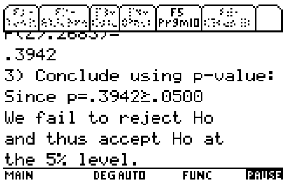
Contents:

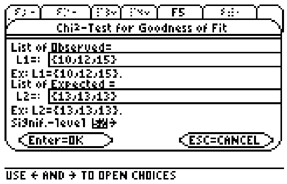
Statistical Tests - Step by Step
Enter data or statistics, select the level of significance and view how the appropriate statistical test is performed until the conclusion of the test: reject or fail to reject Ho.

- Z-Test for mean where sigma is known (1 or 2 samples).
- T-Test for mean where sigma is unknown (1 or 2 samples).
- Z-Test for proportion (1 or 2 samples).
- Test for Regression Line Slope
- ChiSquare Test for Goodness of Fit
- ChiSquare Test for Independence
- F Test for 2 variances
- ANOVA to test several means
- Non-parametric Tests : Sign Test, Sign Rank Test, Kruskal-Wallis, etc
- Read about Errors of Significant Tests: Type 1 Error, Type 2 Error, Power of a Test.

Statistical Tests - with ease and steps

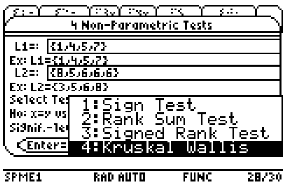


Z Test for proportions



Chi Square Goodness of Fit Test

Non Parametric Tests: Kruskal Wallis Test - with ease and steps



Confidence Intervals - Step by Step

Distributions

Regressions

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- Plot and Analyse Data: Histogram, Scatterplot, x-y Box Plot, modified Box Plot
- Measures of Centre: Mean, Median, Mode, Harmonic Mean, Geometric Mean
- Measures of Spread: Standard Deviation, Variance, Max , Min, Range
- Moving Average, Weighted Average
- Visual Simulation: Law of Large Numbers
- Random Number Generator (customisable)
- Random Variables: Compute Variance, Expected Value and Standard Deviation

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- Automatic Probability Checker for independent events, disjoint events.
- Also solves P(A or B)=P(A)+P(B)-P(A and B) for the unknown.
- Compute Marginal and Conditional Probabilities. Step by Step
- Compute Odds of Events
- Visual Simulation: Law of Large Numbers
- Compute Combinations nCr and Permutations nPr, n!, Counting Principle, Pascal Triangle
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You can find more information at:

http://www.ti89.com/spme/index_spme.htm

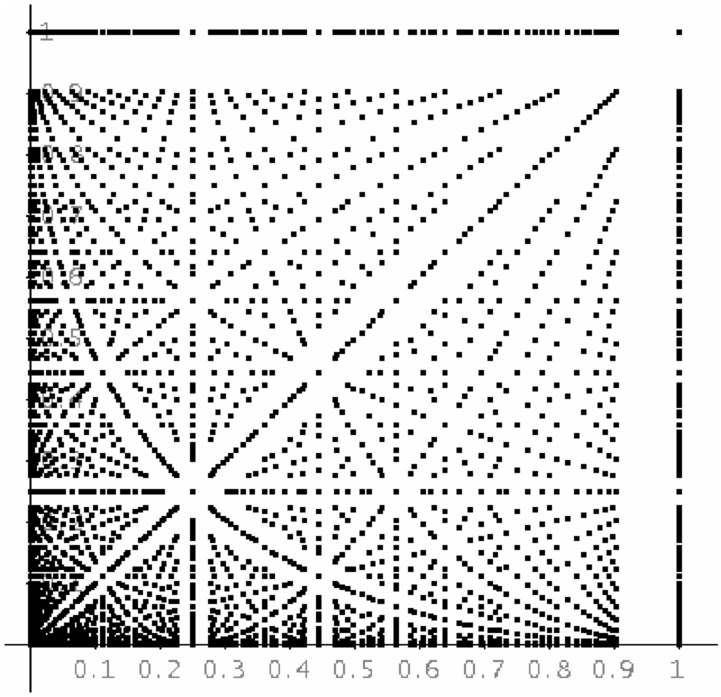
http://www.ti89.com/spme/documentation.htm

Some time ago I received a mail from our member Miltom Lesmes from Bogota, Colombia. The text was very short, but the graphs were of high interest for me. This what he wrote:

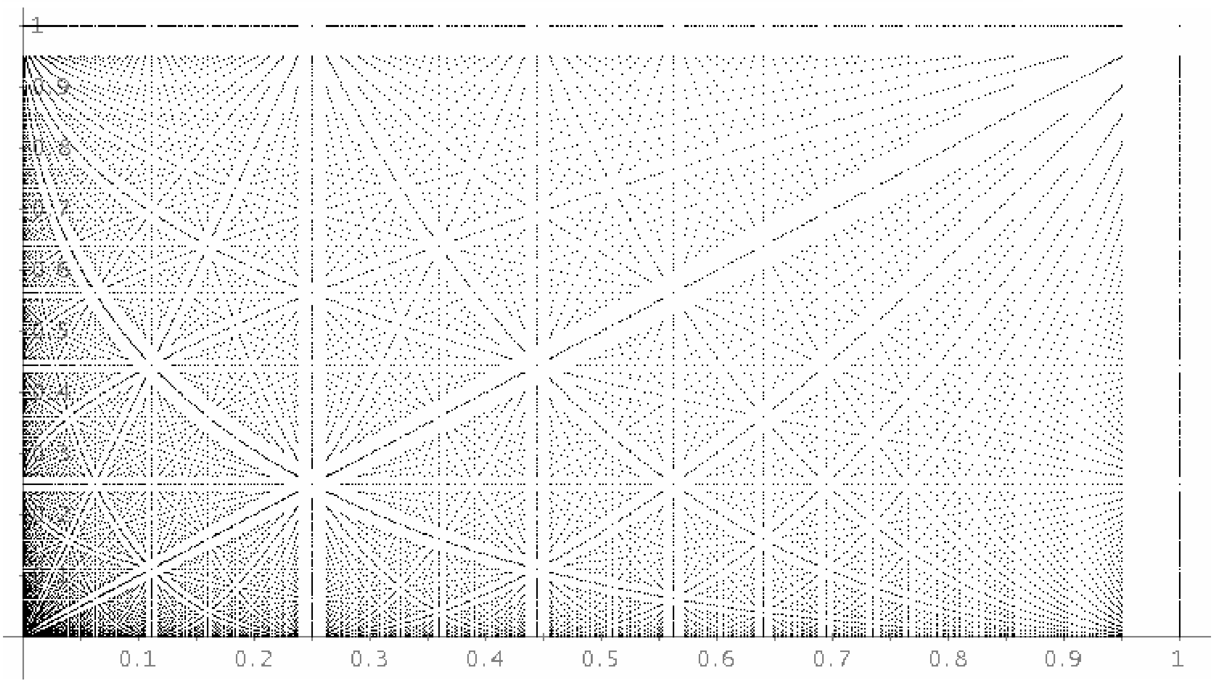
Dear Josef,

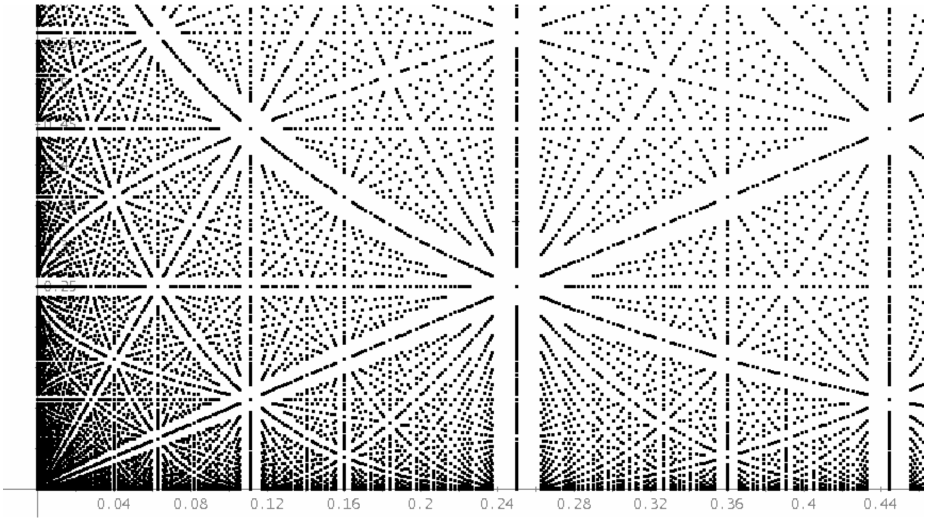
You remember things like this,

$$\#1: \quad \text{VECTOR}\left(\text{VECTOR}\left(\text{VECTOR}\left(\left[\left[\left(\frac{i}{k}\right)^2, \left(\frac{j}{k}\right)^2\right]\right], i, 0, k\right), j, 0, k\right), k, 20\right)$$

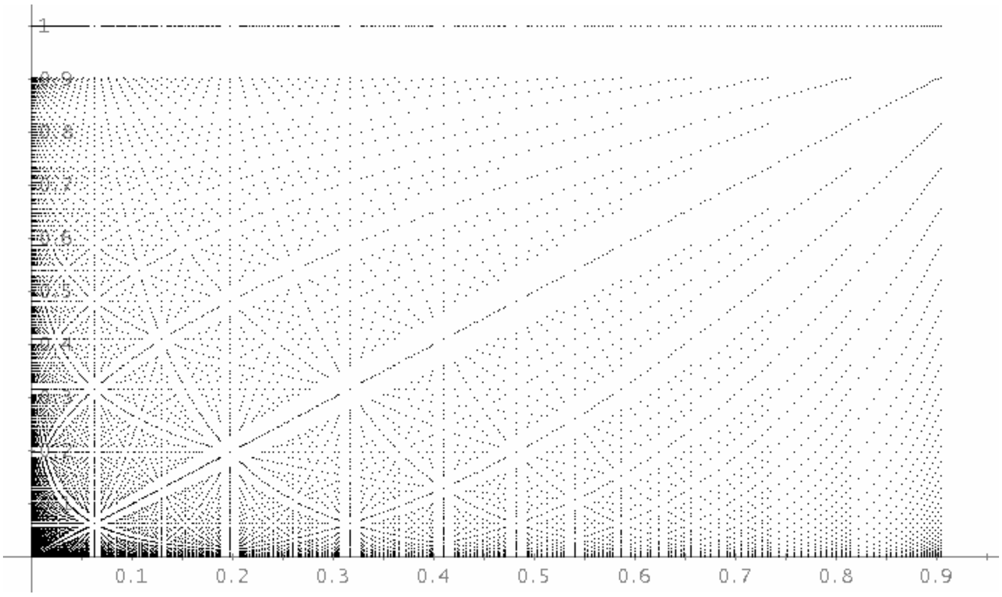


$$\#6: \quad \text{VECTOR}\left(\text{VECTOR}\left(\text{VECTOR}\left(\left[\left[\left(\frac{i}{k}\right)^2, \left(\frac{j}{k}\right)^2\right]\right], i, 0, k\right), j, 0, k\right), k, 40\right)$$



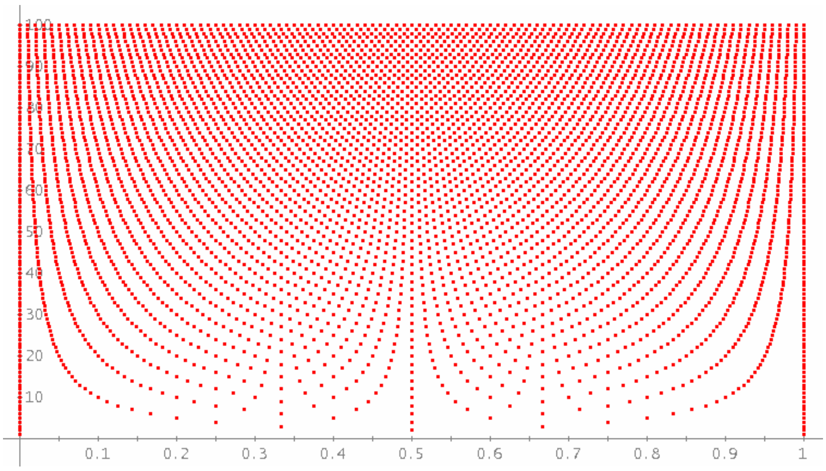


#8: $$\mathrm{VECTOR}\left(\mathrm{VECTOR}\left(\mathrm{VECTOR}\left(\left[\left(\frac{i}{k}\right)^4, \left(\frac{j}{k}\right)^4\right], i, 0, k\right), j, 0, k\right), k, 40\right)$$

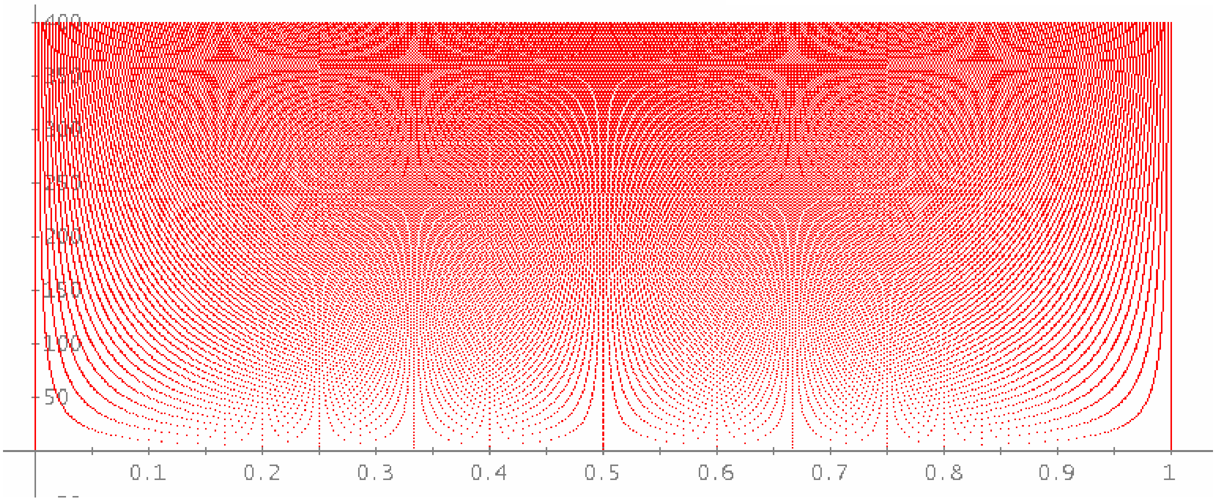


Now Farey Sequences:

#9: $$\mathrm{VECTOR}\left(\mathrm{VECTOR}\left(\left[\frac{i}{k}, k\right], i, 0, k\right), k, 1, 100\right)$$



#11: $\text{VECTOR}\left(\text{VECTOR}\left(\left[\frac{i}{k}, k\right], i, 0, k\right), k, 1, 400\right)$



To be honest, I didn´t know much more about Farey sequences than that they exist, that Johann Wiesenbauer wrote about them in one of his Titbits (Titbits 11, DNL#27, 1997) and that there is a number theoretic function FAREY(n) implemented in DERIVE. So I informed about Farey sequences and found some interesting details in CRC Concise Encyclopedia of Mathematics from Eric Weisstein.
Josef

The Farey Sequence F_n for integer $n > 0$ is the set of irreducible rational numbers a/b with $0 \le a \le b \le 1$ and $(a,b) = 1$ arranged in increasing order.

$$F_1 = \left\{\frac{0}{1}, \frac{1}{1}\right\}$$

$$F_2 = \left\{\frac{0}{1}, \frac{1}{2}, \frac{1}{1}\right\}$$

Some authors (like Albert Rich in DERIVE) do not include $\frac{0}{1}$.

$$F_3 = \left\{\frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1}\right\}$$

...

