THE DERIVE - NEWSLETTER #78

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THE BULLETIN OF THE

USER GROUP

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July 2010

Interesting and recommended websites:

Find a collection of animations from Canada (Genevieve Savard, Montréal)

http://www.seg.etsmtl.ca/Math/Animations/index.html

Hello Josef, how are you doing?
Maybe you already know this: Albert Rich (and David Jeffrey) launched a new website :

http://www.apmaths.uwo.ca/RuleBasedMathematics.

Take a look at it.   It is so nice to see that our "good old Derive" – at least the spirit– is still alive!
Michel

See also:
http://groups.google.com/group/sci.math.symbolic/topics

http://www.apmaths.uwo.ca/~arich/

Guillermo Bautista collected a list of free pdf- and e-books:

http://math4allages.wordpress.com/tag/free-ebooks-pdf/

http://math4allages.wordpress.com/math-and-multimedia-in-facebook/

Another nice website:

http://www.mathcasts.org/

The World Lecture Hall:

http://wlh.webhost.utexas.edu/results.cfm?count=1&from=find&keywords=Mathematics

Math Department of North Carolina State University

http://www.math.ncsu.edu/index.php

Finite Math Online (Videos)

http://www.math.ncsu.edu/ma114/index.html

Using the Web to Teach Mathematics - Undergraduate Courses

http://www-math.ucdenver.edu/w4t/undergrad/undergrad

Mathematics at Dartmouth

http://www.math.dartmouth.edu

and specially

http://www.math.dartmouth.edu/~doyle/

Browse the On Line Tutorials and On Line Utilities

http://www.zweigmedia.com/RealWorld/

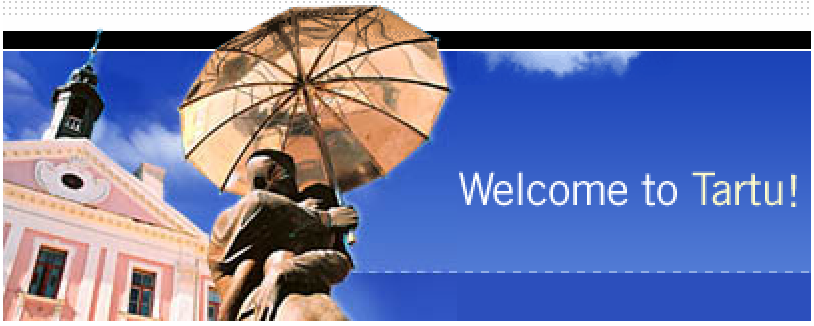
Find more Links on page 3

Dear DUG Members,

I know that this DNL is long overdue. It was a very busy June and July – preparing lectures for TIME 2010, exchanging emails concerning the contributions in this newsletter, and some days – hopefully well deserved – holidays in the mountains.

You can find some pictures from TIME 2010 on the last page of this DNL. I'll give an extended report in the next newsletter. I can assure that TIME 2010 in Málaga was a great conference which was excellent organized by José Luis Galan and his team (many thanks to Gabriel, Pedro, Yolanda and the students of the university).

It was a great pleasure for all of us that we could offer an invitation for TIME 2012 in the closing session of TIME 2010. Marina Lepp and Eno Tönisson provided a tempting presentation inviting us to come to Tartu, Estonia in July 2012.



I collected – and checked – some websites which might be of interest for you.
Please have a look to RUBI (A Rule Based Integrator), published by Albert Rich (and David Jeffrey) which is really impressive. You might meet RUBI again going to

http://groups.google.com/group/sci.math.symbolic/

At TIME 2010 I could attend an exciting lecture given by Ramon Eixarch. The WIRIS folks showed how to convert DERIVE files (mth- and dfw-files as well) to WIRIS. They explained some difficulties and problems and invited all interested people to test the program. You are invited to download the respective WIRIS-Version for free (until December) and report your experiences:

http://wiris.com/portal/project/derive

You will find two of my first experiments on pages 38 - 43.

Roland Schröder shows another activity for students. It is funny that I used a related simulation very often as an introductory example for probability theory. So I could not resist append this together with the NSpireCAS realisation which gave the opportunity for demonstrating its convincing statistics tools. Many thanks to Guido Herweyers for a very valuable advice wrt the spreadsheet treatment of the problem.

With my best regards until next time

The *DERIVE-NEWSLETTER* is the Bulletin of the *DERIVE* & CAS-*TI User Group*. It is published at least four times a year with a contents of 40 pages minimum. The goals of the *DNL* are to enable the exchange of experiences made with *DERIVE*, *TI*-CAS and other CAS as well to create a group to discuss the possibilities of new methodical and didactical manners in teaching mathematics.

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Contributions:
Please send all contributions to the Editor. Non-English speakers are encouraged to write their contributions in English to reinforce the international touch of the *DNL*. It must be said, though, that non-English articles will be warmly welcomed nonetheless. Your contributions will be edited but not assessed. By submitting articles the author gives his consent for reprinting it in the *DNL*. The more contributions you will send, the more lively and richer in contents the *DERIVE* & CAS-*TI Newsletter* will be.

Next issue: September 2010
Deadline 15 September 2010

Preview: Contributions waiting to be published

Some simulations of Random Experiments, J. Böhm, AUT, Lorenz Kopp, GER
Wonderful World of Pedal Curves, J. Böhm
Tools for 3D-Problems, P. Lüke-Rosendahl, GER
Financial Mathematics 4, M. R. Phillips
Hill-Encription, J. Böhm
Simulating a Graphing Calculator in *DERIVE*, J. Böhm
Henon & Co, J. Böhm
Do you know this? Cabri & CAS on PC and Handheld, W. Wegscheider, AUT
An Interesting Problem with a Triangle, Steiner Point, P. Lüke-Rosendahl, GER
Overcoming Branch & Bound by Simulation, J. Böhm, AUT
Diophantine Polynomials, D. E. McDougall, Canada
Graphics World, Currency Change, P. Charland, CAN
Cubics, Quartics – Interesting features, T. Koller & J. Böhm
Logos of Companies as an Inspiration for Math Teaching
Exciting Surfaces in the FAZ / Pierre Charland´s Graphics Gallery
BooleanPlots.mth, P. Schofield, UK
Old traditional examples for a CAS – what´s new? J. Böhm, AUT
Truth Tables on the TI, M. R. Phillips
Advanced Regression Routines for the TIs, M. R. Phillips
Where oh Where is It? (GPS with CAS), C. & P. Leinbach, USA
Embroidery Patterns, H. Ludwig, GER
Mandelbrot and Newton with *DERIVE*, Roman Hašek, CZ
Snail-shells, Piotr Trebisz, GER
A Conics-Explorer, J. Böhm, AUT
Practise Working with times
Huffman-Code with DERIVE and TI-CAS, J. Böhm, AUT
Tutorials for the NSpireCAS, G. Herweyers, BEL
Some Projects with Students, R. Schröder, GER
Dirac Algebra, Clifford Algebra, D. R. Lunsford, USA
Nonlinear Regression on the TI-NSpireCAS, M. R. Phillips, USA

and others

Impressum:
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More Links which might be of interest:

Visit the Computer Algebra Group at Simon Fraser University (Browse the Preprints)
http://www.cecm.sfu.ca/CAG/index.shtml

Graph Theory Tutorials (Univ. of Tennessee Martin)
http://www.utm.edu/departments/math/graph/

Web Educator's Library Collection of Mathematical Explorations
http://www.math.metrostate.edu/welcome/

Cardano: An Adventure in Algebra in 8 Parts
http://mathdl.maa.org/images/upload_library/4/vol2/microworlds/library/cardano.html

Dear ICMI Colleagues,
Here is the next issue of The Montana Mathematics Enthusiast [available free at the following links]
http://www.math.umt.edu/TMME/vol7no2and3/index.html

University of Nebraska-Lincoln: Mathematics needed for Mechanics, Engineering Statics, Mechanics of Elastic Bodies and Engineering Dynamics
http://emweb.unl.edu/

Lessons, Tutorials and Lecture Notes
http://archives.math.utk.edu/tutorials.html

An Optimization Course (University of Cambridge)
http://www.statslab.cam.ac.uk/~rrw1/opt/index98.html

Fibonacci Numbers and the Golden Section
http://www.maths.surrey.ac.uk/hosted-sites/R.Knott/Fibonacci/fib.html

Martin „Osterhase" (Easter Bunny)

Martin Osterhase informs us that under

http://groups.google.com/group/sci.math.symbolic/browse_thread/thread/6967a38606c67815

Version 1.4 of his opus "PSLQ for Derive" is ready for downloading.

(PSLQ is an algorithm which finds Integer Relations between real numbers $x_1, x_2, \dots, x_n$ such that $a_1x_1 + a_2x_2 + \dots + a_nx_n = 0$, with not all $a_i = 0$. Follow the thread then you will find respective pdf-files. There is a lot of information about PSLQ in the Internet, Josef)

Michel Beaudin, Montréal, Canada

… Also, version 2 of Nspire CAS software (on the PC) impressed me a lot, yesterday! For example, I defined 2 functions

f1(x) = a*x^2 +3*x+1 and f2(x) = 2*sin(b*x)

in the "Y editor" in function graphic mode and use the slider bar. With the new "math menu" for calculus (like finding the point of intersection of 2 curves), I note that the coordinates of the intersection point I found are updated when the slider bars (or in animate mode) are changed! Let's hope that 3D plotting AND 2D implicit plotting AND differential equations plotting will be added into version 3 (for example). Then, for us at ETS, it will become a very interesting tool. I just hope it won't be too long!

Michel

Michel Beaudin, Montréal, Canada

Did you note that Nspire CAS OS2 now solves (exactly) cubic equations? For example, try solve(x^3-3x-1=0,x). You will see that OS2 can find the 3 real roots, in exact arithmetic and seems to use Viete's formula. But the way the answers are written not as good as Derive will do. I emailed Elena Smirnova and also Gosia Brothers and told them how to get very easily the nice answers of Derive. The same occurs when only one root is real (try x^3 + 3x-1 = 0): Cardano's formula is used but, again, the answer of Nspire CAS is "ugly" compared to Derive (no radicals on the denominator in Derive: radicals in the denominator in Nspire ... and in Maple!).

TI-NspireCAS:

$$\text{solve}\left(x^3-3\cdot x-1=0,x\right)$$

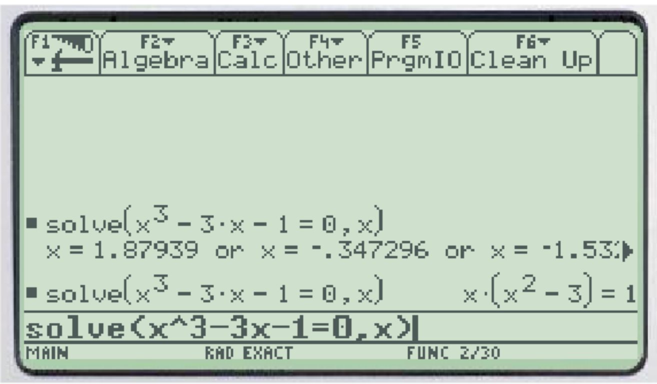
$$x=\frac{-\left(\cos\left(\frac{2\cdot\pi}{9}\right)+\sin\left(\frac{2\cdot\pi}{9}\right)\cdot\sqrt{3}+1\right)}{2\cdot\cos\left(\frac{\pi}{9}\right)} \text{ or } x=\frac{-1}{\cos\left(\frac{2\cdot\pi}{9}\right)+\sin\left(\frac{2\cdot\pi}{9}\right)\cdot\sqrt{3}+1} \text{ or } x=2\cdot\cos\left(\frac{\pi}{9}\right)$$

DERIVE:

$$\text{SOLVE}(x^3 - 3\cdot x - 1 = 0,\ x)$$

$$x = -\ 2\cdot\text{COS}\left(\frac{2\cdot\pi}{9}\right) \vee x = 2\cdot\text{COS}\left(\frac{\pi}{9}\right) \vee x = -\ 2\cdot\text{SIN}\left(\frac{\pi}{18}\right)$$

TI Voyage 200:



A Game with 3 Dice

Roland Schröder, Celle, Germany

Rolling three dice with different colours we can distinguish 6^3 = 216 different results. The possible sums of the three dice 3, 4, 5, …, 17, 18 occur with frequencies H(3), H(4), H(5), …, H(17), H(18). The value table which relates each sum to its frequency – the so called frequency distribution – can be obtained manually only with some effort. Here the computer will help to make work easier.

Solving the problem with DERIVE we will use the important VECTOR-fuction. A VECTOR is nothing else than a list of objects which can be presented by DERIVE e.g. numbers, variables, expressions, equations but also true vectors.

`VECTOR(a + b, b, 1, 6)` or `VECTOR(a + b, b, 6)`

gives the sequence (list) of all expressions a + b, with b running through the numbers from1 to 6. Simplifying this expression we obtain the sextuple:

`[a + 1, a + 2, a + 3, a + 4, a + 5, a + 6].`

Such sextuples can be elements of a vector of vectors. Try:

`VECTOR(VECTOR(a + b, a, 1, 6), b, 1, 6).`

Here we have a vector (list) of vectors (lists). In mathematics we call a vector of vectors a “matrix”.

$$\begin{bmatrix} 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 5 & 6 & 7 & 8 & 9 \\ 5 & 6 & 7 & 8 & 9 & 10 \\ 6 & 7 & 8 & 9 & 10 & 11 \\ 7 & 8 & 9 & 10 & 11 & 12 \end{bmatrix}$$

The above matrix shows the possible sums (6^2 = 36 results are possible). We can read off the frequencies, e.g. the sum 8 can appear 5 times (2+6, 6+2, 3+5, 5+3, 4+4). As we want to play with three dice we need to extend the VECTOR-expression by a third summand c and a third VECTOR-command with c running from 1 to 6:

`VECTOR(VECTOR(VECTOR(a + b + c, a, 1, 6), b, 1, 6), c, 1, 6).`

The result is too bulky for printing here. (You may do it in your DERIVE session). Instead of this we will count all possible occurences of the sum k. Like doing it manually we inspect all matrices and mark the numbers k. Then we count our marks.

For the marking we can use the IF-function of DERIVE: if a + b + c = k,
then write 1,
else write 0.

This is in DERIVE syntax: `IF(a + b + c = k, 1, 0).`

This shall be applied on the whole VECTOR of the six matrices from above. Function J does the job:

```
J(k):=VECTOR(VECTOR(VECTOR(IF(a + b + c = k, 1, 0), a, 6), b, 6), c, 6)
```

gives for k = 8 six matrices, which show a One on all places where a+b+c=8 and a Zero on all remaining places.

```
J(8)
```

$$
\left|\begin{array}{cccccc|cccccc|cccccc}
0&0&0&0&0&1&0&0&0&0&1&0&0&0&0&1&0&0\\
0&0&0&0&1&0&0&0&0&1&0&0&0&0&1&0&0&0\\
0&0&0&1&0&0&0&0&1&0&0&0&0&1&0&0&0&0\\
0&0&1&0&0&0&0&1&0&0&0&0&1&0&0&0&0&0\\
0&1&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
\hline
0&0&1&0&0&0&0&1&0&0&0&0&1&0&0&0&0&0\\
0&1&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
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0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{array}\right|
$$

Now we have to add the Ones column for column, row for row, and matrix for matrix, which gives the absolute frequency:

```
H(k):= ∑(∑(∑(J(k)))).
H(8) = 21
```

The value table of function H(k) is given by:

```
VECTOR([k, H(k)], k, 3, 18).
```

Transposing the table (exchanging columns and rows) gives a result which is easier to print. (The Transpose-Operator is the ` character!)

```
VECTOR([k, H(k)], k, 3, 18)`
```

$$
\begin{bmatrix}
3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18\\
1 & 3 & 6 & 10 & 15 & 21 & 25 & 27 & 27 & 25 & 21 & 15 & 10 & 6 & 3 & 1
\end{bmatrix}
$$

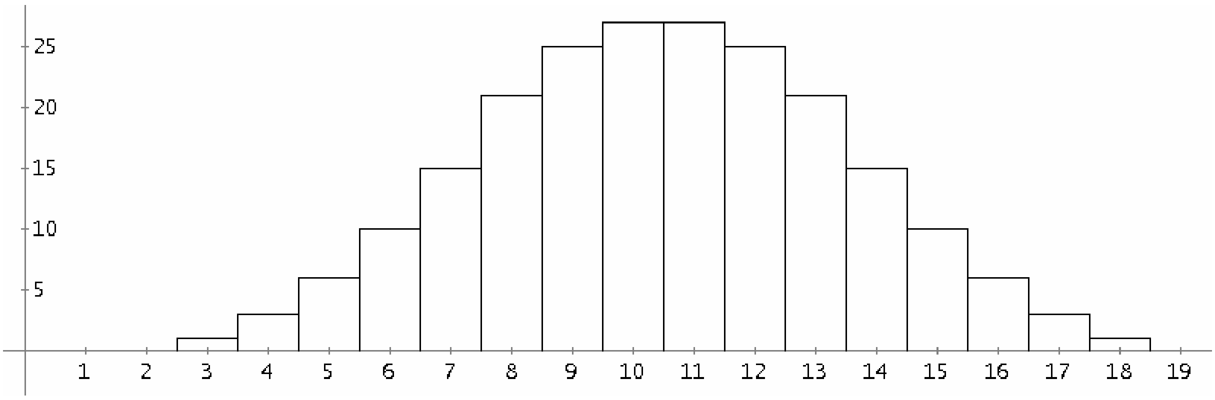
The histogram for representing the frequencies of the sum of the dice can be generated as a VECTOR (list, sequence) of rectangles (= boxes) and then be plotted.

```
BOX(k) ≔ [k - 1/2, 0; k - 1/2, H(k); k + 1/2, H(k); k + 1/2, 0]
BOXES ≔ VECTOR(BOX(k), k, 3, 18)
```

Highlight the last expression, switch to the Plot-Window and plot the histogram showing the absolute frequencies; for plotting the relative frequencies you have to replace H(k) by H(k)/216 in the BOX(k) – function.

```
BOXR(k) ≔ [k - 1/2,0;k - 1/2,H(k)/216;k + 1/2,H(k)/216;k + 1/2,0]
BOXESR ≔ VECTOR(BOXR(k), k, 3, 18)
```

The histogram of the absolute frequencies:



A Game with 3 Dice (2)

Josef Böhm, Würmla, Austria

It is mere chance that I used this game many years as an introductory – and motivating – problem for probability theory. I asked my students as follows:

Assume I am allowed to offer you a game of chance: I am the banker. You can roll three dice and I will pay the winnings depending on the sum of the dice according to the following table:

Sum	0 – 12	13	14	15	16	17	18
Payment	0 €	1€	2€	3€	4€	5€	6€

What would you be willing to pay for one game (having the chance to win up to 6 €)?

What do you think did the students offer? What would you offer?

I recommend to bring some dice and let the students play the game. Within 30 minutes you can collect many data if there are some groups rolling, counting and noting the results (sums and possible payments of the banker).

You can gather all results in order to obtain an approximation for the distribution of the sums and of the payments and then calculate the average of the sums and of all the payments. How much must I charge for one game that I will gain some profit on a long-term basis? I am quite sure that some of the students will now reconsider their offers.

Until now we didn´t need any technology. In the next step we will perform a simulation of this game which allows playing it some thousand times.

I will present the DERIVE version and the TI-92/V200 version as well.

We start with DERIVE and write a fine program for simulating the GAME.

```
#1:   [lsums :=, lwins :=]

      game(num, dummy, roll, sum, win, i := 0) :=
        Prog
          dummy := RANDOM(0)
          lsums := []
          lwins := []
          Loop
            i :+ 1
#2:         If i > num exit
            roll := VECTOR(RANDOM(6) + 1, k, 3)
            sum := ∑(roll)
            win := IF(sum < 13, 0, sum - 12)
            lsums := ADJOIN(sum, lsums)
            lwins := ADJOIN(win, lwins)
          RETURN "ready, sums in lsums, winnings in lwins"
```

The program is not very difficult and it should be possible to develop it together with the students. (Note that `lsums` and `lwins` are global variables!)

```
#3:   game(1000) = ready, sums in lsums, winnings in lwins
```

Simplifying `game(1000)` delivers the results of 1000 rolls collected in two lists: `lsums` is the list of the dice sums, and `lwins` is the list of the banker´s payments. These two lists can be used for several statistics investigations including graphic representations. I recommend applying the many useful statistics tools provided in DNLs #45, #46 and #50.

