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THE BULLETIN OF THE

DERIVE®

USER GROUP

+ CAS-TI

Contents:

December 2009

Interesting and recommended websites:

Lehrstuhl für Didaktik der Mathematik Uni Erlangen, Prof. Thomas Weth
www.didmath.ewf.uni-erlangen.de/Homepage/hp_weth.htm

Die Zeitschrift für MathematiklehrerInnen (Universität Salzburg und PH Salzburg)
www.mathematikimunterricht.at/Newsletter/index_news.html

The Journal of Symbolic Geometry
journal.geometryexpressions.com/

Documents of Maths & Stats & OR
www.ltsn.gla.ac.uk/headocs/

Among others a resource for background pictures
staff.spd.dcu.ie./oreillym/geometry.htm

Teach Engineering, Resources for K12
www.teachengineering.com/index.php

A publication of the Institute of Electrical and Electronics Engineers
www.ieee.org/web/education/preuniversity/tispt/lessons.html

Online proceedings of the CADGME 2009 Conferrence is available at:
www.risc.uni-linz.ac.at/about/conferences/cadgme2009/

The first Journal of Mathematical Modeling and Application of the Reference Center for Mathematical Modeling in Teaching - CREMM. The first number is on line at: furb.br/ojs/index.php/modelling

Teaching Math through Culture – a very interesting site!
www.rpi.edu/~eglash/csdt.html

Just recently I was informed by Philip Yorke (former DERIVE dealer for UK) about a very rich resource of A-Level Math (Exam questions and others). Many thanks Philip for this valuable notice. Have a look – or more than only one!
www.mathsnetalevel.com

Please inform me about interesting websites that we can share our resources, Josef.

Dear DUG Members,

This is DNL#76 containing the remaining lectures of the special DERIVE session at ACA09 in Montreal. With DNL#77 we will proceed publishing original contributions intended for publication in the DERIVE Newsletter.

The User Forum is not very extended in this issue because the articles needed a lot of pages. I received a mail from "Santa Claus" about a lot of DERIVE bugs. You will find them in the next DNL.

As now WINDOWS 7 is on the market I have been asked several times if DERIVE will work under WINDOWS 7. Yes, it does. The respective information provided by DUG-Members is given in the User Forum. Many thanks to Günter and Peter.

Among many others Phil Todd joined the DUG this year. Phil is author of Geometry Expressions, which is an excellent piece of software, GE enables exporting of results to DERIVE and to TI-Nspire as well. We will demonstrate this in 2010.

We have good news for you. Bernhard Kutzler gave permission to put all Conference Proceedings which were published by bk-teachware on the ACDCA-website (www.acdca.ac.at). We will start uploading as soon as possible. Philip Yorke who published the Proceedings of the famous Krems Conferences (1992 and 1993) also permitted to upload the Proceedings. This will be more work, because we don't´have them in electronic form. I´ll scan the books and produce pdf-files. (This will need some time, so please be patient.)

Have a look at the recommended websites. I checked them just now – they all are valid at the moment.

We wish a Merry Christmas and a Happy New Year 2010
Noor and Josef Böhm

Jose Luis Galan and Josef are planning TIME 2010 (right).

Finally I´d like to remind you that deadline for submitting a paper for the TIME2010 conference is 15 March. The website provides some information about accomodation and registration fees.

The Conference website is www.time2010.uma.es.

The *DERIVE-NEWSLETTER* is the Bulletin of the *DERIVE* & CAS-*TI User Group*. It is published at least four times a year with a contents of 40 pages minimum. The goals of the *DNL* are to enable the exchange of experiences made with *DERIVE*, *TI*-CAS and other CAS as well to create a group to discuss the possibilities of new methodical and didactical manners in teaching mathematics.

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Contributions:
Please send all contributions to the Editor. Non-English speakers are encouraged to write their contributions in English to reinforce the international touch of the *DNL*. It must be said, though, that non-English articles will be warmly welcomed nonetheless. Your contributions will be edited but not assessed. By submitting articles the author gives his consent for reprinting it in the *DNL*. The more contributions you will send, the more lively and richer in contents the *DERIVE* & CAS-*TI Newsletter* will be.

Next issue: March 2010
Deadline 15 February 2010

Preview: Contributions waiting to be published

Some simulations of Random Experiments, J. Böhm, AUT, Lorenz Kopp, GER
Wonderful World of Pedal Curves, J. Böhm
Tools for 3D-Problems, P. Lüke-Rosendahl, GER
Financial Mathematics 4, M. R. Phillips
Hill-Encription, J. Böhm
Simulating a Graphing Calculator in *DERIVE*, J. Böhm
Henon, Mira, Gumowski & Co, J. Böhm
Do you know this? Cabri & CAS on PC and Handheld, W. Wegscheider, AUT
Steiner Point, P. Lüke-Rosendahl, GER
Overcoming Branch & Bound by Simulation, J. Böhm, AUT
Diophantine Polynomials, D. E. McDougall, Canada
Graphics World, Currency Change, P. Charland, CAN
Cubics, Quartics – interesting features, T. Koller & J. Böhm
Logos of Companies as an Inspiration for Math Teaching
Exciting Surfaces in the FAZ / Pierre Charland´s Graphics Gallery
BooleanPlots.mth, P. Schofield, UK
Old traditional examples for a CAS – what´s new? J. Böhm, AUT
Truth Tables on the TI, M. R. Phillips
Advanced Regression Routines for the TIs, M. R. Phillips
Where oh Where is IT? (GPS with CAS), C. & P. Leinbach, USA
Embroidery Patterns, H. Ludwig, GER
Mandelbrot and Newton with *DERIVE*, Roman Hašek, CZ
Snail-shells, Piotr Trebisz, GER
A Conics-Explorer, J. Böhm, AUT
Coding Theory for the Classroom?, J. Böhm, AUT
Tutorials for the NSpireCAS, G. Herweyers, BEL
Some Projects with Students, R. Schröder, GER
Runge-Kutta Unvealed, J. Böhm, AUT
The Horror Octahedron, W. Alvermann, GER
RK6, Heinrich Ludwig, GER

and others

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DERIVE and Windows 7

Peter Hofbauer, Horn, Austria

Hi Josef,

DERIVE is running under Windows 7 without any problems. The Online Help cannot be shown (like under Vista). But there is a patch available:

Download the respective patch from

http://www.microsoft.com/downloads/details.aspx?familyid=258AA5EC-E3D9-4228-8844-008E02B32A2C&displaylang=de

(for 64-bit or 32-bit operation systems).

Best regards
Peter

Note: Some screen resolutions may cause troubles (parts of the display are cut off and inserted at other places, …) Looks very strange. What to do? Decrease the resolution, then it should work properly.

Günter Schödl, Wr. Neusdtadt, Austria

Hallo Josef!

Derive runs properly under Windows 7. All what one has to do is installing a patch for the Online-Help. This can be found at www.microsoft.at under

KB917607
Windows6.1-KB917607-x64.msu for 64Bit Systems
Windows6.1-KB917607-x86.msu for 32Bit Systems

Best regards
Günter

Gerhard Hagen, Friesach, Austria

Dear all,
I ask for help to solve a "DERIVE-mystery". I found the following function in an interesting article of the "Scientific News" and wanted to reproduce it with DERIVE:

$$\Delta v(x)=\begin{cases}\sqrt{2a\cdot x} & 0\le x<\frac{a}{2}\cdot T_R^{\;2}\\ a\cdot T_R & \frac{a}{2}\cdot T_R^{\;2}\le x<v_0\cdot T_R-\frac{a}{2}\cdot T_R^{\;2}\\ \sqrt{2a\cdot v_0\cdot T_R-2a\cdot x} & v_0\cdot T_R-\frac{a}{2}\cdot T_R^{\;2}\le x<v_0\cdot T_R=d\\ 0 & d=v_0\cdot T_R\le x\end{cases}$$

```
        Δv(x, a, tr, v0) :=
          If 0 ≤ x < a/2·tr^2
             √(2·a·x)
             If a/2·tr^2 ≤ x < v0·tr - a/2·tr^2
#1:             a·tr
                If v0·tr - a/2·tr^2 ≤ x < v0·tr
                   √(2·a·v0·tr - 2·a·x)
                   If v0·tr ≤ x
                      0
```

This looks quite good, the "mystery" appears by substituting special values for the parameters, eg a = 7, tr = 1 and v0 = 10.

$$\#2:\quad \Delta v(x,\ 7,\ 1,\ 10) = \text{IF}\left(0 \le x < \frac{7}{2},\ \sqrt{(2\cdot 7\cdot x)},\ \text{IF}(7 \le x < 10 - 7,\ 7,\ \text{IF}(10 - 7 \le x < 10,\ \sqrt{(2\cdot 7\cdot 10 - 2\cdot 7\cdot x)},\ \text{IF}(10 \le x,\ 0)))\right)$$

Within the first IF-clause a/2 is correctly replaced by 7/2 but starting with the second IF all remaining a/2 are replaced by 7 (= a). Can you help me?

DNL: I don´t know why DERIVE is behaving so strange, but there is help. I rewrote the function - using an auxiliary variable – as follows. Now it works as expected.

```
      Δv(x, a, tr, v0, a_) :=
        Prog
          a_ := a/2
          If x < 0
             ?
             If x < a_·tr^2
#1:             √(2·a·x)
                If x < v0·tr - a_·tr^2
                   a·tr
                   If x < v0·tr
                      √(2·a·v0·tr - 2·a·x)
                      0
```

$$\#2:\quad \Delta v(x,\ 7,\ 1,\ 10) = \text{IF}\left(x < 0,\ ?,\ \text{IF}\left(x < \frac{7}{2},\ \sqrt{(2\cdot 7\cdot x)},\ \text{IF}\left(x < 10 - \frac{7}{2},\ 7,\ \text{IF}(x < 10,\ \sqrt{(2\cdot 7\cdot 10 - 2\cdot 7\cdot x)},\ 0)\right)\right)\right)$$

Judith Lindenberg, Austria

Dear Josef,

We define nv(x,m,s):= … the density function of the normal distribution and as pnv(m,s,a,b) the area under the density between a and b.

TI-Nspire has no problems solving (pnv(400, 10, 400-d,400+d)=0.95 , d) but it seems to be unable to solve for the mean: nsolve(pnv(m, 2, 495,∞)=0.95 , m). Any advice?

DNL:

Dear Judith,

Numerical solving is sometimes tedious – even for CAS (without any special tricks DERIVE needs 12.5 sec to find m = 498.2897). But you can assist your system by adding restrictions for the numerical search. There are some possibilities for TI-Nspire:

As 0.95 is greater 0.5 we should know that mean m is greater than 495. A rough estimation could be m ~ 500, so enter your guess as follows:

nSolve(pnv(m,2,495,∞)=0.95,m=500) or nSolve(pnv(m,2,495,∞)=0.95,m=600).

There is another – even easier way – remembering the wonderful |-operator from the TI-92/Voyage 200 you can enter:

nSolve(pnv(m,2,495,∞)=0.95,m)|m>495 (m > 400 does it, too).

Merry Christmas,
Josef

A TOOLBOX WITH DERIVE

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Abstract: An analysis of some of the characteristics of the European Higher Education Area (EHEA), its difference in relation to the current University teaching system and the role that new technologies might play in this new scenario has been performed. This paper suggests a new possibility in use of technologies: The design of a “toolbox” with DERIVE instructions about topics in a usual Calculus course.

Introduction

The implementation of the EHEA (see [1, 2]) implies new teaching methods taking into account that the students now are the centre of the learning process. The role of teachers changes and they must be able to guide their students’ work (see [6, 7, 8, 9]).

Teachers are currently being required to change the traditional teaching model in order to adapt to “learning based on competences”. It is necessary to define the competences to be acquired by the students after attending a course on a certain subject, and to design the activities according these competences.

Such change may fail if a considerable amount of effort, imagination, common sense, and hope is not devoted to it. The system inertia and the difficulty involved in designing effective activities must be borne in mind, since teachers generally have extensive experience in preparing expository lessons, with more or less of students, and although we have worked hard to find the best way of introducing and presenting concepts and results, we do not have any experience in guiding the search.

The new teaching model implies more autonomous work by students. To see this it is merely necessary to analyze the new structure of studies, which are articulated in 60 yearly ECTS (European Credit Transfer System) credits, where each ECTS credit must reflect between 25 and 30 hours of the students’ “overall work”, only 10 to 15 of which must necessarily involve classroom attendance.

This is why teachers find it challenging to design a mathematical course for engineering students (for example a course on Calculus of One Variable or a course on Linear Algebra) taking these determining factors into account.

In our opinion we are doomed to design mathematical courses in which *magisterial* lessons (theory and problems solved in detail by the teacher), *practical workshops* (problem solved and Mathematical laboratories based in a CAS), and *tutorial activities* must be blended so that students can acquire the required competences.

Laboratory classes must be designed with clear goals. Our proposal is that Computer Algebra System (CAS) could help in the automatic performance of certain tasks involved in the problem-solving process. For this, it would sometimes be necessary to use certain functions or commands which might already be integrated to the system or which might have been prepared by the teacher, or even created by the students themselves.

Working in a more autonomous way allows students to access computing technologies outside conventional training, which is why it is essential that they acquire the skills to make optimum use of them. The advantages of CAS must be boosted: visualization, computation facilities, the possibility of experimenting…, avoiding possibly damaging effects such as the lack of a critical attitude when considering the computer response, an inability to interpret the results, etc.

In any case, in order to be effective all the activities suggested must be designed without letting the intended goals out of sight, mainly taking into account the students at whom they are aimed.

1. A box of mathematical tools

Mathematical subjects, which are usually programmed within the first years of Engineering studies, have as their main goal the initiation of students into the language of Science and Technology and their preparation in the correct use of certain algorithms in problem solving.

Teachers often complain that outside the context of the subject (for example, in later years) students are not able to use the mathematical skills acquired during the basic years of their training. They seem to have a kind of "mental laziness" that prevents them from remembering and using what they have learnt, and in many cases they do not have a fast and effective way to find the information or the appropriate methods either.

The use of technology in the classroom is increasing and in certain cases students are even asked to define some tools that allow them to automate certain simple tasks, such as for example the implementation of a function to calculate the tangent to a curve $y = f(x)$ at a point and subsequently use it in other problems. In general, however, when teachers suggest to students that they should define a method aimed at automating a mathematical task they merely wish to help them understand the corresponding algorithm and they do not usually expect students to design their own resources or use the implemented functions in following years.

One educational activity is to encourage students to create their own well-organized “Toolbox” for solving mathematical problems. This toolbox is no more than a file or collection of files of utilities, programmed in the characteristic programming language of a CAS (Derive, Maple, Maxima) or even on a calculator with symbolic or graphic capacities such as the TI92 or Casio ClassPad 300.

