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DERIVE®

USER GROUP

+ CAS-TI

C o n t e n t s:

March 2009

ACA 2009 http://aca2009.etsmtl.ca/

The meeting is run in a standard format where sessions are held in one or more 2 to 3hour blocks of time. These blocks typically consist of halfhour talks or a onehour overview and halfhour talks. The halfhour slot includes time for questions.

Session Proposals: Individuals are invited to organize a session. Proposals should be sent to the conference program chairs, Michel Beaudin (**michel.beaudin@etsmtl.ca**) and Michael Wester (**wester@math.unm.edu**). The duties of the session organizers are: • Submit a proposal for the session.

- Invite the speakers.
- Maintain a web page describing the session and providing talk abstracts. Examples can be found on the ACA main website at **http://math.unm.edu/aca.html**.

The scientific quality of a special session is the sole responsibility of the session organizers. Each session organizing team will issue a call for papers/contributions stating the terms for submitting contributions, including the deadline each team has set for receiving. Please check the list of approval sessions at **http://aca2009.etsmtl.ca/** and consult the link to the session that addresses your interests.

See ACA 2009 website for more details: **http://aca2009.etsmtl.ca/**

Applications and Libraries development in DERIVE

Special Session of ACA 2009

José Luis Galán García; Pedro Rodríguez Cielos; Gabriel Aguilera Venegas; Josef Böhm

http://www.derive.aca2009.uma.es/ ; aca_derive @ctima.uma.es

June 25-28, 2009
École de technologie supérieure (ETS)
Université du Québec, Montréal, Québec, Canada.

Description of Applications and Libraries development in DERIVE

http://www.derive.aca2009.uma.es/

DERIVE™ 6 is a powerful system for doing symbolic and numeric mathematics. It processes algebraic variables, expressions, equations, functions, vectors, matrices and Boolean expressions in the same way as a scientific calculator processes numbers.

Problems in the fields of arithmetic, algebra, trigonometry, calculus, differential equations, linear algebra, complex analysis and propositional calculus can be solved with a click of the mouse. Plots of mathematical expressions in two and three dimensions using various coordinate systems can be easily performed. Furthermore, the use of the slider bar utility allows beautiful presentations of plot movement and makes it possible to study plots depending on different parameters.

The seamless integration of numeric, algebraic and graphic capabilities makes DERIVE 6 an excellent tool not only for learning or teaching but also for doing mathematics in many applications.

Although DERIVE is no longer being developed, and TI-NSPIRE CAS has been proclaimed as its successor, there are many professionals who are still using DERIVE.

The main purpose of this Special Session is to share the applications and libraries developed in DERIVE or TI-NSPIRE CAS. Thus, papers about the use of DERIVE in different disciplines, the development of specific libraries, experiences involving DERIVE are welcome. Lectures using TI-NSPIRE CAS and the TI-92/VOYAGE 200 are also welcome.

Papers presented in this session will be published in a special issue of The Derive Newsletter (http://www.austromath.at/dug/).

See Session website for more details: http://www.derive.aca2009.uma.es/

Dear DUG Members,

I send you my best regards from a wonderful springtime in Austria.

You may believe it or not, this is the first issue of volume 19 of our DERIVE and CAS-TI Newsletter. Inspecting the list of future contributions you can imagine that we have materials for some more volumes.

Roland Schröder from Celle, Germany, sent a wonderful collection of short projects for classroom investigations. His first one is a treatment of an ancient way for multiplying integers – which offers a nice connection to information technology of today. Just recently I received a mail from Roland announcing a German publication of his collection. Please inform yourself on the back page.

Guido Herweyers from Belgium offers a series of introductory papers for the use of NspireCAS. Many thanks for that. We will start publishing his papers with the next DNL.

I have the pleasure to announce two included papers from colleagues who intended to give a talk at TIME 2008. Their talk was accepted but unfortunately they both could not participate. Hildegard Urban-Woldron from Austria and Michael de Villiers from South Africa provided their papers for our newsletter. I am very grateful for that and I am sure that you will enjoy the "Science" and the "Function and Proof" as well. Michael de Villiers sent an exciting copy of his "Some Adventures in Euclidean Geometry" to Würmla. In the first chapter he describes a "Classroom Episode". I was very enthusiastic about this chapter and asked Michael for permission to translate it for the Austrian Teachers. You can find the translated version on www.acdca.ac.at in the very near future.

Thanks to Hildegard and Michael, hope to see you at our next Conference (Malaga 2010).

The Information Page shows an invitation for participating at ACA09 (Applications of Computer Algebra). There is a special "DERIVE Strand". All colleagues who want to keep the spirit of DERIVE alive are very welcome. We also welcome all friends who would like to help transferring the spirit and the features of DERIVE to TI-NspireCAS. Colleagues who have no experience with DERIVE but with the TI-CASs are cordially invited to share their applications and libraries with us.

You might miss Johann Wiesenbauer´s Titbits in this DNL. He wanted to have a perfect paper about the "Quadratic Sieve" and this took some days more than intended. So I finished the DNL without including his contribution. But I have a new folder DNL74 and started a new document dnl74.doc which includes Titibits(37) from Johannes. Many thanks, Hannes, you will have no problems with the next deadline.

Finally I´d like to inform you that our phone number has changed.

The new number is **++43-06604070480**. The old number will be valid for some time, but I don´t have the FAX-machine connected any longer. We didn´t receive a FAX for almost two years.

My wife and I wish you a pretty spring (fall on the other side of the globe).

The *DERIVE-NEWSLETTER* is the Bulletin of the *DERIVE* & CAS-*TI User Group*. It is published at least four times a year with a contents of 40 pages minimum. The goals of the *DNL* are to enable the exchange of experiences made with *DERIVE*, *TI*-CAS and other CAS as well to create a group to discuss the possibilities of new methodical and didactical manners in teaching mathematics.

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Contributions:
Please send all contributions to the Editor. Non-English speakers are encouraged to write their contributions in English to reinforce the international touch of the *DNL*. It must be said, though, that non-English articles will be warmly welcomed nonetheless. Your contributions will be edited but not assessed. By submitting articles the author gives his consent for reprinting it in the *DNL*. The more contributions you will send, the more lively and richer in contents the *DERIVE* & CAS-*TI Newsletter* will be.

Next issue: June 2009
Deadline 15 May 2009

Preview: Contributions waiting to be published

Some simulations of Random Experiments, J. Böhm, AUT, Lorenz Kopp, GER
Wonderful World of Pedal Curves, J. Böhm
Tools for 3D-Problems, P. Lüke-Rosendahl, GER
Financial Mathematics 4, M. R. Phillips
Hill-Encription, J. Böhm
Simulating a Graphing Calculator in *DERIVE*, J. Böhm
Henon & Co, J. Böhm
Do you know this? Cabri & CAS on PC and Handheld, W. Wegscheider, AUT
An Interesting Problem with a Triangle, Steiner Point, P. Lüke-Rosendahl, GER
Overcoming Branch & Bound by Simulation, J. Böhm, AUT
Diophantine Polynomials, D. E. McDougall, Canada
Graphics World, Currency Change, P. Charland, CAN
Cubics, Quartics – interesting features, T. Koller & J. Böhm
Logos of Companies as an Inspiration for Math Teaching
Exciting Surfaces in the FAZ / Pierre Charland´s Graphics Gallery
BooleanPlots.mth, P. Schofield, UK
Old traditional examples for a CAS – what´s new? J. Böhm, AUT
Truth Tables on the TI, M. R. Phillips
Advanced Regression Routines for the TIs, M. R. Phillips
Where oh Where is IT? (GPS with CAS), C. & P. Leinbach, USA
Embroidery Patterns, H. Ludwig, GER
Mandelbrot and Newton with *DERIVE*, Roman Hašek, CZ
Snail-shells, Piotr Trebisz, GER
A Conics-Explorer, J. Böhm, AUT
Practise Working with times
Huffman-Code with DERIVE and TI-CAS, J. Böhm, AUT
Tutorials for the NSpireCAS, G. Herweyers, BEL
Some Projects with Students, R. Schröder, GER

and others

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Peter Lüke-Rosendahl, Germany

Hello Josef,

… I like to read your revised reprints. I am just trying your „Discussion of a Curve“ from revised DNL#15. I found out that “terrace points” (= points with $= f' = f'' = 0 \; = 0$) are not recognized properly. The reason might by that within an IF-condition sign(1) and sign(0) are not distinguished …

In his next mail Peter sent his proposal using the fact that the sign of the first derivative changes in a turning point, but does not change in an extremal value with slope = 0.
I implemented Peter´s idea and by the way I tried to improve the “program” from revised DNL15 to also consider points with vanishing higher derivatives (giving “flat points” …)

Many thanks, Peter for the fruitful discussion.

Guido Herweyers, Belgium

Dear Josef,
I attach a paper which we used at our Introductory Workshops for TI-Nspire. If you find it suitable for the DERIVE and CAS-TI Newsletter, then I can send some more materials for publication in our journal.

Best regards from Belgium,

Guido

DNL:
Dear Guido,
Thanks for your paper. They are really very suitable and I am looking forward to receiving the other papers. It would be great to include them into future issues of the DNL.

Many thanks, Peter for the fruitful discussion.

Nils Hahnfeld, Virgin Islands

Did you ever make music with the TI-89? Try http://www.ticalc.org/pub/89/asm/sound/
I´d like to hear “Leise rieselt der Schnee” (a German Christmas song about soft falling snow), because we don´t have any.

More User Forum on the back page

Elliptic Integrals as an easy to follow Iteration Process
(easy to follow, but not easy to prove!)

Browsing in my old books on programming I found a nice example for performing an iteration algorithm in the field of integration. In [1] is written that Gauß used the method of arithmetic – geometric mean to calculate an elliptic integral of the first kind:

Gauß showed that for $Ell\_I(a,b) = \int_0^{\pi/2} \frac{dx}{\sqrt{a^2 \sin^2 x + b^2 \cos^2 x}}$ the value of the (ellipstic) integral remains unchanged having performed the transformation

$$a' = \sqrt{a \cdot b},\; b' = \frac{a+b}{2}.$$

Then $Ell\_I(a,b) = Ell\_I(a',b')$. Proceeding until a' and b' are very close, then $Ell\_I(a',a') = \dfrac{\pi}{2a'}$.

This is a nice occasion to demonstrate either the ITERATES-function together with appropriate LIST operations in DERIVE and/or working in SEQUENCE-Mode on the TIs.

$$\text{ell\_int}(a, b) := \int_0^{\pi/2} \frac{1}{\sqrt{a^2 \cdot \text{SIN}(x)^2 + b^2 \cdot \text{COS}(x)^2}}\,dx$$

ell_int(1, 2)

1.078257823

0.125 sec

$$\text{ell\_appr}(a, b) := \frac{\pi}{2 \cdot \left(\text{FIRST}\left(\text{REVERSE}\left(\text{ITERATES}\left(\left[\sqrt{v_1 \cdot v_2}, \frac{v_1 + v_2}{2}\right], v, [a, b]\right)\right)\right)\right)_1}$$

ell_appr(1, 2)

1.078257823

0.000 sec

ell_int(35, 58)

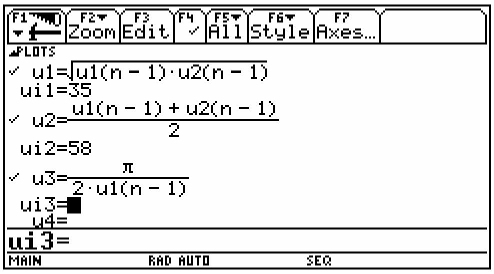
0.03431566042

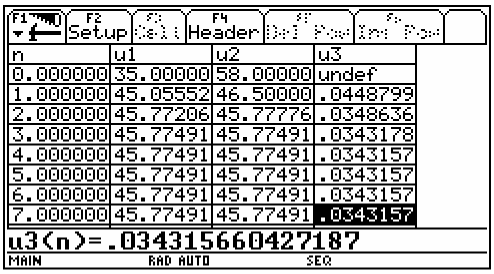
ell_appr(35, 58)

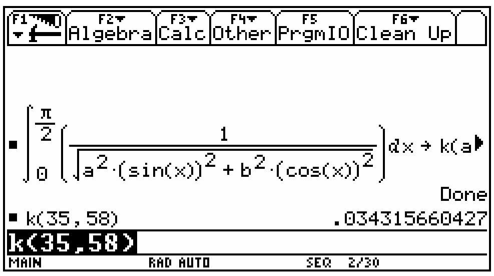
0.03431566042

Compare the calculation times.

Here is the iteration using the sequence mode on the TI-92/Voyage 200







Josef Böhm

[1] *D. Herrmann*, Programmieren von Mikrocomputern 11, Vieweg 1984

Prof. de Villiers intended to attend TIME 2008 in South Africa. Although being in south Africa he couldn´t make it to the Conference and give his lecture. I found his abstract very interesting and asked him later if he would provide his paper – or a similar one – for our newsletter. I am very grateful that he answered immediately and gave permission to reprint one of his many publications. His ideas are not fixed to one piece of software. You can replace Sketchpad with all dynamic geometry programs (Cabri, …). I recommend visiting Prof. de Villiers´website. It is a rich resource for all of you who are loving geometry. Josef

http://mysite.mweb.co.za/residents/profmd/homepage.html

Excerpt from Introduction to De Villiers, M. (1999). **Rethinking Proof with Sketchpad**.
Key Curriculum Press.

THE ROLE AND FUNCTION OF PROOF WITH SKETCHPAD*

Michael de Villiers, University of Durban-Westville

Introduction

The problems that students have with perceiving a need for proof is wellknown to all high school teachers and is identified without exception in all educational research as a major problem in the teaching of proof. Who has not yet experienced frustration when confronted by students asking "*why do we have to prove this?*" The following conclusion by Gonobolin (1954:61) exemplifies the problem:

> "*... the pupils ... do not ... recognize the necessity of the logical proof of geometric theorems, especially when these proofs are of a visually obvious character or can easily be established empirically.*"

According to Afanasjewa in Freudenthal (1958:29) students' problems with proof should not simply be attributed to slow cognitive development (for example, an inability to reason logically), but also that students may not see the **function** (meaning, purpose and usefulness) of proof. In fact, several recent studies in opposition to Piaget have shown that very young children are quite capable of logical reasoning in situations that are real and meaningful to them (Wason & Johnson-Laird, 1972; Wallington, 1974; Hewson, 1977; Donaldson, 1979). Furthermore, attempts by researchers to teach logic to students have frequently provided no statistically significant differences in students' performance and appreciation of proof (Deer, 1969; Walter, 1972; Mueller, 1975). More than anything else, it seems that the fundamental issue at hand is the appropriate motivation of the various functions of proof to students.

The question is, however, "*what functions does proof have within mathematics itself which can potentially be utilized in the mathematics classroom to make proof a more meaningful activity?*" The purpose of this section is to describe some important functions of proof, and briefly discuss some implications for the teaching of proof.

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* This section is a revised version of an earlier article by the author titled "The role and function of proof in mathematics," *Pythagoras*, Nov 1990, 24, 17-24. It is reproduced here with permission of the Association for Mathematics Education of South Africa (AMESA).