But you can do without as follows:

```
#4:   SELECT(k = 4, k, lsums) = [4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4]

#5:   DIM(SELECT(k = 4, k, lsums)) = 14

#6:   VECTOR(DIM(SELECT(k = i, k, lsums)), i, 3, 18)

#7:   [4, 14, 34, 42, 59, 116, 119, 122, 131, 98, 91, 77, 51, 24, 15, 3]
```

#8: $\text{VECTOR}\left(\left[i, \frac{\text{DIM(SELECT}(k = i, k, \text{lsums}))}{\text{DIM(lsums)}}\cdot 100\right], i, 3, 18\right)'$

We have 14 outcomes of the sum 4, #6 gives the list of the absolute frequencies (the H(k)s in Roland's contribution), and #8 produces a table of the relative frequencies (in %):

#9:

$$\begin{bmatrix} 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 \\ 0.4 & 1.4 & 3.4 & 4.2 & 5.9 & 11.6 & 11.9 & 12.2 & 13.1 & 9.8 & 9.1 & 7.7 & 5.1 & 2.4 & 1.5 & 0.3 \end{bmatrix}$$

This was not the question. The question was, what I – as the banker – should charge for one game to have no loss on the long run. So let's calculate the average of all winnings:

```
#10:  AVERAGE(lwins) = 0.587
```

This was one run with 1000 games. We will perform some simulations more:

```
#11:  game(5000) = ready, sums in lsums, winnings in lwins

#12:  AVERAGE(lwins) = 0.605

#13:  game(10000) = ready, sums in lsums, winnings in lwins

#14:  AVERAGE(lwins) = 0.583

#15:  AVERAGE(lsums) = 10.5047
```

10000 rolls needed 28.3 seconds. It seems to be that I should ask for more than 60 Euro Cent for one game. Then I would make some profit.

The average sum is about 10.5.

I always hoped that this result made the students curious how to find the exact value of the average(s). Then you could proceed with Roland's counting procedure or/and without technology (which is not too difficult):

sum 18 = € 6.00	(6,6,6)	1 permutation
sum 17 = € 5.00	(6,6,5)	3 permutations
sum 16 = € 4.00	(6,6,4), (5,5,6)	3+3 = 6 permutations
sum 15 = € 3.00	(5,5,5), (6,4,5), (6,6,3)	1+6+3 = 10 permutations, etc.

How to use the TI handhelds?

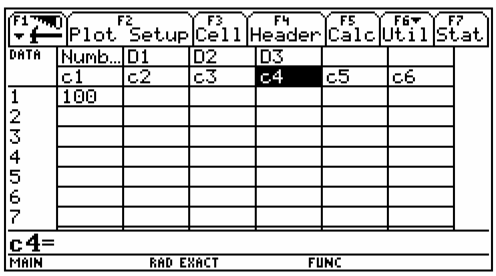
It is no problem to transfer the program to the TI-92 / V200. I believe that it has more didactical value to use the `Data/Matrix Editor` or the `Stats/List Editor Application` than running only a ready made program.

Open a new Data sheet, say `dice`.

Enter the headers of the first four columns.

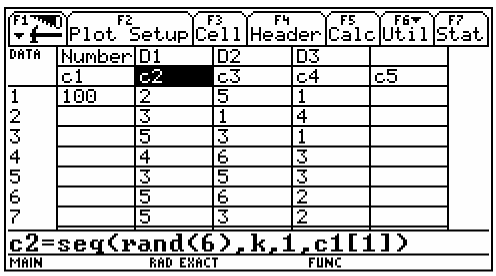
To have the simulation as general as possible we enter the number of experiments in cell `r1c1` = `c1[1]`.

Columns c2, c3 and c4 are provided for the dice.



We generate sequences of integer random numbers from {1,2,3,4,5,6} in columns c2, c3 and c4 consisting of `c1[1]` elements.

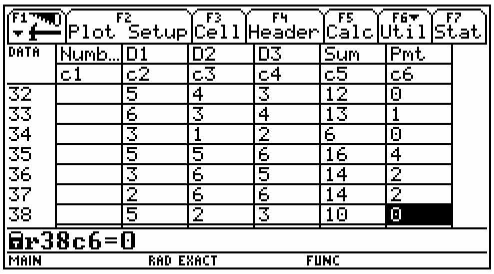
(Note the difference in generating integer random numbers compared with DERIVE.)



Column `c5 = c2 + c3 + c4`, which is the sum of the dice.

Column `c6` is a little bit complicated, because we cannot do it the easy way, like

`c6=when(c5≤12,Ø,c5-12)`.



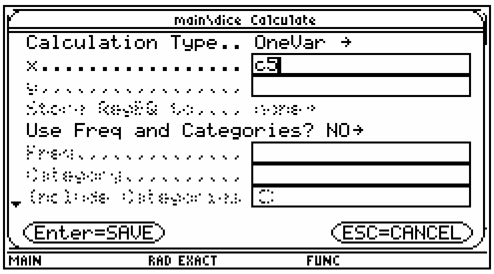
We have to address the single elements and create another sequence:

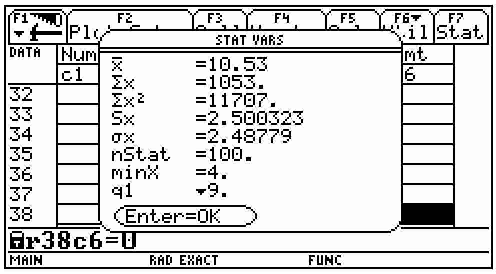
`c6=seq(c5[k]≤12,Ø,c5[k]-12),k,1,c1[1]).`

I recommend turnig Auto-calculate off, otherwise each key press would cause a new calculation and deliver new results – and would take some time.

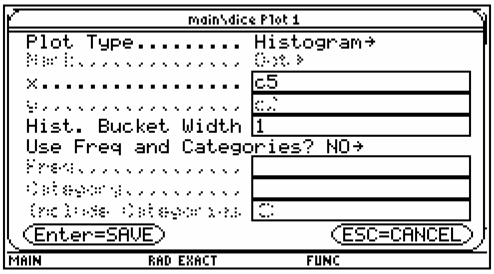
The TIs offer a lot of tools for performing statistics investigations:

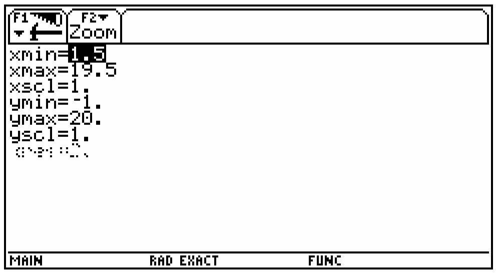
For investigating the distribution of the sums press F5 and choose `OneVar` as `Calculation Type`:

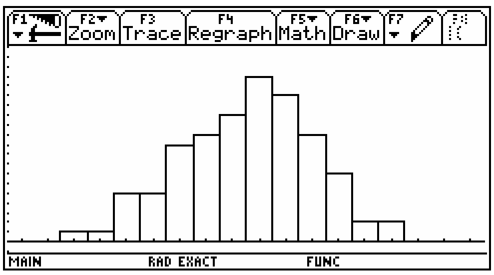


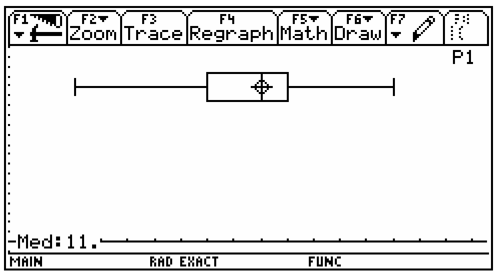


Via F2 we have access to graphic representations as histogram and box plot:

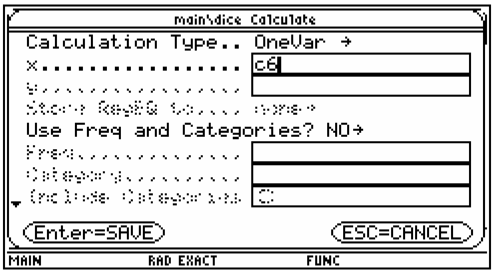


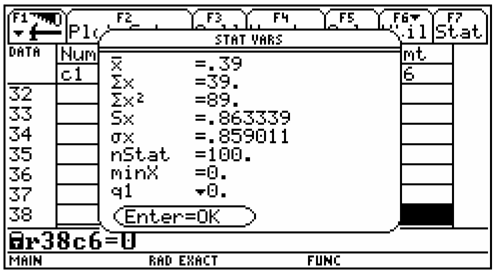






I come back to my special game asking for the bet. What we have to do is easy enough, just change the column in the Calculate feature and repeat the procedure.

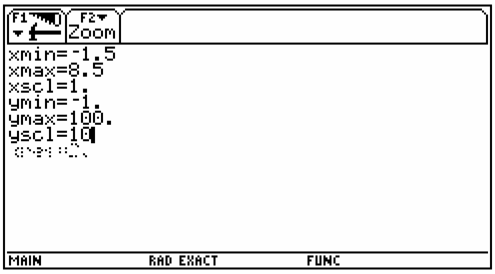


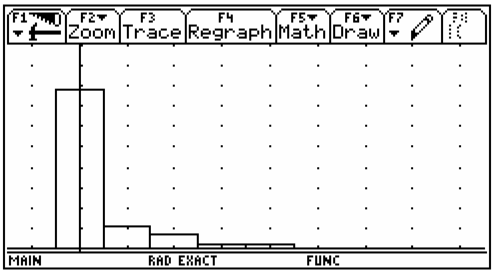


The average of 0.39 is much less than the DERIVE result(s) from above. We had only 100 rolls. Assuming that 20 students are in your class, you can collect the averages to calculate an average of 2000 games. What is the outcome now?

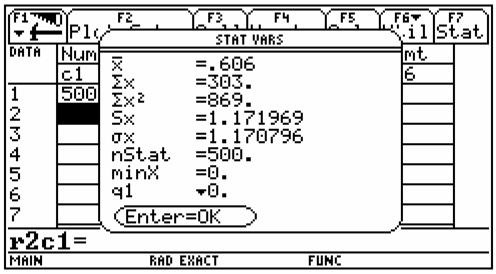
Please take in account that you have to set different randseeds in the home screen. Otherwise it could happen, that all student will get the same random numbers.

The histogram shows that the gamblers would loose their bet in 78 games out of 100.





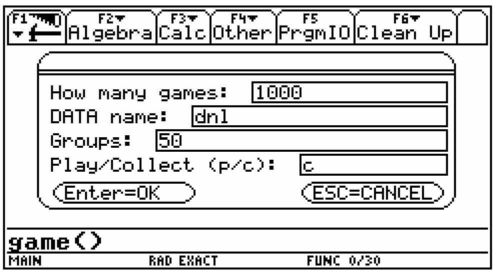
It does not take too much time to simulate 500 games. The average payment is now 0.606. And this fits quite good to the DERIVE results.

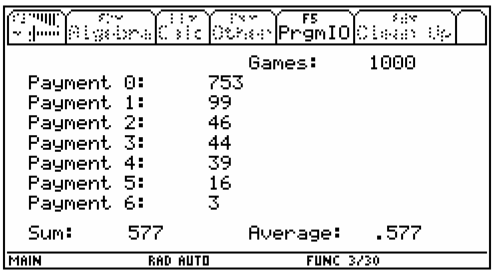


The exact expected value of the payment is

$$E(\text{payment}) = \frac{0\cdot 160 + 1\cdot 21 + 2\cdot 15 + 3\cdot 10 + 4\cdot 6 + 5\cdot 3 + 1\cdot 6}{216} = \frac{126}{216} \approx 0{,}583\,.$$

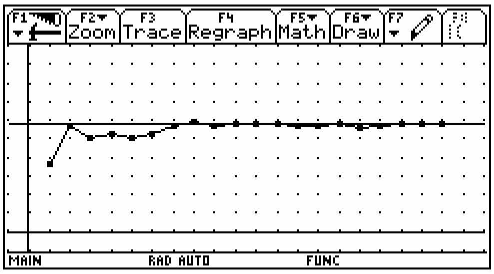
This could be a program with a nice graphic representation. Enter `game()` in the home screen and do 1000 throws of the dice.





After every series of 50 games we calculate the cumulated average and let it collect in list c2 of the DATA sheet , in c1 we will find the number of the games.

The result is now much closer to the theoretical expected value.



The xyline graph demonstrates how the cumulated averages converge towards the expected value (horizontal line).

This is the TI-program.

You may try to convert it to a NSpire program.

The next pages show a possible realisation with TI-NspireCAS.

It would be great if you would send us your specific realisation for publication in the DNL.

```
game()
Prgm
Local  w1,w2,w3,sp,i,sw,lg,sg,θn,l1,l2,gdat,θs,dn
{0,0,0,0,0,0,0}→lg:0→sg
{}→l1:{}→l2
ClrIO
Dialog
Request  "How many games",sp
Request  "DATA name",dn
Request  "Groups",θn
Request  "Play/Collect (p/c)",θs
EndDlog
expr(sp)→sp:expr(θn)→θn
ClrIO
Output  0,120,"Games:"
Output  10,10,"Payment 0:"
Output  20,10,"Payment 1:"
Output  30,10,"Payment 2:"
Output  40,10,"Payment 3:"
Output  50,10,"Payment 4:"
Output  60,10,"Payment 5:"
Output  70,10,"Payment 6:"
Output  85,10,"Sum:"
Output  85,120,"Average:"
Output  0,180,0
For  i,1,7
Output  10*i,100,0
EndFor
Output  85,60,0
Output  85,180,"0.00"
Output  40,120,"Start: [ENTER]"
Pause
Output  40,120,"              "
For  i,1,sp
rand(6)→w1:rand(6)→w2:rand(6)→w3
w1+w2+w3→sw
sw-11→sw
If  sw≤0:1→sw
lg[sw]+1→lg[sw]:sg+sw-1→sg
Output  0,180,i
Output  10*sw,100,lg[sw]
Output  85,60,sg
Output  85,180,"           "
Output  85,183,approx(sg/i)
If mod(i,θn)=0 Then
  augment(l1,{i})→l1
  augment(l2,{sg/i*1.})→l2
EndIf
If  θs="p": Pause
EndFor
NewData  #dn,l1,l2
EndPrgm
```

The NSpire-version gives the opportunity to demonstrate some nice features of the latest version. The first screen shot shows the program together with some explanations.

© On the right hand side you can find one possible realisation of the GAME–simulation.

© Information and simulation procedure will be performed in the Notes;
(klick on the green right arrow in the icon bar above)

© Page 3 offers graphic representations of the distribution of the dice sums and the payments. Just click on the variable at the bottom and change it as you like.

© Page 4 shows the acumulated averages of the payments and demonstrates the convergence towards the expected value of the GAME $\left(=\frac{126}{216}\sim 0.583\right)$. The WINDOW–values must be adjusted.

© On Page 5 you can find a spreadsheet realisation of the simulation. Change the number of experiments in cell A1. Ctrl+R gives a new simulation.

4/99

game 16/16

```
Define game(n,gs,a)=
Prgm
Local d1,d2,d3,sw,w
n_:=n
RandSeed a
lsum:={}:lwin:={}:lavg:={}
lgs:=seq(k·gs,k,1,n/gs)
For i,1,n
d1:=randInt(1,6):d2:=randInt(1,6):d3:=randInt(1,6)
sw:=d1+d2+d3
lsum:=augment(lsum,{sw})
w:=when(sw≤12,0,sw−12)
lwin:=augment(lwin,{w})
If mod(i,gs)=0 Then
lavg:=augment(lavg,{mean(lwin)})
EndIf
EndFor
lavg:=approx(lavg)
EndPrgm
```

This is the main page: The Notes including some Math Boxes (blue and red). You can change the program arguments in the blue printed expression, press enter and you will obtain another simulation. The next pages offer graphic representations.

The GAME

Edit the variables as follows: game(number of games, grouping for cumulated average payment, randomseed = integer number), e.g game(200,20,12345)

game(2000,50,123456) ▸ *Done*

Result:

payment 0 = 1481

payment 1 = 196

payment 2 = 130

payment 3 = 104 Average payment after 2000 games = 0.58

payment 4 = 57

payment 5 = 30

payment 6 = 2

The following lists are available for further use in **Data & Statistics**, **Lists & Spreadsheet** and **Graphs**.