The teacher should suggest a series of essential tools, depending on the corresponding subject, that the box should contain. Students must to define the corresponding functions, test them, and add those they find appropriate. In addition, they must complete their work by writing a “brief user’s manual” for their tools.

If they have created a good toolbox, and they have also understood the algorithms well, they will then have their own resource, which they will be able to use in other subjects during the same or following years.

Before finishing with the general ideas on toolboxes it should be noted that nearly all CAS offer a very thorough toolbox and that the creation of new tools, or the modification of those already available, will only be necessary or convenient for reasons of ease of use or as a teaching strategy in order to adapt them to the user’s specific needs.

2. A toolbox of Calculus

In this section we propose a toolbox that can be done by students of Calculus in the first course of Engineering, whose reference text book may be [3 or 5].

Before present our toolbox, is necessary to make it clear that in DERIVE are implemented the most of the instructions for the study of Calculus of One Variable. Thus, the calculation of limits, derivatives, integrals or Taylor polynomials is “basic” using the DERIVE menu or instructions.

Following the strategy referred to in the previous section, a *supplementary* toolbox has been created that can be used in a *more specific way*.

With the toolbox our aim is to extend the use of the CAS, in our case DERIVE, and we therefore try to use, as far as possible, its symbolic, numerical, and graphic capacities.

Some of the utilities are “improvements” to DERIVE instructions (for example the **TANGENT2** method, where a distinction is made between differentiable and non- differentiable functions). Other tools are analogous to known instructions integrated in DERIVE, whose syntax is complicated. For this reason we prefer students to automate the algorithms according to their own criterion, so that they get a more continuous use when they have to apply these concepts in other topics. In addition, this automation “guarantees” that they have understood the corresponding concepts.

Our toolbox could be designed with several compartments, one for each section of Calculus of One Variable course:

- Complex Numbers.
- Limits and Continuity.

- Differentiability.
- Integral Calculus (including numerical integration).
- Numerical Methods (for solving nonlinear equations).
-

Below we briefly explain the tools.

2.1. Complex Numbers

The Complex Number compartment includes tools for:

- Convert a complex number to exponential form.
- Plot a complex number as a pair of real numbers.
- Compute a list with the nth roots of a complex number.
- Find the n vertices of a regular polygon.

The following example allows one to calculate the nth roots of a complex number and, taking advantage of the graphic capacities of DERIVE, plot a start.

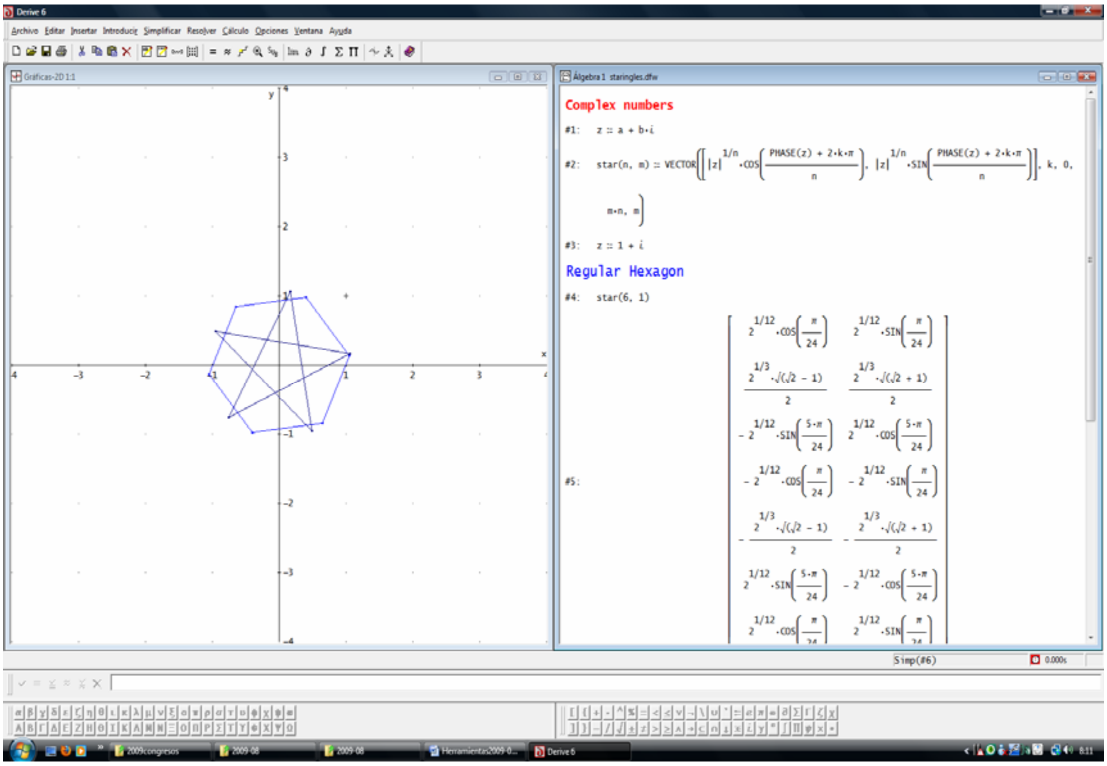


Figure 1: A regular hexagon with a six-pointed start

2.2. Differential calculus

The utilities performed allow the analysis of the continuity or discontinuity of the function at a point (studying the one sided limits, analyzing the equality of the obtained values and comparing with the value of the function at the point). The differentiability of a function at a point is studied following the same strategy.

The calculation of the tangent line has been implemented, with additional information to the **TANGENT** function of DERIVE, since it tells us when the function is not differentiable at the point (see figure 2).

```
The tangent

#1:   f(x) :=

                      f(a + h) - f(a)
#2:   DL(a) := lim    ───────────────
               h→0-          h

                      f(a + h) - f(a)
#3:   DR(a) := lim    ───────────────
               h→0+          h

      TANGENTE2(a) :=
        If DR(a) - DL(a) = 0 ∧ (1/DR(a))^2 > 0
#4:        y = f(a) + DR(a)·(x - a)
           "f has no derivative"

      f(x) :=
        If x ≤ 0
#5:        x
           x^2

#6:   TANGENTE2(0)

#7:                                                   f has no derivative

With the DERIVE instruction TANGENT the result is not correct

#8:   TANGENT(f(x), x, 0)

#9:                                                             x
```

Figure 2: TANGENT and TANGENT2 functions

Taking advantage of DERIVE graphical capacities and of the structure of the IF instruction, the monotonicity and concavity or convexity of a sufficiently differentiable function can be analyzed.

For example, with the instruction **CRECE(x):= IF(F'(x)>0,F(x))** it is possible to represent the curve $y = F(x)$ in the intervals where F is increasing.

The graph of figure 3 has been obtained by applying the instruction **CRECE(x)** with $F(x) = x^3 - x$

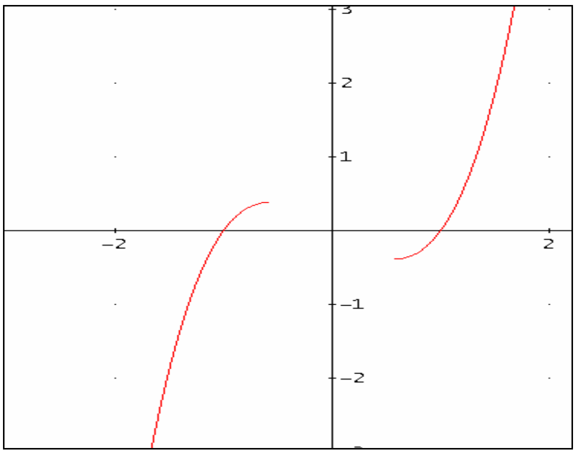


Figure 3: Intervals where $F(x) = x^3 - x$ is increasing

Other functions about Differential Calculus in the toolbox may be:

ROLLE_POINT for finding a point according Rolle's Theorem hypotheses.

LAGRANGE_POINT for finding a point according Lagrange's Theorem hypotheses.

These functions can work in exact or approximate way, because it is necessary solving equations.

2.3. Integral calculus

In order to introduce the Riemann integral we can use (see [4]) the DERIVE instructions: **AreaUnder–Curve** and **LEFT_RIEMANN**. Furthermore we have included in the toolbox the **DER_RIEMANN** for calculating the sum for right rectangles associated to Riemann sums. We also have implemented the **rect_izq** and **rect_der** functions for plotting the left and right rectangles associated to Riemann sums

We promote our students to define simple instructions for computing lengths, areas and volumes with a syntax more recognizable that **POLAR_ARC_LENGTH, PARA_ARC_LENGTH, POLAR_AREA, VOLUME_OF_REVOLUTION, AREA_OF_REVOLUTION,** etc.

For instance, to calculate the length of an arc of the curve $y = f(x)$, it is possible to define the function **LEXP** and theoretically find the length of any arc of curve.
The figure 4 shows the DERIVE implementation and the calculation of the parabola's length $y = x^2$ between the abscissas 1 and 2.

#1: f(x) :=

#2: $\text{LEXP}(a, b) := \int_a^b \sqrt{1 + f'(x)^2}\, dx$

The parabola length

#3: $f(x) := x^2$

#4: LEXP(1, 2)

#5: $$\frac{\text{LN}(\sqrt{85} - 2\cdot\sqrt{17} + 4\cdot\sqrt{5} - 8)}{4} + \sqrt{17} - \frac{\sqrt{5}}{2}$$

And the approximate value

#6: 3.167840904

Figure 4: The length of an arc of curve

Finally, students can define tools for numerical integration using the Composite-Trapezoidal rule and Composite-Simpson rule (see figure 5).

#1: f(x) :=

$$\#2:\quad \mathrm{TRAP}(a, b, n) := \frac{b-a}{n}\cdot\left(\frac{f(a)+f(b)}{2} + \sum_{i=1}^{n-1} f\left(a + \frac{i\cdot(b-a)}{n}\right)\right)$$

$$\#3:\quad \mathrm{SIMPSON}(a, b, n) := \frac{b-a}{3\cdot n}\cdot\left(f(a)+f(b)+4\cdot\sum_{i=1}^{n/2} f\left(a+\frac{(2\cdot i-1)\cdot(b-a)}{n}\right)+2\cdot\sum_{i=1}^{n/2-1} f\left(a+\frac{2\cdot i\cdot(b-a)}{n}\right)\right)$$

$$\#4:\quad f(x) := \frac{x\cdot e^{-2\cdot x}}{x^2+4}$$

$$\#5:\quad \int_{-1}^{1} f(x)\, dx$$

$$\#6:\quad \int_{-1}^{1} \frac{x\cdot e^{-2\cdot x}}{x^2+4}\, dx$$

#7: [TRAP(-1, 1, 10), SIMPSON(-1, 1, 10)]

#8: [-0.4342879366, -0.4215187544]

Figure 5: Numerical Integration

2.4. Numerical calculus for solving nonlinear equations

The methods usually taught to students for solving equations are: Bisection, Newton and fixed-point. DERIVE has the **NEWTON** and **FIXED_POINT** algorithms integrated, so we can propose that students define a function to implement the bisection method. An algorithm for this method could as follows:

```
H(a, b) :=
  If F((a + b)/2)·F(a) < 0
     [a, (a + b)/2]
     If F((a + b)/2)·F(b) < 0
        [(a + b)/2, b]
        [(a + b)/2, (a + b)/2]

BISECC(a, b, n) := ITERATES(H(v , v ), v, [a, b], n)
                               1   2
```

Figure 6: Bisection Method

The instruction **BISSEC(a,b,n)** (see figure 6) computes the interval obtained after n iterations of Bisection method, to solve the equation $f(x) = 0$ in the interval $[a,b]$.

3. Using the toolbox to solve technical problems

Students are encouraged to use the toolbox to solve technical problems. Here we set out one example, taken from an Environmental Sciences exam that has been used by our engineering students.

Specifically, the equation to be solved (which appears together with its solution in the figure 6) provides the height of the chimney of a thermoelectrical station in a rural area with the required quality standards.

"

Numerical methods

```
#1:   F(x) :=

#2:   PUNTO_FIJO(x0, n) := ITERATES(F(x), x, x0, n)

#3:   F(x) :=
```

#4: $\mathrm{NW}(a, n) := \mathrm{ITERATES}\left(x - \dfrac{F(x)}{F'(x)}, x, a, n\right)$

```
      H(a, b) :=
        If F((a + b)/2)·F(a) < 0
           [a, (a + b)/2]
#5:        If F((a + b)/2)·F(b) < 0
              [(a + b)/2, b]
              [(a + b)/2, (a + b)/2]
```

#6: $\mathrm{BISECC}(a, b, n) := \mathrm{ITERATES}(H(v_1, v_2), v, [a, b], n)$

Taken from an exam in my University (April 2009).
Topic: Environmental techniques

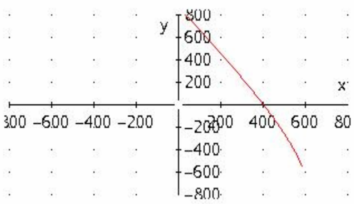
The goal: To find the minimum height of a chimney of a thermoelectric central in a rural area to be concordant with the quality environmental standards. "

"

The equation to be solved for finding the height:
H=1.6*575.29^(1/3)(10(588-H)^(2/3))/5

#7: $F(x) := \dfrac{1.6 \cdot 575.29^{1/3} \cdot (10 \cdot (588 - x))^{2/3}}{5} - x$

#8: $F(H) := \dfrac{1.6 \cdot 575.29^{1/3} \cdot (10 \cdot (588 - H))^{2/3}}{5} - H$



Sketch the graph to approach the initial values of bisection method:

#9: BISECC(200, 500, 10)

#10: $\begin{bmatrix} 200 & 500 \\ 350 & 500 \\ 350 & 425 \\ 387.5 & 425 \\ 387.5 & 406.25 \\ 396.875 & 406.25 \\ 401.5625 & 406.25 \\ 401.5625 & 403.90625 \\ 401.5625 & 402.734375 \\ 402.1484375 & 402.734375 \\ 402.1484375 & 402.4414062 \end{bmatrix}$

Figure 7: The height of a chimney

For engineering purposes the chimney height is around 400 meters.

Conclusion

The use of a toolbox adapted to each student's individual needs containing instructions that the students themselves find useful must, we believe, surely reinforce the learning process, since students participate actively and the use of the CAS is not limited to what is sometimes "blind" use of the CAS instructions.