#1: $\text{FAREY}(6) = \left[\frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, 1\right]$

In DNL#27 you can find Johann´s function to create this sequence.

The sequence is named after Farey, a geologist who mentioned them 1816, but they were found earlier in 1806 by Haros.

The number of the elements of the sequence F_n is given by $N(n) = \sum_{k=1}^{n} \phi(k)$.

The elements of the sequence have two interesting properties:

If $\frac{a}{b}, \frac{a'}{b'}, \frac{a''}{b''}$ are consecutive elements, then $\frac{a'}{b'} = \frac{a + a''}{b + b''}$ and $|a'b - ab'| = 1$.

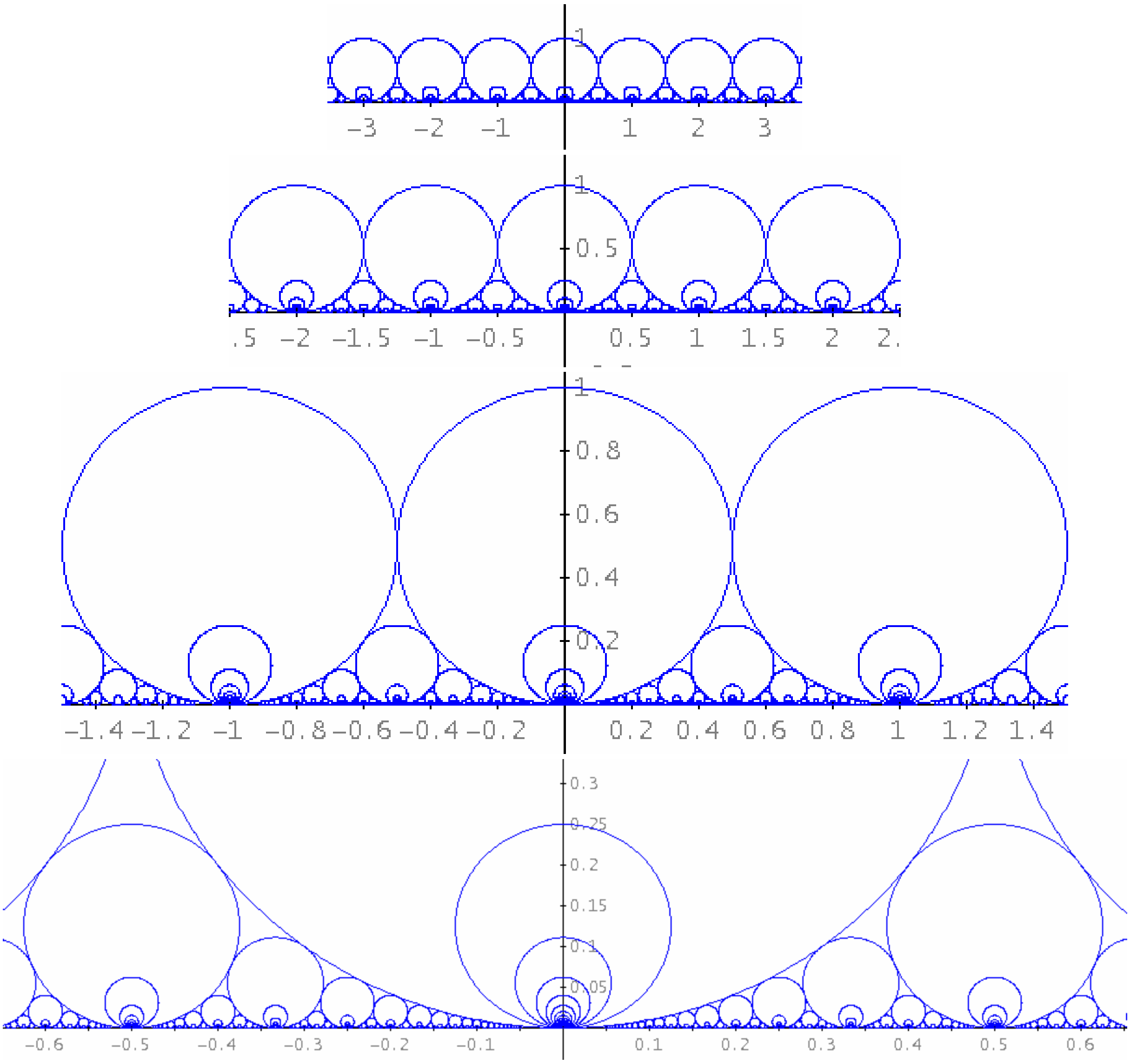
I found under "related topics" that "FORD CIRCLES" provide a method of visualizing the Farey Sequence.

> Take any two integers h and k, then the circle with its centre at $\left(\frac{h}{k}, \frac{1}{2k^2}\right)$ and its radius $r = \frac{1}{2k^2}$ is a "Ford circle".

You can draw as many Ford circles as you want with any hs and ks and none of their respective circles will intersect. This was interesting enough to plot the Ford circles with DERIVE:

$$\mathrm{VECTOR}\left(\mathrm{VECTOR}\left(\left[\frac{1}{2\cdot j^2}\cdot \mathrm{COS}(t) + \frac{i}{j}, \frac{1}{2\cdot j^2}\cdot \mathrm{SIN}(t) + \frac{1}{2\cdot j^2}\right], j, 1, 20\right), i, -10, 10\right)$$

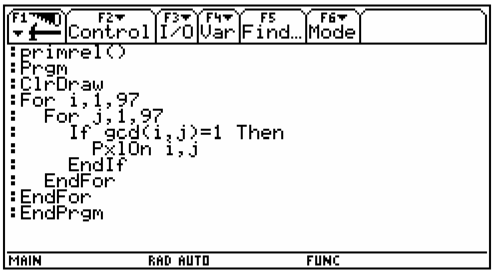
Plot the circles and zoom in:

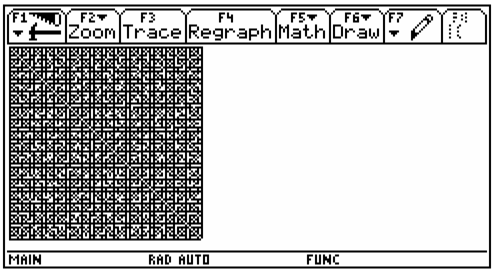


Milton continues in his paper:

Build a matrix with entries *m,n* and 1 if gcd(m,n)=1, zero other cases.