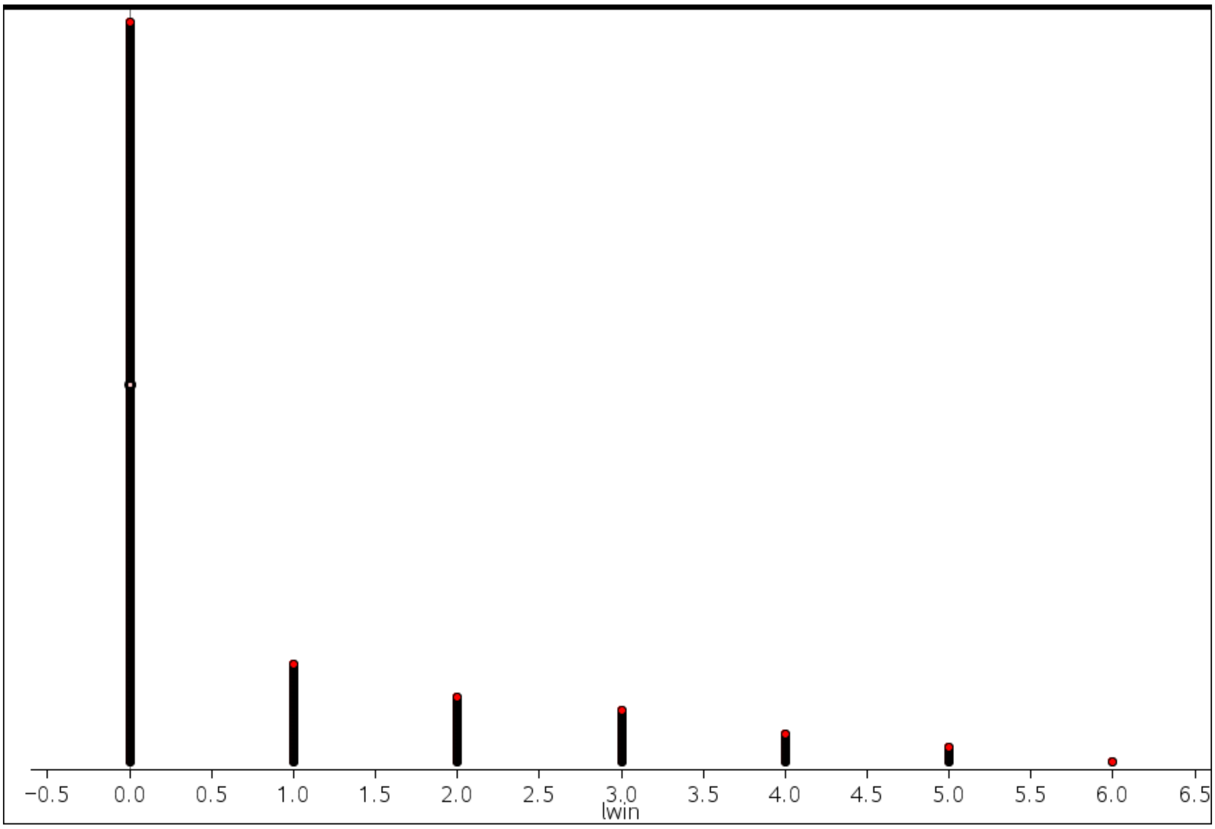
lsum = list of dice sum

lwin = list of payments

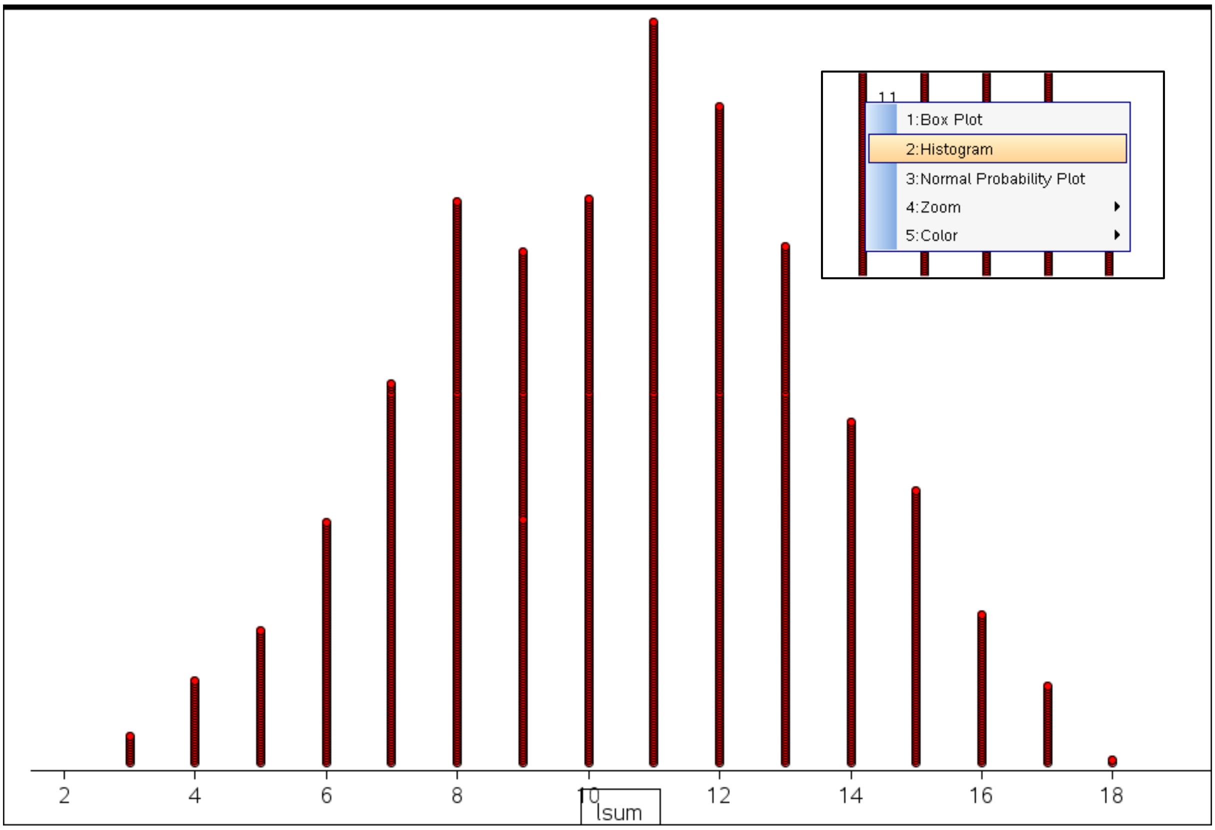
lavg = list of cumulated average payments according to the given group size

lgs = list of the group boundaries

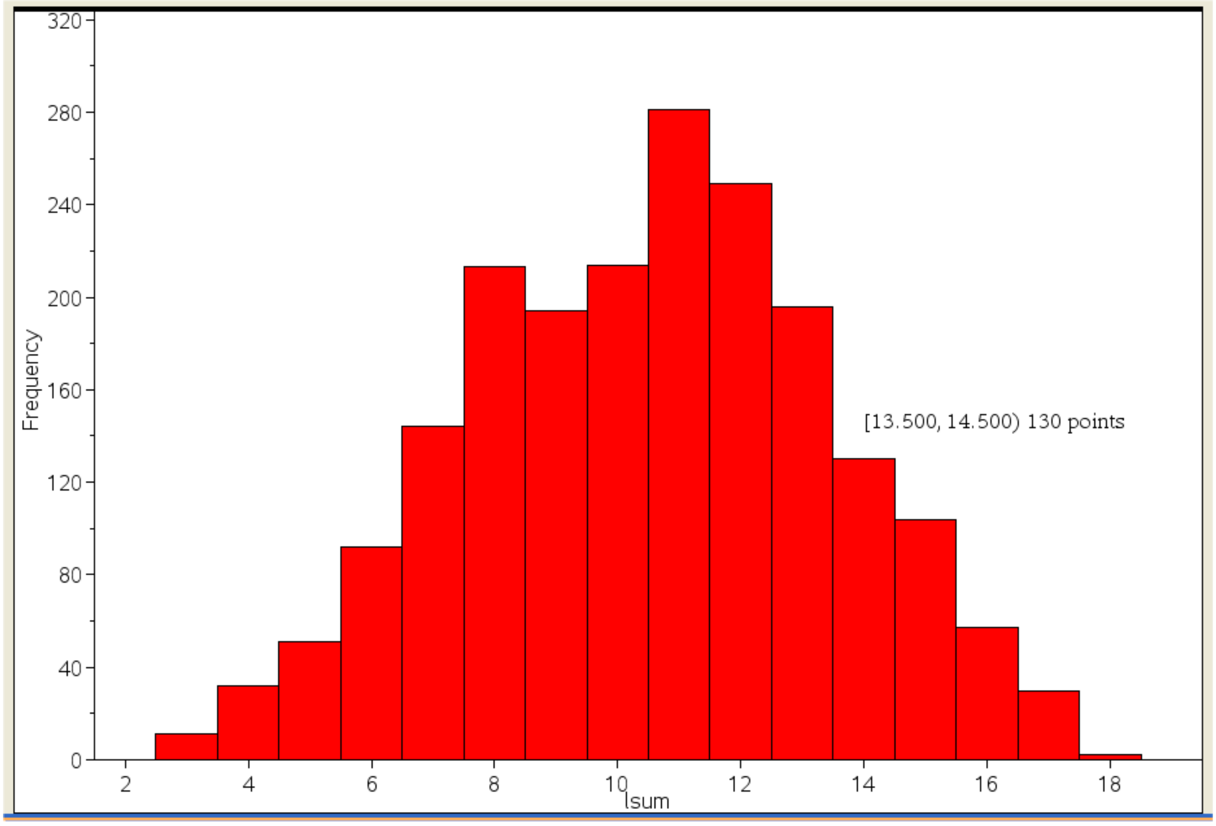
In the Data & Statistics Application you can create various diagrams (Dot Plot, Histogram or Box Plot). Just Click on the variable name and change the provided list.



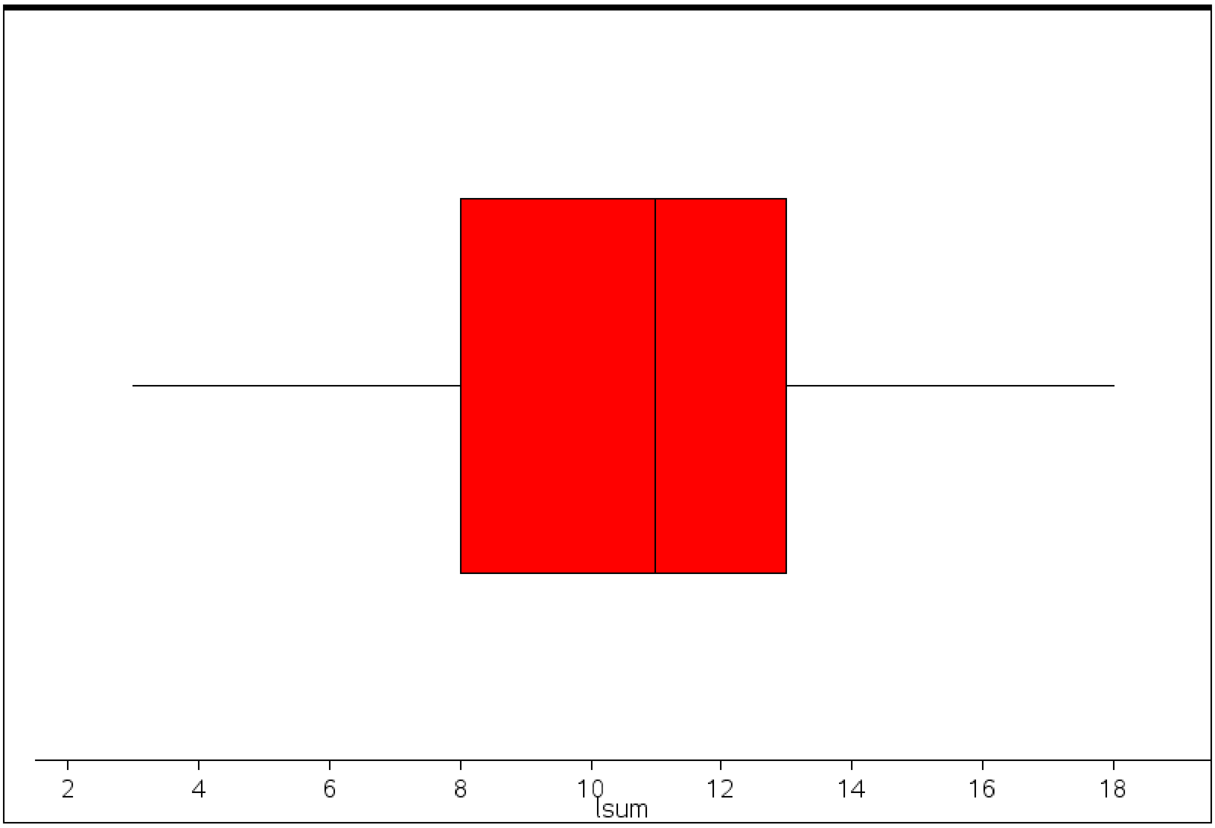
One step gives the distribution of the dice sums. A right mouse click on the diagram opens a window which allows to change the form of the diagram (e.g. to a Histogram).



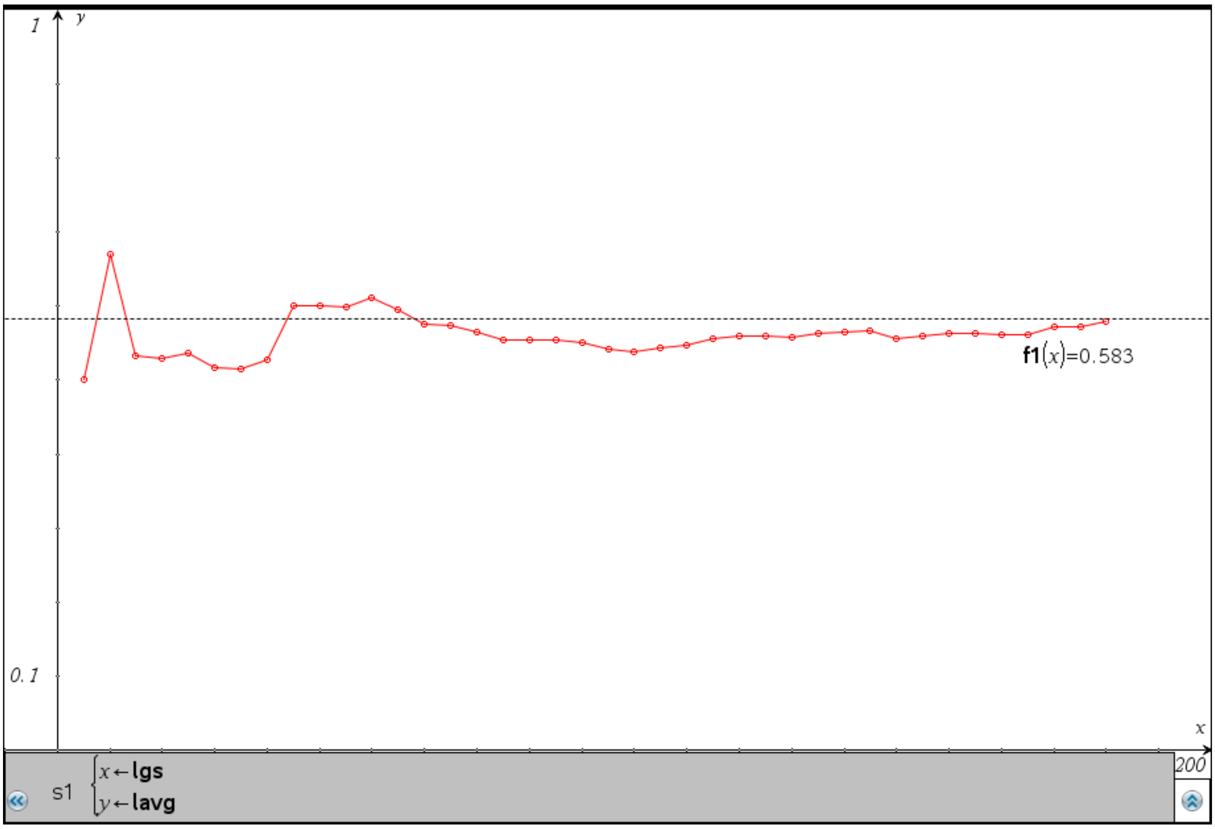
This is the Histogram:



and here is the Box Plot:



In Graphs we show the convergence to the expected value. The WINDOW-values can be adjusted via the WINDOW-settings or you can use the comfortable Zoom Data option.



The spreadsheet simulation:

	A rolls	B die_1	C die_2	D die_3	E dice_sum	F payment	G average	H	I
◆		=randint(1	=randint(1	=randint(1	=die_1+die_2	=seq(when(c			
1	1000	1	6	5	12	0	0.59		
2		3	4	3	10	0			
3		6	4	4	14	2			
4		5	3	1	9	0			
5		1	4	4	9	0			
6		6	6	3	15	3			
7		1	5	5	11	0			
8		5	5	1	11	0			
9		6	5	4	15	3			
10		5	2	6	13	1			
11		1	2	6	9	0			
12		6	2	5	13	1			
13		3	2	6	11	0			
14		2	2	1	5	0			
15		3	4	2	9	0			
16		6	3	2	11	0			
17		1	1	3	5	0			

G1 =approx(mean(**payment**))

Headers of cols B, C and D are: randint(1,6,a1),

header of col F is: seq(when(dice_sum[k]≤12,0,dice_sum[k] – 12),k,1,a1).

Titbits(38) - On Some Problems Related with Binary Sequences

Johann Wiesenbauer, Vienna, Austria

Let's start with a simple hat problem that some of you might already know. It's about a team of 3 players who are randomly assigned blue or red hats so that each player can see the hats of the others, but not his own. They are supposed to make either a guess about the colour of their hats or pass, and will win as a team, if and only if at least one is right and none is wrong. The thing is that they can talk about the best strategy before the test, but as soon as it starts, no communication whatsoever is allowed. Well, it's obvious, that if any previously selected member of the team is chosen to guess the colour of his hat and the others pass, they will have a fifty-fifty chance to win. But can they do better? If you are looking at this problem for the first time, you might doubt this. Surprisingly enough, there is a strategy that is much better, indeed, and here is where coding theory comes into play.

First of all, we should identify the colours red and blue with 0 and 1, respectively, and give each player a unique number from 1 to 3, so that any assignment of hats is actually one of the 8 equally likely binary triples. Hence it might be a good idea to provide routines for conversions from decimal to binary and the other way round. Usually, as in this example, we will need also a fixed block length k for the binary representation.
(If it is omitted in the routine below, then the length of binary word is chosen in a way that binary representation of the given n just fits in.)

```
      bin(n, k := -1, s_ := "") :=
        Loop
          If k = 0
             RETURN s_
#1:       s_ := APPEND(IF(MOD(n, 2) = 0, "0", "1"), s_)
          n := FLOOR(n, 2)
          If n = 0 ∧ k < 0
             RETURN s_
          k :- 1

#2:   bin(43) = 101011

#3:   bin(43, 8) = 00101011

      dec(s, d_ := 0) :=
        Loop
          If s = ""
#4:          RETURN d_
          d_ := 2·d_ + IF(FIRST(s) = "1")
          s := REST(s)

#5:   dec(101011) = 43

#6:   dec(00101011) = 43
```

Then the set S of all possible assignments is given by

```
#7:   S := MAP_LIST(bin(n, 3), n, 0, 7)

#8:   S := [000, 001, 010, 011, 100, 101, 110, 111]
```

As it is well-known, S becomes a metric space, if we define the distance d(u,v) of two words u and v as the number of positions where they differ.

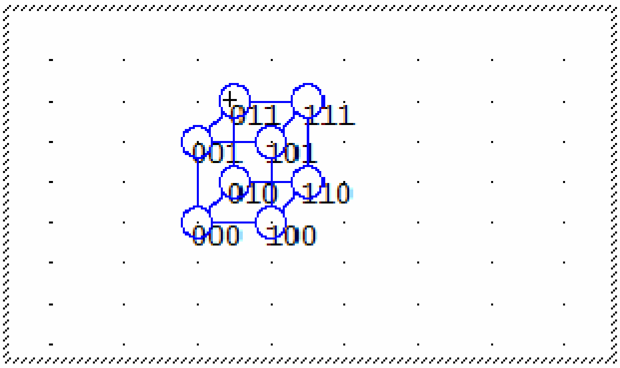
$$\#9: \quad \mathrm{dist}(x, y) := \sum_{k\_=1}^{\mathrm{DIM}(x)} \mathrm{IF}(x_{k\_} \neq y_{k\_})$$

In particular, for each binary word v we can define the sphere with center w and radius r to be the set all of binary words s.t. $d(w,v) \le r$.

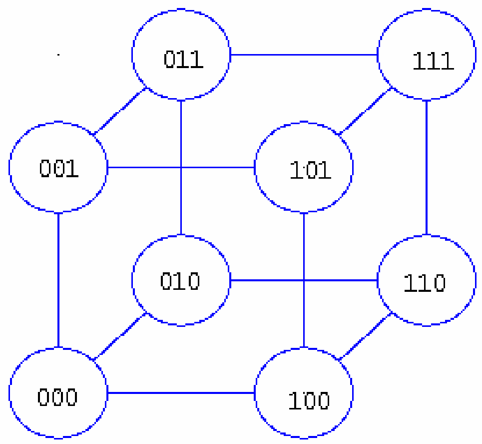
$$\#10: \quad \mathrm{sphere}(v, r) := \mathrm{SELECT}(\mathrm{dist}(v, w) \le r, w, S)$$

What will be needed in the following is that there is a dense packing of our set S with spheres of radius 1, in other words there is a subset C of S, e.g. C = {000,111} (in fact, you can take any two words with maximal distance 3), such that each word in S is in exactly one of the spheres centred at the words in C.

$$\#11: \quad [\mathrm{sphere}(000, 1), \mathrm{sphere}(111, 1)] = \begin{bmatrix} 000 & 001 & 010 & 100 \\ 011 & 101 & 110 & 111 \end{bmatrix}$$



Ok, a little clumsy yet, but if you read this in the Derive file and click on it, it becomes like this:



Actually, what took me most of the time when writing these Titbits were the Derive routines to produce this sort of graphics. Since I will need them also later on and they are quite versatile, I will list them in the following so that you can also make use of them.

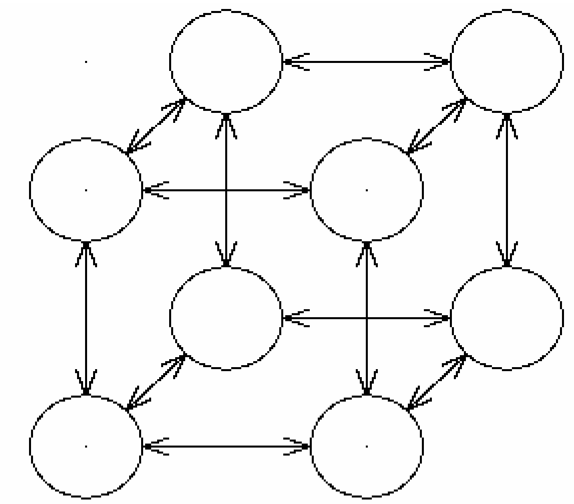
Here connect( ) and circle( ) are two auxiliary routines to draw edges and loops (the latter only in directed graphs, if there are any at all). The meanings of the parameters are:

- u, v in connect() and circle() are vertices given by their coordinates in the x-y-plane
- r is the radius of the circles used for the nodes of the graph
- dir is a Boolean variable that say whether the graph is directed or not
- d and k are referring to the shape of the arrows and usually you will omit them using the default values
- opt in the routine circle can take the values 1,2,3,4 and will provide different shapes of loops
- v and a in graph() are the list of vertices (again given by their coordinates in the x-y-plane) and the adjacency lists of the nodes, while opts is a collection of opt-parameters, which are used in exactly in the given order, if the graph has got any loops.