References

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Contents of the Toolbox

complex numbers.dfw

staringles.dfw

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An Implementation of the Mayan Numbering System in *DERIVE*

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Abstract

The Mayan number system is a base *20*, positional (to be read from top to bottom, not from left to right) system that makes use of a symbol representing zero. It has slightly different variations when used for counting days (in religious and astronomical contexts). Therefore, *20* symbols are needed to represent *0,1,2,…,19*. Of these, the zero was denoted by a shell and the positive ones were represented using dots and horizontal segments. If a number is greater than *20*, the symbols corresponding to units, twentieths, *400's*, *8000's…* are stacked from bottom (units) to top in pure base *20*. We have implemented a procedure that allows to convert numbers between any bases, and that returns the output in (row) vector style; another procedure that builds the *20* Mayan symbols for *0,1,2,…19*; and yet another procedure, that uses the previously mentioned procedures, converts any number from base *10* to base *20* and represents it in the Mayan numbering system. We believe this is an interesting example of synergy among different branches of knowledge (Mathematics, History of Mathematics and Computer Science), that can increase the interest of the students for different topics.

1. Introduction

Let us try to briefly describe the evolution of the Spanish educational system at Secondary Education as regarding Mathematics. Until the mid *70*'s, the curriculum focused on classic plane geometry. After the arrival of *Modern Mathematics*, set theory, relations, correspondences, algebraic structures, number base changes, etc. were studied instead. Now the curriculum of 10-16 year old students focuses on practice and experimentation, oriented to achieve *basic skills*.

Nowadays, possibly as a consequence of an incorrect implementation of the latter ideas, many freshmen at university do not know or do not master basic concepts, procedures and algorithms. Some Schools, like the School of Mathematics of our University, have introduced "0 courses" devoted to those students that can't follow the subjects of the first year at university.

For example, they can't perform number base changes, because they have never studied this topic.

We propose here a unit at Teacher Training level organized in three sessions of 1.5 hours about number base changes. Here, the Mayan numbering system is introduced as a historic justification for the need of working in other bases. Another justification is the use of base 2 (and, indirectly, base 16) by computers. Therefore, this is a multidisciplinary unit, involving Maths, History and Computer Science. The working scheme for the unit is the following:

Studying the problem (including the historic antecedents)

↓

Translating the problem into mathematical language (mathematization)

↓

Solving the problem

↓

Implementing the solution found

There is a didactic antecedent to this unit, at a lower educational level and without using technology: (Roanes, 1971). We do not know of any other similar implementations in any Computer Algebra System (CAS), apart from (URL, 2009), that uses special facilities for inserting captured drawings and doesn't have a didactic orientation, or the very similar (Roanes Lozano & González Redondo, 2009), by the same authors of this paper, but written in the CAS *Maple* (that was successfully experimented during the 2008-2009 course).

2. The Mayan Numbering System

The Mayan civilization developed a characteristic world of astronomical and mathematical knowledge in (and from) their temples, the most appropriated place for a political system organized as priestly autocracy which based its power upon Astrology.

Without any observational instrument (they did not know glass and, consequently, could not manufacture lenses), without sand or water clocks for computing periods of time as hours, minutes or seconds, their smallest unit of time was the *kin* or day, whose duration was measured through the repeated registration of the shadows projected by a wooden bar placed vertically on the ground, known as the *gnomon* (also used by the Greeks for determining the azimuth and the height of the sun).

Taking the day as basis, the subsequent units of time where:

- the *uinal* or month of 20 *kins* (days),
- the *tun* or year of 18 *uinals* ($18 \cdot 20 = 360$ days),
- the *katun* or period of 20 *tuns* ($20 \cdot 360 = 7200$ days),
- the *baktun* or period of 20 *katuns* ($20 \cdot 20 \cdot 360 = 144.000$ days),
- the *pictun* or period of 20 *baktuns* ($20 \cdot 20 \cdot 20 \cdot 360 = 2.880.000$ days),
- etc.

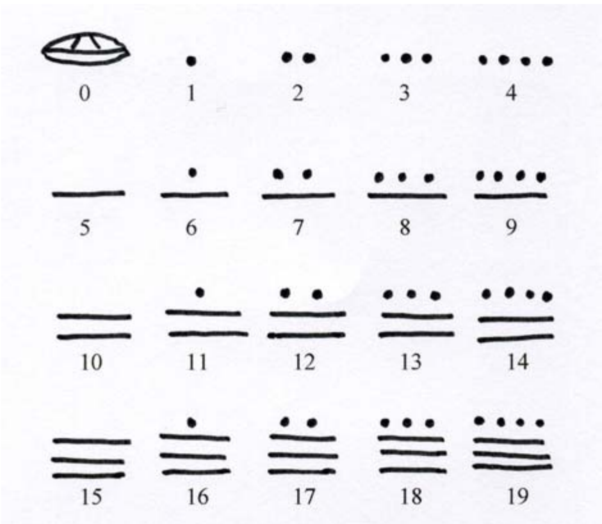
As it seems obvious, Mayan astronomers knew the corrections that had to be made after considering a year of 360 days instead of the solar year of 365.25 days (Grube, 2000).

With this system, and counting long periods of time, they estimated the phases of Venus (which they rounded up to 584 days), Mercury and Mars; the duration of a lunar month in what today we would write as 29.53020 days; and they even registered periods up to 300.000.000 years. We can confirm all these through their commemorative steles and the pages of the few Mayan books (*codices*) which have being preserved (Landa, 1937), where we find an original numbering system with three basic features: it had a base 20, used the place-value system and introduced the zero.

In fact, in association to its religious origin, numerical quantities registered by the Mayas where strictly joined to the god corresponding to each order: the god carrying *kins*, the god carrying *uinals*, the god carrying *tuns*, etc. Their conception of the "zero" was based upon this religious nature: it was a symbol created in order to fill any possible gap in the place where some numerical quantity should be supported by the god corresponding to the said order (Ifrah, 1999).

In short, this system conceived at the temples was perfect for representing dates (elapsed days), prepare ritual celebrations, etc., although with it no arithmetical operation could be performed. For such operations which were well apart from religious rites and more related to administration and commerce, the Mayas put aside the irregular *tun* (year) of 360 *kins*, assuming the natural order $20 \cdot 20 = 400$ proper of a base 20 numbering system.

While in our usual base 10 system we need nine numerals (and the zero), for writing their numerals the Mayas needed nineteen numerals. But the symbols used for each of these nineteen units were very simple: dots and horizontal lines (see Figure 2).



Palenque, Temple of Inscriptions[1]

Figure 2: Mayan numerals.

When we write today a positive four-numerals integer n in our usual base 10 system, $n = wxyz$, where w, x, y, y z can vary from 1 to 9 (or 0), we are really shortening the expression:

$$n = w \cdot 10^3 + x \cdot 10^2 + y \cdot 10^1 + z \cdot 10^0$$

For writing any number greater than nineteen, the Mayas placed the symbols referred in a vertical column containing as many rows as orders of units, including a very special symbol (reminding us of a shell) for filling up an order that could not be left empty of dots or lines:

[1] The pictures were taken at my 3 weeks travel through Mexico in November 2009, Josef.

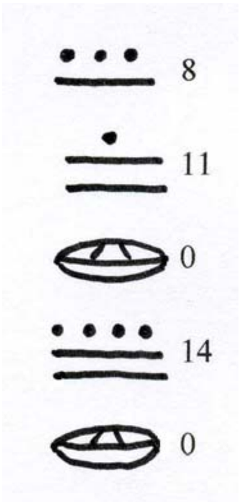
the Mayan zero. Through this notation the Mayas could write as follows what in our system would be a number m:

$$m = \ldots + w \cdot 20^3 + x \cdot 20^2 + y \cdot 20^1 + z \cdot 20^0$$

For example, the number:

$$1{,}368{,}280 = 8 \cdot 20^4 + 11 \cdot 20^3 + 0 \cdot 20^2 + 14 \cdot 20^1 + 0 \cdot 20^0$$

was written using dots, lines and shells as it is shown in Figure 3.



Edzna, Temple of the five Stores

Figure 3: Example of Mayan numerals.

For those interested in further mathematical developments of the Mayas see the book by Romero (2004).

3. What *DERIVE* does about number base changes

Regarding number base changes, the CAS *DERIVE* only provides the possibility to choose the input and output bases (from *2, 8, 10* and *16*) in:

Options > *Mode Settings*

Therefore we'll try to complete its possibilities implementing the corresponding procedures so that it can convert any given integer number into Mayan notation. The mathematical background of each procedure is previously taught or retraced, and the students are guided through the development of the implementations (these details are omitted here for the sake of brevity). As said above, this unit was successfully experimented during the 2008-2009 course with the CAS *Maple*.

4. A *DERIVE* implementation for number base changes

It is straightforward to implement the auxiliary procedure (or function) “integer quotient” in *DERIVE*. As *DERIVE*’s command `MOD` returns the remainder of the integer division, we can follow the integer division definition:

```
              a - MOD(a, b)
quo(a, b) := ---------------
                    b
```

Now the previous procedure can be used as subprocedure by a short recursive procedure that converts numbers from base 10 to any base, and returns the output in (row) vector style:

```
base_bl(n, b) :=
  If n < b
     [n]
     APPEND(base_bl(quo(n, b), b), [MOD(n, b)])
```

Examples:

```
base_bl(399, 20) = [19, 19]
base_bl(400, 20) = [1, 0, 0]
```

5. A *DERIVE* implementation of the Mayan numerals

Representing the Mayan numerals (0 to 19) is tricky and the less interesting procedure from the mathematical point of view.

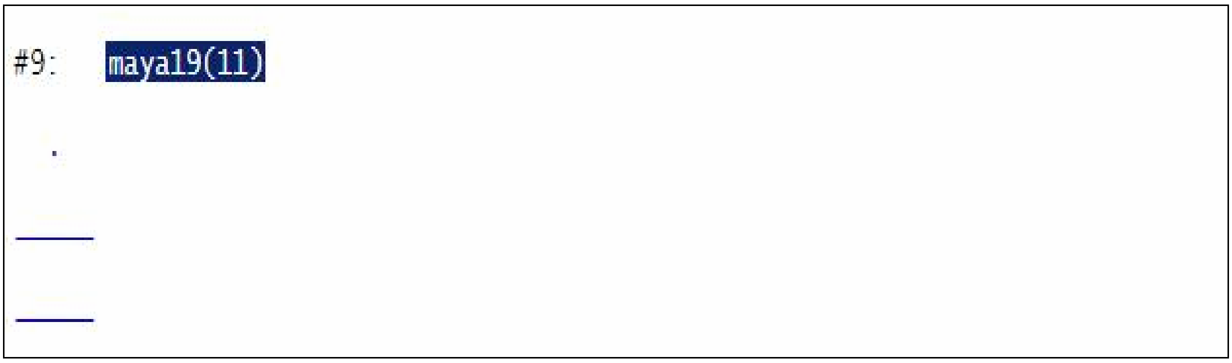
```
maya19(n) :=
  Prog
    If n = 0
       DISPLAY("  θ  ")
       If n = 1
          DISPLAY("  .  ")
          If n = 2
             DISPLAY(" ..  ")
             If n = 3
                DISPLAY(" ... ")
                If n = 4
                   DISPLAY(" ....")
                   If n = 5
                      DISPLAY("_____")
                      maya19_aux(n)
    RETURN " "
```

```
maya19_aux(n) :=
  Prog
    If MOD(n, 5) > 0
       maya19(MOD(n, 5))
    If quo(n, 5) = 1
       maya19(5)
       If quo(n, 5) = 2
          maya_10
          If quo(n, 5) = 3
             maya_15
    RETURN " "

maya_10 := PROG(DISPLAY(_____), DISPLAY(_____))

maya_15 := PROG(DISPLAY(_____), DISPLAY(_____),
                DISPLAY(_____))
```

Example:



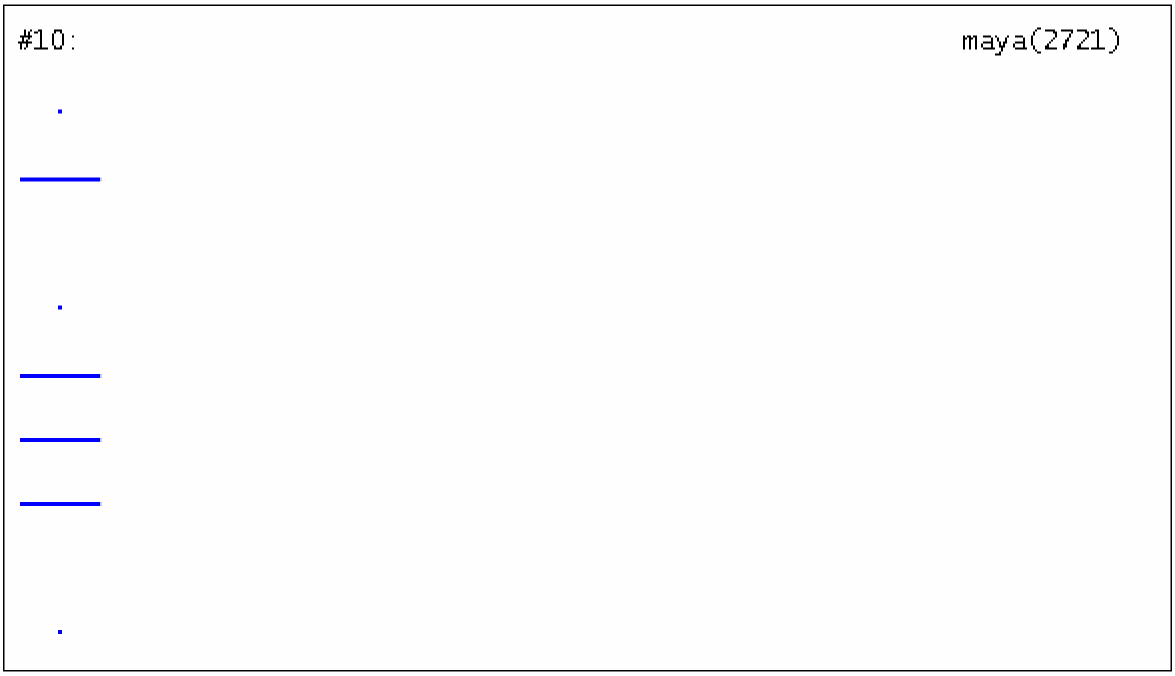
6. A *DERIVE* implementation of a Mayan number system converter

Once we have the procedures of Sections 4 and 5, it is very easy to implement a Mayan number system converter. The idea is to convert first the given number to base 20, to transform the numerals to Mayan notation, and, finally, to stack these numerals.

```
maya(n) :=
  Prog
    maya_aux(base_bl(n, 20))
    RETURN " "

maya_aux(R) :=
  Prog
    If R ≠ []
       maya19(R↓1)
    If R ≠ []
       DISPLAY(" ")
    If R ≠ []
       maya_aux(REST(R))
    RETURN " "
```

Example: $2721 = 6 \times 20 \times 20 + 16 \times 20 + 1$



7. Acknowledgments

This work was partially supported by the research projects *TIN2006-06190* (Ministerio de Educación y Ciencia, Spain) and *UCM2008-910563* (*UCM - BSCH Gr. 58/08*, research group *ACEIA*, Spain).