The functions of proof in mathematics

Traditionally the function of proof has been seen almost exclusively as being to *verify* the correctness of mathematical statements. The idea is that proof is used mainly to remove either personal doubt or the doubt of skeptics, an idea that has one-sidedly dominated teaching practice and most discussions and research on the teaching of proof. For instance, consider the following two quotes:

> "*a proof is only meaningful when it answers the student's* ***doubts****, when it proves what is not obvious.*" (bold added) - Kline (1973:151)
>
> "*the necessity, the functionality, of proof can only surface in situations in which the students meet* ***uncertainty*** *about the truth of mathematical propositions.*" (bold added) - Alibert (1988:31)

Hanna (1989) and Volmink (1990) also appear to define proof only in terms of its verification function as follows:

> "*a proof is an argument needed to* ***validate*** *a statement, an argument that may assume several different forms as long as it is convincing.*" (bold added) - Hanna (1989:20)
>
> "*Why do we bother to prove theorems? I make the claim here that the answer is: so that we may* ***convince*** *people (including ourselves) ... we may regard a* ***proof as an argument sufficient to convince a reasonable skeptic****.*" - Volmink (1990:8; 10)

Although many authors (e.g.Van Dormolen (1977), Van Hiele (1973) and Freudenthal (1973) and others) have argued that one's need for deductive rigour may undergo change and become more sophisticated with time, this is also argued from the viewpoint that the function of proof is mainly that of verification. For example:

> "*... to progress in rigour, the first step is to* ***doubt*** *the rigour one believes in at this moment. Without this* ***doubt*** *there is no letting other people prescribe oneself new criteria of rigour.*" (bold added) - Freudenthal (1973:151)

Many authors have also proposed specific stages in the development of rigour, e.g. Tall (1989:30) proposes three stages in the putting up of a convincing argument, namely the convincing of oneself, the convincing of a friend and the convincing of an enemy. Although extremely useful distinctions, it considers only the verification function of proof.

However, as pointed out by Bell (1976:24) this view of verification/conviction being the main function of proof "*avoids consideration of the real nature of proof*", since conviction in mathematics is often obtained "*by quite other means than that of following a logical proof.*" Therefore the actual practice of modern mathematical research calls for a more complete analysis of the various functions and roles of proof. Although I lay claim to neither completeness nor uniqueness, I have found the following model for the functions of proof useful in my research over the past few years. It is a slight expansion of Bell's (1976) original distinction between the functions of verification, illumination and systematization. The model is presented here (in no specific order of importance) and discussed further on:

* *verification* (concerned with the truth of a statement)
* *explanation* (providing insight into why it is true)
* *systematisation* (the organization of various results into a deductive system of axioms, major concepts and theorems)

* ** discovery (the discovery or invention of new results)*
* ** communication (the transmission of mathematical knowledge)*
* ** intellectual challenge (the self-realization/fulfillment derived from constructing a proof)*

Proof as a means of verification/conviction

With very few exceptions, mathematics teachers seem to believe that only proof provides certainty for the mathematician and that it is therefore the only authority for establishing the validity of a conjecture. However, proof is not necessarily a prerequisite for conviction—to the contrary, conviction is probably far more frequently a prerequisite for the finding of a proof. (For what other weird and obscure reasons would we then sometimes spend months or years trying to prove certain conjectures, if we weren't already convinced of their truth?)

The well-known George Polya (1954:83-84) writes:

> "*... having verified the theorem in several particular cases, we gathered strong inductive evidence for it. The inductive phase overcame our initial suspicion and gave us a strong **confidence** in the theorem. Without such **confidence** we would have scarcely found the courage to undertake the proof which did not look at all a routine job. When you have satisfied yourself that the theorem is **true**, you start **proving** it.*" (bold added)

In situations like the above where conviction prior to proof provides the motivation for a proof, the function of the proof clearly must be something other than verification/conviction.

In real mathematical research, personal conviction usually depends on a combination of intuition, quasi-empirical verification and the existence of a logical (but not necessarily rigourous) proof. In fact, a very high level of conviction may sometimes be reached even in the absence of a proof. For instance, in their discussion of the "heuristic evidence" in support of the still unproved twin prime pair theorem and the famous Riemann Hypothesis, Davis & Hersh (1983:369) conclude that this evidence is "*so strong that it carries conviction even without rigorous proof.*"

That conviction for mathematicians is not reached by proof alone is also strikingly borne out by the remark of a previous editor of the Mathematical Reviews that approximately one half of the proofs published in it were incomplete and/or contained errors, although the theorems they were purported to prove were essentially true (Hanna, 1983:71). Research mathematicians, for instance, seldom scrutinize the published proofs of results in detail, but are rather led by the established authority of the author, the testing of special cases and an informal evaluation whether "*the methods and result fit in, seem reasonable...*" (Davis & Hersh, 1986:67). Also according to Hanna (1989) the reasonableness of results often enjoy priority over the existence of a completely rigourous proof.

When investigating the validity of a new, unknown conjecture, mathematicians usually do not only look for proofs, but also try to construct counter-examples at the same time by means of quasi-empirical testing, since such testing may expose hidden contradictions, errors or unstated assumptions.In this way counter-examples are sometimes produced, requiring mathematicians to reconstruct old proofs and construct new ones. In the attaining conviction, the failure to disprove conjectures empirically plays just as important a role as the process of deductive justification. It appears that there is a logical, as well as a psychological, dimension to attaining certainty. Logically, we require some form of deductive proof, but psychologically it seems we need some experimental exploration or intuitive understanding as well.

Of course, in view of the well-known limitations of intuition and quasiempirical methods themselves, the above arguments are definitely not meant to disregard the importance of proof as an indispensable means of verification, especially in the case of surprising non-intuitive or doubtful results. Rather it is intended to place proof in a more proper perspective in opposition to a distorted idolization of proof as the only (and absolute) means of verification/conviction.

Proof as a means of explanation

Although it is possible to achieve quite a high level of confidence in the validity of a conjecture by means of quasi-empirical verification (for example, accurate constructions and measurement, numerical substitution, and so on), this generally provides no satisfactory explanation why the conjecture may be true. It merely confirms that it is true, and even though considering more and more examples may increase one's confidence even more, it gives no psychological satisfactory sense of illumination—no insight or understanding into how the conjecture is the consequence of other familiar results. For instance, despite the convincing heuristic evidence in support of the earlier mentioned Riemann Hypothesis, one may still have a burning need for explanation as stated by Davis & Hersh (1983:368):

> "*It is interesting to ask, in a context such as this, why we still feel the need for a proof ... It seems clear that we want a proof because ... if something is true and we can't deduce it in this way, this is a sign of a lack of understanding on our part. We believe, in other words, that a proof would be a way of understanding* ***why*** *the Riemann conjecture is true, which is something more than just knowing from convincing heuristic reasoning that it* ***is*** *true.*"

Gale (1990:4) also clearly emphasizes as follows, with reference to Feigenbaum's experimental discoveries in fractal geometry, that the function of their eventual proofs was that of explanation and not that of verification at all:

> "*Lanford and other mathematicians were not trying to validate Feigenbaum's results any more than, say, Newton was trying to* ***validate*** *the discoveries of Kepler on the planetary orbits. In both cases the validity of the results was never in question. What was missing was the* ***explanation****. Why were the orbits ellipses? Why did they satisfy these particular relations? ... there's a world of difference between validating and explaining.*" (bold added))

Thus, in most cases when the results concerned are intuitively self-evident and/or they are supported by convincing quasi-empirical evidence, the function of proof for mathematicians is not that of verification, but rather that of explanation (or the other functions of proof described further on).

In fact, for many mathematicians the clarification/explanation aspect of a proof is of greater importance than the aspect of verification. For instance, the well-known Paul Halmos stated some time ago that although the computer-assisted proof of the four colour theorem by Appel & Haken convinced him that it was true, he would still personally prefer a proof which also gives an "*understanding*" (Albers, 1982:239-240). Also to Manin (1981:107) and Bell (1976:24), explanation is a criterion for a "good" proof when stating respectively that it is "*one which makes us wiser*" and that it is expected "*to convey an insight into why the proposition is true.*"

Proof as a means of discovery

It is often said that theorems are most often first discovered by means of intuition and/or quasi-empirical methods, before they are verified by the production of proofs. However, there are numerous examples in the history of mathematics where new results were discovered or invented in a purely deductive manner; in fact, it is completely unlikely that some results (for example, the non-Euclidean geometries) could ever have been chanced upon merely by intuition and/or only using quasi-empirical methods. Even within the context of such formal deductive processes as axiomatization and defining, proof can frequently lead to new results. To the working mathematician proof is therefore not merely a means of verifying an already-discovered result, but often also a means of exploring, analyzing, discovering and inventing new results (compare Schoenfeld, 1986 & De Jager, 1990).

For instance, consider the following example. Suppose we have constructed a dynamic kite with *Sketchpad* and connected the midpoints of the sides as shown in Figure 1 to form a quadrilateral EFGH. Visually, EFGH clearly appears to be a rectangle, which can easily be confirmed by measuring the angles. By grabbing any vertex of the kite ABCD, we could now drag it to a new position to verify that EFGH remains a rectangle. We could also drag vertex A downwards until ABCD becomes concave to check whether it remains true. Although such continuous variation can easily convince us, it provides no satisfactory explanation why the midpoint quadrilateral of a kite is a rectangle. However, if we produce a deductive proof for this conjecture, we immediately notice that the perpendicularity of the diagonals is the essential characteristic upon which it depends, and that the property of equal adjacent sides is therefore not required. (The proof is left to the reader).

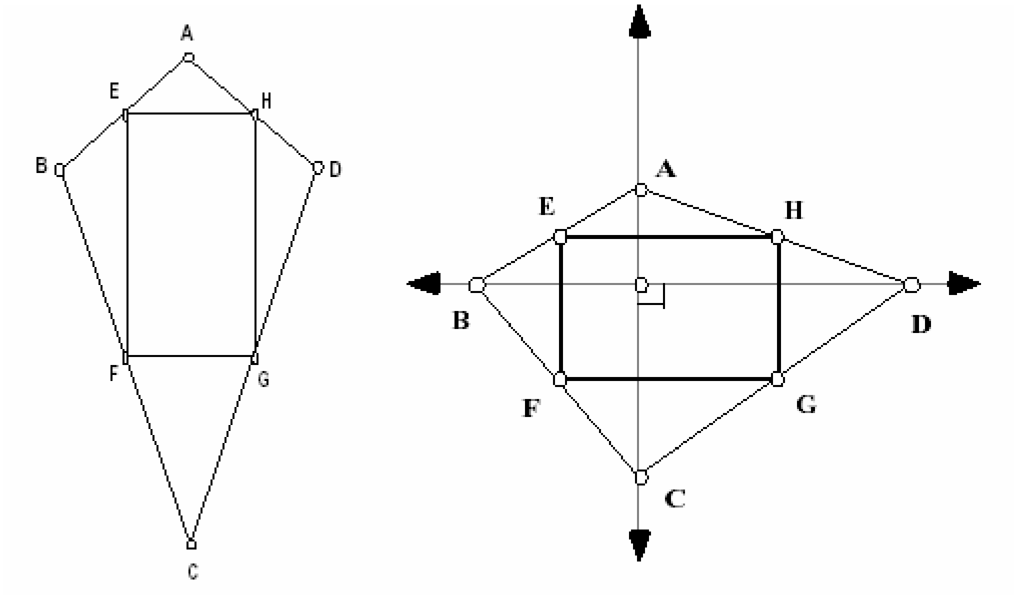

Figure 1

In other words, we can immediately generalize the result to any quadrilateral with perpendicular diagonals (a perpendicular quadrilateral) as shown by the second figure in Figure 1. In contrast, the general result is not at all suggested by the purely empirical verification of the original hypothesis. Even a systematic empirical investigation of various types of quadrilaterals would probably not have helped to discover the general case, since we would probably have restricted our investigation to the familiar quadrilaterals such as parallelograms, rectangles, rhombi, squares and isosceles trapezoids.

The Theorem of Ceva (1678) was probably discovered in a similar deductive fashion by generalizing from an "areas" proof for the concurrency of the medians of a triangle, and not by actual construction and measurement (see De Villiers, 1988). However, new results can also be discovered *a priori* by simply deductively analysing the properties of given objects. For example, without resorting to actual construction and measurement it is possible to quickly deduce that AB + CD = BC + DA for the quadrilateral ABCD circumscribed around a circle as shown in Figure 2 by using the theorem that the tangents from a point outside a circle to the circle are equal.

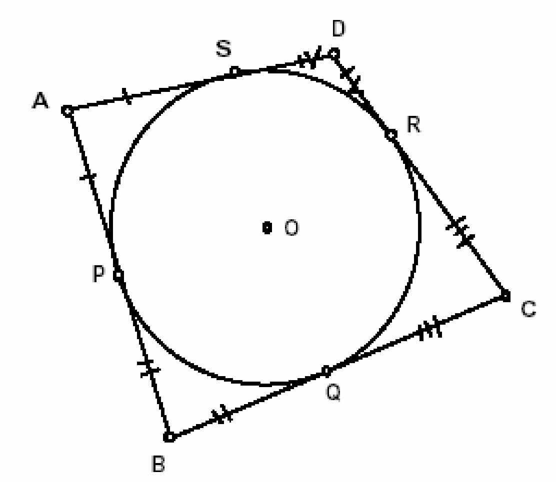

Figure 2

Proof as a means of systematisation

Proof exposes the underlying logical relationships between statements in ways no amount of quasi-empirical testing nor pure intuition can. Proof is therefore an indispensable tool for systematizing various known results into a deductive system of axioms, definitions and theorems. Some of the most important functions of a deductive systematization of known results are given as follows by De Villiers (1986):

- *It helps identify inconsistencies, circular arguments and hidden or not explicitly stated assumptions.*
- *It unifies and simplifies mathematical theories by integrating unrelated statements, theorems, and concepts with one another, thus leading to an economical presentation of results.*
- *It provides a useful global perspective or bird's eyeview of a topic by exposing the underlying axiomatic structure of that topic from which all the other properties may be derived.*
- *It is helpful for applications both within and outside mathematics, since it makes it possible to check the applicability of a whole complex structure or theory by simply evaluating the suitability of its axioms and definitions.*
- *It often leads to alternative deductive systems that provide new perspectives and/or are more economical, elegant, and powerful than existing ones.*

Although some elements of verification are obviously also present here, the main objective clearly is not "*to check whether certain statements are really true*", but to organize logically unrelated individual statements that are already known to be true into *a coherent unified whole*. Due to the global per spective provided by such simplification and unification, there is of course also a distinct element of illumination present when proof is used as a means of systematization. In this case, however, the focus falls on global rather than local illumination. Thus, it is in reality false to say at school when proving selfevident statements such as that the opposite angles of two intersecting lines are equal, that we are "making sure". Mathematicians are actually far less concerned about the truth of such theorems, than with their systematization into a deductive system.

Proof as a means of communication

Several authors have stressed the importance of the communicative function of proof, for example:

> "*... it appears that proof is a form of* ***discourse****, a means of communication among people doing mathematics.*" (bold added) - Volmink (1990:8)
>
> "*... we recognize that mathematical argument is addressed to a human audience, which possesses a background knowledge enabling it to understand the intentions of the speaker or author. In stating that mathematical argument is not mechanical or formal, we have also stated implicitly what it is ... namely, a* ***human interchange*** *based on shared meanings, not all of which are verbal or formulaic.*" (bold added) - Davis & Hersh (1986:73).