When using these routines for the first time you should study the example below, which shows how the graphics above was produced, and also make extensive use of default values, which are reasonable most of the time. There is one thing you must still do manually though, which is writing the names of the nods in the graphics using the function key F12.

```
       connect(u, v, r := 0.2, dir := false, d := 20, k := 2, u_, v_, w_, x_, y_) :=
         Prog
           w_ := [(r/ABS(u - v) - 1/2)·(t/π)·(u - v) + (u + v)/2]
           If ¬ dir
              RETURN w_
           u := r/(k·ABS(u - v))·(u - v)
#12:       x_ := COS(π·d/180)
           y_ := SIN(π·d/180)
           v_ := [u·[x_, -y_], u·[y_, x_]]
           w_ := ADJOIN(connect(v + k·u, v + k·u + v_, 0), w_)
           v_ := [u·[x_, y_], u·[-y_, x_]]
           ADJOIN(connect(v + k·u, v + k·u + v_, 0), w_)

       circle(v, r, opt := 1, v_, w_) :=
         Prog
           If opt = 1
              w_ := [v - [r, r] + r·[COS((5·π - 3·t)/4), SIN((5·π - 3·t)/4)]]
           If opt = 2
              w_ := [v - [-r, r] + r·[COS((7·π + 3·t)/4), SIN((7·π + 3·t)/4)]]
#13:       If opt = 3
              w_ := [v + [r, r] + r·[COS((π + 3·t)/4), SIN((π + 3·t)/4)]]
           If opt = 4
              w_ := [v - [r, -r] + r·[COS((3·π + 3·t)/4), SIN((3·π + 3·t)/4)]]
           v_ := [r·[0.05, 2.1], r·[-0.05, 2.1], r·[-0.05, -2.1], r·[0.05, -2.1]]↓opt
           APPEND(w_, connect(v - v_, v, r, true))
```

```
graph(v, a, r := 0.2, dir := false, d := 20, k := 2, opts := [], a_, k_ := 0, o_, s_, v_, w_) :=
  Prog
    s_ := VECTOR([v_ + r·[COS(t), SIN(t)]], v_, v)
    v_ := v
    Loop
      If v_ = []
         RETURN s_
      k_ :+ 1
      a_ := FIRST(a)
      If MEMBER?(k_, a_)
#14:         Prog
           a_ := a_ \ {k_}
           o_ := 1
           If opts ≠ []
              [o_ := FIRST(opts), opts := REST(opts)]
           s_ := APPEND(s_, circle(v↓k_, r, o_))
      w_ := APPEND(VECTOR(connect(FIRST(v_), v↓a__, r, dir, d, k), a__, a_))
      s_ := APPEND(s_, w_)
      v_ := REST(v_)
      a := REST(a)

#16:  a1 := [{2, 4, 5}, {1, 3, 6}, {2, 4, 7}, {1, 3, 8}, {1, 6, 8}, {2, 5, 7}, {3, 6, 8}, {4, 5, 7}]

#17:  graph(v1, a1, 0.2)
```



You might think I have forgotten about our hats problem, but I haven't and I don't want to keep you on tenterhooks anymore. Actually the solution is quite simple: Each player should check by what he can see if the actual binary triple, corresponding to the assignment of hats, could be one of the two centers 000 and 111 of our spheres above, i.e., in the chosen subset *C*. If the answer is no, which is the "normal case" as it were, he should pass, otherwise he is supposed to guess, whether it is the center actually or one of the 3 others words in the same sphere. Of course, he will guess the colour of his hat in a way s.t. the resulting word is NOT in *C*, as the chances of being right is 3/4 in this way. Note that at least one of the 3 players is given this chance as it is a dense spheres packing, and they will make a wrong guess in case the actual triple is a in our chosen subset *C*.

Usually the winning strategy is given in the following way easy to remember: Pass, if you see 2 different colours (hence, the current assignment can't be 000 and 111 anymore) and guess, if you see two hats of the same colour, namely the other colour then.

As you might guess there is a nice generalization of this problem to the case of n people, where n is of the Form n=2^m-1. It's easy to see that only for an n of this form there can be again a dense packing with spheres of radius 1, as only in this case the number of binary words in a sphere is a power of 2, which is a necessary condition as this number must be a divisor of the total number 2^n of words. But is it already sufficient? Some of you might already know that this condition also guarantees the existence of such a dense sphere packing, indeed, but what might come as a surprise (at least, I became aware of this fact only recently) that there is also an amazingly simple greedy algorithm of computing a possible set *C* of centers of the sphere (of course, there are many).

As shown below, all you have to do is to start with the empty set *C* and adjoin a new word, if and only if it both has a distance of at least 3 to all words already in *C* and is minimal in the lexicographic order among all words this property (or what amounts to the same thing, it's decimal equivalent is minimal w.r.t. to this property).

```
        densepacking(m, c_ := {}, s_) :=
          Prog
            s_ := {0, ..., 2^m - 1}
            Loop
#18:          If EVERY(dist(c__, bin(FIRST(s_), m)) ≥ 3, c__, c_)
                 c_ := ADJOIN(bin(FIRST(s_), m), c_)
              s_ := REST(s_)
              If s_ = {}
                 RETURN c_

#19:    densepacking(3) = {000, 111}

#20:    densepacking(7)
#21:    {0000000, 0000111, 0011001, 0011110, 0101010, 0101101, 0110011,

          0110100, 1001011, 1001100, 1010010, 1010101, 1100001, 1100110,

          1111000, 1111111}

#22:    DIM(densepacking(7)) = 16

#23:    MAP_LIST(dec(k_), k_, densepacking(7))

#24:    {0, 7, 25, 30, 42, 45, 51, 52, 75, 76, 82, 85, 97, 102, 120, 127}
```

Hence, if n is of the form n = 2^m-1, then a team of n players could use the set of corresponding to that n (for example, the list of 16 code words above, if n = 7), and carry out exactly the same strategy: Only if one of the players can tell, from what he sees, that the assignment could be a codeword, he will guess, namely that it is not a codeword, otherwise he will pass. Only if they are unlucky and it is a codeword indeed, they will lose. The probability for this case is only 1/2^m though.

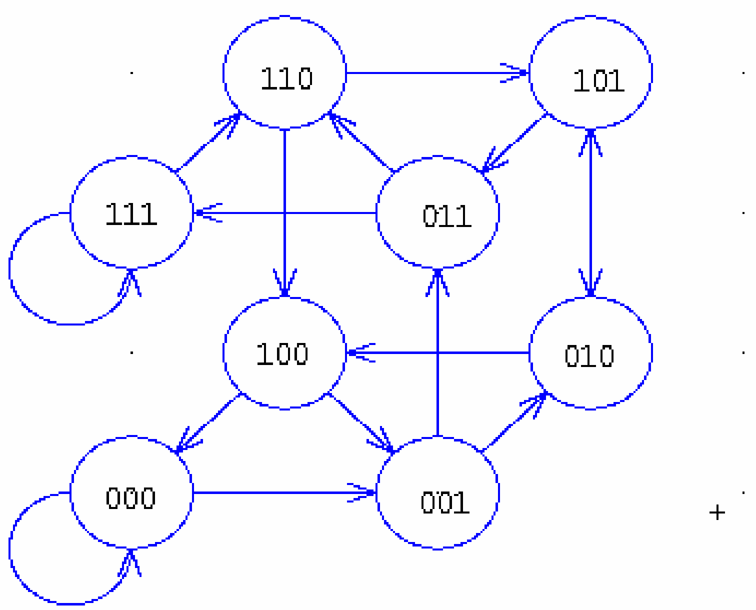
Note that usually all will pass except for one, who will guess the colour of his hat. There is a nice morale from this story, which Berlekamp put in the following way:
"If the evidence suggests someone on your team knows more than you, you should keep your mouth shut".

As the experts here know, there are many things you could learn from this example, in fact, it could be the start of a lecture on coding theory, but I don't have the time and the space to do so. I don't want to quit though without another nice example of the graphic routines above, which leads to the following directed graph.

#25: $v3 := \begin{bmatrix} 0 & 1 & 1.5 & 0.5 & 1.5 & 0.5 & 0 & 1 \\ -0.5 & -0.5 & 0 & 0 & 1 & 1 & 0.5 & 0.5 \end{bmatrix}$,

#26: a3 := [{1, 2}, {3, 8}, {4, 5}, {1, 2}, {3, 8}, {4, 5}, {6, 7}, {6, 7}]

#27: graph(v3, a3, 0.2, true)



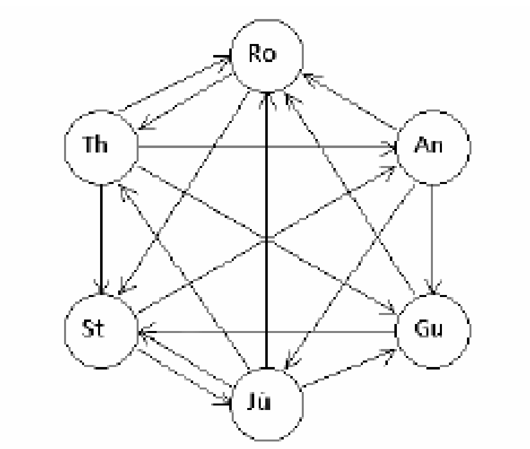
The graph is closely connected to the following question: If there is a wheel with a window s.t. only 4 binary digits are visible in this and after rotating the wheel a little bit (in the literal sense of the word!) the first bit will disappear and at the end a new one (either 0 or 1) will appear, is it possible to arrange the 16 bits in such a way that each binary word of length 4 occurs exactly once? I leave you with this question alone, just one hint: Find an Euler tour in the graph above, where the edges (not the vertices!) describe those rotations by 1 bit, and use it to construct that sequence.

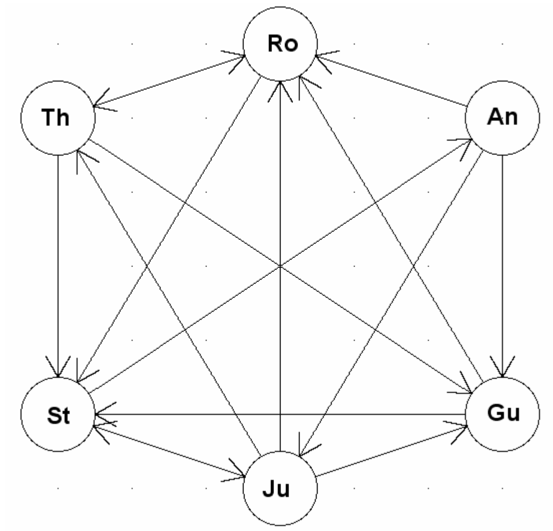
I met Johann at TIME 2010 and he offered to give some instructions how to use his tool plotting directed and undirected graphs. Unfortunately we didn´t find time enough, so I tried it on my own.
I took some graphs from our textbook “Mathe mit Gewinn” which seemed suitable for applying this tool. My first attempts failed – but then suddenly I had the flash in my brain – and it worked.

You will find the original from the textbook at the left and an the right the DERIVE-product.

$$\#29:\quad v4 := \begin{bmatrix} 0 & 3 & 3 & 0 & -3 & -3 \\ -3 & -2 & 2 & 3 & 2 & -2 \end{bmatrix},$$

```
#30:  a4 := [{2, 6, 4, 5}, {4, 6}, {2, 1, 4}, {5, 6}, {4, 2, 6}, {3, 1}]

#31:  graph(v4, a4, 0.5, true, 30, 1.5)
```

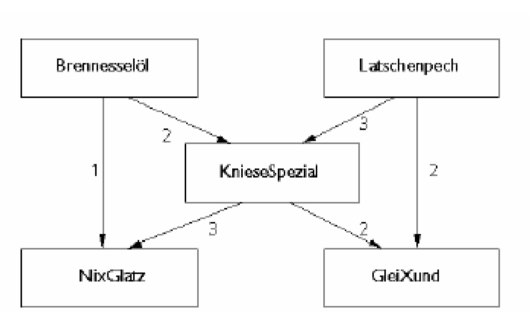




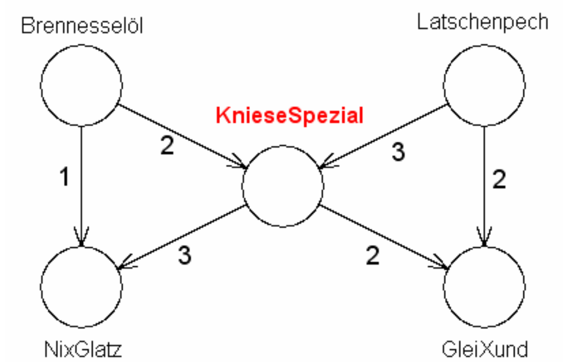
Note that I did not use the default values for k (angle in deg) and d (parameter for the barbs).

$$\#32:\quad v5 := \begin{bmatrix} 0 & 1 & -1 & -1 & 1 \\ 0 & 0.5 & 0.5 & -0.5 & -0.5 \end{bmatrix},$$

```
#33:  a5 := [{4, 5}, {1, 5}, {1, 4}, {}, {}]

#34:  graph(v5, a5, 0.2, true)
```



Kniesebeins Graph zur Mischung der Elixiere



Nodes which are end points of the graph (nodes #4 (NixGlatz) and #5 (GleiXund) are included in the adjacens list as empty sets {}.

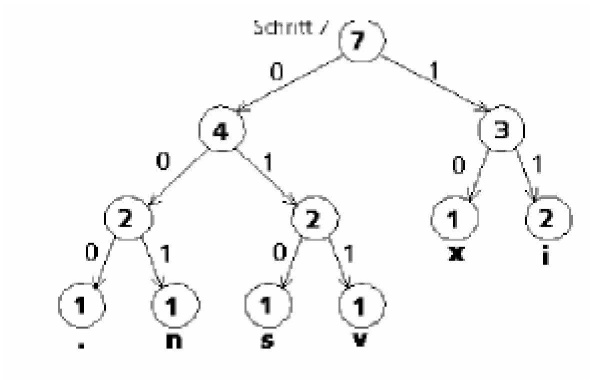
More Examples

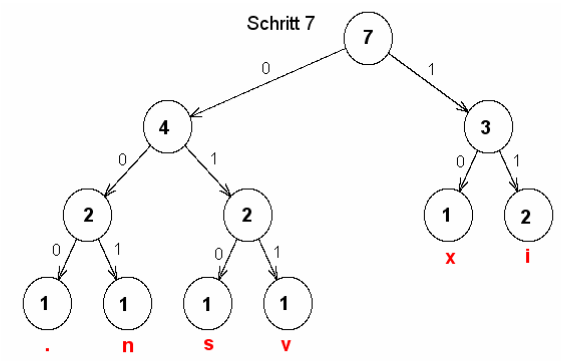
This is a part of a Huffman-Tree (from Coding Theory):

#35: huff7_v := $\begin{bmatrix} 0 & -2.5 & 1.5 & -3.5 & -1.5 & 1 & 2 & -4 & -3 & -2 & -1 \\ 3 & 2 & 2 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$,

#36: huff7_a := [{2, 3}, {4, 5}, {6, 7}, {8, 9}, {10, 11}, {}, {}, {}, {}, {}, {}]

#37: graph(huff7_v, huff7_a, 0.3, true)



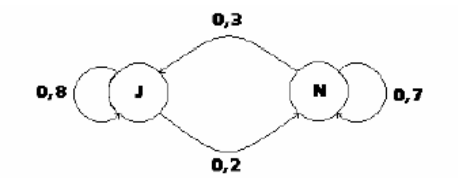


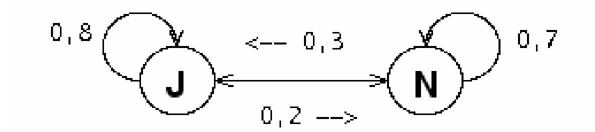
If the arrows have different weights in both directions then we would need two parallel arrows, which is not so easy to achieve. I helped myself by including the direction in the labels. Note the opts-list (last parameter) for assigning the appropriate loops (= "circle-arrows").

#37: v6 := $\begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}$

#38: a6 := [{1, 2}, {1, 2}]

#39: graph(v6, a6, 0.3, true, 20, 2, [4, 3])

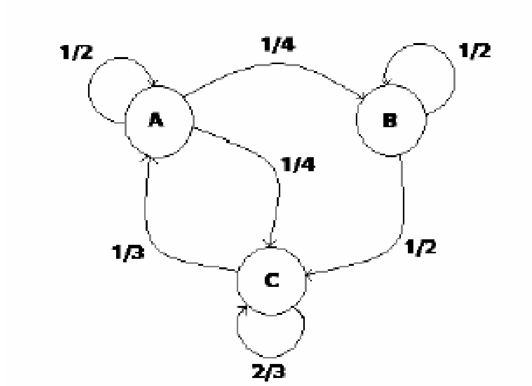


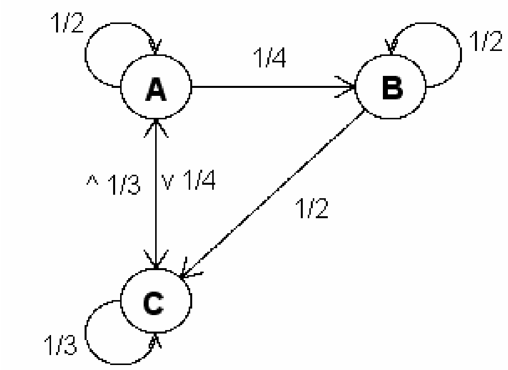


#40: v7 := $\begin{bmatrix} -1 & 0 \\ 1 & 0 \\ -1 & -2 \end{bmatrix}$

#41: a7 := [{1, 2, 3}, {2, 3}, {1, 3}]

#42: graph(v7, a7, 0.3, true, 30, 1.5, [4, 3, 1])





SPACE-FILLING CURVES

G P Speck, Wanganui, New Zealand

27Apr2010

From: G P Speck
37 Lincoln Rd
Wanganui, NZ

Dear Joseph:

I recall that you were always interested in the graphing features of DERIVE. I have just come across some elementary space-filling graphs (using the "Greatest Integer Step Function") that I discovered by myself decades ago, and I checked to see what DERIVE would do in reproducing these graphs. I thought that I would get a message such as "Sorry cannot plot." But to my surprise DERIVE produced very good graphs of these functions in many cases. In other cases too many stray "spaghetti lines" (which had to be ignored) were present in the DERIVE graphs. The graphs consist of solid squares of points which can be positioned along a line, a parabola, a circle, a hyperbola etc.

I think the comments in the preceding paragraph should be enough so that you would know whether or not you have seen such graphs, either in some text or in some submissions to you as Editor of DERIVE. Perhaps such graphs are now "old hat".

In any case, I am under no illusions that I am the only one who has discovered these graphs over the years.

Thus, if you have not seen graphs as described above and want me to forward some to you I will do so.

I am currently tutoring students sitting scholarship exams and they find DERIVE more than useful in preparation for these exams. Unfortunately this tutoring takes me away from considering interesting reading in DERIVE articles and elsewhere.

Some time ago I was preparing an article on SUDOKU after correspondence with you. I had almost wrapped up the article when other considerations intervened. My conclusion in the article was that thousands upon thousands of SUDOKU problems could be created from a couple hundred given in a newspaper. I must get back to finalizing that article.