8. Conclusions

We do not know of any other similar implementations in any CAS, apart from (URL, 2009), that uses special facilities for inserting graphics, and the very similar (Roanes-Lozano & Gonzalez-Redondo, 2009), written in *Maple*.

We believe this is an interesting example of synergy among different branches of knowledge (Mathematics, History of Mathematics and Computer Science), that can increase the interest of students for different topics.

The only negative issue we find is that parts of the implementation are a bit tricky (Section 5), something that can distract students from the mathematical goals of the unit. Perhaps this part could be provided to the students implemented beforehand.

The "Observatory", Chichén Itzá

Uxmal, Temple of the Wizard

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These are the Spanish presenters of the ACA09 – DERIVE session.

Eugenio Roanes Lozano

José Luis Galan

Agustín de la Villa

Pedro Rodrigues Cielos

The second part of the Gallery is on page 48.

CAS-Tools for Exercising

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Abstract

Even in times of a CAS a lot of basic manipulating skills – up to a not commonly agreed amount – seem to remain necessary in mathematics education. The teacher can provide programs and functions to encourage self responsible learning, training and repeating these skills. It is the teacher – and sometimes the curriculum and the given "standards" – which level of difficulty is appropriate.

I will show some examples reaching from calculating GCD and LCM via factorizing and expanding expressions up to basics of calculus using various computer algebra systems. This could raise a discussion about possible fields in math curriculum which might be suitable for CAS-supported exercising. And this could also raise a discussion about advantages and disadvantages of using these or similar tools.

Let me start with a view "some years" back. It was in 1994 when I received a diskette from my friend Jan Vermeylen (from Belgium) – no email at these times– containing three files[10]:

`QUADR_EQ.MTH`

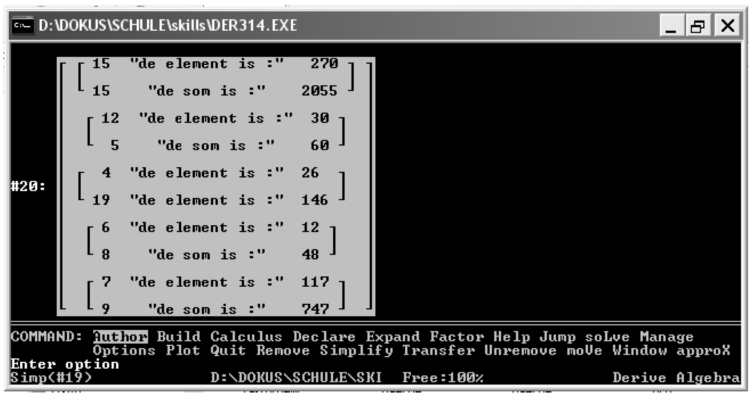
`POLY_FAC.MTH`

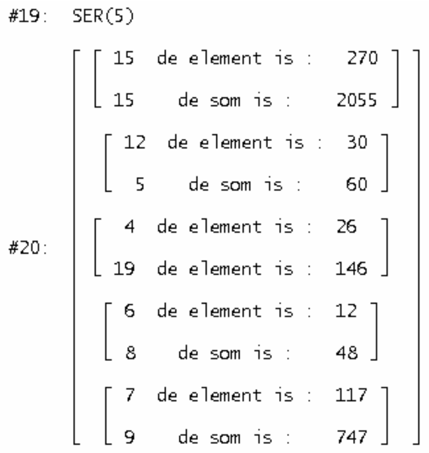
and

`REKRIJ.MTH`

QUADR_EQ	1 KB	MTH-Datei	16.07.1994 15:02
REKRIJ	2 KB	MTH-Datei	14.07.2008 10:35

Let´s try if REKRIJ works with the recent version of DERIVE, too:





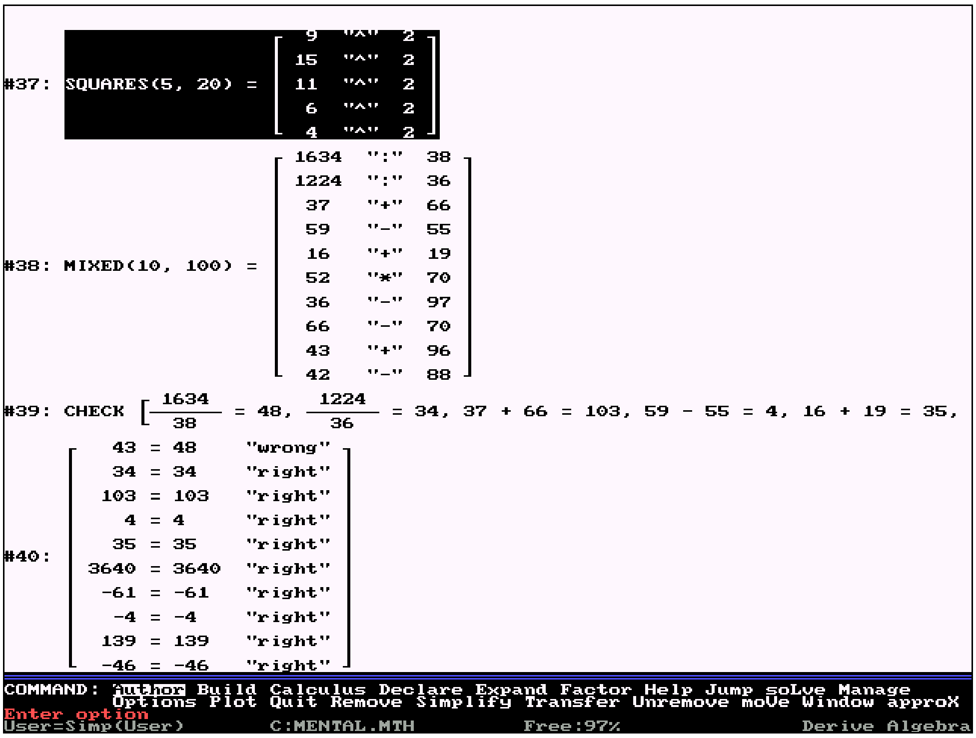
REKRIJ with DERIVE 3.14 and with DERIVE 6.1

I left the file in its original Flemish (= Dutch) version. All problems are connected with arithmetic series. Examples like these could and still can be found in nearly all textbooks.

The first example should be read as: the 15th element of an arithmetic series is 270 and the sum of the first 15 elements is 2055. Find the series (1st element and difference).

Solutions were not provided. I will come back to a modern version of this tool later in the paper.

There was another attempt providing training tools for elementary calculation skills presented by Heinz Rainer Geyer[12]:



Sometimes I like to reanimate my old DOS-versions of DERIVE.

Allow a time step to the handheld devices:

My school was the first College for Business Administration in Austria where CAS was used in all classes. When the TI-92 appeared on the market we – the math teachers – asked the parents to buy a device for their children for enabling a modern and attractive math education. It was in these times when I remembered another old BASIC-program which was later replaced by a DERIVE tool for exercising solving quadratic equations applying the rule of Vieta[2, 3].

$\text{vieta}() = (z^2 - 14\cdot z = -13)$

ant(1, 13) = richtig!

$\text{vieta}() = (z^2 + 5\cdot z = 0)$

ant(0, -5) = richtig!

$\text{vieta}() = (i^2 - 19\cdot i = -48)$

ant(-3, -16) = falsch, sondern: 3, 16

$\text{vieta}() = (v^2 + v = 72)$

ant(8, -9) = richtig!

$$\text{results}() = \begin{bmatrix} \text{Aufgaben gestellt:} & 4 \\ \text{davon richtig:} & 3 \\ \text{das sind in \%:} & 75 \end{bmatrix}$$

And here is the TI-Version:

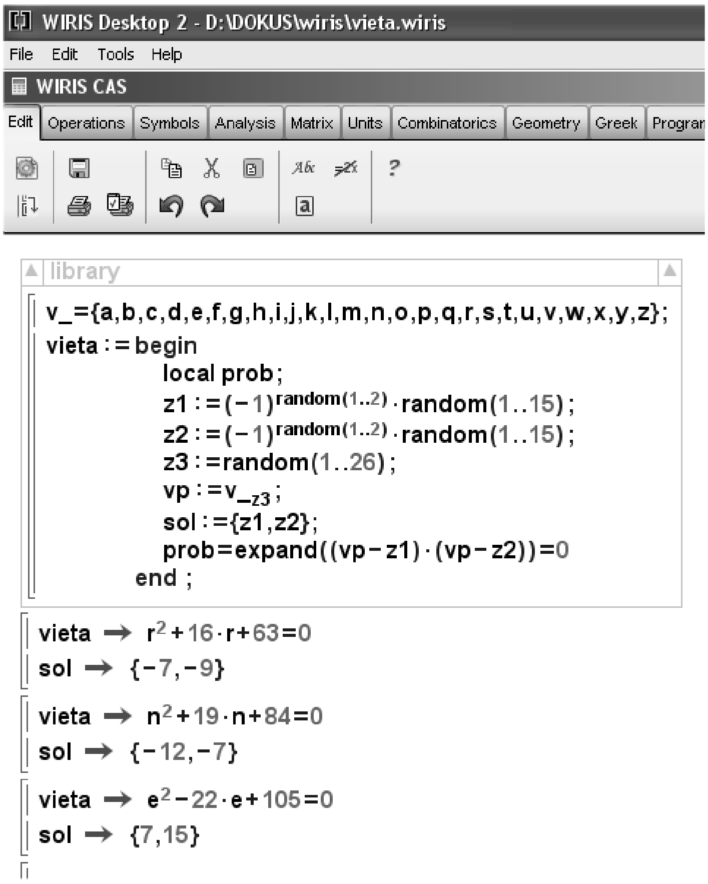
```
Problems: 1, correct 1
g^2 + 18·g + 80 = 0
1. Solution:
-10
2. Solution:
-8
right
End = ESC. next = any
MAIN    DEG AUTO    FUNC 0/30    BUSY
```

```
Problems: 3, correct 3
c^2 - 10·c - 11 = 0
1. Solution:
-11
2. Solution:
1
sorry, false   x1 = 11, x2 = -1
End = ESC. next = any
MAIN    DEG AUTO    FUNC 0/30    BUSY
```

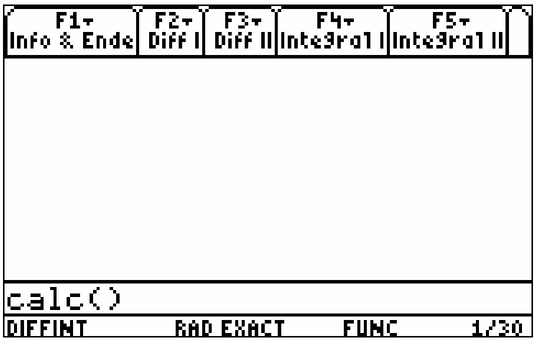
The idea remained the same, the tools changed. The pupils used the program during the breaks in school, often in the bus or railway on their way to or from school and sometimes only just for fun. One female pupil told me that she used her "VIETA" to concentrate before learning for other subjects.

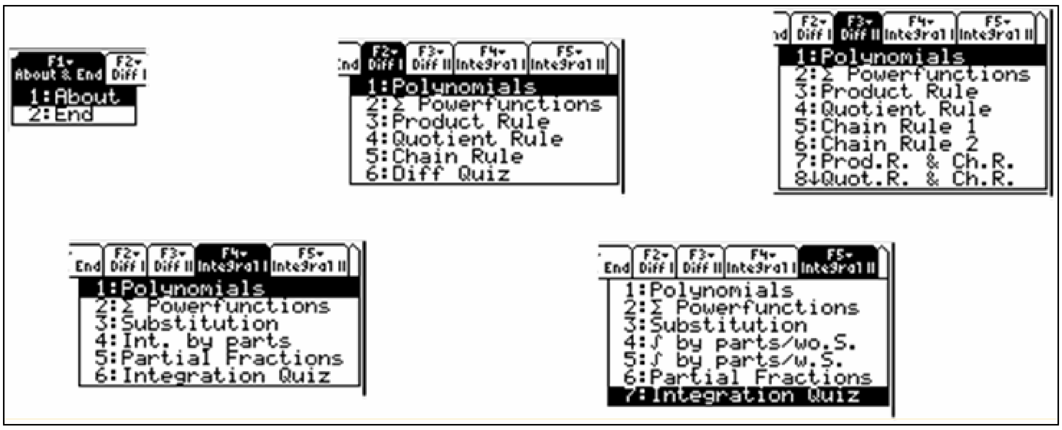
Just to demonstrate that the basic ideas can be realized using (almost) all tool I tried to program my “VIETA” with WIRIS, a CAS which recently has become popular in some Austrian schools. It is not the tool which decides what can be done, it is always the user (= teacher) who forms the software according to his or her didactical intentions and needs.
See the WIRIS screen shot on the next page.

I´d like to know in how many CA-systems I will program VIETA in the future?



Let´s have another jump: Even in times of CAS the basic rules of differentiation and integration should be mastered by the students. **Calc()** – a program package for the TI-handheld devices CAN help[1].
It is in your responsibility as teacher up to which level you think that the manipulating skills in calculus are indispensable.





All tasks are random generated. I´ll demonstrate the basic idea by using the respective DERIVE tool which was created when we had the opportunity to teach math in the PC lab working with DERIVE.