Similarly, Davis (1976) has also mentioned that one of the real values of proof is that it creates a forum for critical debate. According to this view, proof is a unique way of communicating mathematical results between professional mathematicians, between teachers and students, and among students themselves. The emphasis thus falls on the social process of reporting and disseminating mathematical knowledge in society. Proof as a form of social interaction therefore also involves subjectively negotiating not only the meanings of concepts concerned, but implicitly also of the criteria for an acceptable argument. In turn, such a social filtration of a proof in various communications contributes to its refinement and the identification of errors, as well as sometimes to its rejection by the discovery of a counter-example.

Proof as a means of intellectual challenge

To mathematicians proof is an intellectual challenge that they find as appealing as other people may find puzzles or other creative hobbies or endeavours. Most people have sufficient experience, if only in attempting to solve a crossword or jigzaw puzzle, to enable them to understand the exuberance with which Pythagoras and Archimedes are said to have celebrated the discovery of their proofs. Doing proofs could also be compared to the physical challenge of completing an arduous marathon or triathlon, and the satisfaction that comes afterwards. In this sense, proof serves the function of **self-realization** and **fulfillment**. Proof is therefore a testing ground for the intellectual stamina and ingenuity of the mathematician (compare Davis & Hersh, 1983:369). To paraphrase Mallory's famous comment on his reason for climbing Mount Everest: *We prove our results because they're there.* Pushing this analogy even further: it is often not the existence of the mountain that is in doubt (the truth of the result), but whether (and how) one can conquer (prove) it!

Finally, although the six functions of proof above can be distinguished from one another, they are often all interwoven in specific cases. In some cases certain functions may dominate others, while in some cases certain functions may not feature at all. Furthermore, this list of functions is by no means complete. For instance, we could easily add an *aesthetic* function or that of *memorization* and *algorithmization* (Renz, 1981 & Van Asch, 1993).

Teaching proof with Sketchpad

When students have already thoroughly investigated a geometric conjecture through continuous variation with dynamic software like *Sketchpad*, they have little need for further conviction or verification. So verification serves as little or no motivation for doing a proof. However, I have found it relatively easy to solicit further curiosity by asking students *why* they think a particular result is true; that is to challenge them to try and *explain* it. Students quickly admit that inductive verification merely confirms; it gives no satisfactory sense of illumination, insight, or understanding into how the conjecture is a consequence of other familiar results. Students therefore find it quite satisfactory to then view a deductive argument as an attempt at explanation, rather than verification.

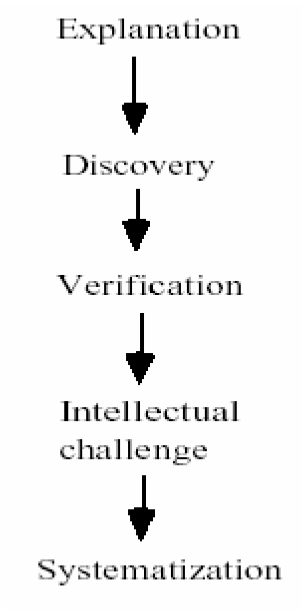

Figure 3

It is also advisable to introduce students early on to the discovery function of proof and to give attention to the communicative aspects throughout by negotiating and clarifying with your students the criteria for acceptable evidence, the underlying heuristics and logic of proof. The verification function of proof should be reserved for results where students genuinely exhibit doubts. Although some students may not experience proof as an intellectual challenge for themselves, they are able to appreciate that others can experience it in this way. Furthermore, in real mathematics, as anyone with a bit of experience will testify, the purely systematization function of proof comes to the fore only at an advanced stage, and should therefore be with-held in an introductory course to proof. It seems meaningful to initially introduce students to the various functions of proof more or less in the sequence given in Figure 3, although not in purely linear fashion as shown, but in a kind of spiral approach where other earlier introduced functions are revisited and expanded. The chapters of this book are organized according to this sequence, and a few approaches to spiraling through the sequence are suggested in the Foreword.

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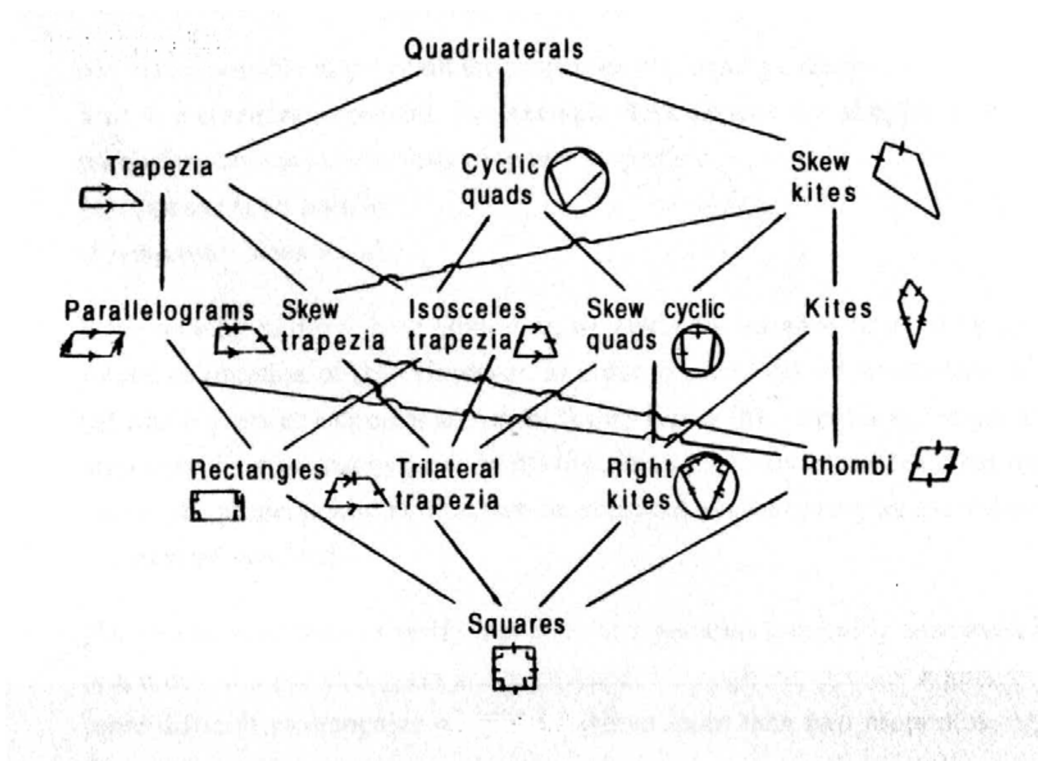
Michael sent a great booklet about investigations of quadrangles. The first chapter is a fictual report of a geometry class on quadrilaterals and their properties. I asked Michael for permission to translate this chapter for our teachers and he agreed, many thanks for that. You can find this translation on the ACDCA-website www.acdca.ac.at.

Part of this discussion between teacher and students is finding the hierarchy and connections between the different kinds of quadrilaterals (quads). This is the proposed structure:



USING SCIENCE AS A TOOL FOR LEARNING MATHEMATICS

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ABSTRACT
Mathematics, as the language of numbers, is an important tool in science classes, but science is not generally considered as a tool for teaching mathematics. This article presents examples incorporating science concepts and problem solving in math classes using a motion detector (Calculator Based Ranger, CBR) and technology from Texas Instruments (TI-Nspire-CAS-handheld or. TI-Nspire-CAS-computer software). Real world data collection tools and Nspire introduce students to many fascinating concepts in mathematics and give them interactive ways to visualize relationships and patterns and enhance critical thinking. The author is investigating the mathematical and pedagogical potential of using TI technology (Graphical calculator, Voyage, Nspire) in combination with Vernier sensors and probes as devices to collect various kinds of data and of using the software to serve as a powerful analysis tool, helping students to build mathematical models. Experiences have been made in grade 9 to 11 (15- to 17-year old students) are reported. The use of technology seems to effectively enhance students' learning. Students are actively engaged in learning as they make predictions, take measurements, analyze their data and make decisions about presenting their work. They are challenged to display their individual talents and mathematical abilities in real world problem solving situations.

1. INTRODUCTION AND FRAMEWORK

The use of technology in mathematics is for the purpose of enhancing teaching and learning. "*Technology should be used in the teaching and learning of mathematics and science when it allows one to perform investigations that either would not be possible or would not be as effective without its use.*"(BRYAN, 2006, p. 231).

With science-related experiments, mathematics concepts and skills can be effectively strengthened while illustrating real world applications. Equipment to collect and process data allows students to have more time to perform repeated data collection trials and for conceptual analysis of the experimental data. Research indicates that the use of sensors/probes is effective, particularly in the area of graphical interpretation. Thornton and Sokoloff (1990) found in an early study that students using real-time graphs improved their kinematics graphing skills and their understanding of the qualitative aspects of motion they observed, compared to students using delayed-time graphs.
Currently, new technological tools are widespread in most classrooms now, but just the presence of technology cannot by itself bring educational change. With regard to pedagogy, teachers who want to use technology effectively face new challenges. The author's interest is the impact of new technologies in the curriculum and the study of the consequences of this impact on the teaching and learning processes. Students' cognitive processes are analyzed when they are confronted with an open problem (i.e. when not provided with a predefined algorithm to solve it). The interest is on the evolution of students' cognitive abilities from an empirical approach to various mathematical activities, which involve experiences such as observing, noticing details, modifying and identifying invariants, to more abstract ones, which lead to applied mathematical knowledge enabling the student to "make sense" of the information and doing some research on their own.

The current article, showing issues related to the effective teaching and learning of Mathematics with real-world data using technology, is part of the author's framework of research; its goal is to form a theoretical didactic basis of learning and teaching theories for new media. The use of graphical calculators does not require new didactics, but teachers have to be aware that learning with graphical calculators follows its own rules and functions differently compared to the classical methods of teaching. This highlights challenges and limits when organizing the learning process with data collecting and analyzing activities. The development of students' skills depends both on pedagogical interventions and on the creation of appropriate learning objects. The technology can only serve as a "raw material". Understanding of teaching and learning processes and assessment of the potential of new media are the theoretical foundation of the author's practical work in answering the question "How does the practice of teaching change by the use of new media and technology?" The success of teaching with new media and technology has to be evaluated as well as the concepts forming the basis of these new methods.

It is assumed that students will have a stronger relationship with data measured by themselves instead of reading them in a textbook. Compared to the traditional instruments used in the classroom, e.g. thermometer or stop watches, more data can be more precisely acquired and the shape of the corresponding curves is obtained easier and faster. Thus, students need less time for data acquisition and have more time at their disposal for analysis, investigation and interpretation of data. Students can investigate reproducibility and variability in the so-called what-would-be-if scenarios, which is an additional benefit. Students can analyze the data both algebraically and graphically and associate these relationships with mathematical functions. Thus they can use the data to model representative functions and discover the physical meaning of different coefficients and parameters.

2. TECHNOLOGY-SUPPORTED MATHEMATICS ACTIVITIES

The following three practical examples were used in the classroom and demonstrate the methodical–didactical potential of the combination of simple data acquisition and advanced data processing with the help of Nspire (see. Fig. 1). Concepts of physics can be tested by visualization and interpretation of data and mathematical models can be developed to describe physical experiments. The CBR (Calculator Based Ranger) in combination with Nspire offers a learning environment for experimenting, modelling, analyzing and visualizing real world data. The students involved in the study were Austrian Secondary School students (15-17 years old). The tasks used in the study were real world open problems, which differ from traditional tasks of the form "prove that". The students are asked to explore the situations, make conjectures and finally prove them.

Figure 1. TI Nspire (a learning environment for mathematics and science)

ACTIVITY 1: THE INCLINED PLANE – WHAT GALILEO WASN'T ABLE TO DO

Students determine, like Galileo had done in the early part of the 17th century, the mathematical relationship between the angle of an incline and the acceleration of a cart down a ramp by using a motion detector to measure the speed and acceleration of a cart rolling down an incline. By extrapolating the acceleration vs. the sine of the track angle graph they mathematically obtain the value for free fall acceleration and discuss the validity of extrapolation the acceleration value to an angle of 90°.
Galileo was able to measure the acceleration only for small angles, because his time devices were not precise enough, and used these data in extrapolation to determine a useful value for the acceleration of free fall.
Students experience that their experimental equipment is precise enough to make things visible Galileo could not even could dream of.

Before conducting this experiment, students should have been introduced how to use the sine and cosine functions to resolve vectors into perpendicular components.

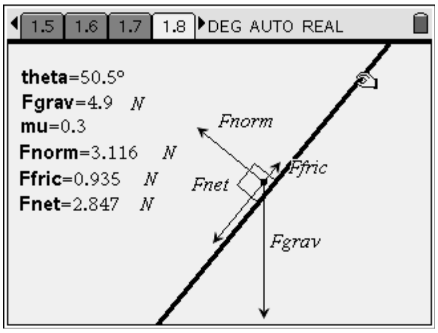


Figure 2. Representation of forces as vectors in a free-body diagram

In this activity, students first use a free-body force-diagram to investigate the forces acting on a mass placed on an inclined plane.

Students explore the relationships between the coefficient of friction, the critical angle, the gravitational force, and the normal force and shall predict the acceleration as a function of the ramp angle and finally compare their predictions to their experimental results.

Students' exploration can be guided by questions:

- What is the relationship between the angle of an inclined plane and the normal force/the gravitational force/the frictional force on an object resting on the plane?
- What happens to an object on an inclined plane when the net force is greater than zero?
- What is the relationship between the magnitude of the normal force and the magnitude of the gravitational force?

Students then vary the angle of the inclined plane and observe the changes in the forces acting on the object. They also discover the effect of changing the coefficient of friction.
A next question can lead them to a real world experiment to evaluate and compare their mathematical calculations with real world data: Newton's second law of motion can be expressed with the equation $F = ma$. What is the acceleration of a 200 g object for a coefficient of friction of 0.3 and an angle of 4°?

By performing the experiment and analyzing the data they get further interesting insights in mathematics and science concepts. Students perform a series of experiments much as Galileo did with inclined planes. As the angle of inclination of the inclined plane is increased, the object's acceleration also increases. As the angle of inclination approaches 90°, the value of the acceleration becomes closer and closer that of gravitational acceleration.

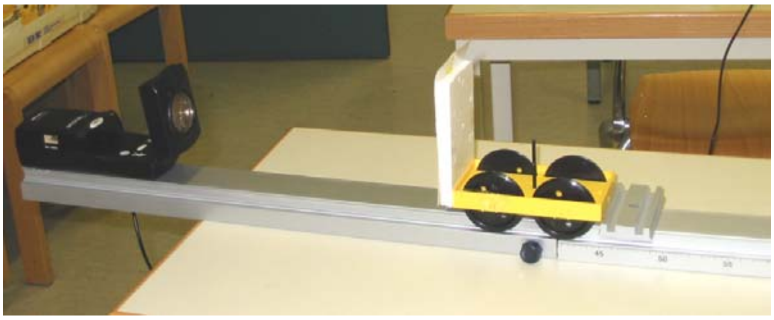


Figure 3. Acceleration of a cart on an inclined plane

Students collect distance and time data for a cart rolling down an incline using a CBR (see Fig. 3 on the left side) connected to a TI Nspire handheld or using Nspire computer software.
Then they construct the best-fit parabola for the data and construct a tangent to the parabola.

They capture data about the slope of the parabola at various points and graph the results. Then they use those results to explore the relationship between the displacement and velocity functions and between the angle of the inclined plane and the acceleration of the cart.