Hope all is going well with you, Joseph.

Sincerely,

G P Speck

Dear friend, 29 April 2010

Many thanks for your extended mail.
I am always interested in graphics (with or without using DERIVE). It would be great if you could send one or the other samples of your findings. I am quite sure that it would fit into one of the next DNLs.

We had an article on SUDOKUS – but only for solving them – in one of our earlier Titbits. A contribution on how to create problems would form a wonderful completion.

Best regards from the other side of the globe,
Josef

SPACE-FILLING CURVES VIA GREATEST INTEGER FUNCTION (GIF)

Josef: The following concerns our email exchange on space filling curves.

I think you will find the 21 pages labelled P1 through P21 and stapled together self-explanatory. I will welcome your comments on the GIF graphs displayed. Have you seen such graphs in mathematics texts or published elsewhere? (I'll be surprised if you haven't.) They are certainly elementary in the extreme.

The two `VECTOR(VECTOR)` charts labelled Chart I and Chart II may require some comment. It's obvious that we needn't have print-outs of charts such as Chart I and Chart II but rather can view them from the computer screen. However, for purposes of explaining the use of these charts without sitting beside some one at a computer and viewing the screen it's an expeditious alternative to have these print-outs before us.

The `VECTOR(VECTOR)` statement in the upper left-hand corner of each Chart produces its matrix as shown upon using Simplify. The statement in general is of the form

`VECTOR(VECTOR(F(x,y),x,a,b,ε),y,c,d,ε)`

where the F(x,y) is the function in a F(x,y) = 0 representation of a curve whose graph we wish to analyze using DERIVE. Thus the curve whose graph we wish to analyze in Graph I (page 36) is given by

`GIF(y) - GIF(x)^3 = 0`

and the associated `VECTOR(VECTOR)` statement is `GIF(y) - GIF(x)^3`.

Hence the 30×30 = 900 entries in Chart I are values of `GIF(y) - GIF(x)^3` at the 900 points (x,y) given by x varying from -1 thru 1.9 in steps of 0.1 along with y varying from -1 thru 1.9 in steps of 0.1. From Chart I we see further that there are 300 points along a diagonal at which `GIF(y) - GIF(x)^3 = 0`. These are the only points covered by Chart I at which `GIF(y) - GIF(x)^3 = 0` or equivalently, at which `GIF(y) = GIF(x)^3`.

If we now view the reproduced DERIVE graph of `GIF(y) = GIF(x)^3`, we see that all of our considerations in viewing Chart I have a direct bearing on testing whether or not it is "high likely" that the DERIVE graph as shown is a "reasonable" one. Via Chart I we have tested 100 evenly distributed (x,y) points in each of the three squares nearest the origin on the DERIVE graph for `GIF(y) = GIF(x)^3` and found that each of these (x,y) points does satisfy `GIF(y) = GIF(x)^3`.

I have stated previously that the lines that DERIVE displays between the filled squares of the graph `GIF(y) = GIF(x)^3` are spurious. This is easy to establish (with some reservations due to the impossibility of reading coordinates on a DERIVE graph REALLY ACCURATELY!) by defining `F(x,y):= GIF(y) - GIF(x)^3` and testing some points on these lines; e.g. testing F(2,6), F(-1,-4), F(-1,-3), etc. None of these is equal to zero as would have to be the case if the point being tested to lie on the graph of F(x,y) = 0.

Now returning to our Charts I and II. Most of the same types of comments that were made on Chart I re `GIF(y) - GIF(x)^3` can be made on Chart II (page 37) re `GIF(x)^2 - GIF(y)^2 - 1` as well. However Chart II emphasizes even more strongly than Chart I that while the 0's in any Chart matrix are certainly the most significant numbers in such a matrix, the non-zero numbers therein can be quite significant as well. For example, in Chart II five of the nine basic squares are filled with -1's. This tells us that if we replace the `VECTOR(VECTOR)` function with the original function with 1 added to it, the resulting Chart matrix will have the five basic squares which formerly contained all -1's now contain all 0's, with the effect that the new graph will have five associated space filled unit squares in the positions where the -1's used to appear! This is not so surprising when we note that the new equation we are dealing with is `(GIF(x) - GIF(y))·(GIF(x) + GIF(y)) = 0`! This gives two GIF "lines" intersecting as shown on the reproduced graph (see page 34). That the CHART II stair step lines emanating from the two filled squares are spurious can be established as before by defining

`F(x,y):=GIF(x)^2 - GIF(y)^2 - 1`

and testing points on these lines: e.g. F(2,2), F(3,2.5), F(-2.5,-2), etc.

I have found the production of matrices of functional values as used here particularly helpful in dealing with “thorny” graphs of complicated functions (much more complicated than the space-filling graphs considered in this paper).

Thus, Josef, you can see from the examples and analysis methods given herein that DERIVE can often give a quite decent graphical representation of many space-filling curves given via the Greatest Integer Function. In other cases the representations contain spurious points and lines but still give some useful information. In any case the graphs are “interesting” and easy to produce using DERIVE. Take any standard (or even not so standard) Calculus text curve given in the form $F(x,y) = 0$, replace x with GIF(x) and y with GIF(y) to get F(GIF(x),GIF(y)) = 0, and see what DERIVE will produce!

Let me know what you think of the material herein, Josef.
We can proceed from there.

Sincerely,

George Speck

SPACE FILLING CURVES via GIF

First, some preliminary remarks: Since the content of this paper is well within the scope of secondary school mathematics, some of the comments herein will be directed at secondary school mathematics students.

It is assumed that you will be viewing a DERIVE screen in which the left half of the screen will be a 2D-Plot window and the right half of the screen will be an Algebra window containing comments, expressions, definitions, etc., some of which you are now viewing. If there is any question on starting with these two windows see Chapt 2 of the DERIVE 6 manual or the DERIVE 5 manual.

To prepare the 2D-Plot window on your left for plotting the initial GIF "curves" to be defined, click on the 2D-Plot Titlebar, click on Set, click on Plot Range, Click on Length/Center and enter the following

Horizontal: Length 18, Center 0, Intervals 18
Vertical: Length 18, Center 0, Intervals 18

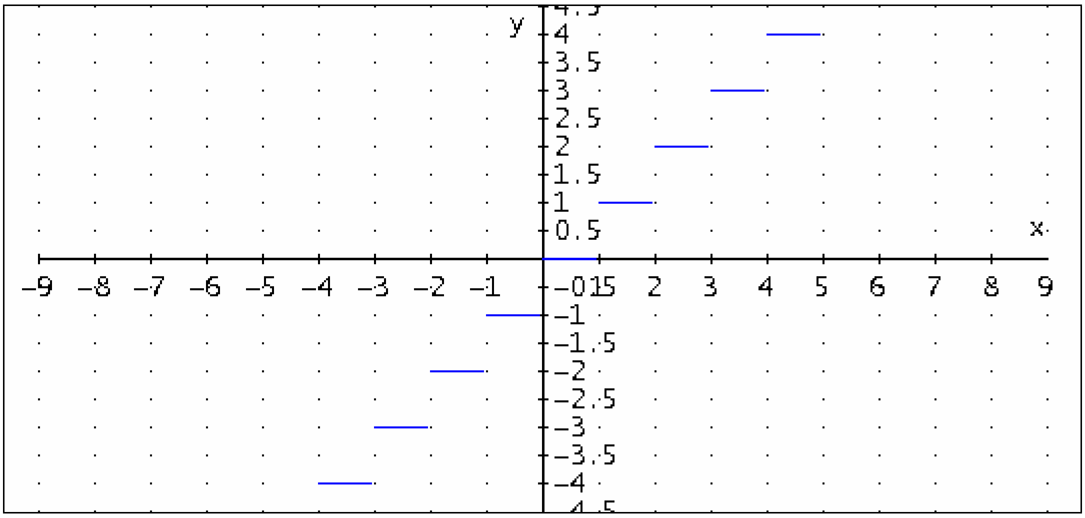
The "Greatest Integer Function" (American terminology) referred to in the title of this paper is equivalent to the DERIVE FLOOR step function, where FLOOR(x) denotes the greatest integer less than or equal to x for each real number x.

Now define GIF(x) as FLOOR(x) in the line following and place y = GIF(x) in the next line.

```
#1:   GIF(x) ≔ FLOOR(x)
#2:   y = GIF(X)
```

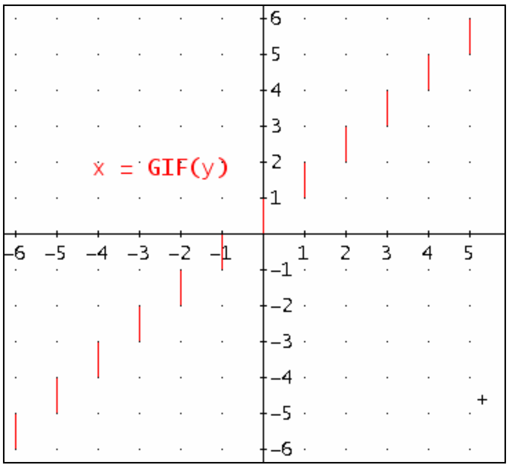
To display a portion of the graph of y = GIF(x) in the 2D-Plot window: highlight line #2, click on the Plot window TitleBar, click Options-Display-Color-Next color: Blue, click OK to close Display Options, click Insert-Plot, and then Insert the Annotation y=GIF(x) into the Plot window.

Finally, Embed the Plot into this Algebra window by clicking File-Embed in the Plot window to produce the Embedded Plot following.



Double click on the embedded plot above to see its image in a cleared 2D-plot window on the left. Drag the image to fill the plot window. See the plot above.

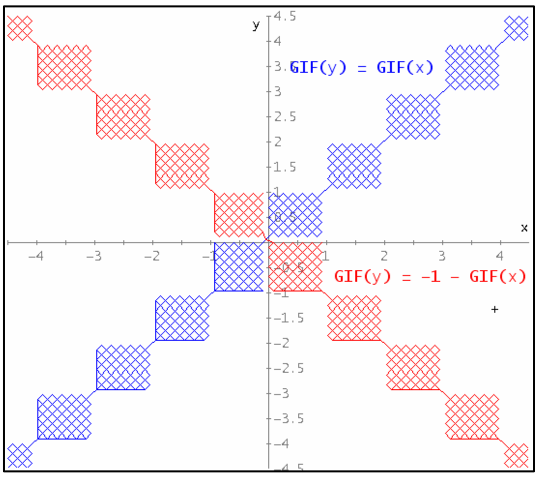
Proceed in a similar manner to that employed in producing the graph of y = GIF(x) above to produce the Embedded Plot of x = GIF(y).



Double click on the embedded plot to see its image in the cleared 2D-plot window on the left. Drag the image to fill the plot window.

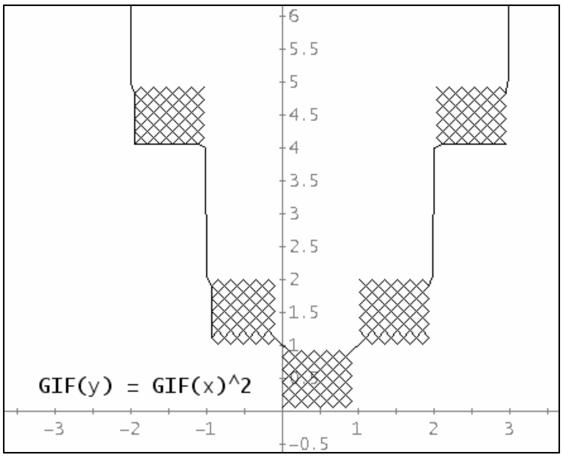
Produce the space-filling lines plotting both following equations (both x and y are replaced respectively by GIF(x) and GIF(y) to produce our first examples of GIF-Space-Filling Curves.

```
#3:    y = x

#4:    GIF(y) = GIF(x)

#5:    y = −1 − x

#6:    GIF(y) = −1 − GIF(x)
```



Produce the following embedded plot for the equation immediately above in a manner similar to the production of the prior graphs. Its equation could be thought of as being formed from the equation of the parabola y = x^2 by replacing the y by GIF(y) and the x by GIF(x).
Thus we could describe it as a GIF "parabola".

#7: $\mathrm{GIF}(y) = \mathrm{GIF}(x)^2$

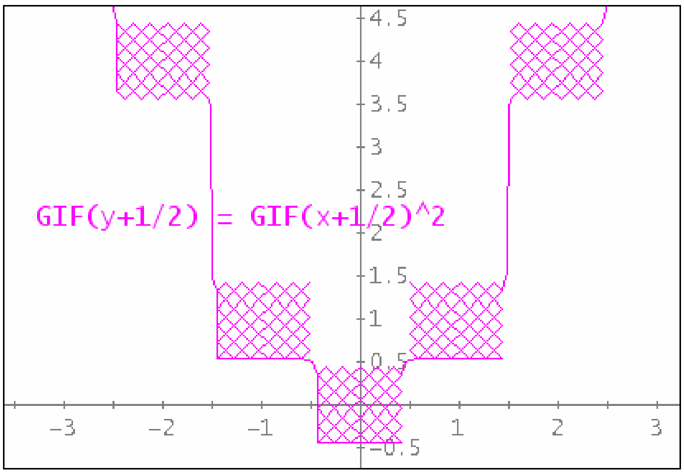


When we double click on the Embedded Plot to produce its full-scale equivalent (after dragging) in the 2D-plot window on the left, the five shaded square regions are distinctly square-like, in contrast to the corresponding five small "blobs" on the embedded version.

The tendril-like lines between separated squares are characteristic of many of the GIF graphs that I have produced. These lines are spurious and are to be ignored. They can be eliminated in an elementary but a bit tedious way, so I am not going to proceed with this matter in this article. Also, I have used a matrix of computations in many examples to bolster the contention that the tendril-lines are spurious. The two short tendrils pointing upwards at the top of the current graph indicate that some square regions are present above the visible plot region.

GIF(y+1/2) = GIF(x+1/2)^2 below represents the graph of GIF(y) = GIF(x)^2 above translated 1/2 unit to the left and down 1/2 unit as shown in the embedded blue graph following the equation. Elementary transformations such as translations, reflections and rotations continue to be able to be applied to space filling curves to modify their positions in the expected manner. (See another transformation on page 32.)

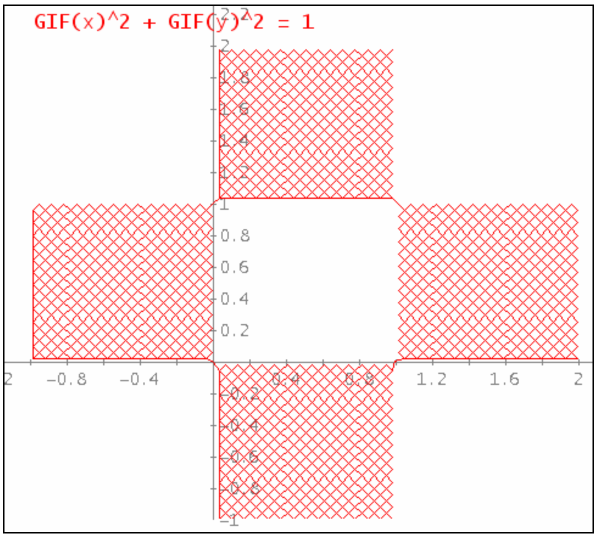
8: $\mathrm{GIF}\left(y + \frac{1}{2}\right) = \mathrm{GIF}\left(x + \frac{1}{2}\right)^2$

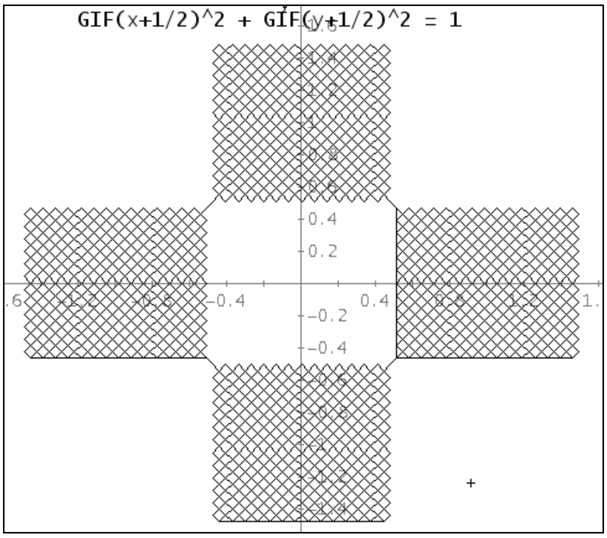


The next equation produces a space filling GIF "circle" as you might surmise. The "circle" is embedded following the equation. Then we will have a translation of this “GIF” circle which centers the “circle” about the origin.

#9: $\mathrm{GIF}(x)^2 + \mathrm{GIF}(y)^2 = 1$

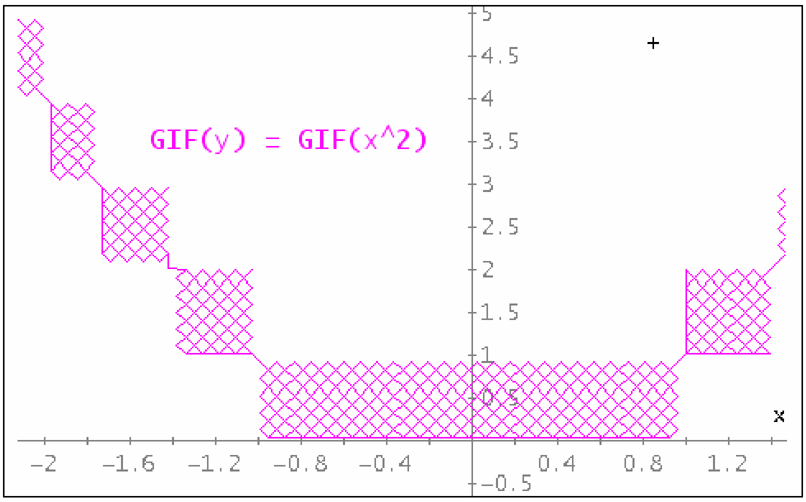
#10 $\mathrm{GIF}\left(x + \frac{1}{2}\right)^2 + \mathrm{GIF}\left(y + \frac{1}{2}\right)^2 = 1$





Now at this early point it is desirable to view what we have suggested to do in general to produce an unlimited number of space filling curves using the GIF function: i.e., given an equation of the form F(x,y) = 0, replace x with GIF(x) and y with GIF(y), then plot F(GIF(x),GIF(y)) = 0 to see what the result will be. We could continue in this manner and get many interesting space filling curves, but we are not limited to this approach. For example, if our initial curve of the general form F(x,y) = 0 is also of the special form y - f(x) = 0, then we could consider the equation GIF(y)-GIF(f(x))=0 and its graph. Thus we open up another unlimited number of GIF space filling curve types for our consideration. As an illustration consider our first space filling curve derived from y-x^2 = 0. We could write simply GIF(y) - GIF(x^2) = 0 and see what the plot would be. We will do that now. This plot has shaded regions which are non-square rectangles.

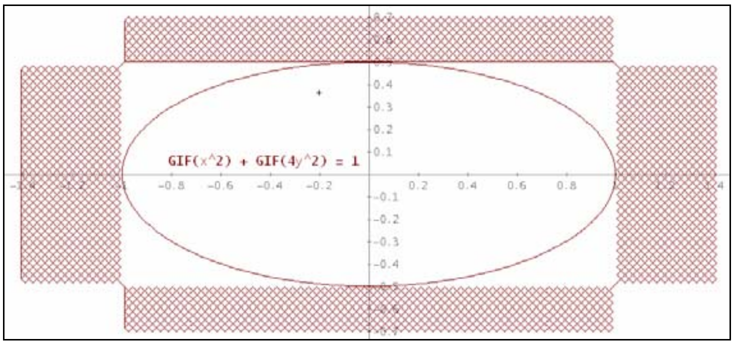
#11: $\mathrm{GIF}(y) = \mathrm{GIF}(x^2)$



In the following example we start with the equation of an ellipse and then apply GIF to each of the terms on the left of the ellipse equation. We then graph the resulting equations.