I provided one program for training the differentiation rules and another one for exercising the integration rules. Both of them are offering two levels of difficulty. This is the "start"-screen for integration.

```
2 levels of difficulty: (1) and (2):

Level 1:

pol1= ... polynomials,             pf1= ... Σ of power functions

subst1= ... Int. by substitution,  part1= ... Int. by parts

pfr1= ... partial fractions,       quiz1= ... random problem

n tasks of same type: pol1b(n), pf1b(n), ..., quiz1b(n)


Level 2:

pol2= ... polynomials,             pf2= ... Σ of power functions

subst2= ... Int. by substitution,  part2= ... Int. by parts

parts= ... Int. parts & inc. substitution

pfr2= ... partial fractions,       quiz2= ... random problem

quiz= ... random problem out of all levels

n tasks of same type: pol2b(n), pf2b(n), ..., quiz2b(n)


check(Integral)= or check([Int1; Int2; ...]) gives the answer (correct/true).

ans= returns the correct antiderivative
```

Let's have 5 integrals of level 1 to exercise integration by substitution together with the correct answers. The students should do the integration by hands and then double check their results.

$$\#48:\quad \text{subst1b(5)} = \begin{bmatrix} \dfrac{99\cdot 2^{1/3}}{16\cdot(33\cdot x + 7)^{2/3}} \\ 20\cdot e^{4\cdot x - 2\cdot x^2}\cdot(1 - x) \\ \dfrac{4\cdot(64\cdot x + 15)\cdot \text{COS}\left(\dfrac{16\cdot x^2}{3} + \dfrac{5\cdot x}{2} + \dfrac{7}{5}\right)}{9} \\ \dfrac{7\cdot(94\cdot x + 27)\cdot \text{COS}\left(\dfrac{47\cdot x^2}{6} + \dfrac{9\cdot x}{2} - 1\right)}{12} \\ 20\cdot \text{COS}(16\cdot x) \end{bmatrix} \qquad \#49:\quad \text{ans} = \begin{bmatrix} \dfrac{9\cdot 2^{1/3}\cdot(33\cdot x + 7)^{1/3}}{16} \\ 5\cdot e^{4\cdot x - 2\cdot x^2} \\ \dfrac{8\cdot \text{SIN}\left(\dfrac{16\cdot x^2}{3} + \dfrac{5\cdot x}{2} + \dfrac{7}{5}\right)}{3} \\ \dfrac{7\cdot \text{SIN}\left(\dfrac{47\cdot x^2}{6} + \dfrac{9\cdot x}{2} - 1\right)}{2} \\ \dfrac{5\cdot \text{SIN}(16\cdot x)}{4} \end{bmatrix}$$

The next figure will show a level 2 quiz (5 problems). This is a random selection of all possible problems.

When I was a teacher and needed problems for a test I did not prepare the problems at home but I ran my problem generator files and eg simplified twice **quiz1b(10)** for two groups of students in the class. So they could not blame me for providing extra difficult tasks for the test – the computer made the selection.

$$\#54:\quad \mathrm{quiz2b}(5) = \begin{bmatrix} 15\cdot\sqrt{x} + \dfrac{1}{4\cdot x^2} \\[2ex] \dfrac{12\cdot x}{5} \\[2ex] 3\cdot x^3 + \dfrac{13}{2\cdot x^{2/3}} \\[2ex] \dfrac{119\cdot \mathrm{COS}\left(\dfrac{17\cdot x}{6}\right)}{54} \\[2ex] \dfrac{56\cdot x^5}{5} - 6\cdot x^4 - \dfrac{x^3}{5} - \dfrac{7\cdot x^2}{5} - \dfrac{5\cdot x}{4} + \dfrac{1}{3} \end{bmatrix}$$

$$\#56:\quad \mathrm{check}\begin{bmatrix} \dfrac{15\cdot x^{3/2}\cdot 2}{3} - \dfrac{1}{4\cdot x} \\[2ex] \dfrac{6\cdot x^2}{5} \\[2ex] \dfrac{3\cdot x^4}{4} + \dfrac{39}{2}\cdot x^{1/3} \\[2ex] \dfrac{\dfrac{119}{54}\cdot 6}{17}\cdot \mathrm{SIN}\left(\dfrac{17\cdot x}{6}\right) \\[2ex] \dfrac{56\cdot x^6}{6} - \dfrac{6\cdot x^5}{5} - \dfrac{x^4}{20} - \dfrac{7\cdot x^3}{15} - \dfrac{5\cdot x^2}{8} + \dfrac{x}{3} \end{bmatrix} = \mathrm{wrong}$$

Do you find the mistake?

DERIVE has a wonderful feature – its "*STEPWISE SIMPLIFICATION*". Unfortunately no other CAS which is on the market now has a similar option. If the student has a wrong integral and he/she does not know how to obtain the right answer then he/she can ask DERIVE to *stepwise integrate*.

I´ll illustrate this applying stepwise integration on the second problem of the list of integrals by substitution from above:

$$\#50:\quad \int 20\cdot e^{4\cdot x - 2\cdot x^2}\cdot(1 - x)\,dx$$

$$\int a\cdot F(x)\,dx \rightarrow a\cdot\int F(x)\,dx$$

$$\#51:\quad -20\cdot\int e^{-2\cdot x^2 + 4\cdot x}\cdot(x - 1)\,dx$$

$$\int e^{a\cdot F(x)}\cdot F'(x)\,dx \rightarrow \frac{e^{a\cdot F(x)}}{a}$$

$$\#52:\quad 5\cdot e^{-2\cdot x^2 + 4\cdot x}$$

$$\#53:\quad 5\cdot e^{4\cdot x - 2\cdot x^2}$$

There is a similar DERIVE program for exercising, repeating or deepening the differentiation rules. Most of my programs which were produced in the last years show the same structure. The user hast to load the program as a utility file and then simplify the command **start**. Then he is presented the instructions how to run the tool.

It might be a good moment now to stop demonstrating more examples but asking which fields in school mathematics might need some exercising?

Which manipulating (and other?) skills (techniques, competences?) could (could not), should (should not) be trained, deepened and/or extended supported by CAS (or other technologies)?

My list is more or less CAS-oriented but I hope that I can extend it to other technologies and that I will be able to reach one or the other goals of higher competences than "only" manipulating skills.

This is my list:

Working with fractions
Expanding and Factorizing Expressions
Power Rules
Quadratic Equation
Completing the Square
Simultaneous Linear Equations
Working with complex numbers
Long Division of Polynomials
GCD & LCM
*Working with unit*s (**time**, *length, area, ….*)
Basic Problems with Linear Functions
Working with Vectors and Matrices
Set Theory
Truth Tables (Boolean Expressions)
Analysing Conics
Basic Problems for Financial Mathematics
Solving Triangles
Limits
Arithmetic *& Geometric Sequencies and Series*
Do you remember? Jan Vermeylen 1990!
Investigation of Sequences
Differentiation and Integration Rules
<u>Discussion of Curves</u>
Taylor Series*, Fourier Series*
Typical Forms of Differential Equations
Implicit Differentiation
Conversion between number bases
Recognising Function Types from their graphs
Sketching derivative and/or antiderivative to a given function graph
<u>Finding Polynomial Functions</u>

All the items in bold letters are – more or less – ready. The underlined ones do not provide randomly generated problems but help solving given problems. All the items in italics are on my TO-DO-list.

I in the following I will show a selection of screen shots. All tools from above are done using my favourite CAS, DERIVE. But it is possible to convert all ideas to any other CAS which is programmable like MATHEMATICA, MAPLE, MAXIMA, WIRIS, TI-NspireCAS, …

In my abstract I mentioned GCD and LCM. So I´ll start with exercising calculating GCD and LCM of numbers:

```
#11:  start

Trainer für ggT kgV und kgV – Trainer for gcd and lcm

ggtkgv= liefert eine Aufgabe – gcdlcm= offers one task

res= gibt die richtige Antwort aus – res= delivers the correct answer.
```

#12: $\text{gcdlcm} = \begin{bmatrix} \text{Find gcd and lcm of} \\ [36828,\ 468,\ 1800,\ 18972] \end{bmatrix}$

#13: $\text{res} = \begin{bmatrix} \text{GCD =} & 36 \\ \text{LCM =} & 406949400 \end{bmatrix}$

See next the the "modern" version of the arithmetic series program (remember Jan Vermeylen´s file from the DOS times!) This version provides the solutions, too.

```
Trainer for arithmetic series

ar_s = offers a problem,

the last row shows the questions.

ans = gives the correct answer.

#7:
```

#8: $\text{ar\_s} = \begin{bmatrix} a_8 & 51 & \\ s_{12} & 504 & \\ a_1 & \text{difference} & a_{12} \end{bmatrix}$

#9: $\text{ans} = \begin{bmatrix} a_1 & 9 \\ \text{difference} & 6 \\ a_{12} & 75 \end{bmatrix}$

#10: $\text{ar\_s} = \begin{bmatrix} a_1 & -\frac{16}{15} & \\ \text{difference} & -4 & \\ s_x & -436 & \\ x & a_x & a_9 \end{bmatrix}$

#11: $\text{ans} = \begin{bmatrix} x & 15 \\ a_{15} & -\frac{856}{15} \\ a_9 & -\frac{496}{15} \end{bmatrix}$

In 2000 bk-teachware published my Mathe-Trainer containing packages for the TI-92. There was one package **algebra()** which was intended for my students to exercise - among others - expanding powers of binomials[1]. Some years later when I could use DERIVE, version 5 was on the market and with it the possibility for programming.

Two utility files were produced:

expand.mth and **factor.mth**

#1: LOAD(D:\DOKUS\SCHULE\skills\expand.mth)

#2: start

Expanding Expressions

sq= gives a task for squaring a binomial

cu= gives a task for cubing a binomial

tr= gives a task for squaring a trinomial

pr= asks for a product of two binomials

di= sum times difference of two monomials

quiz= presents a random problem of the above tasks

sqb(n), cub(n), ..., quizb(n) gives n tasks of the same type

res?= gives the correct answer and ch(answer)= analyses single results

The students are offered the instructions.

This might by the start of a session exercising cubing binomials:

#4: $cu = -(2\cdot n + 3\cdot u)^3$

#5: $ch(-8\cdot n^3 - 36\cdot n^2\cdot u - 54\cdot n\cdot u^2 - 9) =$ check cubes!

#6: $res? = -8\cdot n^3 - 36\cdot n^2\cdot u - 54\cdot n\cdot u^2 - 27\cdot u^3$

#7: $$cub(3) = \begin{bmatrix} (p + 7\cdot s)^3 \\ (2\cdot g + v)^3 \\ (8\cdot m - t)^3 \end{bmatrix}$$

#8: $$res? = \begin{bmatrix} p^3 + 21\cdot p^2\cdot s + 147\cdot p\cdot s^2 + 343\cdot s^3 \\ 8\cdot g^3 + 12\cdot g^2\cdot v + 6\cdot g\cdot v^2 + v^3 \\ 512\cdot m^3 - 192\cdot m^2\cdot t + 24\cdot m\cdot t^2 - t^3 \end{bmatrix}$$

As you can see, I tried to include some error analysis for the most common types of errors.

(This is a nice programming challenge – specially collecting the "common types of errors" together with the students.)

```
#1:   LOAD(D:\DOKUS\SCHULE\skills\factor.mth)

#2:   start

Program for training factorizing expressions

fo= ... factor out,                pf= ... partially factor out
sq= ... complete square            cu= ... complete cubic
di= ... difference of squares      cc= ... sum or difference of cubics
tr= ... trinomial                  quiz= ... random problem

With fob(n),pfb(n),sqb(n), ..., quizb(n) n tasks of the same type will be offered
ch= shows the correct result

poly= gives a factorizable polynomial
polyb(n)= gives n polynomials
ch= factorizes for rational zeros,
chr= for irrational and chc= for complex zeros
```

#3: $\text{quizb}(5)=\begin{bmatrix} -630\cdot p^2\cdot s^{10}\cdot t^2-1680\cdot p^2\cdot s^7\cdot t^5-1120\cdot p^2\cdot s^4\cdot t^8 \\ -360\cdot h^6\cdot u^2\cdot w^4-180\cdot h^5\cdot u^2\cdot w^5+540\cdot h^4\cdot u^2\cdot w^6 \\ 432\cdot f^3\cdot k\cdot l^2+768\cdot f\cdot k^4\cdot l^2+336\cdot f\cdot k\cdot l^5 \\ 675\cdot c^3\cdot l^3\cdot n^3+1350\cdot c^2\cdot l^6\cdot n^3+675\cdot c\cdot l^9\cdot n^3 \\ 80\cdot b^4\cdot u^2\cdot y^3+500\cdot b^2\cdot u^4\cdot y^3 \end{bmatrix}$

#4: $\text{ch}=\begin{bmatrix} -70\cdot p^2\cdot s^4\cdot t^2\cdot(3\cdot s^3+4\cdot t^3)^2 \\ 180\cdot h^4\cdot u^2\cdot w^4\cdot(w-h)\cdot(2\cdot h+3\cdot w) \\ 48\cdot f\cdot k\cdot l^2\cdot(9\cdot f^2+16\cdot k^3+7\cdot l^3) \\ 675\cdot c\cdot l^3\cdot n^3\cdot(c+l^3)^2 \\ 20\cdot b^2\cdot u^2\cdot y^3\cdot(4\cdot b^2+25\cdot u^2) \end{bmatrix}$

The instruction informs that this file can also be used for training factorizing polynomials on different levels of knowledge: starting with only rational zeros one can proceed to irrational zeros and then finish with complex zeros depending on the age of the students.

$$\#9:\quad \text{polyb}(5)=\begin{bmatrix} x^5-8\cdot x^4+8\cdot x^3 \\ -x^6-15\cdot x^5-48\cdot x^4+64\cdot x^3 \\ 36\cdot x^6-258\cdot x^5+234\cdot x^4+108\cdot x^3 \\ -\dfrac{8\cdot x^3}{5}+\dfrac{104\cdot x^2}{25}-\dfrac{194\cdot x}{25}+\dfrac{208}{25} \\ 81\cdot x^4-717\cdot x^3-\dfrac{323\cdot x^2}{3}-3\cdot x \end{bmatrix}$$

$$\#10:\quad \text{ch}=\begin{bmatrix} x^3\cdot(x^2-8\cdot x+8) \\ x^3\cdot(1-x)\cdot(x+8)^2 \\ 6\cdot x^3\cdot(x-6)\cdot(2\cdot x-3)\cdot(3\cdot x+1) \\ \dfrac{2\cdot(8-5\cdot x)\cdot(4\cdot x^2-4\cdot x+13)}{25} \\ \dfrac{x\cdot(x-9)\cdot(9\cdot x+1)\cdot(27\cdot x+1)}{3} \end{bmatrix}$$

$$\#11:\quad \text{chr}=\begin{bmatrix} x^3\cdot(x+2\cdot\sqrt{2}-4)\cdot(x-2\cdot\sqrt{2}-4) \\ x^3\cdot(1-x)\cdot(x+8)^2 \\ 12\cdot x^3\cdot(x-6)\cdot\left(x-\dfrac{3}{2}\right)\cdot(3\cdot x+1) \\ \dfrac{2\cdot(8-5\cdot x)\cdot(4\cdot x^2-4\cdot x+13)}{25} \\ 9\cdot x\cdot(x-9)\cdot\left(x+\dfrac{1}{27}\right)\cdot(9\cdot x+1) \end{bmatrix}$$

$$\#12:\quad \text{chc}=\begin{bmatrix} x^3\cdot(x+2\cdot\sqrt{2}-4)\cdot(x-2\cdot\sqrt{2}-4) \\ x^3\cdot(1-x)\cdot(x+8)^2 \\ 12\cdot x^3\cdot(x-6)\cdot\left(x-\dfrac{3}{2}\right)\cdot(3\cdot x+1) \\ \dfrac{8\cdot(8-5\cdot x)\cdot\left(x-\dfrac{1}{2}+\sqrt{3}\cdot i\right)\cdot\left(x-\dfrac{1}{2}-\sqrt{3}\cdot i\right)}{25} \\ 9\cdot x\cdot(x-9)\cdot\left(x+\dfrac{1}{27}\right)\cdot(9\cdot x+1) \end{bmatrix}$$