A sample data set for a ramp at an angle of approximately 1.7° is shown in Fig.4 (see dotted points). Students should describe the shape of the graph and find out that the data appear to lie along a parabola. After students have made their predictions, they enter their predicted formulae $y = x^2$) into the formula bar and adjust the graph by hand until the parabola matches the data as closely as possible. *Why does the parabola not fit the whole set of experimental data?*

For exploring the velocity of the cart, students have different possibilities: They construct a tangent to their best-fit parabolas and capture slope and time data or they simply plot a velocity versus time graph.

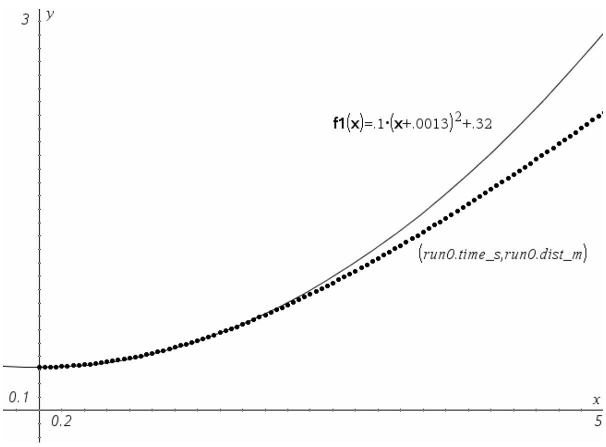


Figure 4. Acceleration of a cart on an inclined plane

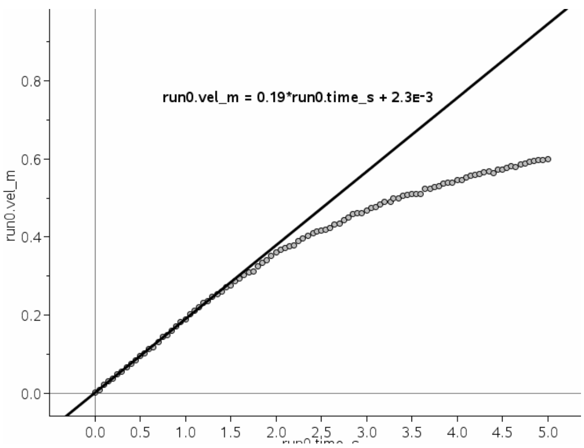


Figure 5. Velocity - time graph for the motion of the cart

Next students plot the velocity versus time data using the TI Nspire application "data and statistics" and find the best fitting movable line describing the data or simply use linear regression.

How does the acceleration coefficient for the equation for this line (see Fig. 5) compare with that in the equation for f1(x) in Fig. 4?

What does this indicate about the relationship between the equation for displacement and the equation for velocity?
What would a plot of acceleration vs. time look like? What would a plot of velocity vs. displacement look like?

The data collected in activity 1 provide students with the information they need to calculate average velocity of the vehicle and to reach conclusions about its instantaneous velocity at any point and its change in velocity or acceleration and how velocity and acceleration depend on the angle of the ramp.

For small angles they can find out, that they must not assume the acceleration is constant, even when they have been told to ignore air resistance when solving physics problems involving free fall. The use of technology enables students not only to make air resistance "visible" and perform different experiments but encourages students to build mathematical models including air resistance as a drag force for simulating real world phenomena and getting deeper insights: The motion of a cart on an inclined plane does not follow simple formulas used in school physics.

Under ideal circumstances the force F_{net} (see Fig. 1) is calculated by $F_{net} = m \cdot g \cdot \sin(\theta)$ and because θ does not change for a certain ramp - related ramp F_{net} and therefore the acceleration $a = \frac{F_{net}}{m} = g \cdot \sin(\theta)$ only depends on g and the angle of the inclined plane. But in the real world forces such as friction or air drag result in deviations from the pure model:

- If the cart begins to move, wheel-related friction is a force working in the opposite direction and proportional to the perpendicular force F_{norm}.
- Also the velocity-dependent air drag works in the opposite direction; its value is $F_{air} = -k \cdot v^2$.
- Factor k depends on the cross-section area and on the cart design.

Cart acceleration is then not a constant value, but decreases with increasing velocity according to:

$a(v) = g \cdot \sin(\theta) - a_{fric} - \frac{k}{m} \cdot v^2$ The time-distance relation can be numerically calculated for the actual cart movement with „Lists & Spreadsheet" (an Nspire menu option), taking into account also air drag.

In summary, students model the cart movement on the inclined ramp and take also friction and air drag into account. They obtain a better fit of the model by variation of the air drag parameter k. With this feature, NSpire expands the range of measurable processes in mechanics and supports the formation of models.

ACTIVITY 2: AIR RESISTANCE AND TERMINAL VELOCITY – FINDING THE RIGHT MODEL

In activity 2 students will collect data on the rate at which coffee filters fall exploring the effect of air resistance on a falling object. Coffee filters are light enough to reach terminal velocity in a short distance. They will attempt to determine the terminal velocity for different numbers of falling filters and explore the relationship between mass and terminal velocity by choosing between two models for the drag force, which is opposite to the direction of motion and assumed to depend on the velocity of the falling object. Students shall find the right model by analyzing the data: Is the drag force proportional to the velocity or is it proportional to the square of the velocity?

First students draw a free body diagram of a falling coffee filter recognizing that there are only two forces (gravity and air resistance) acting an the filter. Once the terminal velocity has been reached, the acceleration and the net force are zero.

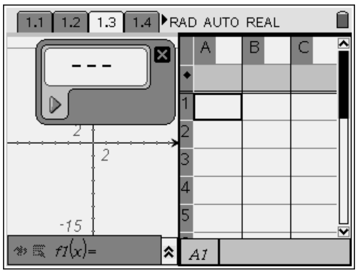


Figure 6. Data collection with CBR and TI Nspire

The activity is designed to be student-centered by guiding the students through the main steps of the activity with a worksheet.
Students begin data collection by releasing one filter and plot a distance vs. time graph in the Data & Statistics application. The graph should have a long linear region representing the time during which the filter was falling at constant velocity.
Using the Movable Line tool in the Data & Statistics application students produce the line that best fits their data and record the slope of the line as the terminal velocity.
Then students repeat the experiment with 2, 3, 4, 5, … coffee filters and begin exploring the relationship between mass, drag force and terminal velocity.

They plot terminal velocity vs. mass and in another graph terminal velocity squared vs. mass and use the Regression tool to fit the data to a straight line. Both data sets are very close to linear and that makes it difficult for students to determine which data set gives the best fit to the linear relationship.

The activity illustrates how technology can be used within learning environments to connect rich mathematical content with the learner's real environment – in this case free fall with air drag as opposed to the ideal models of free fall in a vacuum, exploring mathematical models for explaining physics phenomena and removing misconceptions by building useful and sophisticated models.

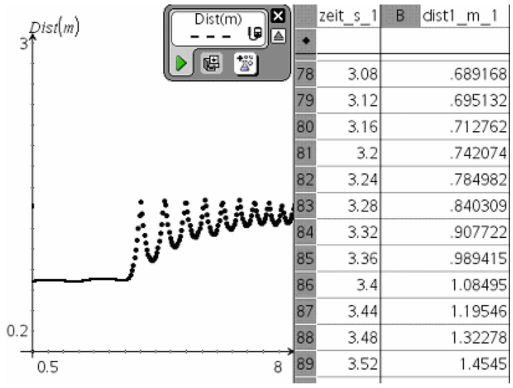
ACTIVITY 3: THE BOUNCING BALL AND HOW HIGH WILL IT BOUNCE?

A ball is dropped from a height of 1 m and the height of the bouncing ball is continuously measured with a distance measuring device connected to TI NSpire and the data collected are analyzed. The measured movement of the ball is described as a function of time and the gravity law is derived. With energy calculations, insights can be gained where energy is lost during bouncing.

The ball is a freely falling and bouncing object where air friction is neglected. Therefore only gravity affects the ball's movements which show that acceleration is approximately constant. The time-distance graphs are parabolic functions, which can be described by the quadratic equation $y = a(x-b)^2+c$ where the highest point is described by the coordinates (b, c) with c as the maximum height and b as the corresponding time. The parameter a represents mathematically the shape of the parabola and depends physically on the degree of acceleration caused by gravity, which is constant during the experiment. The curves obtained for the time-distance graphs of the individual bounces are first adjusted manually – by determining the parameters b and c and by varying parameter a.

In the classroom the following questions can be asked:

- What is the highest speed of the ball and when does it occur?
- What is the acceleration during falling?
- Which function describes the distance (height) of the ball?
- Is there a model to describe the height of the ball?
- How can the total distance of the ball be determined?
- What processes determine the “bouncing back” of the ball from the floor?
- In which way the rebound height decreases from one bounce to the next?
- Can you determine how high a ball will rebound on each bounce and make predictions about its motion?



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80	3.16	.712762
81	3.2	.742074
82	3.24	.784982
83	3.28	.840309
84	3.32	.907722
85	3.36	.989415
86	3.4	1.08495
87	3.44	1.19546
88	3.48	1.32278
89	3.52	1.4545

Figure 7. Data collection with CBR and TI Nspire

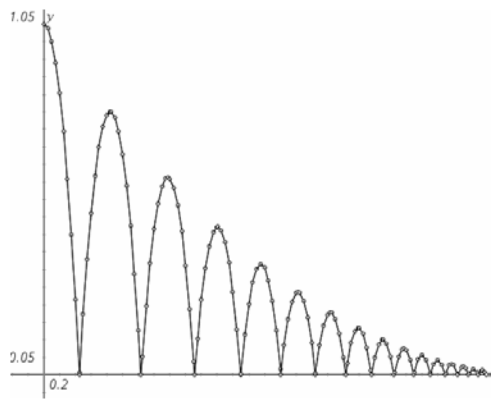

Figure 8. Height of the ball as a function of time

Looking at the time-distance-diagram of the ball, it is recognized that the ball first falls and then bounces from the floor. Then it moves up, slowed down by gravity until it falls down again. This movement corresponds to repeated vertical throws. Therefore both phases of movement, i.e. up and down, can be described by quadratic functions. For this, the data for a complete bounce have to be selected from the total set of data. From this section the parameters for the ball’s movement can be obtained.

After selecting an individual bounce and quadratic regression, the function describing the ball’s movement is obtained with the help of regression analysis. With the help of the application “Graphs & Geometry” the best fitting curve can be found by hand too. How are the functions of the second and the third parabola similar, how are they different? How are the parameters connected to the movement of the ball?

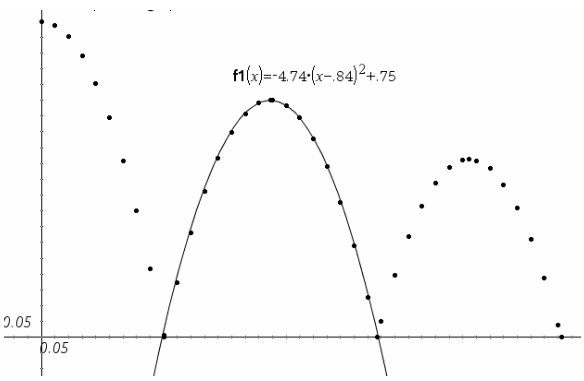

Figure 9. Quadratic function for the ‘middle” parabola

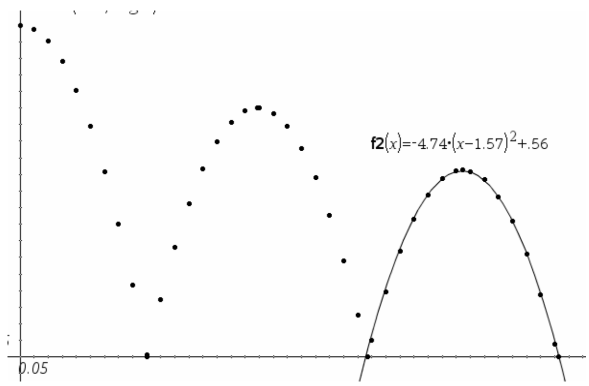

Figure 10. Quadratic function for the ‘third” parabola

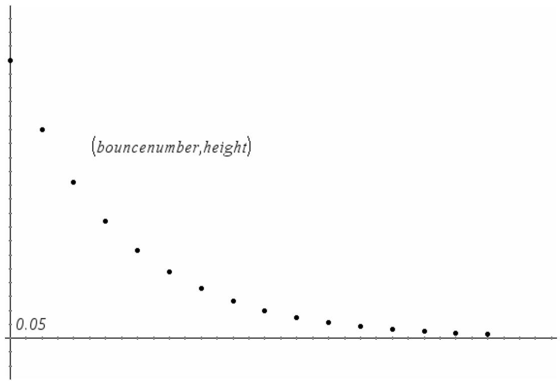

Figure 11. 'Maximum height' as a function of the 'bouncenumber'

The maximum height decreases exponentially from bounce to bounce for each ball and its initial height. For $y = h \cdot p^x$, y is the current height, h is the initial height, p is a constant depending on the properties of the ball and the floor and x is the number of the bounce.

For $x = 0 \rightarrow y = h$ (the initial height of the ball, from which it has been dropped). The coefficients of the equation describing the exponential function are determined from the data obtained. The experiment can be repeated with different balls, heights and floor types.

The bouncing ball experiment can be studied in a unit on geometric sequences. Students can also use the data to explore quadratic and exponential functions. If students try to model a function that fits this data by using regression or finding the curve fitting the data best by hand, they have to be aware, that the data are discrete.

Students can determine recursive and explicit formulas for geometric sequence that models their data. They determine the constant ratio between successive bounces by dividing the height of the second bounce by the height of the first, the third bounce by the second, and so on. The they determine the average ratio and use it in the explicit and recursive formulas.

The recursive formula for the data in Fig. 11 is: $h_1 = 1$, $h_n = 0.75 \cdot h_{n-1}$, for integers $n \geq 2$ (n represents the bounce number).

The explicit formula for the data is: $h_n = 1 \cdot (0.75)^{n-1}$ for integers $n \geq 1$.

A further challenge could be to find a function that models the data 'maximum height as a function of time'.

3. CONCLUSIONS AND IMPLICATIONS

Summarizing the technology-supported mathematics activities, simple experiments and questions motivate a rich discourse and activity in the classroom and further exploration.

How do students construct mathematical ideas in technological environment tools (like TI NSpire and the motion detector)? The students were at first challenged by the exercises, and the investigations also generated surprise moments that motivated students to focus their thinking. When exploring these problems, students developed functions and even algebraic expressions and explained their conclusions. Thus, the lessons involved multiple stages of investigation: prediction, testing, rejection or extension of hypotheses, discovering and exploring the underlying mathematics, and making generalizations and proving results.

The students were actively engaged during these moments, negotiating and communicating ideas with the teacher and with other students. But successful learning is an individual and active process, based on intrinsic motivation, interest and active interaction with the learning environment. The proactive learner searches for his own way, looks out for questions and challenges and should be able to find appropriate answers in a well-built learning environment. The challenge for the teacher is to embed technology tools in the general syllabus and into the class room. Implementing this technology in the class room leads to a substantial change in the learning process. Passive reception of lectured information gives way to an active, individual construction of knowledge. Critical for these new learning processes are learning situations which offer guidance as well as room for individual exploration.