#12: $x^2 + 4\cdot y^2 = 1$

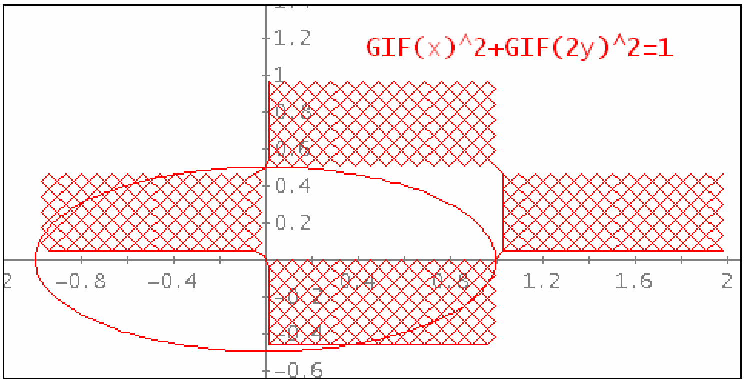
#13: $\mathrm{GIF}(x^2) + \mathrm{GIF}(4\cdot y^2) = 1$



As an alternative to the immediately preceding graph, starting with x^2 + (2y)^2 = 1 we could have chosen to produce yet another GIF graph with the following GIF equation and its embedded graph.

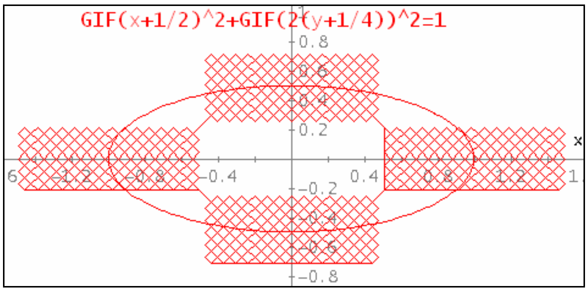
#12: $x^2 + 4\cdot y^2 = 1$

#15: $\mathrm{GIF}(x)^2 + \mathrm{GIF}(2\cdot y)^2 = 1$



Hence, under a translation that takes the center of the preceding graph to the origin, we have the following equation and its embedded graph.

#16: $$\mathrm{GIF}\left(x + \frac{1}{2}\right)^2 + \mathrm{GIF}\left(2\cdot\left(y + \frac{1}{4}\right)\right)^2 = 1$$

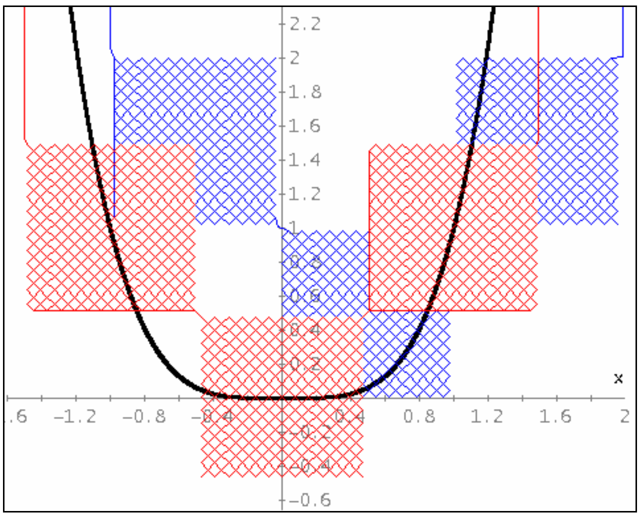


Three “Quartics” give a nice picture:

17: $\mathrm{TABLE}(x^4, x, -1.5, 1.5, 0.001)$

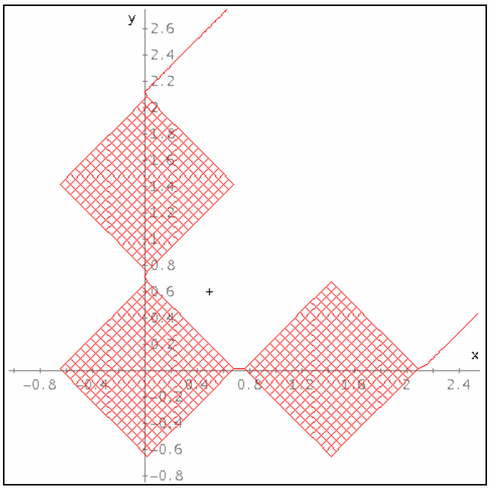
18: $\mathrm{GIF}(y) = \mathrm{GIF}(x)^4$

19: $$\mathrm{GIF}\left(y + \frac{1}{2}\right) = \mathrm{GIF}\left(x + \frac{1}{2}\right)^4$$



Composition of a translation [1/2, 1/2] followed by a rotation thru 45° on GIF(y) = GIF^4(x).

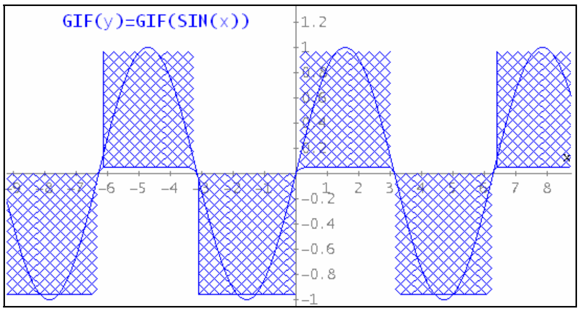
$$\mathrm{GIF}\left(\frac{x}{\sqrt{2}} - \frac{y}{\sqrt{2}} + \frac{1}{2}\right)^4 = \mathrm{GIF}\left(\frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}} + \frac{1}{2}\right)$$



Additionally, we can certainly construct space filling curves where elementary transcendental functions occur. This is illustrated in the two following examples beginning with y = sin(x) and y = *e*^x from which we produce the two GIF equations given and their graphs.

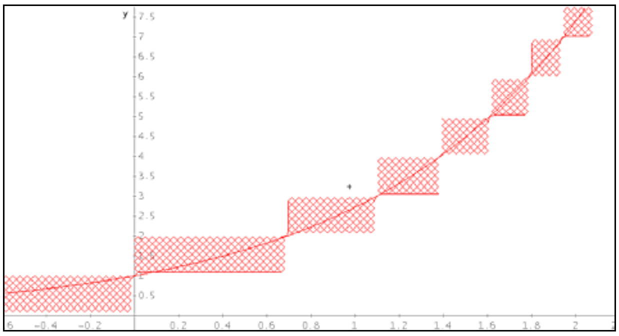
#21: $y = \mathrm{SIN}(x)$

#22: $\mathrm{GIF}(y) = \mathrm{GIF}(\mathrm{SIN}(x))$



#23: $y = e^x$

#24: $\mathrm{GIF}(y) = \mathrm{GIF}(e^x)$

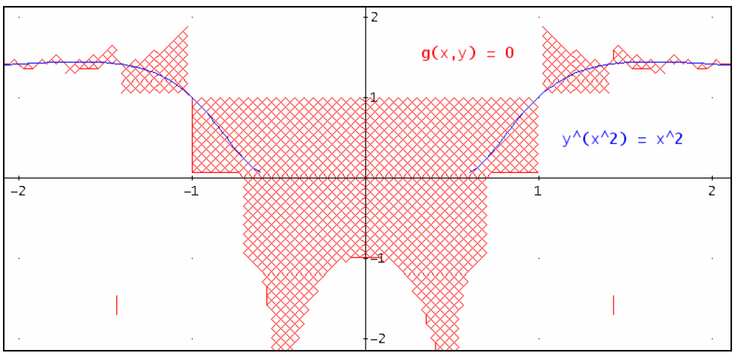


As a final example we offer the following somewhat "bizarre" initial equation y^(x^2) - x^2 = 0 and its GIF counterpart GIF(y^(x^2)) - GIF(x^2) = 0. It is not surprising that a somewhat “bizarre” graph results.

#25: $y^{x^2} - x^2 = 0$

#26: $g(x, y) := \mathrm{GIF}\left(y^{x^2}\right) - \mathrm{GIF}(x^2)$

#27: $g(x, y) = 0$



Now if we double click on the embedded plot above to see its larger and more detailed image in the 2D-plot window on the left, we would like to have DERIVE test a block of points each of which purportedly lies on the graph of g(x,y) = 0, as displayed, to verify that they do indeed lie on the graph. To this end, consider the matrix of points which follows.

```
#28:  VECTOR(VECTOR(g(x, y), x, -0.5, 0.5, 0.1), y, -0.5, 0.5, 0.1)
```

The following 11x11 matrix results from applying Simplify - Basic to the preceding VECTOR(VECTOR) statement. Each of the 121 numbers in the matrix should be zero if the graph as shown is valid. Each of the eleven 1's produced in the matrix corresponds to a point on the y-axis where x=0 and y is between -0.5 and 0.5, and it appears from the graph that g(x,y) = 0 for each such point (x,y).

What is the problem? Is the difficulty to be found in DERIVE programming? The author is currently preparing an article on Indeterminate Forms which answers this question.

#29:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We now consider the matter of multi-colored space filling plots, which many students find pleasing to the eye.

(When multi-colored plots are desired with the 2D-plot window active click Options-Display-Color to be sure that the box preceding "Automatically change color of new plots" is ticked.)

As an example, we consider the initial equation y^2=x^2 and its associated GIF equation GIF(y)^2 = GIF(x)^2.

Next we define the GIFF function as follows (in keeping with correspondence from Josef Bohm) which enables us to restrict the domain of the GIF function to the half-open interval [a,b).

```
       GIFF(a, b, x) :=
#30:     If a ≤ x < b
            FLOOR(x)
```

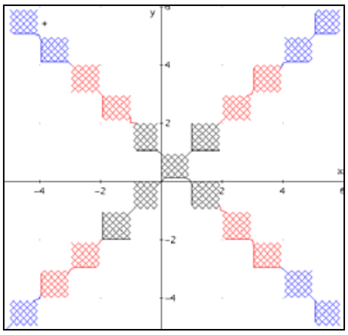
For our example a satisfactory set of (a,b) pairs are (a,b) = (-10,11), (a,b) = (-6,6), and (a,b) = (-3,3) for a tri-colored graph. (This can be seen by viewing a single-colored graph and deciding how we wish to apportion our three colors.)

With these pairs we construct the matrix of three GIFF equations as follows.

#31:
$$\begin{bmatrix} \mathrm{GIFF}(-6, 6, y)^2 - \mathrm{GIFF}(-6, 6, x)^2 = 0 \\ \mathrm{GIFF}(-4, 4, y)^2 - \mathrm{GIFF}(-4, 4, x)^2 = 0 \\ \mathrm{GIFF}(-2, 2, y)^2 - \mathrm{GIFF}(-2, 2, x)^2 = 0 \end{bmatrix}$$

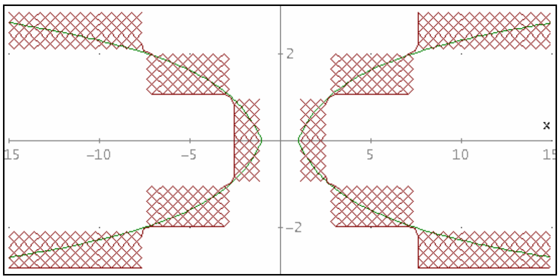
Now with the above matrix highlighted and the 2D-plot window activated, keep clicking on Insert - Plot (or simply press F4) until a plot with colors that you find satisfactory occurs.

(You can get precisely the three colors that you want with the 2D-plot window active by Clicking Options - Display - Color, shutting off "Automatically change color of new plots" and working with "Next color" for each of the three colors.)

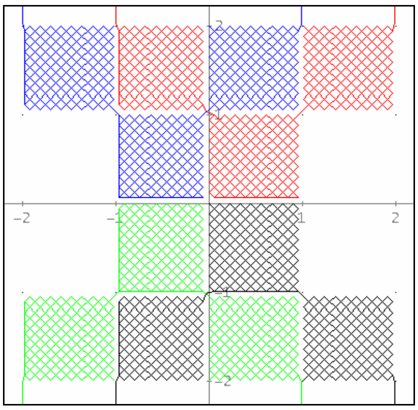


Finally, interested readers may wish to view the additional GIF graphs in the following list. Note: If you click any number preceding any of the following matrices you will get the various graphs within the matrix plotted almost simultaneously using Insert - Plot. Keep clicking to get a change in colors. Also, you can click on any individual line in a matrix to get an individual plot.

#32:
$$\begin{bmatrix} \mathrm{GIF}(|y|) - \mathrm{GIF}(\mathrm{LOG}(|x|)) = 0 \\ |y| - \mathrm{LOG}(|x|) = 0 \end{bmatrix}$$

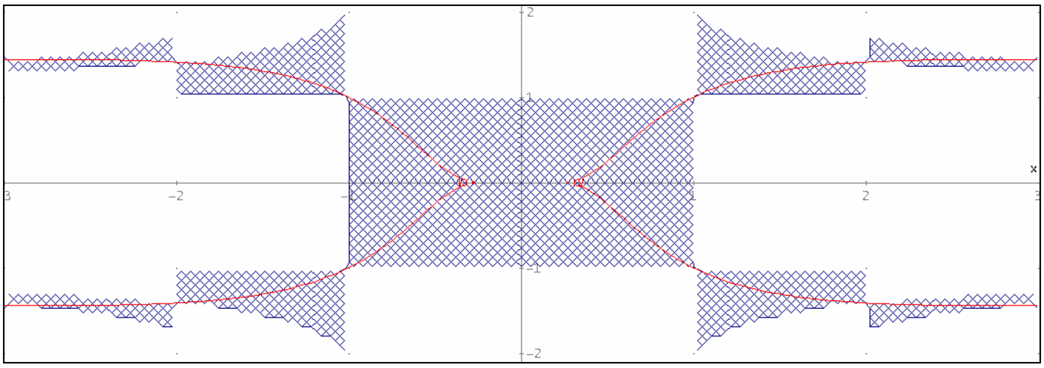


#33:
$$\begin{bmatrix} \mathrm{GIF}(y) - \mathrm{GIF}(x)^4 = 0 \\ \mathrm{GIF}(y) - \mathrm{GIF}(-x)^4 = 0 \\ \mathrm{GIF}(-y) - \mathrm{GIF}(x)^4 = 0 \\ \mathrm{GIF}(-y) - \mathrm{GIF}(-x)^4 = 0 \end{bmatrix}$$



#34:
$$\begin{bmatrix} |y|^{|x|} - |x| = 0 \\ \mathrm{GIF}(|y|^{|x|}) - \mathrm{GIF}(|x|) = 0 \end{bmatrix}$$

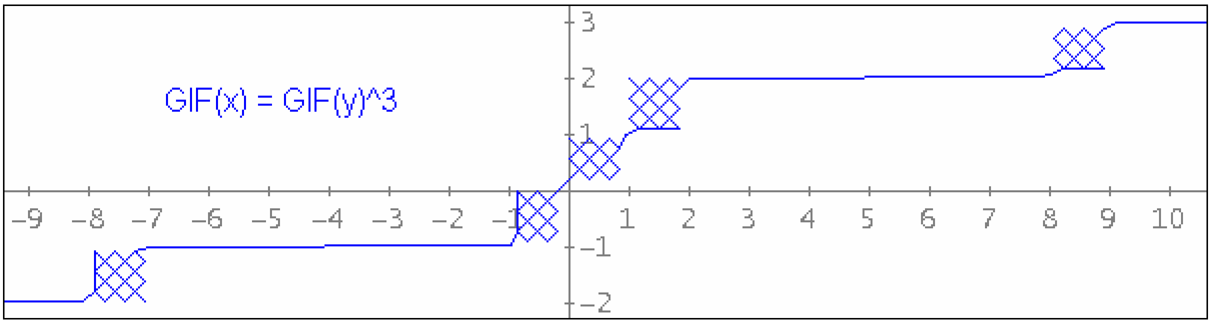
See the graph next page.



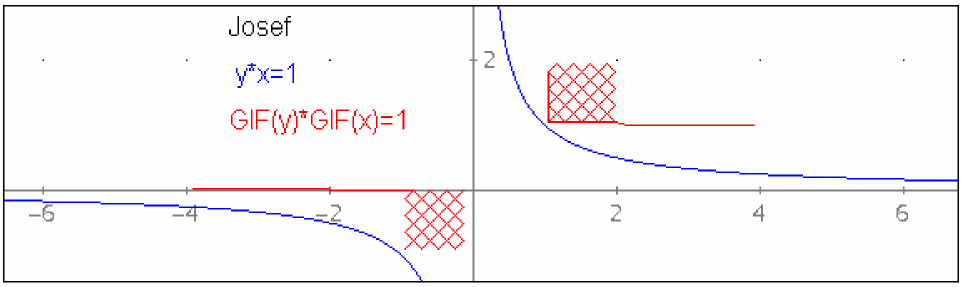
#35: $\mathrm{GIF}\left(y^{2x^2}\right) - \mathrm{GIF}\left(x^2\right) = 0$

Of course there are an unlimited number of GIF graphs easily constructed and easy to investigate with DERIVE's help - but heed this warning: construction and investigation of GIF graphs can become addictive!

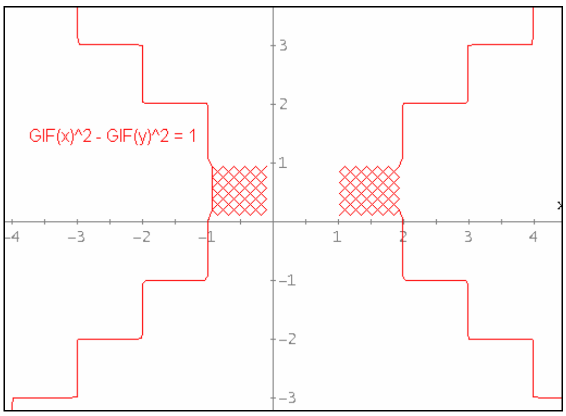
****



I am trying y = 1/x, Josef.



The stairstep lines bring to mind the two branches of a hyperbola, but this is an example which DERIVE handles poorly in contrast to many other examples. The stairstep lines emanating from the two squares are spurious.



Comments of the Editor:

George and I had an extended exchange of e-mails and files (pages 25 – 27, and some others). Here is his final product (starting on page 27). In the first draft of the “Space Filling Curves” George defined functions GIF1, GIF2 and GIF3, which were later replaced by GIFF.

Charts I and II are the remains of his first letter containing more than 20 pages full of printed graphs together with the two charts given below.

Experimenting with GIF and GIFF is exciting and to repeat George´s word “it can become addictive” – like many other graphic investigations.

Chart I: `VECTOR(VECTOR(GIF(y) – GIF(x)^3, x, –1, 1.9, 0.1),y,1.9, –1 ,– 0.1)`

y

2	2	2	2	2	2	2	2	2	2	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0
2	2	2	2	2	2	2	2	2	2	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0
2	2	2	2	2	2	2	2	2	2	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0
2	2	2	2	2	2	2	2	2	2	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0
2	2	2	2	2	2	2	2	2	2	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0
2	2	2	2	2	2	2	2	2	2	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0
2	2	2	2	2	2	2	2	2	2	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0
2	2	2	2	2	2	2	2	2	2	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0
2	2	2	2	2	2	2	2	2	2	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0
2	2	2	2	2	2	2	2	2	2	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
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0	0	0	0	0	0	0	0	0	0	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2
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0	0	0	0	0	0	0	0	0	0	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2

x

Chart 2: `VECTOR(VECTOR(GIF(x)^2-GIF(y)^2-1, x, -1, 1.9, 0.1),y,1.9, -1 ,- 0.1)`

y

-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
0	0	0	0	0	0	0	0	0	0	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	0	0	0	0	0	0	0	0	0	0
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0	0	0	0	0	0	0	0	0	0	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	0	0	0	0	0	0	0	0	0	0
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0	0	0	0	0	0	0	0	0	0	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	0	0	0	0	0	0	0	0	0	0
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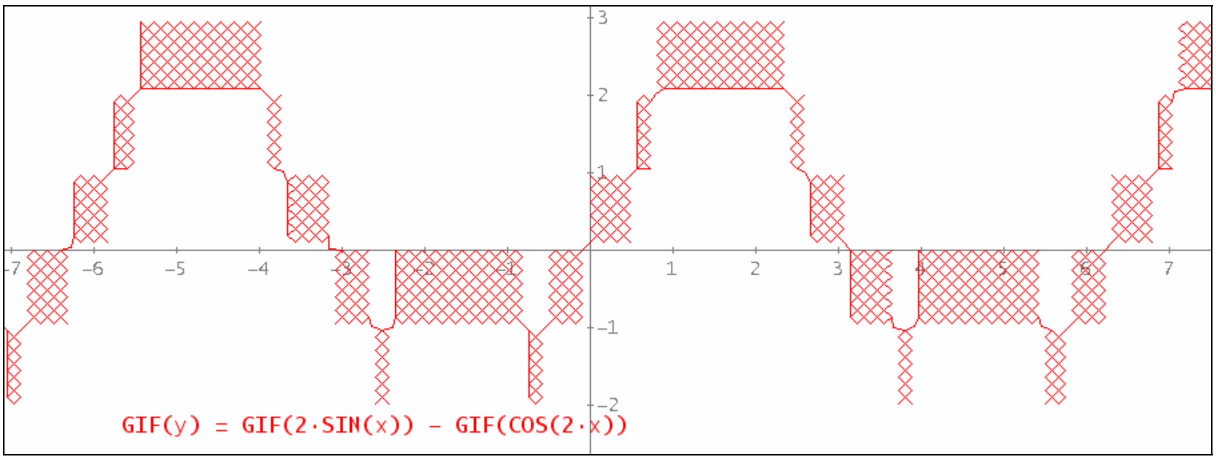
-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-2	-2	-2	-2	-2	-2	-2	-2	-2	-2	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1

x

… it can really become addictive, Josef



DERIVE meets WIRIS

Josef Böhm, Würmla, Austria

As I wrote in the Letter of the Editor, I attended the lecture of Ramon Eixarch at TIME 2010 where he presented the project converting DERIVE files to WIRIS.
I downloaded the respective WIRIS-Version (URL in my Letter) and made some experiments. These are two of my first attempts: I start with a DERIVE Calculus session:

#1: $f(x, a) := \dfrac{x^3 - 3\cdot a\cdot x^2 + 3\cdot a^2\cdot x - 12\cdot x}{8}$

#2: $f(x, 1)$

A family of Curves

#3: VECTOR(f(x, a), a, -5, 5)

#4: $\mathrm{VECTOR}\left(f(x, a), a, -5, 5, \dfrac{1}{2}\right)$

We find the turning points: f´(x,a) = 0

#5: $\dfrac{d}{dx} f(x, a)$

#6: $\dfrac{3\cdot(x^2 - 2\cdot a\cdot x + a^2 - 4)}{8}$

#7: $\dfrac{d}{dx} f(x, a) = \dfrac{3\cdot(x^2 - 2\cdot a\cdot x + a^2 - 4)}{8}$

#8: $\dfrac{3\cdot(x^2 - 2\cdot a\cdot x + a^2 - 4)}{8} = 0$

We eliminate parameter a from equation y = f(x,a) in order to obtain the locus of the turning points:

#9: $y = f(x, a)$

#10: $y = \dfrac{x\cdot(x^2 - 3\cdot a\cdot x + 3\cdot(a^2 - 4))}{8}$

Solving one equation for a and substituting for a in the other equation.

#11: $\mathrm{SOLVE}\left(\dfrac{3\cdot(x^2 - 2\cdot a\cdot x + a^2 - 4)}{8} = 0, a\right)$

#12: $a = x - 2 \vee a = x + 2$

There are two solutions and we subtitute twice:

#13: VECTOR(y = f(x, a), a, [x - 2, x + 2])

#14: $\left[y = \dfrac{x^2\cdot(x - 6)}{8}, y = \dfrac{x^2\cdot(x + 6)}{8}\right]$

This will not be converted (#13 --> #14) or

#15: [y = f(x, x - 2), y = f(x, x + 2)]

#16: $\left[y = \dfrac{x^2\cdot(x-6)}{8},\ y = \dfrac{x^2\cdot(x+6)}{8}\right]$

These are the loci of the turning points.

We would like to work faster for the locus of the inflection points.

#17: $\text{SOLVE}\left(\left[y = f(x, a),\ \left(\dfrac{d}{dx}\right)^2 f(x, a) = 0\right],\ [y, a]\right)$

#18: $\left[y = \dfrac{x\cdot(x^2 - 12)}{8} \wedge a = x\right]$

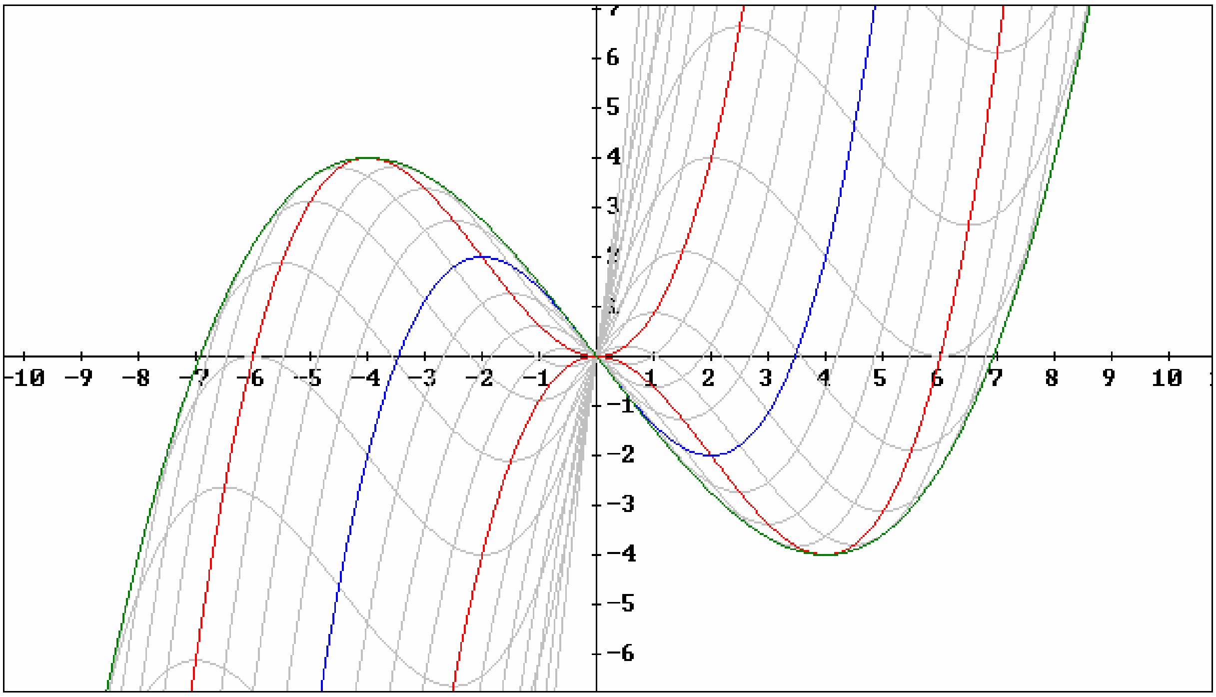
Let's find the enveloppe:

Keeping one position x fixed and looking for the extremal function value depending on parameter a. Hence:

#19: $\text{SOLVE}\left(\left[y = f(x, a),\ \dfrac{d}{da} f(x, a) = 0\right],\ [y, a]\right)$

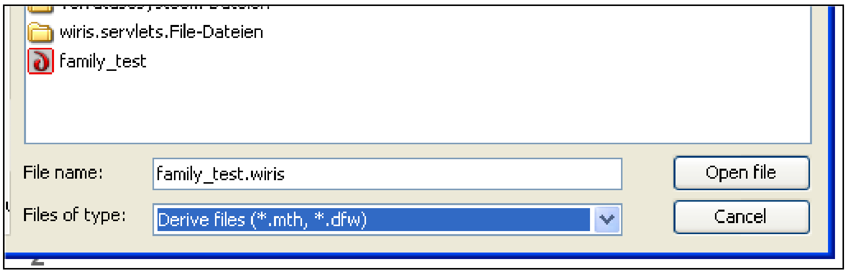
#20: $\left[y = \dfrac{x\cdot(x^2 - 48)}{32} \wedge a = \dfrac{x}{2}\right]$

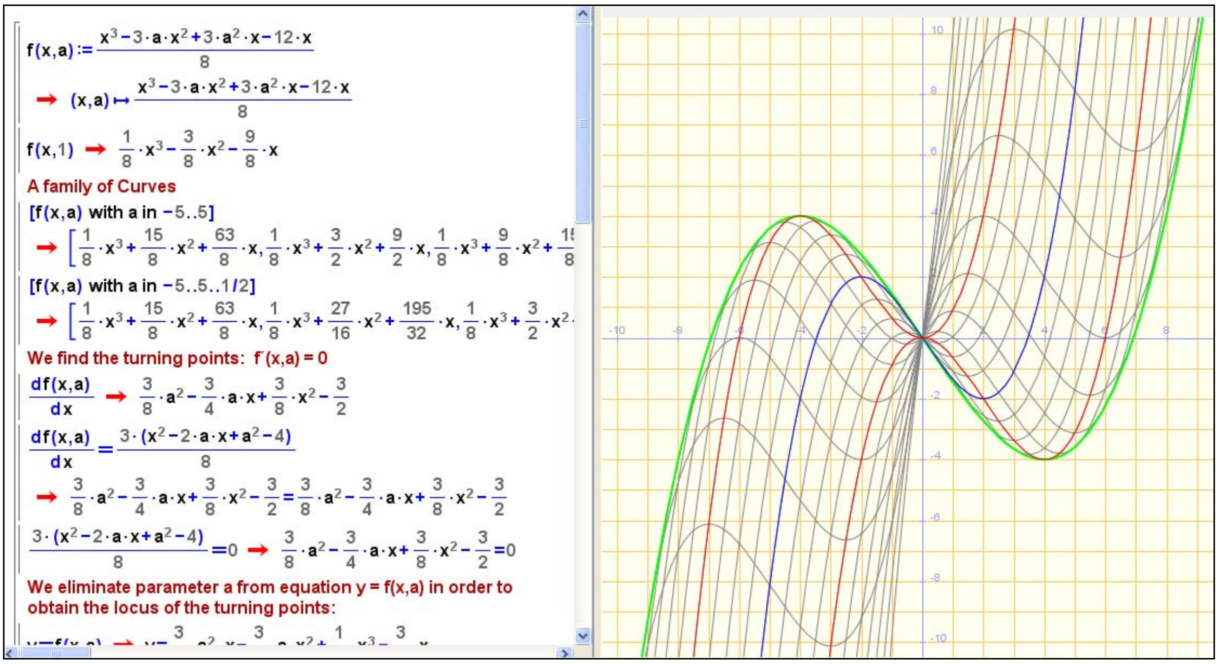
Is this correct?



I saved this file as family_test.dfw and opened the WIRIS test version.

Now I am offered loading wiris-files … but also mth- and dfw-files:





We eliminate parameter a from equation y = f(x,a) in order to obtain the locus of the turning points:

$y=f(x,a) \rightarrow y=\frac{3}{8}\cdot a^2\cdot x-\frac{3}{8}\cdot a\cdot x^2+\frac{1}{8}\cdot x^3-\frac{3}{2}\cdot x$

Solving one equation for a and substituting for a in the other equation.

$\text{solve}\left(\frac{3\cdot(x^2-2\cdot a\cdot x+a^2-4)}{8}=0,a,\mathbb{C}\right) \rightarrow \{\{a=x-2\},\{a=x+2\}\}$

There are two solutions and we subtitute twice:

[with in [x−2,x+2]]

or

$[y=f(x,x-2),y=f(x,x+2)] \rightarrow \left[y=\frac{1}{8}\cdot x^3-\frac{3}{4}\cdot x^2,y=\frac{1}{8}\cdot x^3+\frac{3}{4}\cdot x^2\right]$

These are the loci of the turning points.We would like to work faster for the locus of the inflection points.

$\text{solve}\left(\left[y=f(x,a),\frac{d\left(\frac{d(f(x,a))}{dx}\right)}{dx}=0\right],\{y,a\},\mathbb{C}\right) \rightarrow \left\{\left[a=x,y=\frac{1}{8}\cdot x^3-\frac{3}{2}\cdot x\right]\right\}$

$\left[y=\frac{x\cdot(x^2-12)}{8}\wedge a=x\right] \rightarrow \left[y=\frac{1}{8}\cdot x^3-\frac{3}{2}\cdot x\wedge a=x\right]$

Let's find the enveloppe:Keeping one position x fixed and looking for the extremal function value depending on parameter a. Hence:

$\text{solve}\left(\left[y=f(x,a),\frac{df(x,a)}{da}=0\right],\{y,a\},\mathbb{C}\right) \rightarrow \left\{\left[a=\frac{1}{2}\cdot x,y=\frac{1}{32}\cdot x^3-\frac{3}{2}\cdot x\right]\right\}$

Is this correct?

Perform the plots in WIRIS!!

fam1={y=f(x,a) with a in −5..5..1/2}

$\rightarrow \left\{y=\frac{1}{8}\cdot x^3+\frac{15}{8}\cdot x^2+\frac{63}{8}\cdot x, y=\frac{1}{8}\cdot x^3+\frac{27}{16}\cdot x^2+\frac{195}{32}\cdot x, y=\frac{1}{8}\cdot x^3+\frac{3}{2}\cdot x^2+\frac{9}{2}\cdot x, y=\frac{1}{8}\right.$

plot(

$\left\{y=\frac{1}{8}\cdot x^3+\frac{15}{8}\cdot x^2+\frac{63}{8}\cdot x, y=\frac{1}{8}\cdot x^3+\frac{27}{16}\cdot x^2+\frac{195}{32}\cdot x, y=\frac{1}{8}\cdot x^3+\frac{3}{2}\cdot x^2+\frac{9}{2}\cdot x, y=\frac{1}{8}\cdot x^3+\right.$

,{color=grey})

→ plotter1

Is it possible to plot a family of curves directly? (For example : plot(fam1))

plot($y=\frac{1}{32}\cdot x^3-\frac{3}{2}\cdot x$,{color = green,line_width =2}) → plotter1

plot({$y=\frac{1}{8}\cdot x^3-\frac{3}{4}\cdot x^2, y=\frac{1}{8}\cdot x^3+\frac{3}{4}\cdot x^2$},{color = red}) → plotter1

plot($y=\frac{1}{8}\cdot x^3-\frac{3}{2}\cdot x$,{color=blue}) → plotter1

I added the commands for the plots using the opportunities to include color and line_width within the plot command. You can see the result on page 40.

My next test file was a file – dating from my time as teacher – dealing with financial mathematics where some functions have been defined in a DERIVE-Toolbox for later use:

$$ew(kap, zeit\_, r\_, zp) := kap\cdot r\_^{zeit\_\cdot zp}$$

$$bw(kap, zeit\_, r\_, zp) := \frac{kap}{r\_^{zeit\_\cdot zp}}$$

ew(1000, 5.5, 1.02, 2)

1243.37

$$1000\cdot 1.02^{11}$$

1243.37

$$bwrv(rente, anzahl, rp, r\_, zp) := \frac{rente\cdot(r\_^{-zp\cdot anzahl/rp}-1)}{r\_^{-zp/rp}-1}$$

$$bwrn(rente, anzahl, rp, r\_, zp) := \frac{bwrv(rente, anzahl, rp, r\_, zp)}{r\_^{zp/rp}}$$

$$ewrv(rente, anzahl, rp, r\_, zp) := bwrv(rente, anzahl, rp, r\_, zp)\cdot r\_^{anzahl\cdot zp/rp}$$

$$ewrn(rente, anzahl, rp, r\_, zp) := bwrn(rente, anzahl, rp, r\_, zp)\cdot r\_^{anzahl\cdot zp/rp}$$

$$H(i, ip, rp) := \left(1+\frac{i}{ip\cdot 100}\right)^{ip/rp}$$

$$FMF(cap, f1\_p0, paym, n, e1\_b0, i, ip, rp) := \frac{FLOOR\left(100\cdot RHS\left(SOLVE\left(cap = \frac{paym\cdot(H(i, ip, rp)^n-1)}{H(i, ip, rp)-1}\cdot \ldots\right.\right.\right.}{\ldots}$$

Again I opended the file in WIRIS and received an unevaluated document:

$$\mathrm{ew}(\mathrm{kap},\mathrm{zeit\_},\mathrm{r\_},\mathrm{zp}):=\mathrm{kap}\cdot \mathrm{r\_}^{\mathrm{zeit\_}\cdot \mathrm{zp}}$$

$$\mathrm{bw}(\mathrm{kap},\mathrm{zeit\_},\mathrm{r\_},\mathrm{zp}):=\frac{\mathrm{kap}}{\mathrm{r\_}^{\mathrm{zeit\_}\cdot \mathrm{zp}}}$$

$$\mathrm{ew}(1000,5.5,1.02,2)$$

$$1243.37$$

$$1000\cdot(1.02)^{11}$$

$$1243.37$$

$$\mathrm{bwrv}(\mathrm{rente},\mathrm{anzahl},\mathrm{rp},\mathrm{r\_},\mathrm{zp}):=\frac{\mathrm{rente}\cdot\left(\mathrm{r\_}^{\frac{-\mathrm{zp}\cdot\mathrm{anzahl}}{\mathrm{rp}}}-1\right)}{\mathrm{r\_}^{\frac{-\mathrm{zp}}{\mathrm{rp}}}-1}$$

$$\mathrm{bwrn}(\mathrm{rente},\mathrm{anzahl},\mathrm{rp},\mathrm{r\_},\mathrm{zp}):=\frac{\mathrm{bwrv}(\mathrm{rente},\mathrm{anzahl},\mathrm{rp},\mathrm{r\_},\mathrm{zp})}{\mathrm{r\_}^{\frac{\mathrm{zp}}{\mathrm{rp}}}}$$

$$\mathrm{ewrv}(\mathrm{rente},\mathrm{anzahl},\mathrm{rp},\mathrm{r\_},\mathrm{zp}):=\mathrm{bwrv}(\mathrm{rente},\mathrm{anzahl},\mathrm{rp},\mathrm{r\_},\mathrm{zp})\cdot\mathrm{r\_}^{\frac{\mathrm{anzahl}\cdot\mathrm{zp}}{\mathrm{rp}}}$$

$$\mathrm{ewrn}(\mathrm{rente},\mathrm{anzahl},\mathrm{rp},\mathrm{r\_},\mathrm{zp}):=\mathrm{bwrn}(\mathrm{rente},\mathrm{anzahl},\mathrm{rp},\mathrm{r\_},\mathrm{zp})\cdot\mathrm{r\_}^{\frac{\mathrm{anzahl}\cdot\mathrm{zp}}{\mathrm{rp}}}$$

$$\mathrm{H}(\mathrm{i},\mathrm{ip},\mathrm{rp}):=\left(1+\frac{\mathrm{i}}{\mathrm{ip}\cdot 100}\right)^{\frac{\mathrm{ip}}{\mathrm{rp}}}$$

$$\mathrm{FMF}(\mathrm{cap},\mathrm{f1\_p0},\mathrm{paym},\mathrm{n},\mathrm{e1\_b0},\mathrm{i},\mathrm{ip},\mathrm{rp}):=\frac{\left\lfloor 100\cdot\mathrm{solve}\left(\mathrm{cap}=\frac{\mathrm{paym}\cdot(\mathrm{H}(\mathrm{i},\mathrm{ip},\mathrm{rp})^{\mathrm{n}}-1)}{\mathrm{H}(\mathrm{i},\mathrm{ip},\mathrm{rp})-1}\cdot\left(\frac{\mathrm{H}(\mathrm{i},\mathrm{ip},\mathrm{rp})^{1-\mathrm{e1\_b0}}}{\mathrm{H}(\mathrm{i},\mathrm{ip},\mathrm{rp})^{\mathrm{n}\cdot(1-\mathrm{f1\_p0})}}\right),\mathrm{x},0,\infty,\mathbb{C}\right)_2+0.5\right\rfloor}{100}$$