Having some experience in programming it is not really a problem to transfer the idea(s) to TI-Nspire. The screen shot shows the FACTOR-utility. The students have to do factorization by hands and then check their results by using the provided functions.

$$\text{expand}(dib(3)) \quad \begin{bmatrix} 25\cdot m\cdot v^2\cdot z^6-400\cdot m^5\cdot v^2\cdot z^2 \\ 240\cdot h^2\cdot m^4\cdot p^2-540\cdot h^6\cdot m^2\cdot p^2 \\ 560\cdot k^7\cdot n^3\cdot z^3-875\cdot k\cdot n^3\cdot z^5 \end{bmatrix}$$

$$\text{factor}\left(\begin{bmatrix} 25\cdot m\cdot v^2\cdot z^6-400\cdot m^5\cdot v^2\cdot z^2 \\ 240\cdot h^2\cdot m^4\cdot p^2-540\cdot h^6\cdot m^2\cdot p^2 \\ 560\cdot k^7\cdot n^3\cdot z^3-875\cdot k\cdot n^3\cdot z^5 \end{bmatrix}\right) \quad \begin{bmatrix} 25\cdot m\cdot v^2\cdot z^2\cdot(z-2\cdot m)\cdot(z+2\cdot m)\cdot(z^2+4\cdot m^2) \\ -60\cdot h^2\cdot(3\cdot h^2-2\cdot m)\cdot(3\cdot h^2+2\cdot m)\cdot m^2\cdot p^2 \\ -35\cdot k\cdot n^3\cdot z^3\cdot(5\cdot z-4\cdot k^3)\cdot(5\cdot z+4\cdot k^3) \end{bmatrix}$$

$$\text{expand}(hh()) \quad 900\cdot g^5\cdot h^3-900\cdot g^4\cdot h^3-900\cdot g^3\cdot h^3$$

$$\text{factor}(900\cdot g^5\cdot h^3-900\cdot g^4\cdot h^3-900\cdot g^3\cdot h^3) \quad 900\cdot g^3\cdot(g^2-g-1)\cdot h^3$$

$$\text{expand}(sk()) \quad 224\cdot c^8\cdot k\cdot z-28\cdot c^2\cdot k\cdot z^4$$

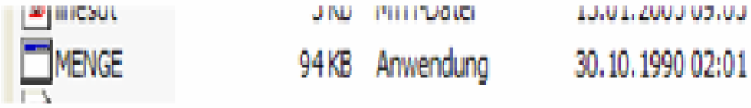
$$\text{factor}(224\cdot c^8\cdot k\cdot z-28\cdot c^2\cdot k\cdot z^4) \quad -28\cdot c^2\cdot k\cdot z\cdot(z-2\cdot c^2)\cdot(z^2+2\cdot c^2\cdot z+4\cdot c^4)$$

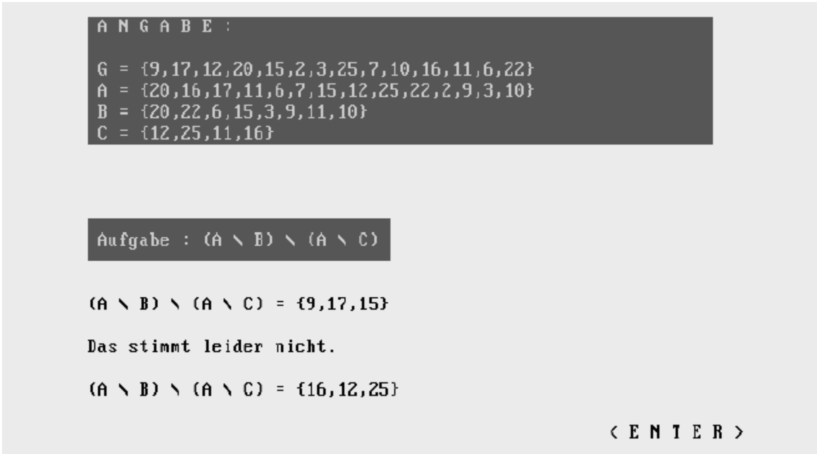
$$\text{expand}(zub(4)) \quad \begin{bmatrix} -6\cdot f\cdot g\cdot s^4-8\cdot g^3\cdot s^4-30\cdot c^2\cdot f\cdot g\cdot s^3-40\cdot c^2\cdot g^3\cdot s^3 \\ -240\cdot n^3\cdot q^4\cdot v^6-540\cdot n^3\cdot q^{10}\cdot v^4 \\ 200\cdot g^5\cdot m^3\cdot p^3+800\cdot g^3\cdot m^3\cdot p^5 \\ 8\cdot c^3\cdot f^9\cdot k-24\cdot c^3\cdot f^6\cdot k^3+18\cdot c^3\cdot f^3\cdot k^5 \end{bmatrix}$$

7/99

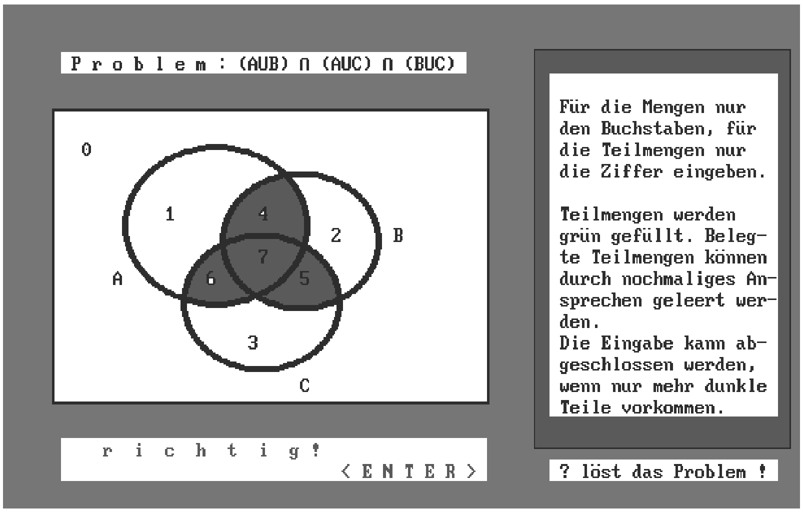
I don´t want to demonstrate the Long Division because this can be found in DERIVE Newsletter 71/72 from 2008[15].

Many many years ago I wrote a BASIC program for teaching and training set theory. It was an “intelligent” program, because I implemented a parser in order to perform any set operation with 2, 3 or four sets of numbers or characters (given or randomly chosen).





The set operations could be visualized.



The nice thing is that students in my former school are still using this “antique” program which is running in the DOS-environment.

It was a challenge for me to produce a similar tool with DERIVE – without facing the problem to implement again a parser – CAS does the job for me.

This is the recent version of set theory exercising and learning. The students have the choice between basics and more complex operations with 2, 3 or 4 sets. The program provides a random generated universal set with the respective subsets and a list of problems.

Here is the section for tasks involving three subsets:

Given is a universal set together with 4 subsets A, B, C and D.

Find the following sets: – Simplify sols for the solutions!

(A ∪ B) ∩ (B ∪ C)	(A ∩ B) ∪ (A ∩ D)
B ∪ (C ∩ D)	A ∩ (C ∪ D′)
(A ∪ C) ∩ (A ∪ D) ∩ (C ∪ D),	(C ∪ D) \ A
(A ∩ B′) ∩ D	(B \ D) \ (B \ A)
A′ \ (B ∪ C)	(A \ B) ∪ (A \ D)
(B \ C) ∪ (B ∩ D)	A′ ∪ C′ ∪ D′
C ∩ (A ∩ B)′	(D′ ∪ (A \ C))′
(A ∪ B)′ ∩ (C ∩ D)′	(C ∪ B) \ (C \ D)

#16: three=

Universal Set:	{2,8,11,13,16,21,24,29,30}
Subset A:	{2,8,13,16,21,30}
Subset B:	{8,11,16,21,30}
Subset C:	{2,8,11,13,16,21,24,29,30}
Subset D:	{8,11,16,21,29,30}

The students shall solve the tasks by paper and pencil and then compare their results with the given solutions. Own operations can be entered. It is also possible to generate sets of random generated sets for self made exercises.

I call the solutions for the tasks given above:

#17: sols=

{2,8,11,13,16,21,30}	{8,16,21,30}
{8,11,16,21,29,30}	{2,8,13,16,21,30}
{2,8,11,13,16,21,29,30}	{11,24,29}
{}	{}
{}	{2,13}
{8,11,16,21,30}	{2,11,13,24,29}
{2,11,13,24,29}	{8,11,16,21,29,30}
{24}	{}

Providing exercises with Venn diagrams was not so easy, but finally I found a way to realize this in a satisfying way.

I show the part with three subsets embedded in a universal set. First see the "instructions" together with a list of provided problems.

Exercising set diagrams (Venn diagrams) with **three sets**

Double click on the provided graph to open the 2D plot window.

Editing A or B or C and plotting shades the respective sets, editing and plotting q1 through q8 shades the distinct disjoint subsets.
This might help to visualize the following set operations:

Sketch the Venn diagrams for the given set operations by paper and pencil -
- or using the technique described above.

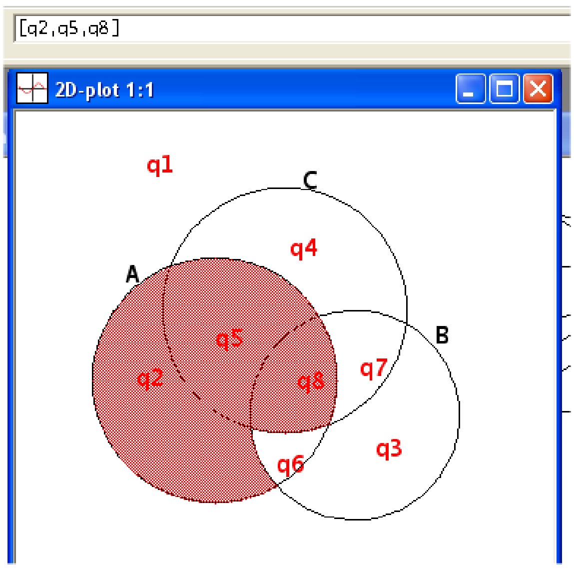
(1)	$(A \cup B) \cap C$	(2)	$(A \cap C) \cup B$
(3)	$(A \cup B) \cap (B \cup C)$	(4)	$(A \cap B) \cup (A \cap C)$
(5)	$A \cap (B \cup C')$	(6)	$(A \cup C) \cap (A \cup B) \cap (B \cup C)$
(7)	$(A \cup C) \setminus B$	(8)	$(A \cap B)' \cup (A \cap C)'$
(9)	$A' \setminus (B \cup C)$	(10)	$(A \setminus B) \cup (A \cap C)$
(11)	$(A \setminus C) \cup (C \setminus B)$	(12)	$A' \cup B' \cup C'$
(13)	$(A \cup B \cup C) \setminus (A \cap B \cap C)$	(14)	$(A \cup B)' \cap (C \cap B)'$
(15)	$(A \setminus C)' \cup (B \setminus C)$	(16)	$((A \cap B) \cup (A \cap C) \cup (B \cap C)) \setminus (A \cap B \cap C)$

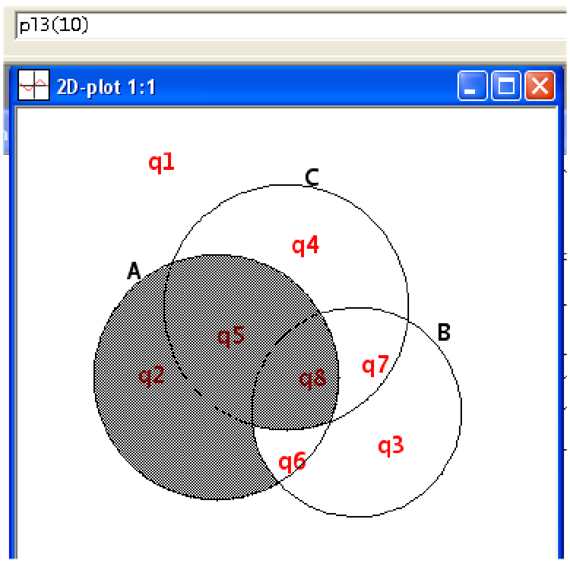
The correct answer will be given by pl3(# of the task), eg. pl3(12) plots $A' \cup B' \cup C'$.

You can visualize your own diagrams but you have to take in account that DERIVE cannot plot set operations but their eqivalent as Boolean expressions. It is easy to convert the set operations to Boolean expressions:

$A \cup B \quad \rightarrow \quad$ A or B or $A \vee B$
$A \cap B \quad \rightarrow \quad$ A and B or $A \wedge B$
$A' \quad \rightarrow \quad$ not A or $\neg A$
$A \setminus B \quad \rightarrow \quad$ A and not B or $A \wedge \neg B$

Let´s assume that student Josef wants to treat problem (10). He marks the disjunct subsets q2 and q5 (= A\B) together with q8 (= A∩B∩C) by entering and plotting [q2,q5,q8], because he thinks that this is the solution. Then he plots pl3(10) to check if he is right or not.





I will skip the Solving Triangles Training (will follow in another DNL) and remind you that the utility for working with times was published in DNL #74 ("What´s the TIME, Grandie?)[16].