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Compared to the traditional instruments used in the classroom in mathematics teaching, more data can be very precisely acquired by using a motion detector in combination with TI Nspire or a graphical calculator and the shape of the corresponding curves is obtained easier and faster. Thus, students need less time for data acquisition and have more time at their disposal for analysis, investigation and interpretation of data.

Currently available media and technological tools have to be transformed with didactic know-how into effective teaching media and teaching tools. The author is aware that the use of technology in a didactically designed learning environment changes substantially the learning and working behavior of students in the class room and outside. Teachers have to learn when and how to use new media and technology and what impact they may have on the students' education. The student should be offered a learning environment where he can move between different levels and within different structures, where he is stimulated to ask his own questions and where he finds also help to answer those questions. The student should not be overwhelmed with a predefined succession of facts, but he should be helped to discover knowledge and to generate his own knowledge with his mind (compare Frank Thissen, 1997).

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Thissen, F., (1997) *Vortrag auf der Learntec 1997*. Veröffentlicht in Beck, U. & Sommer, W. (Hrsg.) *Learntec 1997. Europäischer Kongress für Bildungstechnologie und betriebliche Bildung*, Tagungsband, S. 69-79, Karlsruhe.

Thornton, R.; Sokoloff, D. (1990). Learning motion concepts using real-time microcomputer-based laboratory tools. American Journal of Physics, 58, 858-867.

Web resources (http://education.ti.com)

TI Nspire Physics. Free-Body Forces: Inclined Plane – ID: 8740 © 2007 Texas Instruments Incorporated
TI Nspire Physics. Air Resistance – ID: 8739 © 2007 Texas Instruments Incorporated
Easy Data Collection Activities. How high will it bounce? Activity 13 © 2005 Texas Instruments Incorporated

In revised DNL#15 I presented a "DERIVE-program" for investigating graphs for zeros, turning and inflection points, symmetries, asymptotes, … Our member Peter Lüke-Rosendahl was not completely satisfied with my routines because I didn´t consider the higher derivatives. We had some exchanges of emails and based on his proposals I tried to improve the tool as you can see in the following examples.

I included an auxiliary function to consider NSOLUTIONS and SOLUTIONS as well because of the different treatment of polynomial and non polynomial equations. What I am naming as "flatpoints" are points with f'', f''', $f^{(IV)}$, … = 0.

Many thanks, Peter.

$$f(x) := \frac{x^6 + 3\cdot x^5 - 3\cdot x^4 - 11\cdot x^3 + 6\cdot x^2 + 12\cdot x - 200}{100}$$

discuss

Zeros:

$$x\cdot(x^5 + 3\cdot x^4 - 3\cdot x^3 - 11\cdot x^2 + 6\cdot x + 12) = 200$$

2.33178	0
-3.33178	0

Extremals:

$$f'(x) = 0.03\cdot(2\cdot x^5 + 5\cdot x^4 - 4\cdot x^3 - 11\cdot x^2 + 4\cdot x + 4)$$

1	-1.92	TerrPt
-2	-1.92	TerrPt
-0.5	-2.0339	MIN

Infl.points: slope

$$f''(x) = 0.06\cdot(5\cdot x^4 + 10\cdot x^3 - 6\cdot x^2 - 11\cdot x + 2)$$

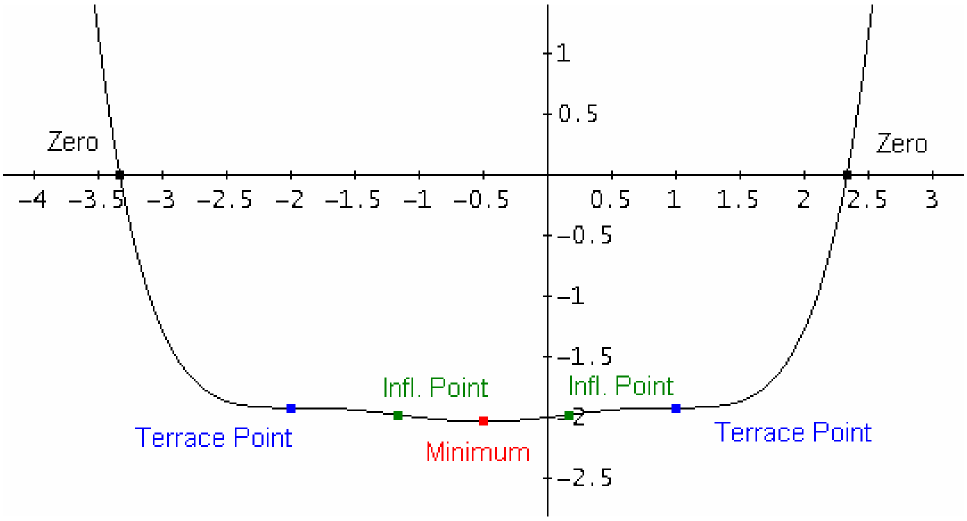
		slope
1	-1.92	0
-2	-1.92	0
0.170820	-1.97832	0.130407
-1.17082	-1.97832	-0.130407

Poles:

none

Limits:

And this is the graph including labelling of the interesting points. (Terrace Point = Inflection point with slope = 0).



$$f(x) := \frac{1}{80}\cdot(x^4 - 16\cdot x^3 + 96\cdot x^2 - 288\cdot x)$$

discuss

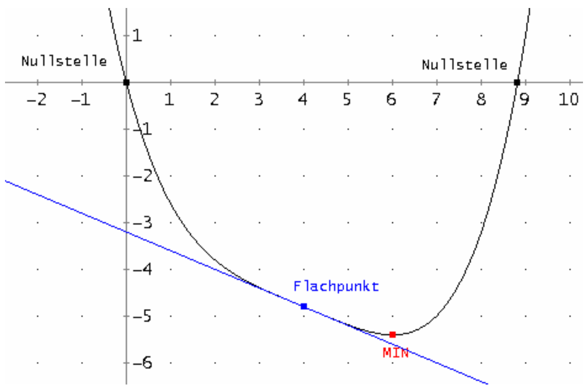
```
Zeros:
x·(x^3 - 16·x^2 + 96·x - 288) = 0
0                                  0
8.81637                            0

Extremals:
f'(x) =   0.05·(x^3 - 12·x^2 + 48·x - 72)
6              -5.4              MIN

Infl.points:                     slope
f''(x) =   0.15·(x^2 - 8·x + 16)
               none

Flatpoints:                      slope
f'''(x) =   0.3·(x - 4)
4              -4.8              -0.4

Poles:
none
```



Here we can find a "flat point". The given table of results make it easy to plot the points of interest together with the tangents if necessary.

The next one is a rational function with a pole and an approximating polynomial function:

$$f(x) := \frac{x^5 \cdot (6 - x)}{100 \cdot (x - 2)}$$

discuss

Zeros:

$$\frac{x^5 \cdot (x - 6)}{x - 2} = 0$$

0	0
6	0

Extremals:

$$f'(x) = - \frac{0.01 \cdot x^4 \cdot (5 \cdot x^2 - 36 \cdot x + 60)}{(x - 2)^2}$$

0	0	TerrPt
2.62020	6.73025	MIN
4.57979	11.0916	MAX

Infl.points: slope

$$f''(x) = \frac{0.04 \cdot x^3 \cdot (5 \cdot x^3 - 42 \cdot x^2 + 120 \cdot x - 120)}{(2 - x)^3}$$

0	0	0
3.51264	8.79371	3.16838

Flatpoints: slope

$$f'''(x) = - \frac{0.12 \cdot x^2 \cdot (5 \cdot x^4 - 48 \cdot x^3 + 180 \cdot x^2 - 320 \cdot x + 240)}{(x - 2)^4}$$

0	0	0

Poles:

$x = 2$

Limits:

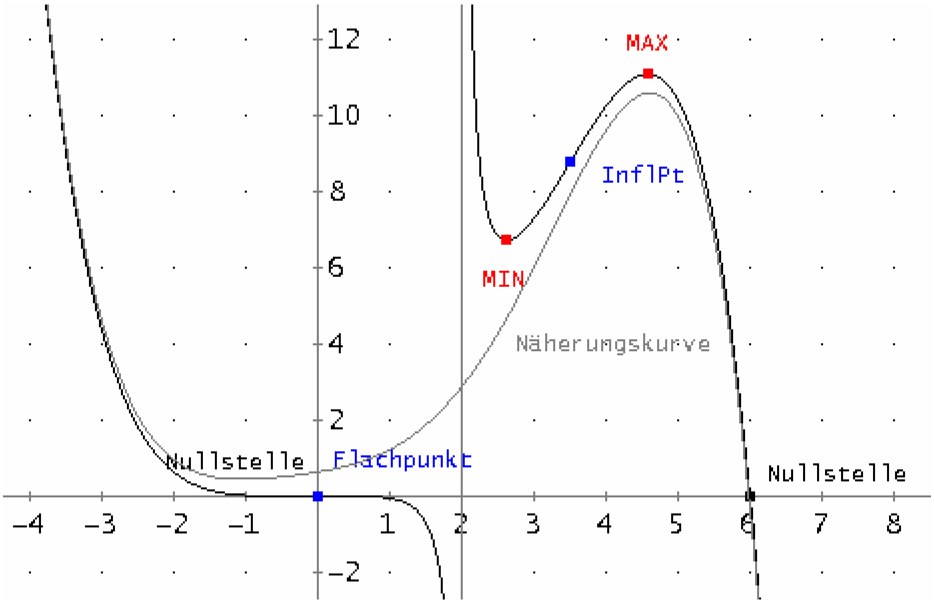
x --> ∞	?
x --> -∞	?

Asymptote:

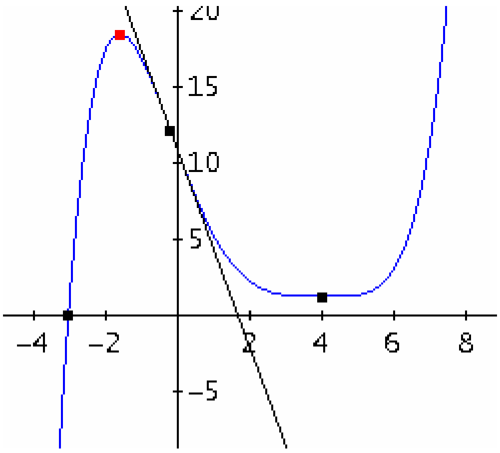
$$y = - 0.01 \cdot x^5 + 0.04 \cdot x^4 + 0.08 \cdot x^3 + 0.16 \cdot x^2 + 0.32 \cdot x + 0.64$$

Symmetry:

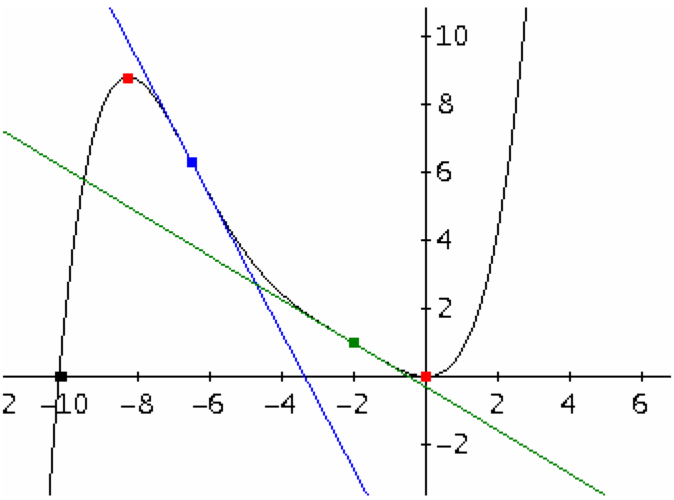
no symm.



$$f(x) := \frac{x^5 - 13\cdot x^4 + 48\cdot x^3 + 32\cdot x^2 - 512\cdot x + 868}{80}$$



$$f(x) := \frac{2\cdot x^5 + 35\cdot x^4 + 200\cdot x^3 + 520\cdot x^2}{1000}$$

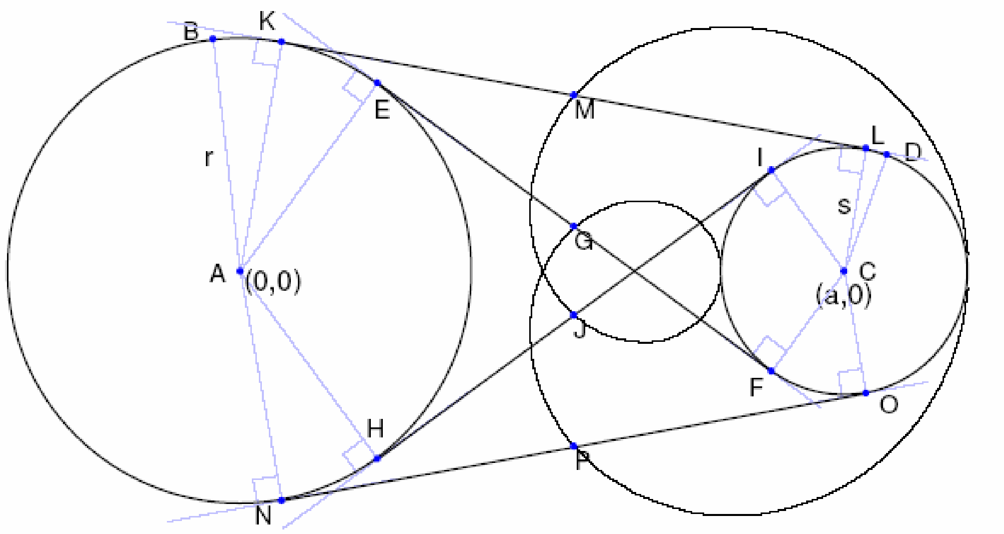


A Challenging Locus

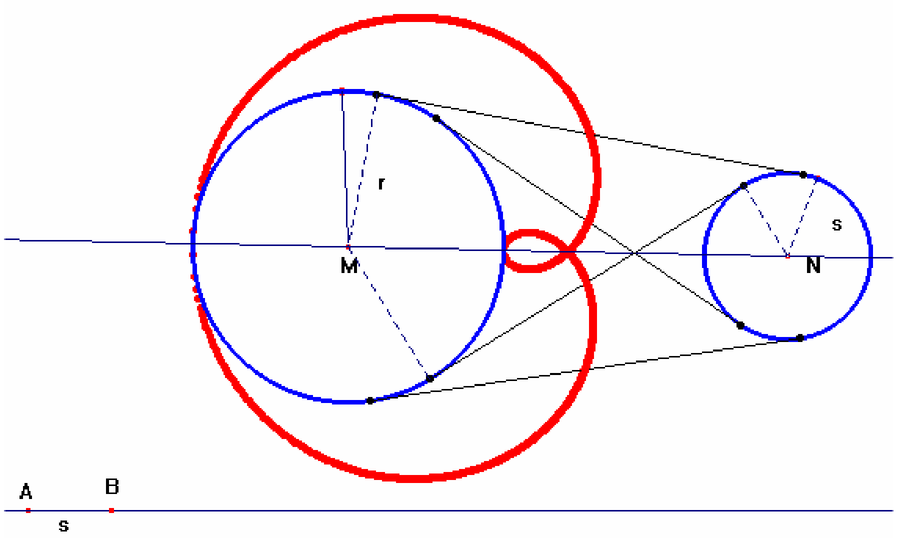
Josef Böhm

In DNL#70 I presented the program *Geometry Expressions* together with two examples. One example showed the implicit form of the locus of centers of common tangents to two circles. *Geometry Expressions* is not only a dynamic geometry program but has a powerful CAS working in the background. Otherwise it would not be possible to deliver the implicit form of this locus (a 4th order equation).