The following program is a representation equivalent to the matrix

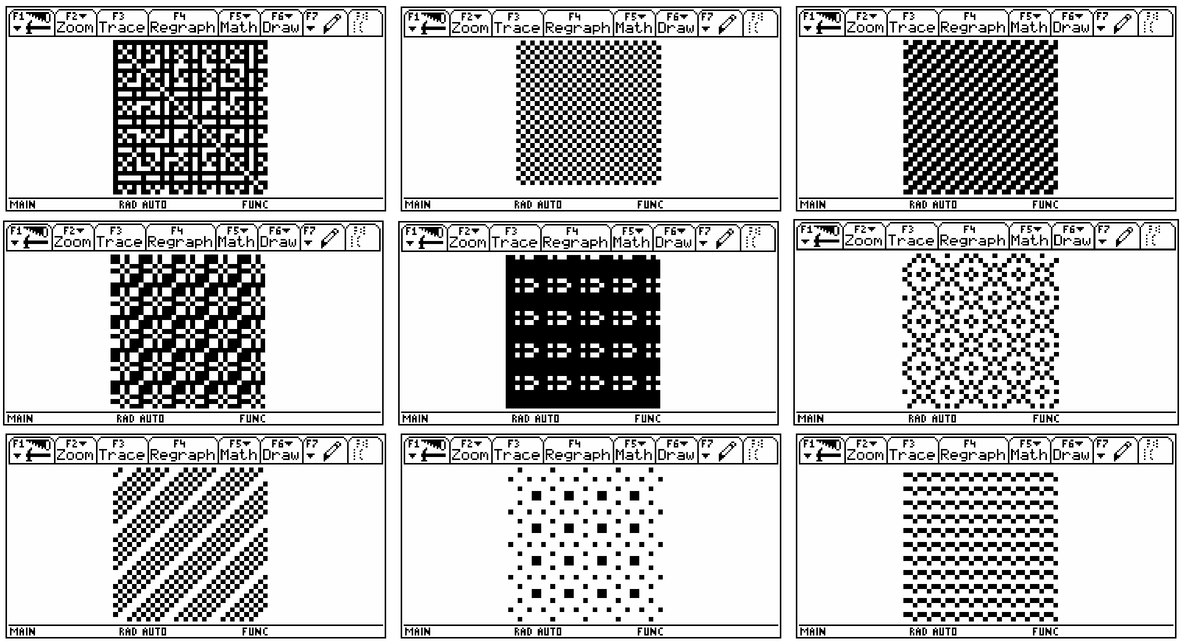




With a little change can we plot larger points – and place the square of interest in the centre of the screen. So we can see the details of this pattern

```
primrel_()
Prgm
ClrDraw
For i,1,33
   For j,1,33
    If gcd(i,j)=1 Then
©   If gcd(i^2+j^2,2)=1 Then
©   If gcd(i^3+j^3,3)=1 Then
©   If gcd(i^3+j^3,7)=1 Then
©   If gcd(i^2+j^3,7)=1 Then
©   If gcd(i^2+j^2,1Ø)=1 Then
©   If gcd(i^3+j^3,1Ø)=1 Then
©   If gcd(i+i^2+j+j^2,3)=1 Then
©   If gcd(i+i^2+j+j^2,7)≠1 Then
©   If gcd(i+j^2,3)≠1 Then
      PxlOn 3*i,3*j+65:PxlOn 3*i,3*j-1+65:PxlOn 3*i,3*j+1+65
      PxlOn 3*i+1,3*j+65:PxlOn 3*i+1,3*j-1+65:PxlOn 3*i+1,3*j+1+65
      PxlOn 3*i-1,3*j+65:PxlOn 3*i-1,3*j-1+65:PxlOn 3*i-1,3*j+1+65
   EndIf
   EndFor
EndFor
```

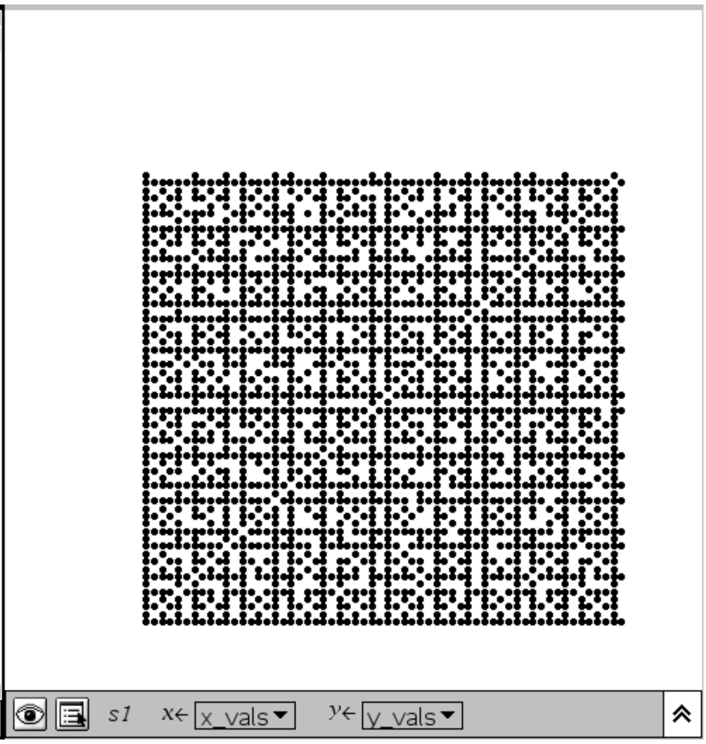
All the command lines starting with the comment-character © can be activated and other conditions can be tried. The results are nice and sometimes really unexpected patterns.



```
primrel()
        x−values in x_vals, y−values
        in y_vals
                                  Done
                                  1/99
```

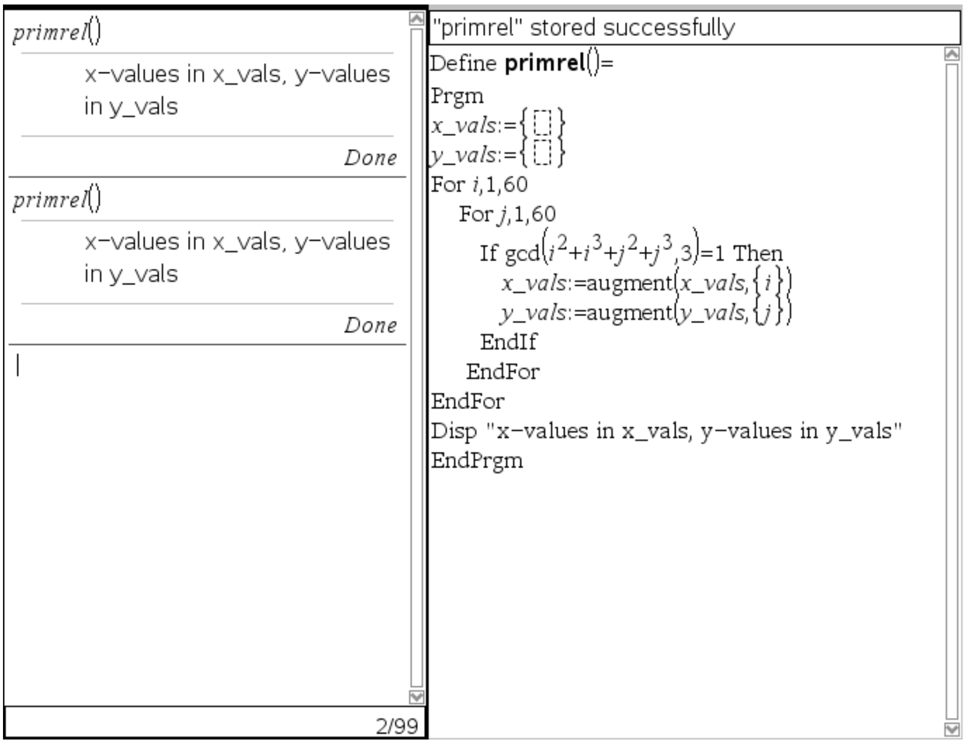
```
"primrel" stored successfully
Define primrel()=
Prgm
x_vals:={ }
y_vals:={ }
For i,1,60
  For j,1,60
    If gcd(i,j)=1 Then
      x_vals:=augment(x_vals,{i})
      y_vals:=augment(y_vals,{j})
    EndIf
  EndFor
EndFor
Disp "x−values in x_vals, y−values in y_vals"
EndPrgm
```

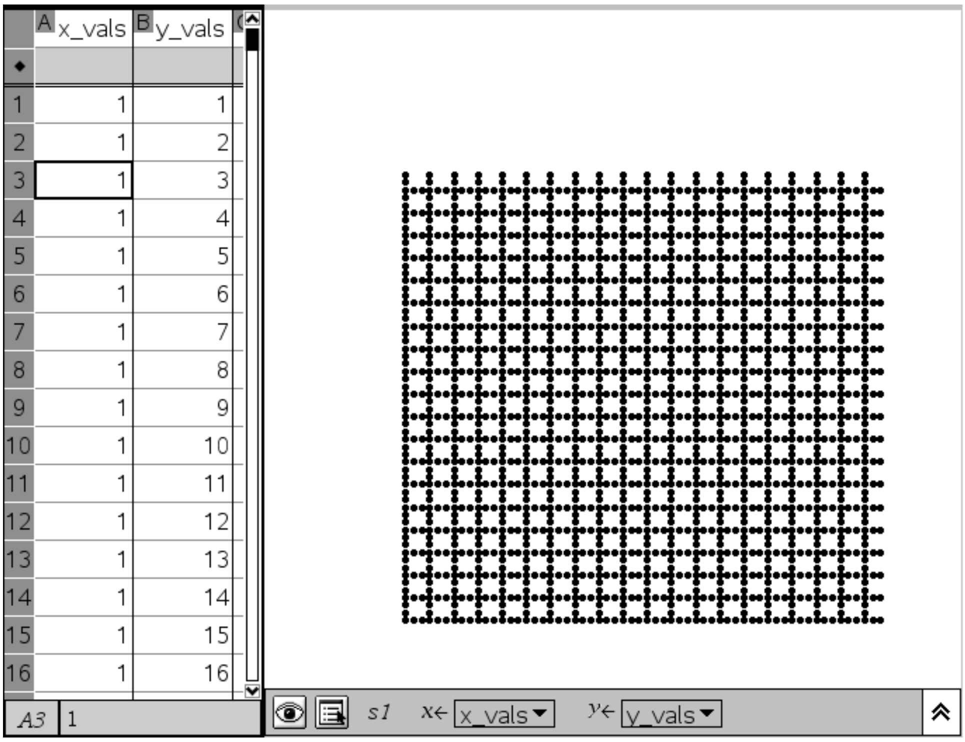
	A x_vals	B y_vals
♦		
1	1	1
2	1	2
3	1	3
4	1	4
5	1	5
6	1	6
7	1	7
8	1	8
9	1	9
10	1	10
11	1	11
12	1	12
13	1	13
14	1	14
15	1	15
16	1	16
A	x_vals	



You have to change some lines in the program to obtain all the other patterns from above. But this is easy work. You can add all the various conditions using the "Comment"-character in the same way as on the Voyage 200.

Finally I give some lines in DERIVE-code. The powerful VECTOR-command is very helpful.
Josef






```
VECTOR(VECTOR(IF(GCD(i, j) = 1, [i, j], ?, ?), i, 100), j, 100)

pattern(u, i_, k_) := VECTOR(VECTOR(IF(GCD(i, k) = u, [i, k], ?, ?), i, i_), k, k_)

pattern(5, 200, 200)

                           2     2
VECTOR(VECTOR(IF(GCD(i   + k  , 3) = 1, [i, k], ?, ?), i, 100), k, 100)
```

This problem arises from the Weierstrass substitution, u = tan(x/2), which introduces singularities not in the integrand. Old calculus texts would some times have the above error (telling you the answer of the definite integral was 3 went it should be 5). New texts carefully choose the a and b so that [a,b] does not contain a singularity of H.

Ralph

Johann Wiesenbauer, Vienna

Hi Ralph,

Yes, f(x) is defined everywhere on the real line and continuous, here you are perfectly right and I'm clearly wrong. For the rest, I never claimed that one of the forms F(x), G(x), H(x) is incorrect, all three are antiderivates of f(x). All I said that F(x) is more "beautiful" than the other two, being continuous on whole R, while the other forms are not. Furthermore, Derive should not yields different results when simplifying expressions at once and step-by-step. As said, this is strictly speaking a bug, though one I really dont mind. Everything else is correct though as regards the computations of Derive, to say it once more.

Sorrry, as to these misunderstandings where at least one was due to a mistake of mine.

Cheers,
Johann

Ralph Freese ralph@MATH.HAWAII.EDU

Hi Johann,

Sorry, I didn't imply anything you said was wrong (except about f(x) having discontinuities, the kind of error I am continuously making ;). The rest of your remarks were right on as were your remarks about the correct general antiderivative of 1/x.

I mainly wanted to give some of the history of the early days when we were programing Derive's integration. As I said Al Rich is the one is responsible for getting rid of the discontinuities. While most people were willing to accept G(x) as an antiderivative of f(x), they really noticed it when Mathematica, Maple and Macsyma all gave wrong answers to definite integrals; only Derive was getting it right. (Of course the others have since fixed it.)

A little more: you can see G(x) and H(x) are not fully correct because
f(x) > 0 everywhere, so any antiderivate is strictly increasing, but G(x) and H(x) are periodic. The easiest way to fix this is to add a step function to G(x), as you noted earlier. But the result is a differentiable function expressed as the sum of two discontinuous functions. Al didn't like this and found a way to transform this into better form, F(x) in this case.

Ralph

Alfonso Jesús Población Sáez alfonso@mat.uva.es

Dear Josef

It is a pleasure for me to send you

¡¡¡¡ Merry Christmas and a very happy New Year 2009 ! ! ! !

If you are still with forces and humour to entertain for awhile, here it is this triangle built by Jim Smoak that represents the coefficients (terms) by parity (81 odd ones in red, and 384 even, in green and black) of trinomial expansion (a + b +c)^29, The grace is to try to identificate if all the terms are in this way and really exists this beautiful symmetry.