One of my latest products is a tool for analysing conics:

```
#1:   LOAD(D:\DOKUS\SCHULE\skills\conics.mth)

#2:   start

Conics Trainer – Trainer für Kegelschnitte

con1 = Center in origin unrotated – Kegelschnitt in Ursprungslage

con2 = shifted origin, unrotated – verschobener Kegelschnitt

con3 = conic shifted and rotated – Kegelschnitt in allgemeiner Lage

con = random choice from above – Zufallsaufgabe aus den obigen


ans = gives the analysis; ant = liefert die Analyse
```

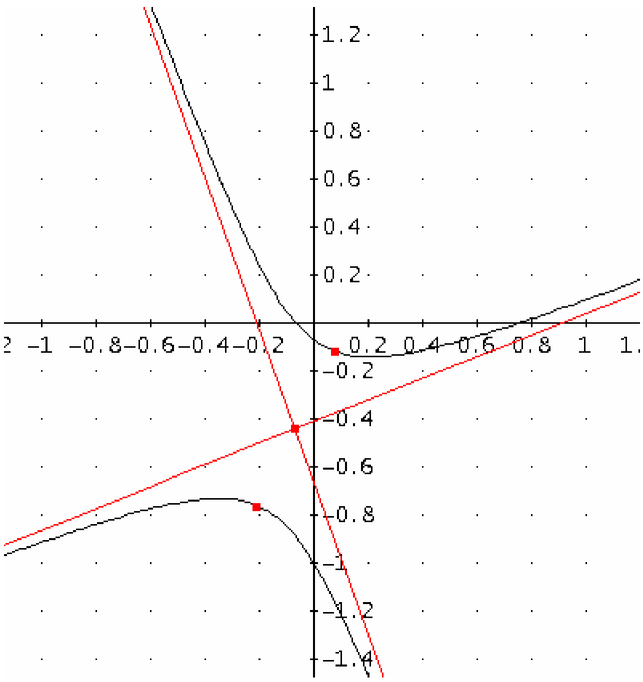
#4: $\text{con3} = (20\cdot x^2 - 38\cdot x\cdot y - 14\cdot x - 14\cdot y^2 - 15\cdot y = 1)$

#5: ans =

Type	Center	Real axis 2a	Imaginary axis 2b
Hyperbola	[−0.06942, −0.4414]	0.7052	0.6266

Real Vertices		Asymptotes	
0.07450	−0.1195	1.372·t − 0.06942	0.6192·t − 0.4414
−0.2133	−0.7634	0.4535·t − 0.06942	− 1.435·t − 0.4414

The student performs the calculation – possibly supported by CAS – and checks the obtained results. As an add on he/she can plot the conic and its center, the vertices and the asymptotes in case of a hyperbola.



I presented a lot of examples from various fields of school mathematics performed with various pieces of soft- and hardware. So you might have made up your own opinion about Pros and Contras for using technology in this very special way.

What are my personal Pluses and Minuses? I´ll start with the **MINUSES**:

- **Very (too?) close to traditional (fundamental) mathematics education**
- **There is a danger that training could become a main activity in math teaching**

I am quite sure that the reader will add some other objections – which I probably will share.

My **PLUSES** are:

+ **The tools could (should) be developed, tested and improved in cooperation with the students, which asks for deeper understanding of the algorithms and techniques**
+ **Encourages self responsible exercising parallel to math education (if necessary)**
+ **Hesitating teachers could be led to the use of CAS – and hopefully will not use CAS for training manipulation skills only but will later change – or at least consider to change – their methods teaching maths.**
+ **Students like it!**

Let me add one more aspect which is very important from my point of view:

The kids/pupils/students have to learn a lot of rules. I believe that nobody will teach calculus without talking about the chain rule. The students then know the rule and have in their minds: "*Outer derivative times inner derivative*", which is ok.

But it is my strong opinion that a special competence is needed to apply this rule in the appropriate way and to know when just this rule must be applied. And this is valid for many cases applying rules – also in daily life. Knowing the rule is not sufficient enough. One must be able to recognise when and how to apply the rule – and not in all cases does the computer all the work.

Exercised enough??

References

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[2] J. Böhm, *Programmieren in Derive*, bk-teachware 2002.

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[4] H. Heugl, E. Lehmann, W. Herget, B. Kutzler, *Unverzichtbare handwerkliche Rechenkompentenzen im CAS-Zeitalter* ,(*Indispensable Competencies in the Era of CAS*), http://www.acdca.ac.at/material/vortrag/kompet1.htm.

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[6] W. Herget, *Rettet die Ideen! – Rettet die Rezepte?*, Hischer Horst (ed.), Rechenfertigkeit und Begriffsbildung – Zu wesentlichen Aspekten des Mathematikunterrichts vor dem Hintergrund von Computeralgebrasystemen. Hildesheim: Franzbecker, pp156-169.

[7] W. Herget,*Wie viel Termumformung braucht der Mensch? – Taschencomputer und Mathematikunterricht*, Amelung Udo (ed.): Der TI-92 im Mathematikunterricht. Pfingsttagung 1998. Zentrale Koordination Lehrerausbildung, ZKL-Texte Nr. 7, Westfälische Wilhelms-Universität, Münster, pp 3-19.

[8] Derive Manual, Texas Instruments.

[9] Voyage 200 Manual, Texas Instruments.

Several articles published in the Derive & CAS-TI Newsletters (DNL):

[10] Jan Vermeylen and J. Böhm, *Vieta at Random*, DNL#20, 1996

[11] J. Böhm, *The Trigonometric Superbox*, DNL#23, 1996

[12] H.-R. Geyer, *MENTAL.MTH*, DNL#27, 1997

[13] J. Böhm, *DERIVE as Problem Generator*, DNL#40, 2000

[14] J. Wiesenbauer, *A Macro for Solving Equations*, DNL#42, 2001

[15] J. Böhm, *Long Division - Step by Step*, DNL#71/72, 2008

[16] J. Böhm, *What´s the Time, Grandie?*. DNL#74, 2009

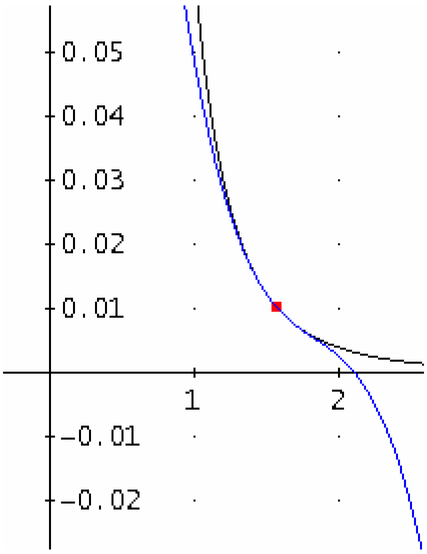
There is some space left. So I cannot resist to add my Taylor-Series-Exercising Tool:

#2: $$\text{tay} = \begin{bmatrix} \text{function} & \text{x0} & \text{degree} \\ \dfrac{1}{16\cdot x^4} & \dfrac{\pi}{2} & 3 \end{bmatrix}$$

#3: $$\text{check}\left(-\frac{160\cdot x^3}{\pi^7} + \frac{280\cdot x^2}{\pi^6} - \frac{168\cdot x}{\pi^5} - \frac{35}{\pi^4}\right) = \text{wrong}$$

#4: $$\text{res} = -\frac{160\cdot x^3 - 280\cdot\pi\cdot x^2 + 168\cdot\pi^2\cdot x - 35\cdot\pi^3}{\pi^7}$$

#5: plot



The student is given a random generated function together with location x0 where to develop a Taylor series of requested degree. **plot** shows the respective graphic representation.

Making Life in an Introductory Linear Algebra Course Easier with DERIVE

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The Schmalkalden University Faculty of Business and Economics moved its introductory linear algebra course from the classroom to the PC lab, and purchased a *DERIVE* license that also allows its use on the students' own PCs. A *DERIVE* utility file was then developed to facilitate exercises with special matrices throughout the course. This utility file allows the computation of zero matrices and vectors, matrices and vectors of ones, random matrices, as well as idempotent and orthogonal matrices, "just-in-time" whenever they are needed during the course. This paper demonstrates how beneficial it is for students to sit in front of a PC in an introductory linear algebra course, from the very basic to the more advanced topics.

1 Some preliminaries

A matrix is a rectangular array of elements in *m* rows and *n* columns. We denote a matrix by a boldface capital letter, e.g. $\boldsymbol{A}$. In *DERIVE*, a matrix can be defined simply by clicking on , setting the number of rows and columns, and entering the elements. An element of a matrix is denoted by the corresponding lowercase letter with a double index, the first index being the row, and the second the column index. For example, the element in the second row and first column of $\boldsymbol{A}$ is denoted by a_{21}.

A matrix with only one column ($n = 1$) is a (column) vector. Hence, throughout the course, a vector has to be defined in *DERIVE* as a matrix with one column; the *vector* data type (e.g. defined by clicking on) is not used. We denote a vector by a boldface lowercase letter, e.g. $\boldsymbol{x}$. Row vectors are not defined; transposed column vectors are used instead. A matrix is transposed by taking its rows (columns) and writing these as columns (rows) of the transposed matrix. The transpose of a matrix or vector is denoted by a prime, e.g. $\boldsymbol{A}'$. Clearly, the transpose of an $m \times n$ matrix is an $n \times m$ matrix. In *DERIVE* the transposition sign is a grave accent which can be entered, for example, by clicking on ` in the symbol list at the bottom right of the *DERIVE* window.

A matrix which has the same number of rows and columns ($m = n$) is called a square matrix. The diagonal from the upper left to the lower right of a square matrix is called the main diagonal.

2 Matrices and Vectors Containing 0s and 1s

It is important for students to develop a good understanding of

- matrices and vectors containing only 0s (zero matrices and vectors),
- matrices and vectors containing only 1s (matrices and vectors of ones), and
- square matrices containing 1s on the main diagonal and otherwise 0s (identity matrices).

While zero matrices and vectors behave similarly to the number $0 \in \mathbb{R}$, matrices and vectors of ones do not behave similarly to the number $1 \in \mathbb{R}$, but rather identity matrices are the matrix algebra analogues to $1 \in \mathbb{R}$. However, it is rather tedious to verify this in exercises. The utility file therefore includes the following five functions:

`O(m,n)`	generates an $m \times n$ zero matrix	(1)
`o(n)`	generates an $n \times 1$ zero vector	(2)
`J(m,n)`	generates an $m \times n$ matrix of ones	(3)
`l(n)`	generates an $n \times 1$ vector of ones	(4)
`I(n)`	generates an $n \times n$ identity matrix	(5)

The names of the functions were chosen such that they are close to commonly used symbols for such matrices: $\boldsymbol{O}$ for zero matrices, $\boldsymbol{J}$ for matrices of ones, $\boldsymbol{I}$ for identity matrices, $\boldsymbol{o}$ for zero vectors, and “l” (lowercase L) for vectors of ones (since this resembles the symbol **1** which is typically used for vectors of ones). It is important for the students to understand that if they use the utility file (i.e. if the file is loaded into *DERIVE*), they cannot use the above names for anything else (e.g. for a matrix), as this would overwrite the function in the utility file. Note that the `I(n)`-function was added simply to have a shortcut notation for *DERIVE’s* built-in function for the computation of identity matrices, `IDENTITY_MATRIX(n)`.

The utility file also contains a function that generates an $m \times n$ random matrix of nonnegative integers (cf. next section). The third parameter of this function determines the highest possible integer such that the following function is actually within the scope of this section:

`RNDU(m,n,1)`	generates an $m \times n$ random matrix of zeros and ones	(6a)

We define the matrix

$$\underset{3\times 3}{\boldsymbol{A}} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

for which we want to check some of the following properties of zero matrices:

$$\underset{m\times n}{\boldsymbol{A}}\,\underset{n\times l}{\boldsymbol{O}} = \underset{m\times l}{\boldsymbol{O}}\,; \qquad \underset{l\times m}{\boldsymbol{O}}\,\underset{m\times n}{\boldsymbol{A}} = \underset{l\times n}{\boldsymbol{O}}\,; \qquad \underset{m\times n}{\boldsymbol{O}} = \underset{m\times 1}{\boldsymbol{o}}\,\underset{1\times n}{\boldsymbol{o}'}\,; \qquad \underset{m\times n}{\boldsymbol{O}}\,\underset{n\times p}{\boldsymbol{O}} = \underset{m\times p}{\boldsymbol{O}}\,.$$

Note that the last of the above properties implies that the product of any square zero matrix with itself is a square zero matrix of the same dimension, making square zero matrices a standard example for idempotent matrices (any square matrix $\boldsymbol{A}$ with $\boldsymbol{AA} = \boldsymbol{A}$ is called an idempotent matrix, cf. section 4). Screenshot 2.1 shows a few exercises with zero matrices.

```
#1:   LOAD(C:\Program Files\TI Education\Derive 6\Math\SpeM.mth)
```

$$\#2: \quad A := \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$\#3: \quad A \cdot O(3,\ 2) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\#4: \quad o(3)' \cdot A = [[0,\ 0,\ 0]]$$

$$\#5: \quad o(2) \cdot o(3)' = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\#6: \quad O(2,\ 2) \cdot O(2,\ 2) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Screenshot 2.1

We continue with matrices and vectors of ones. For the previously defined matrix $\boldsymbol{A}$ we want to check some of the following properties:

$$\underset{1\times n}{\boldsymbol{a}'}\ \underset{n\times 1}{\mathbf{1}} = \mathbf{1}'\boldsymbol{a} = \sum_{i=1}^{n} a_i\ ; \qquad \underset{m\times n}{\boldsymbol{A}}\ \underset{n\times 1}{\mathbf{1}} = \begin{pmatrix} \sum_{j=1}^{n} a_{1j} \\ \vdots \\ \sum_{j=1}^{n} a_{mj} \end{pmatrix};$$

$$\underset{1\times m}{\mathbf{1}'}\ \underset{m\times n}{\boldsymbol{A}} = \begin{pmatrix} \sum_{i=1}^{m} a_{i1} & \cdots & \sum_{i=1}^{m} a_{in} \end{pmatrix};$$

$$\underset{1\times m}{\mathbf{1}'}\ \underset{m\times n}{\boldsymbol{A}}\ \underset{n\times 1}{\mathbf{1}} = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}\ ; \qquad \underset{m\times n}{\boldsymbol{J}} = \underset{m\times 1}{\mathbf{1}}\ \underset{1\times n}{\mathbf{1}'}\ ; \qquad \underset{m\times n}{\boldsymbol{J}}\ \underset{n\times p}{\boldsymbol{J}} = n\ \underset{m\times p}{\boldsymbol{J}}\ .$$

Clearly, the first four properties show that vectors of ones are useful for the computation of sums, namely of the elements of a vector, or the row, or column, sums of a matrix, or the sum of all elements of a matrix. Screenshot 2.2 shows a few exercises with matrices of ones.

#7: $a := \begin{bmatrix} a1 \\ a2 \end{bmatrix}$

#8: $1(2)' \cdot a = [[a1 + a2]]$

#9: $A \cdot 1(3) = \begin{bmatrix} 6 \\ 15 \\ 24 \end{bmatrix}$

#10: $1(3)' \cdot A = [[12,\ 15,\ 18]]$

#11: $1(3)' \cdot A \cdot 1(3) = [[45]]$

#12: $J(2,\ 3) \cdot J(3,\ 4) = \begin{bmatrix} 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 \end{bmatrix}$

Screenshot 2.2

For a change, the summation of the elements of a vector $\boldsymbol{a}$ is done for a 2×1 vector of arbitrary elements, which are denoted by `a1` and `a2`. The result of the multiplication of two matrices of ones is unexpected by many students, as it is *not* another matrix of ones.