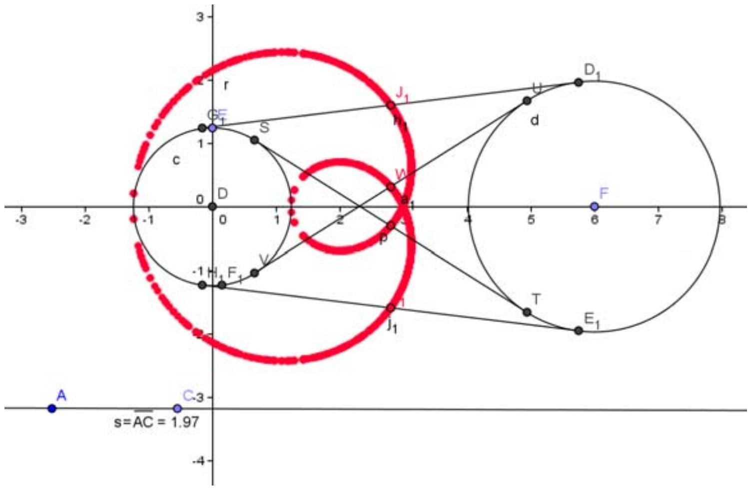
$$\Rightarrow 4\cdot X^4+8\cdot X^2\cdot Y^2+4\cdot Y^4-12\cdot X^3\cdot a-12\cdot X\cdot Y^2\cdot a+a^4-a^2\cdot s^2+Y^2\cdot\left(4\cdot a^2-4\cdot s^2\right)+X^2\cdot\left(13\cdot a^2-4\cdot s^2\right)+X\cdot\left(-6\cdot a^3+4\cdot a\cdot s^2\right)=0$$



This is my reproduction with Cabri. Draging point B changes radius *s* of the circle with centre N. Try to do the construction. It can be that you will face a problem (at the moment when radius *s* > *r*). Then try to resolve this problem!



This is the locus produced with GeoGebra:



I felt inspired to find this equation on my own working with two DERIVE representation forms: the algebraic form in the Algebra Window and the geometric form in the 2D Plot Window.

The mathematics is not so difficult. It covers analytic geometry of secondary school but the manipulating of the expressions would be too boring and tedious not only for students but for the teachers.

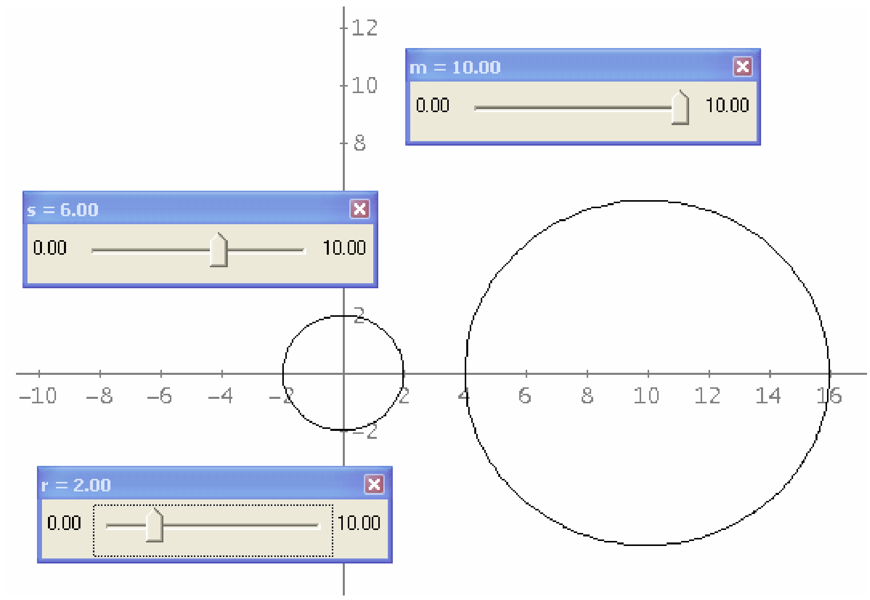
I start with two circles – one of them with varying radius *s*.

#1: $\text{circle\_fixed} := x^2 + y^2 = r^2$

#2: $\text{circle\_var} := (x - m)^2 + y^2 = s^2$

Introduce sliders for s, r and m (0 ... 10 recommended)

Plot both circles (black)



Let´s take any arbitrary line and find the conditions for osculating both circles. The resulting equations containing variables *k* (slope) and *d* (intercept) are called in German "Berühr-bedingung", (in English: "osculating condition"?)

#3: line := $y = k\cdot x + d$

#4: tanpts_f := SOLUTIONS(line ∧ circle_fixed, [x, y])

#5: tanpts_f := $\begin{bmatrix} -\dfrac{\sqrt{r^2\cdot(k^2+1)-d^2}+d\cdot k}{k^2+1} & \dfrac{d-k\cdot\sqrt{r^2\cdot(k^2+1)-d^2}}{k^2+1} \\ \dfrac{\sqrt{r^2\cdot(k^2+1)-d^2}-d\cdot k}{k^2+1} & \dfrac{k\cdot\sqrt{r^2\cdot(k^2+1)-d^2}+d}{k^2+1} \end{bmatrix}$

We need only one point, because the root must become zero in case of osculating.

#6: tanpts_f := $\left[\dfrac{\sqrt{r^2\cdot(k^2+1)-d^2}-d\cdot k}{k^2+1}, \dfrac{k\cdot\sqrt{r^2\cdot(k^2+1)-d^2}+d}{k^2+1}\right]$

#7: cond1 := $r^2\cdot(k^2+1)-d^2 = 0$

The intersection points of the line and the second circle must coincide, too. So we again need only one point and set the discriminant = 0:

#8: tanpts_v := $(\text{SOLUTIONS(line} \wedge \text{circle\_var, [x, y]))}_1$

#9: tanpts_v := $\left[-\dfrac{\sqrt{-d^2-2\cdot d\cdot k\cdot m+k^2\cdot(s^2-m^2)+s^2}+d\cdot k-m}{k^2+1}, -\dfrac{k\cdot\sqrt{-d^2-2\cdot d\cdot k\cdot m+k^2\cdot(s^2-m^2)+s^2}-d-k\cdot m}{k^2+1}\right]$

#10: cond2 := $-d^2-2\cdot d\cdot k\cdot m+k^2\cdot(s^2-m^2)+s^2$

#11: sols := SOLUTIONS(cond1 ∧ cond2, [k, d])

The four pairs of solutions for k and d of the four common tangents are:

#12: sols := $\begin{bmatrix} \dfrac{r+s}{\sqrt{m^2-r^2-2\cdot r\cdot s-s^2}} & -\dfrac{m\cdot r}{\sqrt{m^2-r^2-2\cdot r\cdot s-s^2}} \\ -\dfrac{r+s}{\sqrt{m^2-r^2-2\cdot r\cdot s-s^2}} & \dfrac{m\cdot r}{\sqrt{m^2-r^2-2\cdot r\cdot s-s^2}} \\ \dfrac{r-s}{\sqrt{m^2-r^2+2\cdot r\cdot s-s^2}} & -\dfrac{m\cdot r}{\sqrt{m^2-r^2+2\cdot r\cdot s-s^2}} \\ \dfrac{s-r}{\sqrt{m^2-r^2+2\cdot r\cdot s-s^2}} & \dfrac{m\cdot r}{\sqrt{m^2-r^2+2\cdot r\cdot s-s^2}} \end{bmatrix}$

We calculate the osculation points on the fixed circle

#13: $\text{pts\_f} := \text{VECTOR}(\text{SUBST}(\text{tanpts\_f}, [k, d], \text{sols}_j), j, 4)$

$$\#14:\quad \text{pts\_f} := \begin{bmatrix} \frac{r\cdot(r+s)}{m} & -\frac{r\cdot\sqrt{(m^2 - r^2 - 2\cdot r\cdot s - s^2)}}{m} \\ \frac{r\cdot(r+s)}{m} & \frac{r\cdot\sqrt{(m^2 - r^2 - 2\cdot r\cdot s - s^2)}}{m} \\ \frac{r\cdot(r-s)}{m} & -\frac{r\cdot\sqrt{(m^2 - r^2 + 2\cdot r\cdot s - s^2)}}{m} \\ \frac{r\cdot(r-s)}{m} & \frac{r\cdot\sqrt{(m^2 - r^2 + 2\cdot r\cdot s - s^2)}}{m} \end{bmatrix}$$

and then the respective points on the variable circle

#15: $\text{pts\_v} := \text{VECTOR}(\text{SUBST}(\text{tanpts\_v}, [k, d], \text{sols}_j), j, 4)$

$$\#16:\quad \text{pts\_v} := \begin{bmatrix} \frac{m^2 - s\cdot(r+s)}{m} & \frac{s\cdot\sqrt{(m^2 - r^2 - 2\cdot r\cdot s - s^2)}}{m} \\ \frac{m^2 - s\cdot(r+s)}{m} & -\frac{s\cdot\sqrt{(m^2 - r^2 - 2\cdot r\cdot s - s^2)}}{m} \\ \frac{m^2 + s\cdot(r-s)}{m} & -\frac{s\cdot\sqrt{(m^2 - r^2 + 2\cdot r\cdot s - s^2)}}{m} \\ \frac{m^2 + s\cdot(r-s)}{m} & \frac{s\cdot\sqrt{(m^2 - r^2 + 2\cdot r\cdot s - s^2)}}{m} \end{bmatrix}$$

Connecting the points delivers the segments:

#17: $\text{segments} := \text{VECTOR}([\text{pts\_f}_j, \text{pts\_v}_j], j, 4)$

It is not necessary to simplify expression #17 (bulky expressions)

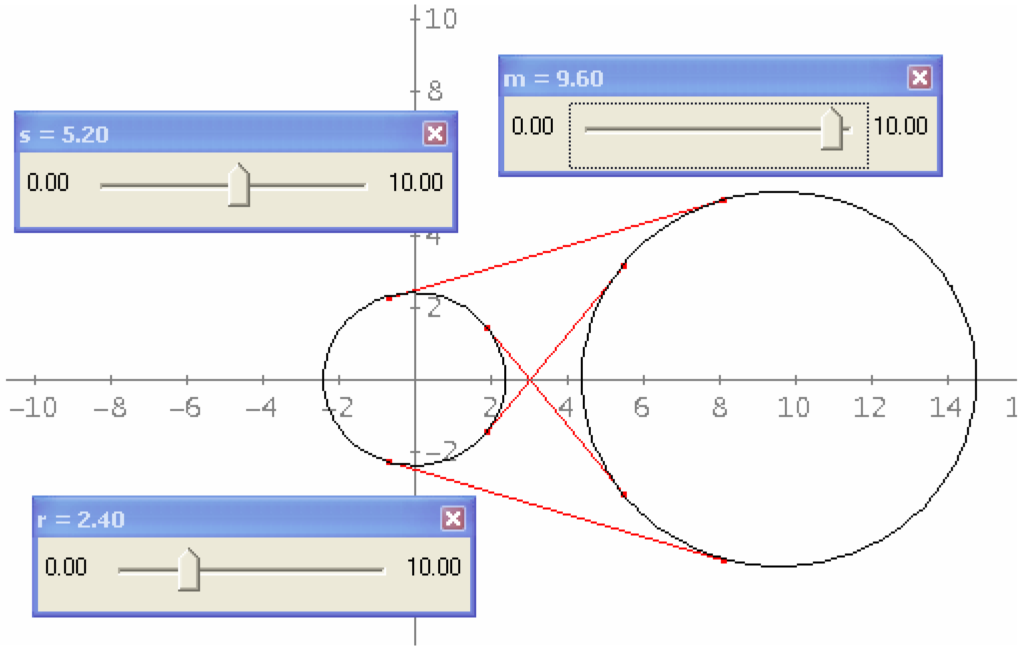
Plot the segments (red)

and then calculate the midpoints of the segments.

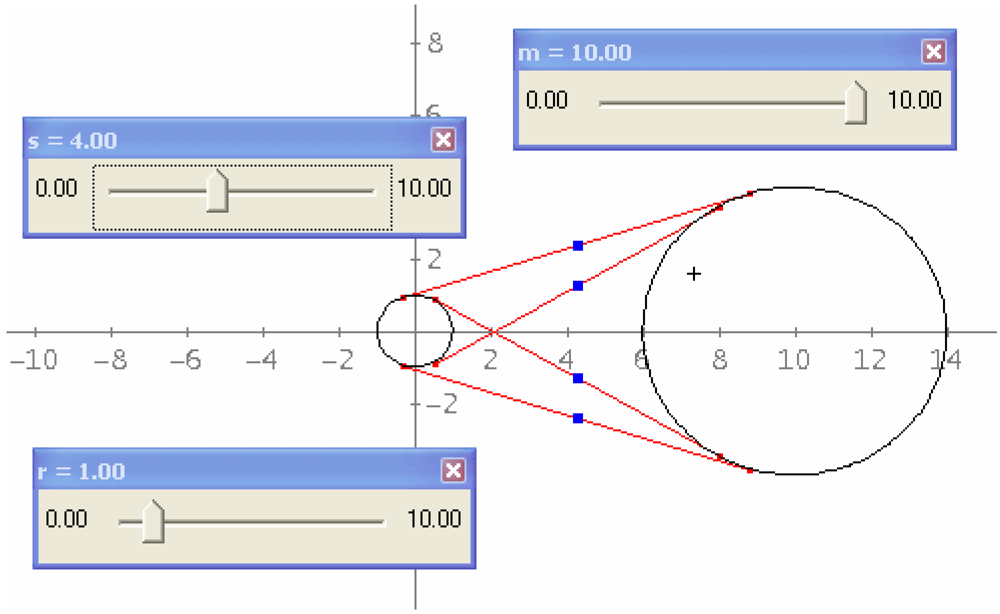
$$\#18:\quad \text{midpoints} := \text{VECTOR}\left(\frac{v_1 + v_2}{2}, v, \text{segments}\right)$$

Plot the midpoints (blue)

This is the parameter form of the locus with parameter s (radius of the variable circle).



The circles together with their common tangents.



The midpoints of the segments.