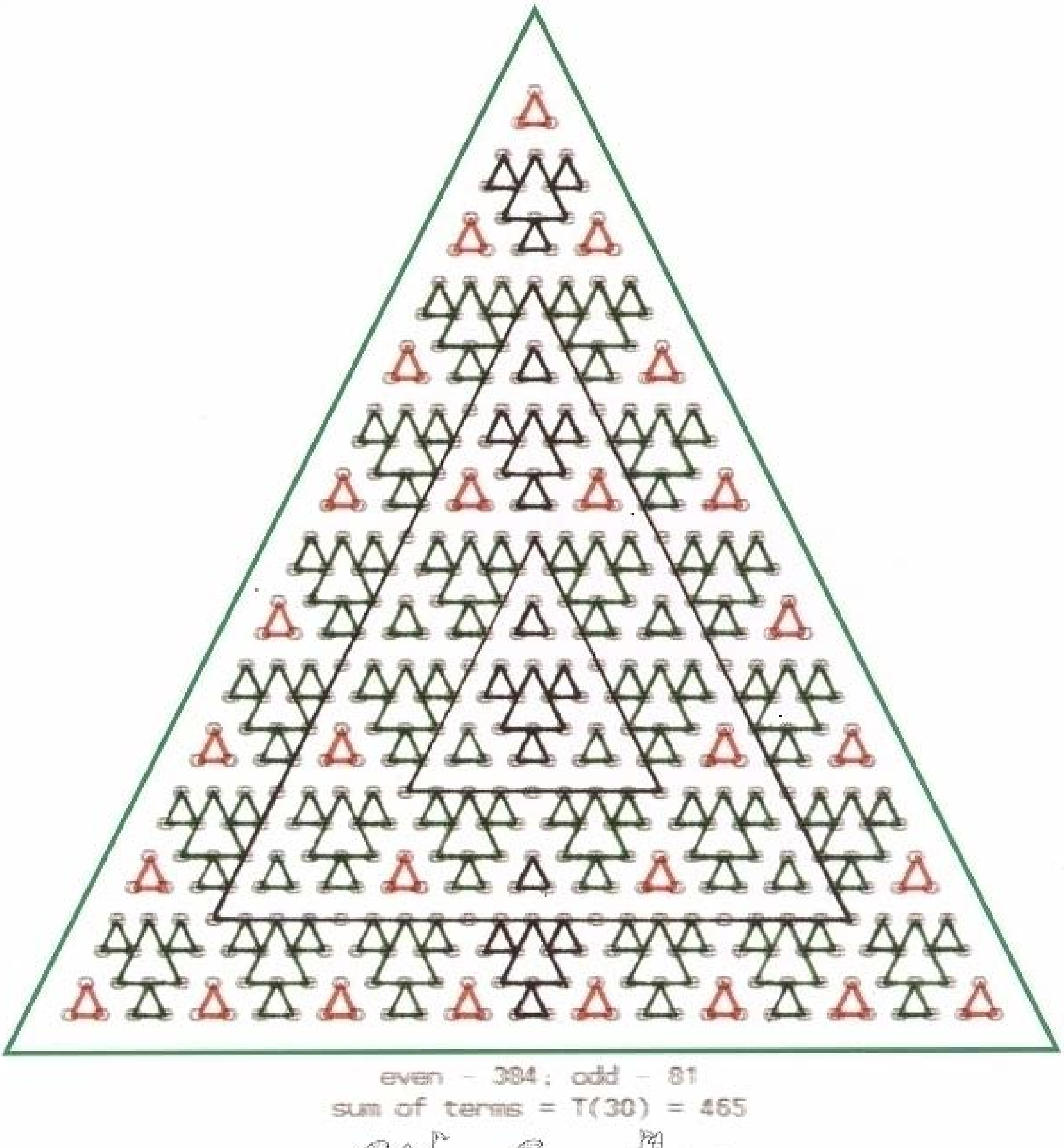
It also has this words: "Whatever the terms, it all adds up to Christmath! Wishing you (and your familiy) the happiest ever!"

Merry Christmath

Trinomial Expansion by Parity

odd powers

(a+b+c)^29



even - 384; odd - 81

sum of terms = T(30) = 465

City Castles

CHRISTMAS TREE

Titbits(36) - Factoring integers with DERIVE

(c) Johann Wiesenbauer, Vienna University of Technology

Is n the positive integer to be factored, usually the first thing one will do is to check it for divisibility by all primes p <=s for some bound s, which is not too big. We use s=1024 as default value for s in the following as this number is also used by DERIVE for the built-in factoring routine. The subsequent routine mindivisor(n,s) yields the smallest prime divisor p<=s, if it exists, and 1 or n otherwise, depending on whether s^2<n or s^2>=n, respectively. Hence for "small" values of n > 1, i.e. with at most 6 digits for our default value of s, this can also be used as a deterministic primality test by checking whether the output is n or not.

```
      mindivisor(n, s := 1024, d_ := 4, t_ := 2) :=
        Loop
          If t_^2 > n
             RETURN n
          If t_ > s
#1:          RETURN 1
          If MOD(n, t_) = 0
             RETURN t_
          t_ :+ d_^IF(t_ ≥ 5)
          d_ := 6 - d_

#2:   TABLE(mindivisor(n), n, 121, 140)`
```

#3:

121	122	123	124	125	126	127	128	129	130	131	132	133	134	135	136	137	138	139	140
11	2	3	2	5	2	127	2	3	2	131	2	7	2	3	2	137	2	139	2

By applying the routine above several times and storing and removing small prime divisors found in this way, we may assume now w.l.o.g. that the number n to be factored hasn't got any "small" divisors anymore. If n>1, it is high time now for a fast probabilistic primality test, such as the subsequent Rabin-Miller test from the Tibits(23) which is included here for the sake of completeness. Here the base a for this test is either any number in the region 0<a<n, a list of such numbers or a negative integer, in which case the list of bases consists of all primes up to |a|.

```
      Rabin_Miller(n, a := 2, a_, s_, t_) :=
        Prog
          If n = 1
             RETURN false
          If EVEN?(n)
             RETURN SOLVE(n = 2)
          If NUMBER?(a)
             If a > 0
                a := [a]
                a := SELECT(PRIME(q_), q_, -a)
          t_ := n - 1
          Loop
#4:         t_ :/ 2
            If ODD?(t_) exit
          Loop
            If a = [] exit
            s_ := t_
            a_ := - ABS(MODS(FIRST(a)^s_, n))
```

```
      Loop
        If a_ = -1
           [a := REST(a), exit]
        s_ :* 2
        If s_ = n - 1
           RETURN false
        a_ := MODS(a_^2, n)
```

```
#5:    Rabin_Miller(1195068768795265792518361315725116351898245581, -31) = true

#6:   SELECT(¬ Rabin_Miller(1195068768795265792518361315725116351898245581, a), a,

       1, 37) = [22, 26, 34, 37]
```

As you can can see the 46-digit number

1195068768795265792518361315725116351898245581

is remarkably "obstinate" as all primes up to 31 are "liars", when it comes to a Rabin-Miller test performed with those bases. Actually a=22 is smallest first positive integer that reveals its compositeness. This is the exception of the rule though and usually one or two Rabin-Miller tests will do. As this number is also hard when it comes to factoring, we consider a smaller example first, namely the Mersenne number n = 2^67-1, which has no prime factors below 2^10, but turns out to be composite as well.

#7: $\text{mindivisor}(2^{67} - 1) = 1$

#8: $\text{Rabin\_Miller}(2^{67} - 1) = \text{true}$

#9: $\text{Rabin\_Miller}(2^{67} - 1, 3) = \text{false}$

We now apply one of the simplest factoring methods, namely the so-called Pollard's rho-method, which is very suitable when trying to find factors of moderate size, say up to about 10^12. Basically, a simple function f on Z_n, like for example f(z) = z^2+1 mod n, is applied to x and twice to y, starting with the same value. All we do is to check after each iteration whether d=gcd(x-y,n) > 1. We return the smaller one of numbers d and n/d, which could be also 1, in case we are very unlucky. We also output the number of iterations, which is expected to lie in the vicinity of 1.2√d.

```
      ρ(n, x := 3, y := 3, d_, i_ := 0) :=
        Prog
          Loop
            i_ :+ 1
            x := MOD(x^2 + 1, n)
#10:        y := MOD(MOD(y^2 + 1, n)^2 + 1, n)
            d_ := GCD(x - y, n)
            If d_ > 1 exit
          DISPLAY(APPEND(STRING(i_), " iterations"))
          MIN(d_, n/d_)
```

#11: $\text{ROUND}(1.2\cdot\sqrt{193707721}) = 16701$

15613 iterations

#12: $\rho(2^{67} - 1) = 193707721$

As the following computation shows, for the found prime factor p = 193 707 721, the number p - 1 splits up into many small primes.

```
#13:        FACTOR(193707721 - 1) = 2^3·3^3·5·67·2677
```

Whenever this is the case, another method by Pollard, his celebrated p-1 method should work very well. For this method, one first select any integer a in the range 1<a<n. In the unlikely case that d=gcd(a,n) turns out to be >1, we return this nontrivial factor d of n, otherwise we replace a by a^q mod n, where q runs through all prime powers below a certain bound s, which is supposed to be input by the user. After each such replacement, we check whether for d=gcd(a-1,n) the condition d > 1 is fulfilled, in which case d is a nontrivial divisor of n, unless we are extremely unlucky and d = n. This is very likely to be successful, if n has got a prime factor p such that p - 1 is a divisor of lcm(1,2,...,s). To increase the chances of a success, we added a second stage after an unsuccessful first stage as described above, where now the latter condition is weakened in the way that only (p - 1)/q is a divisor of lcm(1,2,..,s) for some prime factor q of p -1 with q > s, but q <= t for another bound t. As for this second stage, we introduced another parameter u with default value u=100. It says that rather than computing gcd(a-1,n) after each new a, we form the product b mod n of u subsequent values of a-1 and check the gcd(b,n) > 1 then. If this condition is fullfilled then b will be returned.

In the examples below, this method is applied very successfully to the Mersenne numbers 2^67-1 and 2^257 -1. Note that the latter number has got 88 digits and the detected prime factor p has got 25 digits! Even though it was found in only 13.7s on my PC! By factoring p -1 you can also see why this method has been so incredibly successful for the chosen values of the bounds s and t.

```
      pminus1(n, a, s, t, u ≔ 100, a_, b_ ≔ 1, k_ ≔ 0, p_ ≔ 2, q_) ≔
        Prog
          Loop
            a ≔ MOD(a^p_^FLOOR(LOG(s, p_)), n)
            If GCD(a - 1, n) > 1
               RETURN GCD(a - 1, n)
            p_ ≔ NEXT_PRIME(p_)
            If p_^2 > s exit
          Loop
            a ≔ MOD(a^p_, n)
            If GCD(a - 1, n) > 1
               RETURN GCD(a - 1, n)
            p_ ≔ NEXT_PRIME(p_)
            If p_ > s exit
#14:      a_ ≔ MOD(a^p_, n)
          q_ ≔ NEXT_PRIME(p_)
          Loop
            b_ ≔ MOD((a_ - 1)·b_, n)
            If k_ = 0
               If GCD(b_, n) > 1
                  RETURN GCD(b_, n)
                  k_ ≔ u
            If p_ > t
               RETURN GCD(b_, n)
            a_ ≔ MOD(a_·MOD(a^(q_ - p_), n), n)
            p_ ≔ q_
            q_ ≔ NEXT_PRIME(q_)
            k_ :- 1
```

#15: $\text{pminus1}(2^{67} - 1, 3, 100, 1000) = 1$

#16: $\text{pminus1}(2^{67} - 1, 3, 100, 3000) = 193707721$

#17: $\text{DIM}(2^{257} - 1) = 78$

#18: $\text{pminus1}(2^{257} - 1, 120000, 1200000) = 1155685395246619182673033$

#19: $\text{FACTOR}(1155685395246619182673033 - 1) = 2^3 \cdot 3^2 \cdot 19^2 \cdot 47 \cdot 67 \cdot 257 \cdot 439 \cdot 119173 \cdot 1050151$

There is one important consequence as to RSA we were talking about in the last issue of this series: The prime factors p and q of the modulus n used for RSA must be chosen in a way such that both p -1 and q -1 contain huge prime factors! If one selects p and q at random and of an appropriate size this condition is extremely likely to be fulfilled though! Just to see our routine at work, let's select the prime p in a way that it has got d digits and all prime factors of p-1 have s as an upper bound.

#20:
```
riskyprime(d, s, p_, q_) :=
  Loop
    p_ := 1
    Loop
      q_ := p_·NEXT_PRIME(RANDOM(s))
      If DIM(q_) > d exit
      p_ := q_
    Loop
      If DIM(p_) = d exit
      p_ :* 2
    Loop
      If PRIME(p_ + 1)
         RETURN p_ + 1
      p_ :* 2
      If DIM(p_) > d exit
```

#21: $p := \text{riskyprime}(100, 10^5)$

#22: p :=

2537132963746568510487366523319865111229382728135018892133137399762392085204~
70360697068761139097871 3

#23: FACTOR(p − 1) =

$2^3 \cdot 11083 \cdot 11987 \cdot 26237 \cdot 29927 \cdot 33331 \cdot 36097 \cdot 39953 \cdot 48337 \cdot 54941 \cdot 56299 \cdot 60413 \cdot 63949 \cdot 6$~

$4187 \cdot 68947 \cdot 70009 \cdot 74159 \cdot 74353 \cdot 75367 \cdot 92153 \cdot 94819 \cdot 97327$

#24: $q := \text{NEXT\_PRIME}(\text{RANDOM}(10^{100}))$

#25: q :=

6881910916208059644290378831779626901617763217368545060166875653639019010451~
224050506211194451574871

#26: $n := p \cdot q$

#27: $\text{pminus1}(n, 3, 10^5, 10^6) =$

2537132963746568510487366523319865111229382728135018892133137399762392085204~

70360697068761139097871 3

#28: $\text{pminus1}(11950687687952657925183613157251163518982 45581, 3, 10^5, 10^6) = 1$

As H. Lenstra found out the basic idea of the p-1 method can also be exploited for groups arising from elliptic curves leading to the so-called ECM (=Elliptic Curve Method), a very powerful factoring method that can be used to find factors up to about 40 digits and more. Following ideas by P. Montgomery we use here elliptic curves of the special form

#29: $g \cdot y^2 = x^3 + c \cdot x^2 + x$

Using projective coordinates, i.e. by substituting x/z for x, y/z for y and multplying with z^3, we get the corresponding homogeneous equation

#30: $g \cdot y^2 \cdot z = x^3 + c \cdot x^2 \cdot z + x \cdot z^2$

Here we consider solutions (x,y,z) ≠ (0,0,0), where two triples (x1,y1,z1), (x2,y2,z2) are identified if there is a nonzero scalar t such that (x1,y1,z1) =t(x2,y2,z2). In particular, if z≠0, we can always identify (x,y,z) with (x/z,y/z,1). If z=0, then x=0 as well from #1 and y may be chosen to be 1. This triple (0,1,0) is the "point at infinity", often denoted by O.

In the following, we define the sum U+V of two different (!) points using their difference W:=U -V, which must be known. We drop the y- coordinate, i.e. each point is actually represented by the pair (x,z), since we don't need y for our purposes. All computations are carried out mod n, where n is the number to be factored. Hence, we are not dealing with "true" elliptic curves, because Zn is not a field, and there may be points with a nonzero z that is not invertible mod n. If this is the case, then we have won, as the gcd(z,n) is usually a nontrivial factor of n (unless we are extremely unlucky and gcd(z,n)=n).

Now look at the following very simple (and beautiful!) formula for the addition of U and V, where W, n and c have the meaning above. This is essentially the famous "Montgomery-trick" !

#31: $\text{addh}(u, v, w, n) := \left[\text{MOD}\left(w_2 \cdot (u_1 \cdot v_1 - u_2 \cdot v_2)^2, n\right), \text{MOD}\left(w_1 \cdot (u_1 \cdot v_2 - v_1 \cdot u_2)^2, n\right)\right]$

What about the case U = V, which was excluded above? It's only slightly more complicated!

#32: doubleh(u, n, c) := $\left[\mathrm{MOD}\left(\left(u_1^2 - u_2^2\right)^2, n\right), \mathrm{MOD}\left(4\cdot u_1\cdot u_2\cdot\left(u_1\cdot(u_1 + c\cdot u_2) + u_2^2\right), n\right)\right]$

The following routine can be used to compute the additive power mU of a point U for a positive integer m.

```
multh(u, m, n, c, t_, x0_, z0_, x1_, z1_, x2_, z2_) :=
  Prog
    x0_ := FIRST(u)
    z0_ := FIRST(REST(u))
    x1_ := x0_
    z1_ := z0_
    x2_ := MOD((x0_^2 - z0_^2)^2, n)
    z2_ := MOD(4·z0_·(x0_^2·(x0_ + c·z0_) + x0_·z0_^2), n)
    OutputBase := Binary
    m := NAME_TO_CODES(m)
    OutputBase := Decimal
    Loop
      m := REST(m)
      If m = []
         RETURN [x1_, z1_]
#33:  If FIRST(m) = 48
         Prog
           t_ := x2_
           x2_ := MOD(z0_·(x1_·x2_ - z1_·z2_)^2, n)
           z2_ := MOD(x0_·(x1_·z2_ - t_·z1_)^2, n)
           t_ := x1_
           x1_ := MOD((x1_^2 - z1_^2)^2, n)
           z1_ := MOD(4·z1_·t_·(t_·(t_ + c·z1_) + z1_^2), n)
         Prog
           t_ := x1_
           x1_ := MOD(z0_·(x1_·x2_ - z1_·z2_)^2, n)
           z1_ := MOD(x0_·(t_·z2_ - x2_·z1_)^2, n)
           t_ := x2_
           x2_ := MOD((x2_^2 - z2_^2)^2, n)
           z2_ := MOD(4·z2_·t_·(t_·(t_ + c·z2_) + z2_^2), n)
```

At last, we are ready to implement the ECM. Here σ is a parameter that determines the coefficient c of the elliptic curve as well as the coordinates of a point on that curve. s and t are the bounds for the first and second stage of ECM, respectively. d is a constant used during the second stage and should be of order O(sqrt(t)).

```
ECM(n, σ, s, t, d := 100, a_, c_, g_, p_ := 2, q_ := 2, r_, s_, t_, u_, v_, w_) :=
  Prog
    u_ := MOD(σ^2 - 5, n)
    v_ := MOD(4·σ, n)
    g_ := GCD(4·u_^3·v_, n)
    If g_ > 1
       RETURN IF(g_ < n, g_, 1)
    c_ := INVERSE_MOD(4·u_^3·v_, n)
    c_ := MOD((v_ - u_)^3·(3·u_ + v_)·c_ - 2, n)
    u_ := [MOD(u_^3, n), MOD(v_^3, n)]
    w_ := u_
```

```
          Loop
            w_ ≔ multh(w_, p_^FLOOR(LOG(s, p_)), n, c_)
            p_ ≔ NEXT_PRIME(p_)
            If p_^2 > s exit
          g_ ≔ GCD(w_↓2, n)
          If g_ > 1
             If g_ < n
                RETURN g_
                Loop
                  u_ ≔ multh(u_, q_^FLOOR(LOG(t, q_)), n, c_)
                  g_ ≔ GCD(u_↓2, n)
                  If g_ > 1
                     RETURN IF(g_ < n, g_, 1)
                  q_ ≔ NEXT_PRIME(q_)
          u_ ≔ w_
          q_ ≔ p_
          Loop
            w_ ≔ multh(w_, p_, n, c_)
            p_ ≔ NEXT_PRIME(p_)
            If p_ > s exit
          g_ ≔ GCD(w_↓2, n)
          If g_ > 1
             If g_ < n
                RETURN g_
#34:            Loop
                  u_ ≔ multh(u_, q_, n, c_)
                  g_ ≔ GCD(u_↓2, n)
                  If g_ > 1
                     RETURN IF(g_ < n, g_, 1)
                  q_ ≔ NEXT_PRIME(q_)
          s ≔ 2·CEILING(s, 2) - 1
          u_ ≔ doubleh(w_, n, c_)
          s_ ≔ [doubleh(u_, n, c_), u_]
          Loop
            s_ ≔ ADJOIN(addh(FIRST(s_), u_, FIRST(REST(s_)), n), s_)
            If DIM(s_) = d exit
          s_ ≔ REVERSE(s_)
          t_ ≔ VECTOR(MOD(s_↓k_↓1·s_↓k_↓2, n), k_, 1, d)
          u_ ≔ multh(w_, s - 2·d, n, c_)
          v_ ≔ multh(w_, s, n, c_)
          r_ ≔ s
          Loop
            If r_ ≥ t
               RETURN 1
            a_ ≔ MOD(v_↓1·v_↓2, n)
            w_ ≔ [- v_↓1, v_↓2]
            p_ ≔ r_
            Loop
              p_ ≔ NEXT_PRIME(p_)
              If p_ > r_ + 2·d exit
              q_ ≔ (p_ - r_)/2
              g_ ≔ MOD(g_·(t_↓q_ - a_ - ∏(w_ + s_↓q_)), n)
            g_ ≔ GCD(g_, n)
            If g_ > 1
               RETURN IF(g_ < n, g_, 1)
            w_ ≔ v_
            v_ ≔ addh(v_, s_↓d, u_, n)
            u_ ≔ w_
            r_ :+ 2·d

      ecmfactor(n, b1 ≔ 1000, b2 ≔ 10000, σ0 ≔ 2, d ≔ 100, e_) ≔
        Loop
          WRITE(σ0)
#35:      e_ ≔ ECM(n, σ0, b1, b2, d)
          If e_ > 1
             RETURN [e_, σ0]
          σ0 :+ 1
```

For testing purposes the following routine returns for a given σ the coefficient c of the elliptic curve and a point U on it. (If the algorithm fails, because a certain number x has no inverse mod n, then the gcd(x,n) >1 will be returned.)

```
      params(n, σ, c_, g_, u_, v_) :=
        Prog
          u_ := MOD(σ^2 - 5, n)
          v_ := MOD(4·σ, n)
          g_ := GCD(4·u_^3·v_, n)
#36:      If g_ > 1
             RETURN IF(g_ < n, g_, 1)
          c_ := INVERSE_MOD(4·u_^3·v_, n)
          c_ := MOD((v_ - u_)^3·(3·u_ + v_)·c_ - 2, n)
          [c_, [MOD(u_^3, n), MOD(v_^3, n)]]
```

Assuming that ECM works for a given σ and the bounds s and t, the following routine will use a binary search in order to find the optimal (i.e. smallest possible) bounds s and t.

```
      ECMbounds(n, σ, s := 10000, t := 1000000, s_ := 0, t_ := 0, u_) :=
        Prog
          If ECM(n, σ, s, t) = 1
             RETURN ?
          Loop
            u_ := FLOOR(t + t_, 2)
            If u_ = t_ exit
            If ECM(n, σ, s, u_) = 1
#37:           t_ := u_
               t := u_
          Loop
            u_ := FLOOR(s + s_, 2)
            If u_ = s_
               RETURN [s, t]
            If ECM(n, σ, u_, t) = 1
               s_ := u_
               s := u_
```

Okay, after all those lengthy routines, you certainly want to see them at work at last. Here are just a few examples, but they only represent the proverbial tip of the iceberg, as I have to come to an end after all. Hence actually a lot of experimenting is left to you! I do hope though you enjoy these routines as much as I did when testing them!

#38: $\mathrm{FACTOR}(2^{101} - 1) = 7432339208719 \cdot 341117531003194129$

#39: $\mathrm{ECM}(2^{101} - 1, 4, 1000, 20000) = 7432339208719$

#40: $\mathrm{ECMbounds}(2^{101} - 1, 4, 1000, 20000) = [227, 17800]$

#41: $\mathrm{ECM}(2^{101} - 1, 4, 227, 17800) = 7432339208719$

#42: $\mathrm{ecmfactor}\left(2^{2^{7}} + 1, 5000, 150000\right) = [59649589127497217, 26]$

#43: $\mathrm{ecmfactor}\left(2^{2^{8}} + 1, 2000, 30000\right) = [1238926361552897, 8]$

(Compare the calculation times between the built in FACTOR and Johann´s routine! You will be more than only impressed. Josef)