After realizing that matrices of ones do not behave similarly to the number $1 \in \mathbb{R}$, attention can immediately be directed to identity matrices. For the matrix $\boldsymbol{A}$ and the vector $\boldsymbol{a}$ already defined we want to check the following property (see Screenshot 2.3):

$$\underset{m \times n}{\boldsymbol{A}}\ \underset{n \times n}{\boldsymbol{I}} = \underset{m \times m}{\boldsymbol{I}}\ \underset{m \times n}{\boldsymbol{A}} = \boldsymbol{A}\,.$$

#13: $I(3) \cdot A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

#14: $I(2) \cdot a = \begin{bmatrix} a1 \\ a2 \end{bmatrix}$

#15: $I(2) \cdot I(2) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Screenshot 2.3

It is also shown that the product of the 2×2 identity matrix with itself is the 2×2 identity matrix. As this is true for any identity matrix (which follows immediately from the above-mentioned property by choosing $\boldsymbol{A} = \boldsymbol{I}$), identity matrices are another standard example for idempotent matrices.

3 Random Matrices

Many students appreciate having the option to use additional example matrices to practise certain topics in matrix algebra. The utility file therefore includes the function

`RNDU(m,n,max)` generates an $m \times n$ matrix of random nonnegative integers (6)

The random numbers are from the set $\{0,1,2,\ldots,max\}$, i.e. the third parameter determines the highest possible value; for example, if $max = 1$, the generated matrix contains only 0s and 1s (cf. Screenshot 3.1). The `RNDU`-function should only be called with a positive integer as the third parameter. If $max \leq 0$, an error message is printed, if $max > 0$ but not an integer, no random matrix is generated. Note that the `RNDU`-function uses the built-in *DERIVE* function `RANDOM(n)` with $n = max + 1$.

#2: RNDU(2, 2, 1) = $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$

#3: RNDU(2, 2, 1) = $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

#4: RNDU(2, 2, 1) = $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

Screenshot 3.1

Only the last of the three matrices generated in Screenshot 3.1 has full rank (is nonsingular), the other two have rank 1. As any element of a matrix generated by the `RNDU(2,2,1)`-function is either 0 or 1, and as there are only 16 different 2×2 matrices with two different elements, the probability of getting a matrix having full rank is lower ($^6/_{16}$) than it is of getting a singular matrix ($^{10}/_{16}$). This allows an interesting discussion of the rank of a matrix.

In general, the probability of getting a rank-deficient matrix decreases if the number of rows, the number of columns, or the *max* value is increased. For example, all three matrices in Screenshot 3.2 are of full row rank.

#5: RNDU(2, 3, 9) = $\begin{bmatrix} 7 & 5 & 9 \\ 0 & 9 & 6 \end{bmatrix}$

#6: RNDU(2, 3, 9) = $\begin{bmatrix} 2 & 0 & 6 \\ 1 & 7 & 8 \end{bmatrix}$

#7: RNDU(2, 3, 9) = $\begin{bmatrix} 4 & 2 & 1 \\ 9 & 3 & 0 \end{bmatrix}$

Screenshot 3.2

Note that there is also a built-in *DERIVE* function similar to the `RNDU`-function: the `RANDOM_MATRIX(m,n,s)` function generates an $m \times n$ matrix of random integers from the set $\{-s+1, -s+2, \ldots, -1, 0, 1, \ldots, s-1\}$.

4 Idempotent Matrices

As we have now already come across two examples for idempotent matrices (any square matrix $\boldsymbol{A}$ with the property $\boldsymbol{AA}=\boldsymbol{A}$), namely square zero matrices and identity matrices, we should spend a little more time with these matrices. The utility file includes the function

`CNTR(n)` generates an idempotent $n \times n$ matrix (7)

The formula used in this function,

$$\underset{n\times n}{\boldsymbol{I}} - \tfrac{1}{n} \underset{n\times n}{\boldsymbol{J}} \, ,$$

generates a so-called centering matrix (using an identity matrix and a square matrix of ones). Centering matrices are used, for example, in multivariate statistics. Screenshots 4.1 and 4.2 show what centering matrices look like and also that the 2×2 and 3×3 centering matrices are indeed idempotent.

#2: $\text{CNTR}(2) = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$

#3: $\text{CNTR}(2) - \text{CNTR}(2)\cdot\text{CNTR}(2) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Screenshot 4.1

#4: $\text{CNTR}(3) = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$

#5: $\text{CNTR}(3) - \text{CNTR}(3)\cdot\text{CNTR}(3) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Screenshot 4.2

Obviously, the `CNTR`-function can only generate one particular idempotent matrix of a specific dimension. Therefore, the utility file contains another function for the computation of an idempotent matrix, the `IDEM`-function, which uses a property that holds for the Moore-Penrose inverse $\underset{n\times m}{\boldsymbol{A}^+}$ of a matrix $\underset{m\times n}{\boldsymbol{A}}$:

$\boldsymbol{A}^+\boldsymbol{A}$ and $\boldsymbol{AA}^+$ (as well as $\boldsymbol{I}-\boldsymbol{A}^+\boldsymbol{A}$ and $\boldsymbol{I}-\boldsymbol{AA}^+$) are idempotent matrices.

The Moore-Penrose inverse of any matrix $\underset{m\times n}{\boldsymbol{A}}$ is the (unique) matrix $\underset{n\times m}{\boldsymbol{A}^+}$ satisfying the four conditions

$$\boldsymbol{A}\boldsymbol{A}^+\boldsymbol{A}=\boldsymbol{A}\,;\qquad \boldsymbol{A}^+\boldsymbol{A}\boldsymbol{A}^+=\boldsymbol{A}^+\,;\qquad \left(\boldsymbol{A}^+\boldsymbol{A}\right)'=\boldsymbol{A}^+\boldsymbol{A}\,;\ \left(\boldsymbol{A}\boldsymbol{A}^+\right)'=\boldsymbol{A}\boldsymbol{A}^+.$$

As *DERIVE* does not include a function for the computation of the Moore-Penrose inverse, the utility file includes the two functions

`MPIV(a)`	computes the Moore-Penrose inverse of any $n\times 1$ vector $\boldsymbol{a}$	(8)
`MPI(A)`	computes the Moore-Penrose inverse of any $m\times n$ matrix $\boldsymbol{A}$	(9)

Both functions are described in detail in Schmidt (2003).

If $\boldsymbol{A}$ is a square non-singular matrix, the Moore-Penrose inverse and the inverse $\boldsymbol{A}^{-1}$ coincide, and we have $\boldsymbol{A}^+\boldsymbol{A}=\boldsymbol{A}^{-1}\boldsymbol{A}=\boldsymbol{I}$ and $\boldsymbol{I}-\boldsymbol{A}^+\boldsymbol{A}=\boldsymbol{I}-\boldsymbol{A}^{-1}\boldsymbol{A}=\boldsymbol{O}$, i.e. we are back to the two standard examples for idempotent matrices from section 2. Therefore, the more interesting cases are when $\boldsymbol{A}$ is either square but singular, or non-square. The matrix $\boldsymbol{A}$ from section 2 is such a singular matrix. The Moore-Penrose inverse of $\boldsymbol{A}$ is computed in Screenshot 4.3.

#6: $\quad A := \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

#7: $\quad \text{MPI(A)} = \begin{bmatrix} -\frac{23}{36} & -\frac{1}{6} & \frac{11}{36} \\ -\frac{1}{18} & 0 & \frac{1}{18} \\ \frac{19}{36} & \frac{1}{6} & -\frac{7}{36} \end{bmatrix}$

Screenshot 4.3

The utility file includes the function

`IDEM(A)`	generates an idempotent $n\times n$ matrix from any $m\times n$ matrix $\boldsymbol{A}$	(10)

#8: $\quad \text{IDEM(A)} = \begin{bmatrix} \frac{5}{6} & \frac{1}{3} & -\frac{1}{6} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{6} & \frac{1}{3} & \frac{5}{6} \end{bmatrix}$

#9: $\quad \text{IDEM(J(3, 2))} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$

Screenshot 4.4

The formula used in this function is $\boldsymbol{A}^{+}\boldsymbol{A}$. Screenshot 4.4 shows the idempotent matrix which is generated when matrix $\boldsymbol{A}$ is passed as the parameter, and also the one generated by passing the 3×2 matrix of ones.

Note that finding the rank of an idempotent matrix is a relatively easy task, as rank and trace are identical in this case; for the first example in Screenshot 4.4 we get $\frac{5}{6}+\frac{1}{3}+\frac{5}{6}=2$, for the second $\frac{1}{2}+\frac{1}{2}=1$. We can also quickly double-check this on the PC (Screenshot 4.5):

```
#10:          RANK(IDEM(A)) = 2
#11:          TRACE(IDEM(A)) = 2
#12:          RANK(IDEM(J(3, 2))) = 1
#13:          TRACE(IDEM(J(3, 2))) = 1
```

Screenshot 4.5

5 Orthogonal Matrices

Finally, we want to consider orthogonal matrices (any square matrix $\boldsymbol{A}$ with the property $\boldsymbol{A}'=\boldsymbol{A}^{-1}$). The utility file includes the function

`ORTH(a)` generates an orthogonal $n\times n$ matrix from any $n\times 1$ vector $\boldsymbol{a}\neq\boldsymbol{o}$ (11)

The formula used in this function,

$$\underset{n\times n}{\boldsymbol{I}}-2\underset{n\times 1}{\boldsymbol{a}}\underset{1\times n}{\boldsymbol{a}'},$$

$$\#2:\quad b := \begin{bmatrix} \frac{3}{5} \\ \frac{4}{5} \end{bmatrix}$$

$$\#3:\quad b'\cdot b = [[1]]$$

$$\#4:\quad \mathrm{ORTH}(b) = \begin{bmatrix} \frac{7}{25} & -\frac{24}{25} \\ -\frac{24}{25} & -\frac{7}{25} \end{bmatrix}$$

Screenshot 5.1

generates an orthogonal matrix if $\boldsymbol{a}'\boldsymbol{a}=1$. Hence, the vector $\boldsymbol{a}$ which is passed as the parameter will be transformed within the `ORTH`-function such that it is of length 1. Let us nevertheless start with a vector $\boldsymbol{b}$ that has length 1 anyway, as in Screenshot 5.1.

In order to check if this matrix is indeed orthogonal (cf. Screenshot 5.2), we have to compute both its inverse and transpose, and see if these two matrices are identical. Since for any orthogonal matrix we have

$$\boldsymbol{A}'\boldsymbol{A} = \boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{I} \,,$$

a second method to prove that a matrix $\boldsymbol{A}$ is orthogonal is to show that $\boldsymbol{A}'\boldsymbol{A} = \boldsymbol{I}$.

#5: $\mathrm{ORTH(b)}^{-1} - \mathrm{ORTH(b)}' = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

#6: $\mathrm{ORTH(b)}' \cdot \mathrm{ORTH(b)} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Screenshot 5.2

For a second example (Screenshot 5.3) we choose the 2×1 vector of ones as the parameter of the `ORTH`-function.

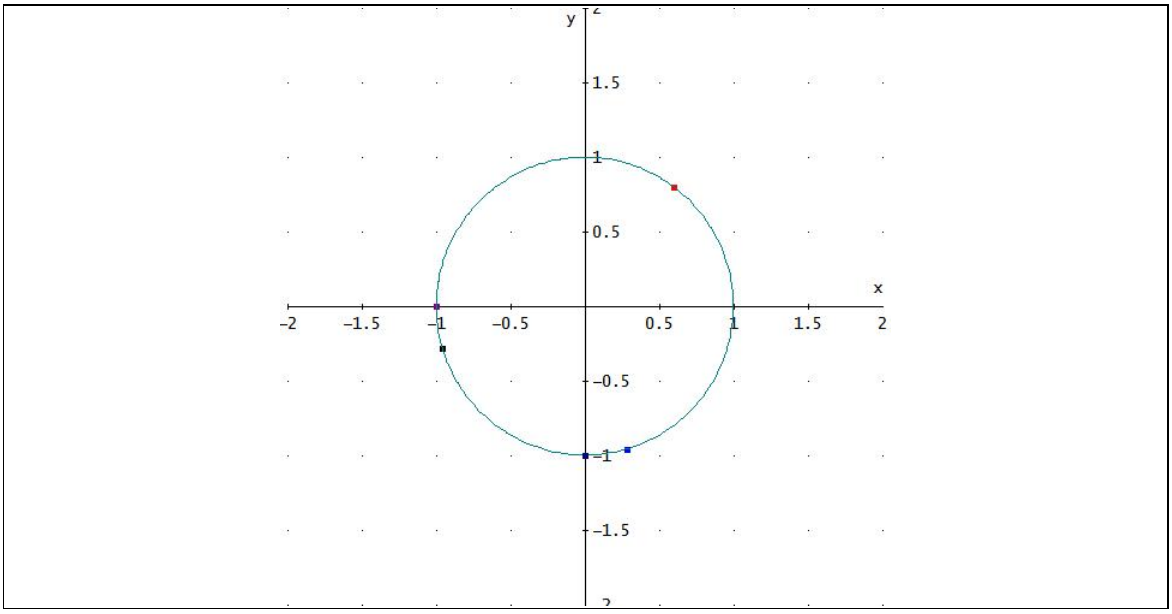
#7: $\mathrm{ORTH(1(2))} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

#8: $\mathrm{ORTH(1(2))}^{-1} - \mathrm{ORTH(1(2))}' = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

#9: $\mathrm{ORTH(1(2))}' \cdot \mathrm{ORTH(1(2))} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Screenshot 5.3

To complete this section we take advantage of the graphical capabilities of *DERIVE*. Any column of an orthogonal matrix has length 1, and any two are pairwise orthogonal. Since both examples are two-dimensional, its column vectors lie on the unit circle and form a right angle (Screenshot 5.4). The graph also shows the vector $\boldsymbol{b}$ from the first example.



Screenshot 5.4

Remarks

This paper is linked to a presentation given on June 26, 2009, at the *ACA 2009 Conference* in Montreal, Canada.
Some portions of this paper are also published in the Proceedings of the *7th Delta Conference* in Gordon's Bay, South Africa.

References

Schmidt, K., 2003, An Introduction to the Moore-Penrose Inverse of a Matrix. *The DERIVE-Newsletter #50*, 12–18.

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ACA09 – DERIVE session presenters, part 2

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