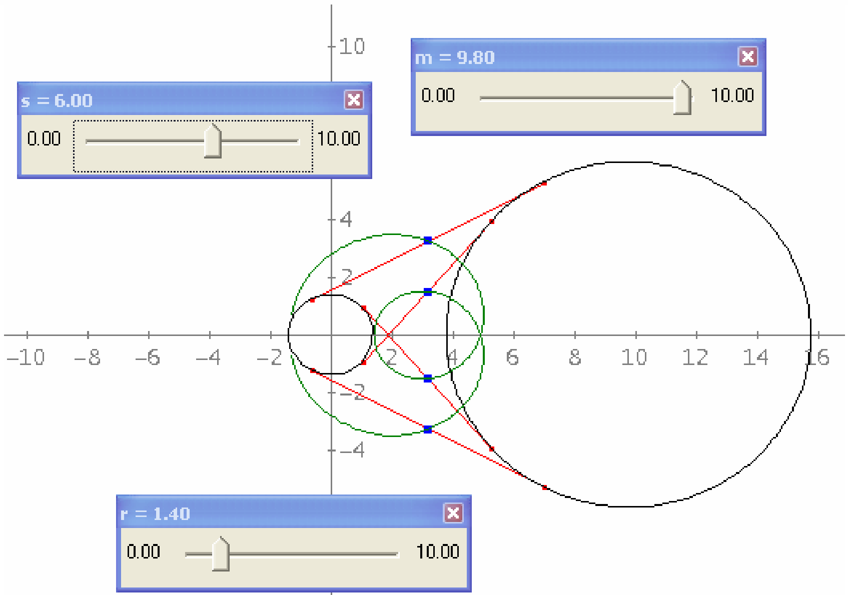
#19: midpoints :=

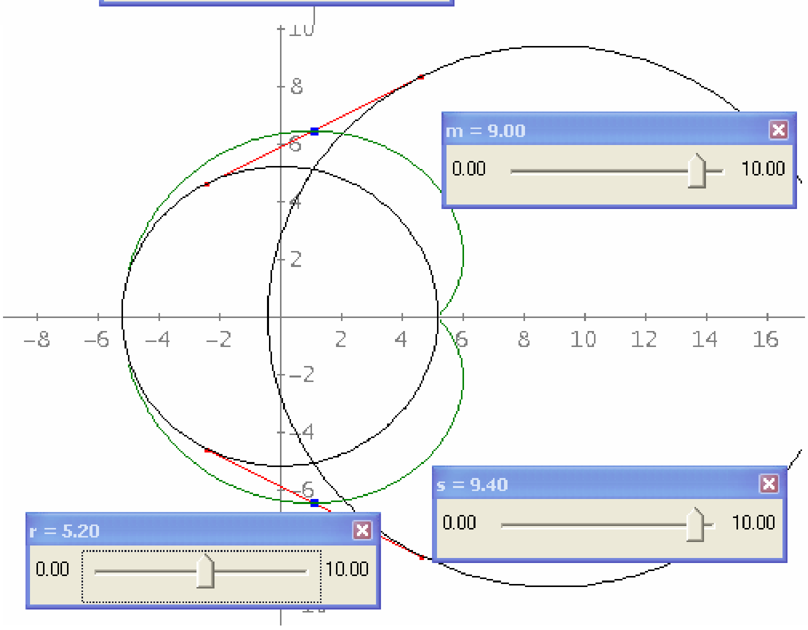
$$\begin{bmatrix} \dfrac{m^2+(r+s)\cdot(r-s)}{2\cdot m} & \dfrac{\sqrt{(m^2-r^2-2\cdot r\cdot s-s^2)\cdot(s-r)}}{2\cdot m} \\ \dfrac{m^2+(r+s)\cdot(r-s)}{2\cdot m} & \dfrac{\sqrt{(m^2-r^2-2\cdot r\cdot s-s^2)\cdot(r-s)}}{2\cdot m} \\ \dfrac{m^2+(r+s)\cdot(r-s)}{2\cdot m} & -\dfrac{\sqrt{(m^2-r^2+2\cdot r\cdot s-s^2)\cdot(r+s)}}{2\cdot m} \\ \dfrac{m^2+(r+s)\cdot(r-s)}{2\cdot m} & \dfrac{\sqrt{(m^2-r^2+2\cdot r\cdot s-s^2)\cdot(r+s)}}{2\cdot m} \end{bmatrix}$$

For plotting the parameter form we have to substitute variable s (which is occupied by the slider) by parameter t.

$$\#20:\quad \begin{bmatrix} \dfrac{m^2+(r+t)\cdot(r-t)}{2\cdot m} & \dfrac{\sqrt{(m^2-r^2-2\cdot r\cdot t-t^2)\cdot(t-r)}}{2\cdot m} \\ \dfrac{m^2+(r+t)\cdot(r-t)}{2\cdot m} & \dfrac{\sqrt{(m^2-r^2-2\cdot r\cdot t-t^2)\cdot(r-t)}}{2\cdot m} \\ \dfrac{m^2+(r+t)\cdot(r-t)}{2\cdot m} & -\dfrac{\sqrt{(m^2-r^2+2\cdot r\cdot t-t^2)\cdot(r+t)}}{2\cdot m} \\ \dfrac{m^2+(r+t)\cdot(r-t)}{2\cdot m} & \dfrac{\sqrt{(m^2-r^2+2\cdot r\cdot t-t^2)\cdot(r+t)}}{2\cdot m} \end{bmatrix}$$

Plot the parameter forms (0 <= t <= 100) (green)





Now let´s try to find an implicit form of the locus

$$\#21: \quad \text{SOLUTIONS}\left(x = \frac{m^2 + (r + t)\cdot(r - t)}{2\cdot m} \wedge y = \frac{\sqrt{(m^2 - r^2 - 2\cdot r\cdot t - t^2)\cdot(t - r)}}{2\cdot m}, [y, t]\right)$$

$$\#22: \quad \begin{bmatrix} -\dfrac{\sqrt{2}\cdot(\sqrt{(-2\cdot m\cdot x + m^2 + r^2)} + r)\cdot\sqrt{(r\cdot\sqrt{(-2\cdot m\cdot x + m^2 + r^2)} + m\cdot x - r^2)}}{2\cdot m} & -\sqrt{(-2\cdot m\cdot x + m^2 + r^2)} \\ \dfrac{\sqrt{2}\cdot(\sqrt{(-2\cdot m\cdot x + m^2 + r^2)} - r)\cdot\sqrt{(-r\cdot\sqrt{(-2\cdot m\cdot x + m^2 + r^2)} + m\cdot x - r^2)}}{2\cdot m} & \sqrt{(-2\cdot m\cdot x + m^2 + r^2)} \end{bmatrix}$$

We have to eliminate parameter *t*. This is done by a little trick applying the solutions-command. As *t* is one of the solution variables, we receive an expression for *y* (the other unknown) which is free of *t*.

The first component looks nice, let´s take it for further manipulations.

$$\#23: \quad y = -\frac{\sqrt{2}\cdot(\sqrt{(-2\cdot m\cdot x + m^2 + r^2)} + r)\cdot\sqrt{(r\cdot\sqrt{(-2\cdot m\cdot x + m^2 + r^2)} + m\cdot x - r^2)}}{2\cdot m}$$

$$\#24: \quad \left(y = -\frac{\sqrt{2}\cdot(\sqrt{(-2\cdot m\cdot x + m^2 + r^2)} + r)\cdot\sqrt{(r\cdot\sqrt{(-2\cdot m\cdot x + m^2 + r^2)} + m\cdot x - r^2)}}{2\cdot m}\right)^2$$

$$\#25: \quad y^2 = \frac{(\sqrt{(-2\cdot m\cdot x + m^2 + r^2)} + r)^2\cdot(r\cdot\sqrt{(-2\cdot m\cdot x + m^2 + r^2)} + m\cdot x - r^2)}{2\cdot m^2}$$

Expanding the numerator of #25 shows that only one root expression is remaining.
One more simplification performs cancellation.

$$\#26: \quad y^2 = \frac{m^2\cdot r\cdot\sqrt{(-2\cdot m\cdot x + m^2 + r^2)} - 2\cdot m^2\cdot x^2 + m^3\cdot x + m^2\cdot r^2}{2\cdot m^2}$$

$$\#27: \quad y^2 = \frac{r\cdot\sqrt{(-2\cdot m\cdot x + m^2 + r^2)} - 2\cdot x^2 + m\cdot x + r^2}{2}$$

We leave the root on the right side and square the whole equation once more.

#28: $(2\cdot y^2 + 2\cdot x^2 - m\cdot x - r^2 = r\cdot\sqrt{(- 2\cdot m\cdot x + m^2 + r^2)})$

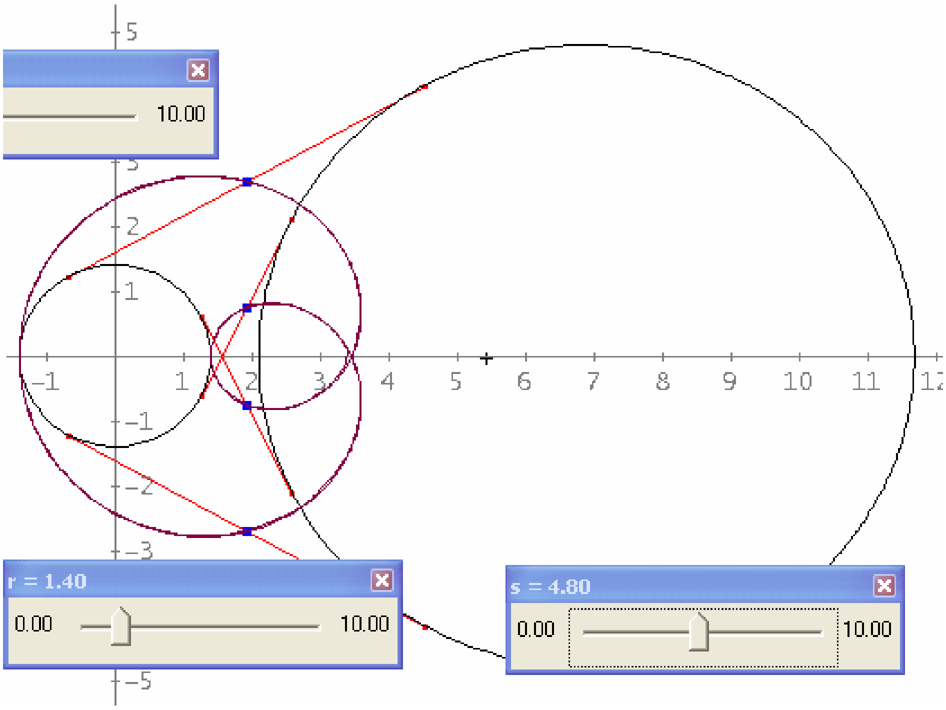
#29: $(2\cdot x^2 - m\cdot x + 2\cdot y^2 - r^2)^2 = - r^2\cdot(2\cdot m\cdot x - m^2 - r^2)$

Now we don´t have any root, we collect alle expressions on one side of the equation

#30: $(2\cdot x^2 - m\cdot x + 2\cdot y^2 - r^2)^2 + r^2\cdot(2\cdot m\cdot x - m^2 - r^2) = 0$

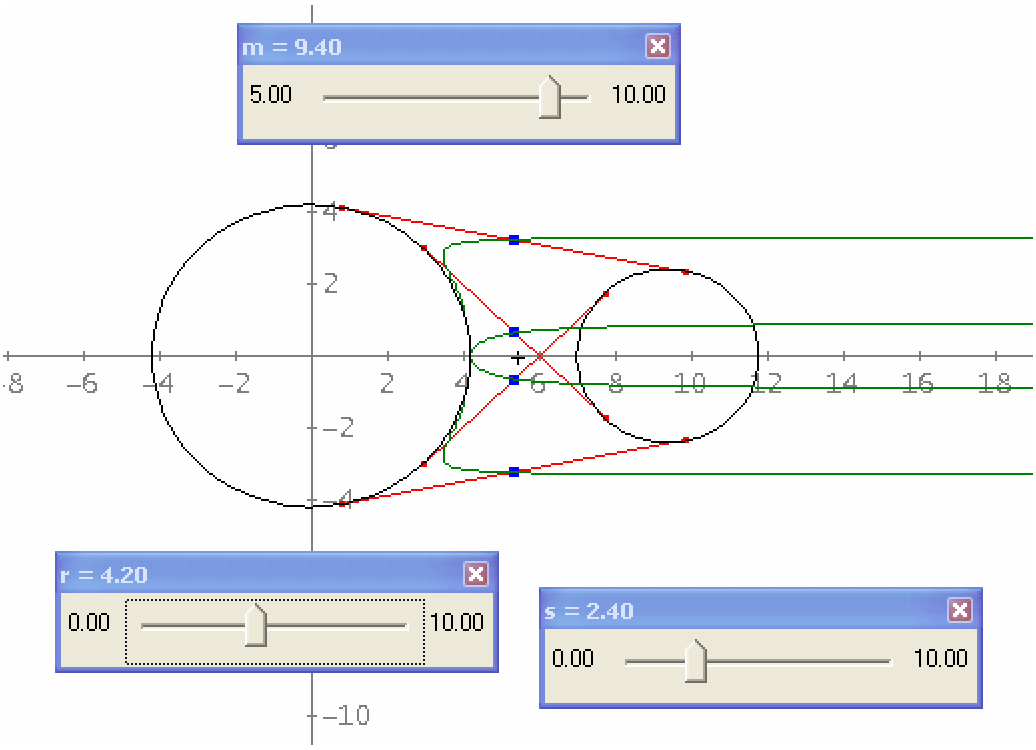
Plot it - the locus is a fourth order curve

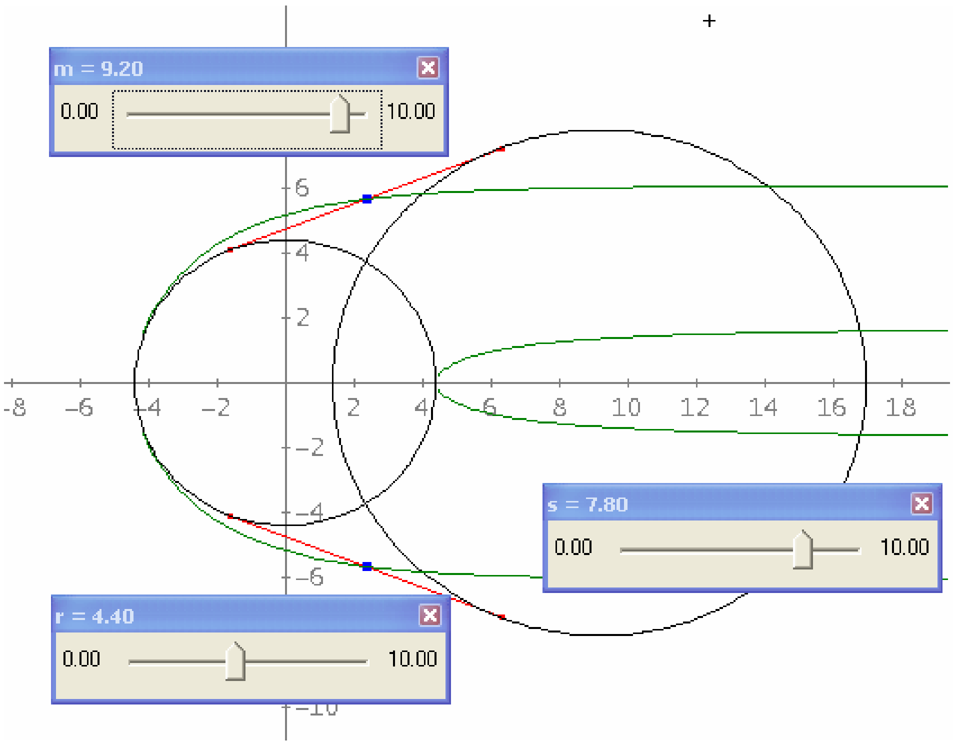
#31: $4\cdot x^4 - 4\cdot m\cdot x^3 + 8\cdot x^2\cdot y^2 + m^2\cdot x^2 - 4\cdot r^2\cdot x^2 - 4\cdot m\cdot x\cdot y^2 + 4\cdot m\cdot r^2\cdot x + 4\cdot y^4 - 4\cdot r^2\cdot y^2 - m^2\cdot r^2 = 0$



Now we take m (centre of the variable circle) as parameter.
Substitute m in expression # 20 by t and plot.