This seemed to work, then I evaluated all expression at once by clicking on the "="-sign (which is here not to see because it is out of the part of the screen shot).

$$\mathrm{ew}(\mathrm{kap},\mathrm{zeit\_},\mathrm{r\_},\mathrm{zp}):=\mathrm{kap}\cdot \mathrm{r\_}^{\mathrm{zeit\_}\cdot \mathrm{zp}}\rightarrow(\mathrm{kap},\mathrm{zeit\_},\mathrm{r\_},\mathrm{zp})\mapsto\mathrm{kap}\cdot \mathrm{r\_}^{\mathrm{zeit\_}\cdot \mathrm{zp}}$$

$$\mathrm{bw}(\mathrm{kap},\mathrm{zeit\_},\mathrm{r\_},\mathrm{zp}):=\frac{\mathrm{kap}}{\mathrm{r\_}^{\mathrm{zeit\_}\cdot \mathrm{zp}}}\rightarrow(\mathrm{kap},\mathrm{zeit\_},\mathrm{r\_},\mathrm{zp})\mapsto\frac{\mathrm{kap}}{(\mathrm{r\_}^{\mathrm{zeit\_}\cdot \mathrm{zp}})}$$

$$\mathrm{ew}(1000,5.5,1.02,2)\rightarrow 1243.4$$

$$1243.37\rightarrow 1243.4$$

$$1000\cdot(1.02)^{11}\rightarrow 1243.4$$

$$1243.37\rightarrow 1243.4$$

$$\mathrm{bwrv}(\mathrm{rente},\mathrm{anzahl},\mathrm{rp},\mathrm{r\_},\mathrm{zp}):=\frac{\mathrm{rente}\cdot\left(\mathrm{r\_}^{\frac{-\mathrm{zp}\cdot\mathrm{anzahl}}{\mathrm{rp}}}-1\right)}{\mathrm{r\_}^{\frac{-\mathrm{zp}}{\mathrm{rp}}}-1}\rightarrow(\mathrm{rente},\mathrm{anzahl},\mathrm{rp},\mathrm{r\_},\mathrm{zp})\mapsto\frac{\mathrm{rente}\cdot\left(\mathrm{r\_}^{\frac{-\mathrm{zp}\cdot\mathrm{anzahl}}{\mathrm{rp}}}-1\right)}{\mathrm{r\_}^{\frac{(-\mathrm{zp})}{\mathrm{rp}}}-1}$$

$$\mathrm{bwrn}(\mathrm{rente},\mathrm{anzahl},\mathrm{rp},\mathrm{r\_},\mathrm{zp}):=\frac{\mathrm{bwrv}(\mathrm{rente},\mathrm{anzahl},\mathrm{rp},\mathrm{r\_},\mathrm{zp})}{\mathrm{r\_}^{\frac{\mathrm{zp}}{\mathrm{rp}}}}\rightarrow(\mathrm{rente},\mathrm{anzahl},\mathrm{rp},\mathrm{r\_},\mathrm{zp})\mapsto\frac{\mathrm{bwrv}(\mathrm{rente},\mathrm{anzahl},\mathrm{rp},\mathrm{r\_},\mathrm{zp})}{\left(\mathrm{r\_}^{\frac{\mathrm{zp}}{\mathrm{rp}}}\right)}$$

$$\mathrm{ewrv}(\mathrm{rente},\mathrm{anzahl},\mathrm{rp},\mathrm{r\_},\mathrm{zp}):=\mathrm{bwrv}(\mathrm{rente},\mathrm{anzahl},\mathrm{rp},\mathrm{r\_},\mathrm{zp})\cdot\mathrm{r\_}^{\frac{\mathrm{anzahl}\cdot\mathrm{zp}}{\mathrm{rp}}}\rightarrow(\mathrm{rente},\mathrm{anzahl},\mathrm{rp},\mathrm{r\_},\mathrm{zp})\mapsto\mathrm{bwrv}(\mathrm{rente},\mathrm{anzahl},\mathrm{rp},\mathrm{r\_},\mathrm{zp})\cdot\mathrm{r\_}^{\frac{\mathrm{anzahl}}{\mathrm{rp}}}$$

$$\mathrm{ewrn}(\mathrm{rente},\mathrm{anzahl},\mathrm{rp},\mathrm{r\_},\mathrm{zp}):=\mathrm{bwrn}(\mathrm{rente},\mathrm{anzahl},\mathrm{rp},\mathrm{r\_},\mathrm{zp})\cdot\mathrm{r\_}^{\frac{\mathrm{anzahl}\cdot\mathrm{zp}}{\mathrm{rp}}}\rightarrow(\mathrm{rente},\mathrm{anzahl},\mathrm{rp},\mathrm{r\_},\mathrm{zp})\mapsto\mathrm{bwrn}(\mathrm{rente},\mathrm{anzahl},\mathrm{rp},\mathrm{r\_},\mathrm{zp})\cdot\mathrm{r\_}^{\frac{\mathrm{anzah}}{\mathrm{rp}}}$$

$$\mathrm{H}(\mathrm{i},\mathrm{ip},\mathrm{rp}):=\left(1+\frac{\mathrm{i}}{\mathrm{ip}\cdot 100}\right)^{\frac{\mathrm{ip}}{\mathrm{rp}}}\rightarrow(\mathrm{i},\mathrm{ip},\mathrm{rp})\mapsto\left(1+\frac{\mathrm{i}}{\mathrm{ip}\cdot 100}\right)^{\frac{\mathrm{ip}}{\mathrm{rp}}}$$

$$\mathrm{FMF}(\mathrm{cap},\mathrm{f1\_p0},\mathrm{paym},\mathrm{n},\mathrm{e1\_b0},\mathrm{i},\mathrm{ip},\mathrm{rp}):=\frac{\left\lfloor 100\cdot\mathrm{solve}\left(\mathrm{cap}=\frac{\mathrm{paym}\cdot(\mathrm{H}(\mathrm{i},\mathrm{ip},\mathrm{rp})^{\mathrm{n}}-1)}{\mathrm{H}(\mathrm{i},\mathrm{ip},\mathrm{rp})-1}\cdot\left(\frac{\mathrm{H}(\mathrm{i},\mathrm{ip},\mathrm{rp})^{1-\mathrm{e1\_b0}}}{\mathrm{H}(\mathrm{i},\mathrm{ip},\mathrm{rp})^{\mathrm{n}\cdot(1-\mathrm{f1\_p0})}}\right),\mathrm{x},0,\infty,\mathbb{C}\right)_2+0.5\right\rfloor}{100}$$

The FMF-function made problems.

On the next page you can see the DERIVE treatment of some calculations applying the provided tools, followed by the WIRIS-evaluation. Respective comments are included.

I sent my findings to WIRIS and I am waiting for an answer.

You can find the URL for joining testing this interesting project in the Letter of the Editor.

```
#13:   Precision := Approximate
#14:   PrecisionDigits := 15
#15:   Notation := Decimal
#16:   bwrv(2000, 42, 4, 1.0325, 2)
#17:   61664.3417275547
#18:   FMF(x, 0, 2000, 42, 0, 6.5, 2, 4)
#19:   61664.34
#20:   bwrn(2500, ∞, 12, 1.035, 1)
#21:   870807.452519285
```

#22: $\mathrm{ew}\left(8.70807\cdot 10^{5},\ \frac{1}{12},\ 1.035,\ 1\right)$

#23: $8.73307\cdot 10^{5}$

#24: $\mathrm{kapw}(zl\_,\ fl\_,\ r\_) := \sum_{i=1}^{\mathrm{DIM}(zl\_)} \frac{zl\__i}{r\_^{fl\__i}}$

Precision=Approximate → **Approximate**
PrecisionDigits=15 → 15
Notation=Decimal → **Decimal**

These settings are not understood by WIRIS (Precision in Calculation and Precision in Notation)!

bwrv(2000,42,4,1.0325,2) → 61664.
FMF(x,0,2000,42,0,6.5,2,4)

or FMF(50000, 0,x, 42, 0, 5, 2, 4) solving for the payment

bwrn(2500,∞,12,1.035,1) → $8.7081\cdot 10^5$
$\mathrm{ew}\left(8.70807\cdot 10^5,\frac{1}{12},1.035,1\right)$ → $8.7331\cdot 10^5$
$8.73307\cdot 10^5$ → $8.7331\cdot 10^5$
$\mathrm{kapw}(zl\_,fl\_,r\_) := \sum i=1 \mathrm{length}(zl\_)\left(\frac{zl\__i}{r\_^{fl\__i}}\right)$ → $(zl\_,fl\_,r\_)\mapsto\left(\Sigma i=1\cdot \mathrm{length}(zl\_)\left(\frac{(zl\__i)}{(r\_^{(fl\_i)})}\right)\right)$

I set the output digits in WIRIS and repeat the caculations from above

precision(8) → 5
bwrv(2000,42,4,1.0325,2) → 61664.342
bwrn(2500,∞,12,1.035,1) → 870807.45
$\mathrm{ew}\left(8.70807\cdot 10^5,\frac{1}{12},1.035,1\right)$ → 873307.

One of our members, Easter Bunny, sent a bundle of DERIVE files containing documented DERIVE bugs connected with several fields of mathematics. Here are two of them. Many thanks to E.B. for the email – exchange. E.B. also provided a program for PSLQ (see page 3).

Bugs in Connection with Integration:

BUG #1

The evaluation of this integral by DERIVE 6.10 is nonsense:

User=Simp(User):

#1: $$\int \mathrm{LN}\left(\frac{1 + x^2}{x}\right) dx = \pm\infty$$

... because the actual antiderivative is finite for all x.

DERIVE 4.07 and DERIVE 5 produce a correct result, however:

User:

#2: $$2\cdot\mathrm{ATAN}(x) + x\cdot\mathrm{LN}\left(\frac{x^2 + 1}{x}\right) - x$$

and so do WIRIS and Maxima ...

The root of the problem lies in this evaluation:

User=Simp(User):

#3: $$\int \mathrm{LN}\left(\frac{1}{x}\right) dx = \pm\infty$$

... where the application of the following integration rule:

User=Simp(User):

#4: $$\int \mathrm{LN}\left(\frac{d\cdot x + c}{b\cdot x + a}\right) dx = -\frac{a\cdot\mathrm{LN}(b\cdot(a\cdot d - b\cdot c)\cdot(b\cdot x + a))}{b} + x\cdot\mathrm{LN}\left(\frac{d\cdot x + c}{b\cdot x + a}\right) + \frac{c\cdot\mathrm{LN}(d\cdot(a\cdot d - b\cdot c)\cdot(d\cdot x + c))}{d}$$

... evidently causes problems when d = 0.

For this integrand too, DERIVE 4.07 and DERIVE 5 return a correct antiderivative:

User:

#5: $$\int \mathrm{LN}\left(\frac{1}{x}\right) dx = x\cdot\mathrm{LN}\left(\frac{1}{x}\right) + x$$

BUG #2

The following integrand is just another way of writing 1/SQRT(1 – a·x); it is therefore real when a·x < 1 and complex when a·x > 1. The antiderivative returned by DERIVE 6.10:

User=Simp(User):

$$\#6:\quad \int \frac{\mathrm{LN}(a\cdot x - 1) - 2\cdot \mathrm{LN}(-\sqrt{(a\cdot x - 1)})}{2\cdot\pi\cdot\sqrt{(a\cdot x - 1)}}\, dx = -\frac{2\cdot\sqrt{(a\cdot x - 1)}\cdot \mathrm{LN}(-\,\mathrm{SIGN}(\sqrt{(a\cdot x - 1)}))}{\pi\cdot a}$$

however, is correct only for a·x > 1. (Stepwise simplification leads to the same result.)

The system seems to make the unwarranted assumption a·x – 1 > 0. In particular, having applied the integration rule:

User:

$$\#7:\quad \int F(\sqrt{(a + b\cdot x)},\ x)\ dx = \frac{2}{b}\cdot \mathrm{SUBST}\left(\int x\cdot F\left(x,\ \frac{x^2 - a}{b}\right) dx,\ x,\ \sqrt{(a + b\cdot x)}\right)$$

it apparently fails to make the variable x a complex variable (until back-substitution).

A correct antiderivative would be:

User:

$$\#8:\quad \frac{\sqrt{(a\cdot x - 1)}\cdot \mathrm{LN}(a\cdot x - 1)}{\pi\cdot a} - \frac{2\cdot\sqrt{(a\cdot x - 1)}\cdot \mathrm{LN}(-\sqrt{(a\cdot x - 1)})}{\pi\cdot a}$$

which is just another way of writing –2/a·SQRT(1 – a·x).

The antiderivative produced by DERIVE 4.07:

likewise is correct only for a·x > 1.

(BTW, do you remember DERIVE for DOS, Josef?)

$$\#2:\quad \int \frac{\mathrm{LN}(a\cdot x - 1) - 2\cdot \mathrm{LN}(-\sqrt{(a\cdot x - 1)})}{2\cdot\pi\cdot\sqrt{(a\cdot x - 1)}}\, dx$$

$$\#3:\quad \frac{2\cdot \hat{\imath}\cdot |a\cdot x - 1|^{3/2}}{a\cdot(1 - a\cdot x)}$$

Definite integrals are consequently also wrong:

User=Simp(User):

$$\#9:\quad \int_0^1 \frac{\mathrm{LN}(a\cdot x - 1) - 2\cdot \mathrm{LN}(-\sqrt{(a\cdot x - 1)})}{2\cdot\pi\cdot\sqrt{(a\cdot x - 1)}}\, dx = \frac{1}{a} - \frac{2\cdot\sqrt{(a - 1)}\cdot \mathrm{LN}(-\,\mathrm{SIGN}(\sqrt{(a - 1)}))}{\pi\cdot a}$$

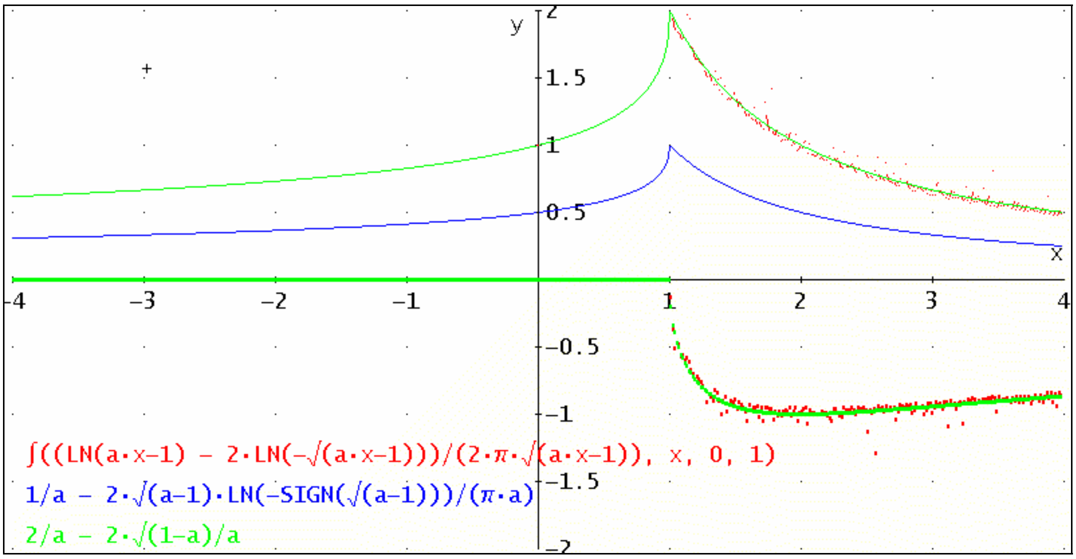
Restricting to a ≤ 1 helps in this case (note the factor of 2):

User:
#10: $a :\in \text{Real } (-\infty, 1]$

User=Simp(User):

$$\#11: \quad \int_0^1 \frac{\text{LN}(a\cdot x - 1) - 2\cdot\text{LN}(-\sqrt{(a\cdot x - 1)})}{2\cdot\pi\cdot\sqrt{(a\cdot x - 1)}}\, dx = \frac{2}{a} - \frac{2\cdot\sqrt{(1 - a)}}{a}$$

2D graphs of LHS(#9), RHS(#9), and RHS(#11) confirm this (with settings: plot real and imaginary parts; no simplification and no approximation before plotting):



Despite the restriction to a ≤ 1, the antiderivative for 0 < x < 1 remains wrong, however:

User:
#12: $x :\in \text{Real } (0, 1)$

User=Simp(User):

$$\#13: \quad \int \frac{\text{LN}(a\cdot x - 1) - 2\cdot\text{LN}(-\sqrt{(a\cdot x - 1)})}{2\cdot\pi\cdot\sqrt{(a\cdot x - 1)}}\, dx = -\frac{\sqrt{(1 - a\cdot x)}}{a}$$

(Here stepwise simplification ends with twice this result.)

The result of applying SUBST_DIFF to this antiderivative is therefore wrong as well (i.e. missing a factor of 2):

User=Simp(User):

$$\#14: \quad \text{SUBST\_DIFF}\left(-\frac{\sqrt{(1 - a\cdot x)}}{a}, x, 0, 1\right) = \frac{1}{a} - \frac{\sqrt{(1 - a)}}{a}$$

User:
#15: $[a :\in \text{Real}, x :\in \text{Real}]$

Beware of analyzing the stepwise reduction of these integrals; opening this can of worms is strongly discouraged! Among the single steps you will find ...

... one step with a doubling effect:

User:
#16: $[a :\in \text{Real } (-\infty, 1], x :\in \text{Real } (0, 1)]$

User:

#17: $$-\frac{\int \frac{2\cdot LN(-\sqrt{(a\cdot x-1)})-LN(a\cdot x-1)}{\sqrt{(a\cdot x-1)}}\,dx}{2\cdot\pi}$$

INT(F(SQRT(a+b·x),x),x) → 2/b*SUBST(INT(x·F(x,(x^2-a)/b),x),x,SQRT(a+b·x))

User:

#18: $$-\frac{2\cdot SUBST\left(\int \frac{2\cdot x\cdot LN(-SIGN(x))}{x}\,dx,\ x,\ \sqrt{(a\cdot x-1)}\right)}{2\cdot\pi\cdot a}$$

(simplify both expressions under the above restrictions)

... one correct step, yet with a halving effect:

User:
#19: [a :∈ Real, x :∈ Real]

User:

#20: $$-\frac{2\cdot SUBST(2\cdot x\cdot LN(-SIGN(x)),\ x,\ \sqrt{(a\cdot x-1)})}{2\cdot\pi\cdot a}$$

SUBST(F(x),x,a) -> F(a)

User:

#21: $$-\frac{4\cdot\sqrt{(a\cdot x-1)}\cdot LN(-SIGN(\sqrt{(a\cdot x-1)}))}{2\cdot\pi\cdot a}$$

(simplify both expressions under the following restrictions)

User:
#22: [a :∈ Real (-∞, 1], x :∈ Real (0, 1)]

... and one incorrect step without ill effect:

User:
#23: [a :∈ Real (-∞, 1], x :∈ Real (0, 1)]

User:

#24: $$-\frac{2\cdot SUBST(2\cdot x\cdot LN(-SIGN(x)),\ x,\ \sqrt{(a\cdot x-1)})}{2\cdot\pi\cdot a}$$

If x>0, SIGN(x) → 1

User:

#25: $$-\frac{2\cdot SUBST(2\cdot x\cdot LN(-1),\ x,\ \sqrt{(a\cdot x-1)})}{2\cdot\pi\cdot a}$$

since LN(-#i) = -#i·π/2 would be correct for a·x < 1 (LN(-1) = #i·π being twice as large).

DUG-Meeting 2010

Conference Dinner

Bärbel Barzel and Colette Laborde

Conference Organizers and Michel Beaudin

Pablo Picasso (in front of the house where he was born)

Paella cooking in Nerja

More pictures will be uploaded on the Conference website.