#32:
$$\begin{bmatrix} \dfrac{r^2 - s^2 + t^2}{2\cdot t} & \dfrac{(s - r)\cdot\sqrt{(- r^2 - 2\cdot r\cdot s - s^2 + t^2)}}{2\cdot t} \\ \dfrac{r^2 - s^2 + t^2}{2\cdot t} & \dfrac{(r - s)\cdot\sqrt{(- r^2 - 2\cdot r\cdot s - s^2 + t^2)}}{2\cdot t} \\ \dfrac{r^2 - s^2 + t^2}{2\cdot t} & -\dfrac{(r + s)\cdot\sqrt{(- r^2 + 2\cdot r\cdot s - s^2 + t^2)}}{2\cdot t} \\ \dfrac{r^2 - s^2 + t^2}{2\cdot t} & \dfrac{(r + s)\cdot\sqrt{(- r^2 + 2\cdot r\cdot s - s^2 + t^2)}}{2\cdot t} \end{bmatrix}$$


m = 9.40
5.00
10.00
r = 4.20
0.00
10.00
s = 2.40
0.00
10.00

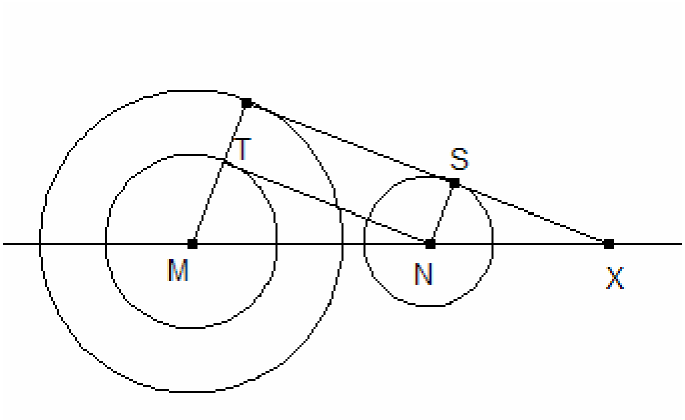

m = 9.20
0.00
10.00
s = 7.80
0.00
10.00
r = 4.40
0.00
10.00

There is another way to find the locus of the midpoints:
Similar triangles support to find point *X*. Then construct the tangent(s) from *X* to the circles and calculate the osculating points.

$MN = m$, $MX = x$, $MT = r - s$, $NS = s$

$$(r - s) : m = r : x$$

$$x = \frac{r \cdot m}{r - s}$$



For the "inner" tangents we receive in a similar way distance *x'*

$$(r + s) : m = r : x'$$

$$x' = \frac{r \cdot m}{r + s}$$

The following lines will become tangents by choosing slope *k* in such a way that we have exact one intersection point between the circles and the lines (same idea as above).

$$\text{tan\_out} := y = k \cdot \left(x - \frac{r \cdot m}{r - s}\right)$$

$$\text{tan\_in} := y = k \cdot \left(x - \frac{r \cdot m}{r + s}\right)$$

$$\text{t\_out\_f} := (\text{SOLUTIONS}(\text{tan\_out} \wedge \text{circle\_fixed}, [x, y]))_1$$

$$\text{t\_out\_f} := \left[\frac{r \cdot (\sqrt{(k^2 \cdot ((r - s)^2 - m^2) + (r - s)^2)} + k^2 \cdot m)}{(k^2 + 1) \cdot (r - s)}, \frac{k \cdot r \cdot (\sqrt{(k^2 \cdot ((r - s)^2 - m^2) + (r - s)^2)} - m)}{(k^2 + 1) \cdot (r - s)}\right]$$

$$\text{t\_out\_v} := (\text{SOLUTIONS}(\text{tan\_out} \wedge \text{circle\_var}, [x, y]))_1$$

$$\text{t\_out\_v} := \left[\frac{s \cdot \sqrt{(k^2 \cdot ((r - s)^2 - m^2) + (r - s)^2)} + m \cdot (k^2 \cdot r + r - s)}{(k^2 + 1) \cdot (r - s)}, \frac{k \cdot s \cdot (\sqrt{(k^2 \cdot ((r - s)^2 - m^2) + (r - s)^2)} - m)}{(k^2 + 1) \cdot (r - s)}\right]$$

$$cc1 := k^2\cdot((r-s)^2 - m^2) + (r-s)^2 = 0$$

Same expression under the root which must become zero:

$$\text{SOLUTIONS}(k^2\cdot((r-s)^2 - m^2) + (r-s)^2 = 0,\ k)$$

$$\left[\frac{r-s}{\sqrt{(m^2-(r-s)^2)}},\ \frac{s-r}{\sqrt{(m^2-(r-s)^2)}}\right]$$

$$t1 := \text{SUBST}\left(t\_out\_f,\ k,\ \frac{s-r}{\sqrt{(m^2-(r-s)^2)}}\right)$$

$$t1 := \left[\frac{r\cdot(r-s)}{m},\ \frac{r\cdot\sqrt{(m^2-(r-s)^2)}}{m}\right]$$

$$t2 := \text{SUBST}\left(t\_out\_v,\ k,\ \frac{s-r}{\sqrt{(m^2-(r-s)^2)}}\right)$$

$$t2 := \left[\frac{m^2 + s\cdot(r-s)}{m},\ \frac{s\cdot\sqrt{(m^2-(r-s)^2)}}{m}\right]$$

$$m1 := \frac{t1 + t2}{2}$$

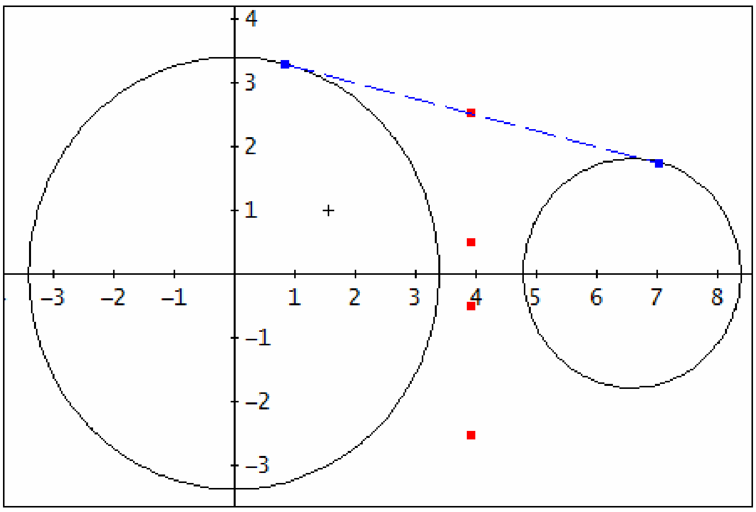
$$m1 := \left[\frac{m^2 + (r+s)\cdot(r-s)}{2\cdot m},\ \frac{\sqrt{(m^2-(r-s)^2)}\cdot(r+s)}{2\cdot m}\right]$$

$$m2 := \left(\text{SUBST}\left(\frac{1}{2}\cdot(t\_out\_f + t\_out\_v),\ k,\ \frac{r-s}{\sqrt{(m^2-(r-s)^2)}}\right)\right)_1$$

$$m2 := \left[\frac{m^2 + (r+s)\cdot(r-s)}{2\cdot m},\ -\frac{\sqrt{(m^2-(r-s)^2)}\cdot(r+s)}{2\cdot m}\right]$$

Repeat the procedure using the other tangents (inside osculating) and finish.

You finally will receive the table #19 from above ...



Multiplication in Abyssinia (or elsewhere!)

Roland Schröder, Celle, Germany

In times before inventing the Indian place value system (8th century A.C.) multiplication was a difficult matter. The reason was that the used digits had no place value and that they could appear several times within one number. What we call multiplication by hand as it our children are learning in elementary school was not possible using Roman numerals. The place values 1, 10, 100, 1000, … were not expressed by positions within the numbers but by own (nowadays redundant) number characters. This makes the Roman numerals to an out dying species, because they miss an important advantage to survive. Those Romans, who were not busy as a wizard of figures used pre-calculated multiplication tables. The following algorithm might explain how these tables were built up.

In the famous Papyrus Rhind (the oldest known collection of arithmetic, 1650 BC) is explained how numbers can be multiplied without applying place values. It works by repeated doubling the first factor and halving the second factor. It is only some decades ago that this art of multiplying was found at the farmers of the Ethiopian highlands.

For calculating 319 × 37, they wrote procedure as follows.

319 × 37

~~638 × 18~~

1276 × 9

~~2552 × 4~~

~~5104 × 2~~

10208 × 1

One can notice that the first factor is indeed doubled at each calculation step, but the second one cannot be halved (if it is an odd number). In this case the second factor is decreased by 1 before halving. The farmer in Abyssinia crosses out all pairs of numbers with an even second factor. Then he adds all first factors of the remaining pairs:

319 +1276 +10208 = 11803

The farmer has found the result: 319 × 37 = 11803.

This looks like sorcery, but it is very easy to explain: Let´s change a little bit what the farmer has written down performing his calculation.

319 × 37 =

= 638 × 18 + 319

= 1276 × 9 + 319

= 2552 × 4 + 1276 + 319

= 5104 × 2 + 1276 + 319

=10208 × 1 + 1276 + 319 = 11803

If the second factor is an odd number then the subtraction by 1 causes an error in the amount of the preceding first factor. This mistake is repaired by adding the factor. The algorithm is recursive and can be described by the following recursion formula.

$A \times B = 2A \times B/2$ (If B is odd)

$2A \times (B - 1)/2 + B$ (else)

DERIVE´s MOD(x,y) gives the remainder of the division x : y. Hence

A × B = <u>2A</u> × <u>(B – MOD(B,2)/2</u> + <u>A</u> × <u>MOD(B,2)</u>.

We build a three column matrix, which shows row after row the <u>first factor</u>, the <u>second factor</u> and the possibly existing <u>summand</u>. We will call this triple of numbers as F(a, b, x):

#1 F(a, b, x): = [2a, (b – MOD(b, 2))/2, a·MOD(b, 2) + x].

The initial triple of the recursion is given by [319, 37, 0] or generally spoken by [u, v, 0]. The next triple is [638, 18, 319] or [2u, (v – 1)/2, u].

Recursion must end if the second factor has become zero. This is done by a statement which will not be explained now:

#2 STOPP(v): = FLOOR(LN(v)/LN(2))+1.

Function G does the complete recursion, which is defined as follows:

#3 G(u, v):= ITERATES(F(a, b, x), [a, b, x], [u, v, 0], STOPP(v))

Lines #1 - #3 must be entered in DERIVE.

We test the algorithm for the multiplication from above (u = 319 and v = 37):

#4: G(319, 37) =

319	37	0
638	18	319
1276	9	319
2552	4	1595
5104	2	1595
10208	1	1595
20416	0	11803

#5: G(37, 319) =

37	319	0
74	159	37
148	79	111
296	39	259
592	19	555
1184	9	1147
2368	4	2331
4736	2	2331
9472	1	2331
18944	0	11803

The number in the 3rd column last row is the product.

Find some additional comments of the editor on the next page:

I have known Roland´s Abyssinian Multiplication as "Egyptian Multiplication". Similar multiplication tables were found written in hieroglyphs. The geographic vicinity and the antique historic connections between the two regions might explain the fact that the farmers in Ethiopia performed multiplication in the same way.

I found this multiplication mentioned in one of Adrian Oldknow´s many papers. He called it "Russian Peasant´s Multiplication". So it seems to be really multi cultural.

Replacing doubling the first number by squaring, adding by multiplying and 0 by 1 gives an efficient way to find powers of numbers:

#6: $P(a, b, x) := \left[a^2, \frac{b - \mathrm{MOD}(b, 2)}{2}, a^{\mathrm{MOD}(b, 2)} \cdot x\right]$

#7: H(u, v) := ITERATES(P(a, b, x), [a, b, x], [u, v, 1], STOPP(v))

#8: $H(23, 15) = \begin{bmatrix} 23 & 15 & 1 \\ 529 & 7 & 23 \\ 279841 & 3 & 12167 \\ 78310985281 & 1 & 3404825447 \\ 6132610415680998648961 & 0 & 266635235464391245607 \end{bmatrix}$

This way to find large powers is used for modular multiplication (*square and multiply*). Large powers are needed in cryptography.

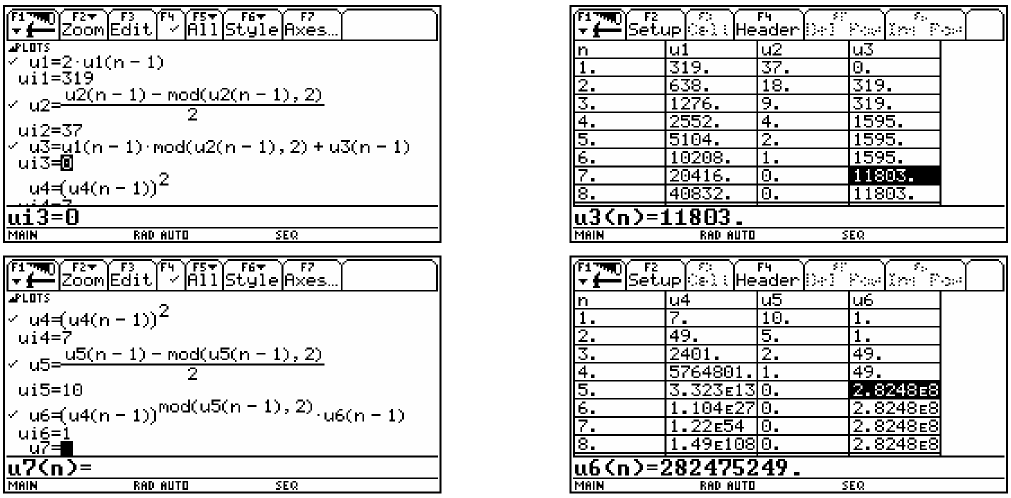
Students might be asked how to explain the stop-condition (expression #2).
Final comment: As there is involved repeated division by 2 students might find a connection to binary numbers:

Convert 37 to a binary number: 1 0 0 1 0 1.

319 × 37	1
~~638 × 18~~	0
1276 × 9	1
~~2552 × 4~~	0
~~5104 × 2~~	0
10208 × 1	1

Do you see the pattern?

Transfer the recursion to the TIs (using the Sequence Mode): 319 . 37 and 7^{10}.



Ein neues Buch von Bernhard Kutzler ist erschienen:

> Bernhard Kutzler, Technologie und das Yin & Yang des Lehrens und Lernens von Mathematik, BK-02, ISBN 978-3-901769-84-9

Mail von Roland Schröder

Lieber Herr Böhm,
Sie schrieben, dass Interesse an den deutschen Übersetzungen meiner im DNL erscheinenden Aufsätze bestehen könnte. Heute kann ich Ihnen mitteilen, dass am Ende dieses Monats ein Büchlein (50 Seiten) herausgegeben wird, in dem alle diese Aufsätze stehen.

Roland Schröder
Experimente mit Computer-Algebra-Systemen
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Es wäre schön, wenn Sie darauf hinweisen könnten (vielleicht zusätzlich in der Bücherecke?).

Herzliche Grüße
Roland Schröder

Nils Hahnfeld, Virgin Islands

Nils and I had a very intense e-mail and TI-files exchange for extending and improving his DEQME-program. The next DNL will present a detailed review. See here some pictures of the menus: