THE DERIVE - NEWSLETTER #80

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THE BULLETIN OF THE

DERIVE®

USER GROUP

+ CAS-TI

C o n t e n t s:

December 2010

It is a pleasure for me to announce two new books:

Our long time member *Paul Drijvers* is editor and contributor of

SECONDARY ALGEBRA EDUCATION
Revisiting Topics and Themes and Exploring the Unknown

ISBN 978-94-6091-333-4 paperback
SensePublishers, October 2010, 236 pages

You can find a preview at:

https://www.sensepublishers.com/product_info.php?manufacturers_id=68&products_id=1141&osCsid=27a01dc4ab7f32825afe0a4b9635b1c2

I recommend visiting Paul´s website: http://www.fi.uu.nl/~pauld/

And there is another one. DUG-Member *Thomas Himmelbauer* – he is author of several DNL-contributions – is not only an excellent mathematician, he also writes mystery novels. Just now his second book has appeared:

TOD IM GYMNASIUM

ISBN 978-3-902784-00-1, Taschenbuch, 205 Seiten
Federfrei 2010

His first book was TOD IN PANNONIEN (Death in Pannonia). The nice thing is that both stories are giving the atmosphere of the crime scene. TOD IN PANNONIEN is happening in the southern part of Burgenland where Thomas lives with his family. Much of the landscape and the people of this region can be found in his books. Southern part of Burgenland is "bilingual", German and Croatian.
TOD IM GYMNASIUM is laid in the atmosphere of an Austrian Secondary school. Reading it you can really smell the "Schulmief" (= "school stink").

Once again I recommend visiting *Michael De Villiers'* homepage.

It was updated November 2010.

http://mysite.mweb.co.za/residents/profmd/homepage4.html

The link given below will lead you immediately to his latest newsletter containing many intersting links and lots of information.

http://mysite.mweb.co.za/residents/profmd/newsletter.html

One of the recommended sites will open a really enjoyable TILING SLIDE SHOW

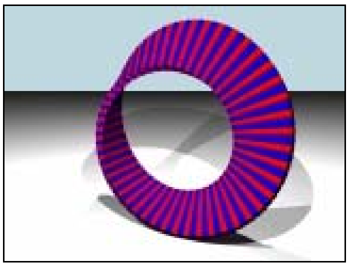
http://www.spsu.edu/math/tiling/tilings.html

Dear DUG Members,

Long, long overdue but now it is ready, DNL#80 can be downloaded. There are two reasons for the delay: The first one is our travel to Tansania in Novem-ber/Dec-ember 2010 and the second one is a very extended and fruitful exchange of emails with Nils Hahnfeld in connection with the DEQME contribution.

Lion in Serengeti

Thanks for many Christmas and New Years wishes. Some of them showed nice graphics. They are included in this letter.



Richard Schorn

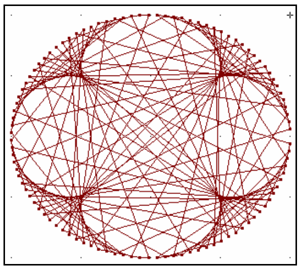
This DNL has only two – but very ex-tended – contributions. I didn't want to split them.

We have an answer to a problem given in DNL#22 and a review of Nils' Differential Equations tool for the TI-89, TI-92, and Voyage 200.

Inspired by DEQME I tried to program a function for stepwise solving one type of DEs with DERIVE and TI-Nspire as well.

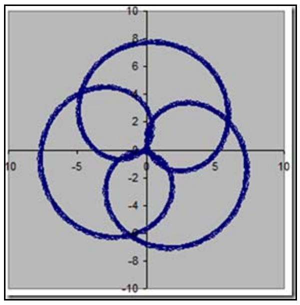
I would like to put your attention to our Book-shelf (left page) and especially to the Modelling Books presented on page 4.

All links recommended an pages 3 and 4 have been checked and they should be valid.



Roland Schröder

I received interesting letters from Roger Folsom and Dietmar Oertel. They will be published next time.



David Sjöstrand

In July ACA 2011 will be held in Houston, TX. There will be an educational session and we will try to have again a special DERIVE and TI-CAS session. I will inform you as soon as possible.

There is another conference in Germany: MNU in Mainz (7 – 11 April). Some DUG-members are giving lectures and work-shops there (R. Albers, K-H. Keunecke, W. Moldenhauer, P. Hofbauer, J. Böhm).

MNU MAINZ 2011

www.bundeskongress-2011.mnu.de

Best regards as ever,

The *DERIVE-NEWSLETTER* is the Bulletin of the *DERIVE* & CAS-*TI User Group*. It is published at least four times a year with a contents of 40 pages minimum. The goals of the *DNL* are to enable the exchange of experiences made with *DERIVE*, *TI*-CAS and other CAS as well to create a group to discuss the possibilities of new methodical and didactical manners in teaching mathematics.

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Contributions:
Please send all contributions to the Editor. Non-English speakers are encouraged to write their contributions in English to reinforce the international touch of the *DNL*. It must be said, though, that non-English articles will be warmly welcomed nonetheless. Your contributions will be edited but not assessed. By submitting articles the author gives his consent for reprinting it in the *DNL*. The more contributions you will send, the more lively and richer in contents the *DERIVE* & CAS-*TI Newsletter* will be.

Next issue: March 2011
Deadline 15 February 2011

Preview: Contributions waiting to be published

Some simulations of Random Experiments, J. Böhm, AUT, Lorenz Kopp, GER
Wonderful World of Pedal Curves, J. Böhm
Tools for 3D-Problems, P. Lüke-Rosendahl, GER
Financial Mathematics 4, M. R. Phillips
Hill-Encription, J. Böhm
Simulating a Graphing Calculator in *DERIVE*, J. Böhm
Henon & Co, J. Böhm
Do you know this? Cabri & CAS on PC and Handheld, W. Wegscheider, AUT
An Interesting Problem with a Triangle, Steiner Point, P. Lüke-Rosendahl, GER
Overcoming Branch & Bound by Simulation, J. Böhm, AUT
Diophantine Polynomials, D. E. McDougall, Canada
Graphics World, Currency Change, P. Charland, CAN
Cubics, Quartics – Interesting features, T. Koller & J. Böhm
Logos of Companies as an Inspiration for Math Teaching
Exciting Surfaces in the FAZ / Pierre Charland´s Graphics Gallery
BooleanPlots.mth, P. Schofield, UK
Old traditional examples for a CAS – what´s new? J. Böhm, AUT
Truth Tables on the TI, M. R. Phillips
Where oh Where is It? (GPS with CAS), C. & P. Leinbach, USA
Embroidery Patterns, H. Ludwig, GER
Mandelbrot and Newton with *DERIVE*, Roman Hašek, CZ & Rob Gough, UK
Snail-shells, Piotr Trebisz, GER
A Conics-Explorer, J. Böhm, AUT
Tutorials for the NSpireCAS, G. Herweyers, BEL
Some Projects with Students, R. Schröder, GER
Dirac Algebra, Clifford Algebra, D. R. Lunsford, USA
Treating Differential Equations (M. Beaudin, G. Piccard, Ch. Trottier)
Structured Combinatorics, D. Oertel, GER
Statistics with TI-Nspire, G. Herweyers, BEL
Cesar Multiplication, G. Schödl, AUT

and others

Impressum:
Medieninhaber: *DERIVE* User Group, A-3042 Würmla, D´Lust 1, AUSTRIA
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The MacTutor History of Mathematics archive
Biographies Index, History Topics Index, Additional Material Index, Famous Curve Index, Mathematicians of the Day
http://www-groups.dcs.st-and.ac.uk/~history/

The German MathCad website
http://www.ptc.com/products/mathcad/

Wolfram Library Archive
http://library.wolfram.com/redir/

Eric's Treasure Trove of Mathematics
http://mathworld.wolfram.com/

Featuring over 2100 applications contributed by the Maplesoft user community
http://www.maplesoft.com/applications/

Homepage of Cliff Pickover
http://sprott.physics.wisc.edu/pickover/home.htm

Topics in Mathematics
In these pages, you will find links to various WWW resources on Mathematics.
They are organized by topics.
http://archives.math.utk.edu/topics/

Mathematical Modules in Chemistry and Biology
http://science.kennesaw.edu/~mburke/modules/

The Geometry Center (University of Minnesota)
Center for Computation and Vizualisation of Geometric Structures. The Geometry Center is now closed. This web site continues as a repository for much of the materials and projects from the Geometry Center.

http://www.geom.uiuc.edu/

Visual Index of all Uniform Polyhedra (R. E. Maeder)
http://www.mathconsult.ch/showroom/unipoly/unipoly

The Math Forum – People Learning Math Together (Drexel University)
http://mathforum.org/
http://mathforum.org/library/

Internet Center for Mathematics Problems
http://www.mathpropress.com/
with among others:
http://www.mathpropress.com/archive/RabinowitzProblems1963-2005.pdf

Art from Code
Enjoy the graphs
http://www.artfromcode.com/

The Spanky Fractal Database
http://www.nahee.com/spanky/index.html

Fractals – Chaos - Attractors a.o.
http://local.wasp.uwa.edu.au/~pbourke/fractals/

Yahoo-Science-Mathematics
http://dir.yahoo.com/Science/mathematics

Euclid's Elements
This edition of Euclid's *Elements* uses a Java applet called the Geometry Applet to illustrate the diagrams.
http://aleph0.clarku.edu/~djoyce/java/elements/elements

An interesting interview about "Darstellende Geometrie" as subject on Secondary schools (in German) can be found at
http://derstandard.at/1295570821131/Interview-Freie-Formen-fordern-neue-geometrische-Modelle

Download free e-books from
http://bookbon.com/uk/student and http://bookbon.com/de/studium
(New publication: Introductory Finite Difference Methods for PDEs)

You are interested in Modelling?

Then I can recommend *Hartmut Bossel*´s 4 books:

Systemzoo 1-3, *Systeme, Dynamik, Simulation*, Books on Demand, Norderstedt (German)
System Zoo 1-3, Systems and Models, Books on Demand, Norderstedt (English)

The books cover models of the following fields: Elementary Systems, Physics, Engineering, Climate, Ecosystems, Resources, Economy, Society and Development. They are based on the modeling software VENSIM PLE which can be downloaded free of charge.

VENSIM PLE (free download for educational use)
Ventana publishes *Vensim* which is used for constructing models of business, scientific, environmental, and social systems.
http://www.vensim.com/download.html

This is nice tutorial for VENSIM PLE:

System Dynamics Resource Page of the Arizona State University
Among others you can find – and download a twenty-three page reference for Vensim PLE (pdf).
http://www.public.asu.edu/~kirkwood/sysdyn/SDRes.htm

There is also a CD available which contains all 100 models which are treated in the System Zoo books:

Systemzoo, co:Tec: www.corec-verlag.de

D-N-L#80	J. Böhm & C. L. Leinbach: Using Rational Airthmetic …	p 5

It is very unusual but you will find the abstract as part of the article on page 9, Josef

Using Rational Arithmetic to Develop a Proof

"What Josef and Carl Saw"

Josef Böhm, Würmla, Austria, and Carl Lewis Leinbach, Gardener, USA

D-N-L#22	Carl's and Marvin's Laboratory 2	p29

Finding a Limit via Geometric Reasoning

Carl Leinbach and Marvin Brubaker, USA

Consider the following sequence of points: $P_n = \begin{cases} [0,0] & n=0 \\ [0,1] & n=1 \\ [1,0] & n=2 \\ \frac{1}{2}P_{n-3} + \frac{1}{2}P_{n-2} & \text{otherwise} \end{cases}$

Notice that this sequence is defined recursively. *DERIVE* allows us to make recursive definitions. We use the IF statement.

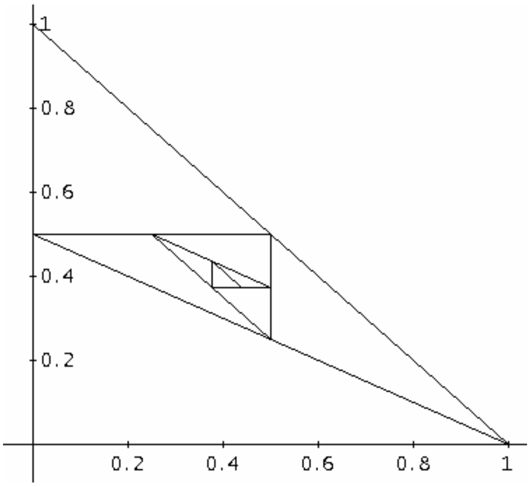
```
P(n):=IF(n=0,[0, 0],IF(n=1,[0,1],IF(n=2,[1,0],1/2P(n-3)+1/2P(n-2))))
```

In this case we had to nest the IF statements three deep. That is because we had three special cases. This function, because of its recursive nature, is slow to evaluate for an n of any size, whatsoever. Nonetheless, author

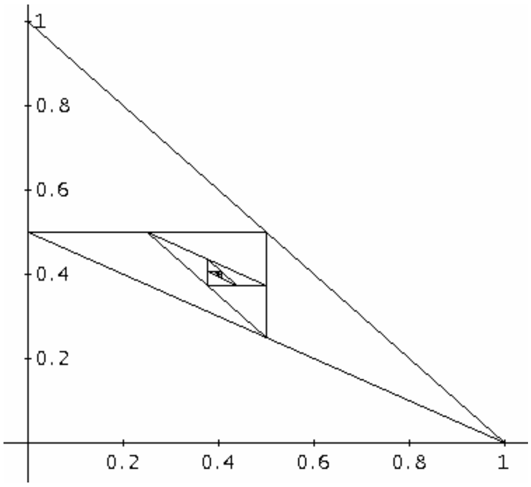
```
VECTOR(P(n),n,0,10)
```

and plot the sequence.

The next figures show the evaluation of the first 10 terms of the sequence and also the first 20 terms. If we move the crosshair on the graph where the plot is dense, i.e., the point of apparent convergence we get a reading of approximately [0.4, 0.4].



VECTOR(P(n),n,0,10)

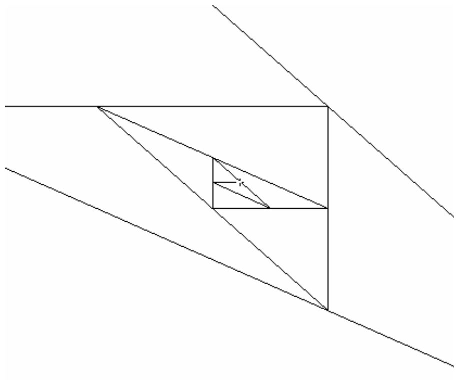


VECTOR(P(n),n,0,20)

We can zoom in and then we read off the coordinates of the crosshair [0.40029, 0.40042].

We can show the last term of the sequence given right above and we get a similar result: [0.40039…, 0.40039 …].

Of course, we had not proved any result. However, the visual evidence is convincing that a limit does exist ([0.4, 0.4]?) and we have a visual illustration of the process of convergence.



$$\text{FIRST(REVERSE(VECTOR(P(n), n, 0, 20)))} = \left[\frac{205}{512}, \frac{205}{512}\right]$$

FIRST(REVERSE(VECTOR(P(n), n, 0, 20))) = [0.400390625, 0.400390625]

The challenge is still there: Proof that the limit is [0.4, 0.4]!

The History of the Lecture

8 January 2010

Dear Carl,

I am now revising DNL#22 which contains Carl's and Marvins's Lab #2, "Finding a Limit via Geometric Reasoning".
I had to change some things due to the fact that DERIVE has changed a lot since 1996. I attach the revised contribution. Hope that you are satisfied with the new form (including a small program).

My question is: do you have a proof for the limit [2/5, 2/5]?

Best regards
Josef

11 January 2010

Josef -

I have not started on Lab 2, but hope to get to it before we leave on Wednesday morning. I have been working on meeting the (now revised) deadline before our Costa Rica trip. I enclosed te vastly revised paper in the hopes that you may find the example that I did on Time Since Death useful for your upcoming workshop. The referees wanted me to make my examples more "beefy", i.e. do some more substantial mathematics and involve the CAS more than I did in the original paper we submitted.

Dear Pat and Carl,

please don´t hurry - the proof is not so important. Enjoy your holidays.

12 January 2010

Josef -

While I was in the doctor's examination room waiting for the doctor to arrive, I tore off a piece of the paper covering the examination table and started to write out terms of the sequence. I got up to 16 terms.

…

…

5. Then prove that the lim(P(4*i)) = 2/5.

At the moment everything is based on my suppositions, not proven fact. I will keep working. Just wanted to keep you up to date.

some days later

Dear Carl,

Thanks for your efforts.

I am on a very similar way - to investigate the pattern of the numerators.

Hi Carl,

I attach my ideas for proving the limit.

27 January 2010

Josef -

I have attached the proof of the limit. I worked on it mainly on the plane ride to Costa Rica and a little bit during our visit to Costa Rica. It took a little more than I expected and as I note there is still one part that I want to clean up. I gave you an outline of that part. It is essential to the argument and I don't like the fact that it gets rather messy with the arithmetic.

3 February 2010

Josef -

I sent you this about a week ago and hadn't heard back. I was wondering what you thought. I think that it could make a good talk on combining the use of the rational arithmetic display of DERIVE to stimulate conjectures for solid mathematical analysis and then developing a proof. This is what we have been talking about for years. What do you think? BTW, I see that your though path and mine crossed at few crucial points. I was thinking that maybe we could develop a joint DNL article or a TIME talk on this type of use of DERIVE. Once again, what do you think?

-Carl

8 February 2010

Josef I have mentioned a joint presentation at Malaga or a DNL article (your choice). Here is how I thought it could go:

History: The DNL #22 Article attributed to Marvin and Carl; a request from Josef for an analytic proof of the limit

Observation Phase: Writing a brief program to examine terms of the sequence; the advantage of the rational arithmetic calculations and print out of DERIVE (and other CAS's)

Conjectures: What Josef saw (even though we worked independently, you were first); what Carl saw; putting conjectures to the test: Using mathematical induction to construct a proof

What do you think? I like the idea, because it uses a skill that we hope to develop amongst our students and uses CAS in much more than a "button pushing mode", which is what some of our antagonists accuse proponents of using CAS in teaching say we are professing.

16/18/20 February 2010

Josef -

Here is the promised draft of the Malaga presentation. Let me know what you think? Once we have the final form for the abstract, I will submit it.

-Carl

Dear Carl,

It looks good, I am busy filling the gap(s) in my PROOF. Maybe that we could add one sentence about possible generalizations (changing the initial values, ...).

I attach a DERIVE file containing a general form for creating our sequence of points together with a nonrecursive way to create the sequence with the requested lim.

Josef

8 March 2010

To time2010@ctima.uma.es

Please, find attached in this mail a (lecture or workshop) proposal for the (ACDCA strand) (TI-Nspire and Derive strand) (Please, indicate the appropriate format and strand).

This is a Lecture Proposal for the TI-Nspire & Derive Strand

Thank you,

Carl Leinbach

Abstract

It all began with an article in DNL #22 entitled **Finding a Limit via Geometric Reasoning** authored by Marvin Brubaker and Carl Leinbach. In that paper the limit of a recursively defined sequence of points was found by connecting successive points with straight lines, thus creating a nested sequence of triangles that seem to converge to the point (0.4, 0.4). While editing an archival edition of DNL #22, Josef correctly pointed out that the paper did not really have a proof of the limit, only a collection of heuristic evidence gained by zooming in on the suspected limit. He wrote to Carl asking if he had a mathematical proof that sequence did, in fact, converge to its claimed limit. Both Josef and Carl began independent work on the problem. Their initial step was the same. They each wrote a small DERIVE program to print out the first *n* terms of the sequence using the CAS's rational arithmetic display of the points. After this their two approaches differed.

In this presentation both Josef and Carl will discuss their approaches to constructing a proof that the sequence converges to its claimed limit, thus supporting the visual evidence. They will also discuss the value of using the Rational Arithmetic to support the discovery of a strategy to accomplish their mathematical goal. If time permits, the presenters will investigate applying their approaches to other sequences of points.

How Carl Attacked The Challenge

Let's suppose that a student had seen the Fibonacci sequence and the proof that the limit of the ratio of successive terms of that sequence converges to the "Golden Mean."

$$\lim_{n\to\infty}\frac{F_{n+1}}{F_n}=\frac{1+\sqrt{5}}{2}$$

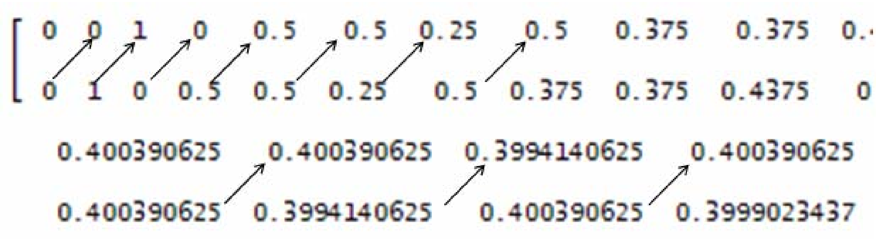
This approach simply can not be mimicked. It leads nowhere. **WHY?**

A next approach might be to try to visualize the terms of the sequence and look for some patterns. Suppose we try to familiarize ourselves with the nature of the sequence without using the features of a CAS, i.e. print out the decimal approximations to the sequence:

0	0	1	0	0.5	0.5	0.25	0.5	0.375	0.375	0.4375	0.375	0.40625	0.40625	0.390625	0.40625	0.3984375	0.3984375	0.40234375	0.3984375	0.400390625	0.400390625	0.3994140625	0.400390625
0	1	0	0.5	0.5	0.25	0.5	0.375	0.375	0.4375	0.375	0.40625	0.40625	0.390625	0.40625	0.3984375	0.3984375	0.40234375	0.3984375	0.400390625	0.400390625	0.3994140625	0.400390625	0.3999023437

What patterns do you see?

Here Is What Carl Saw



Observation 1: Every term of the first sequence lags one term behind the second sequence. Thus, we really only need to deal with one sequence.

Proof: (Using the Principle of Mathematical induction)

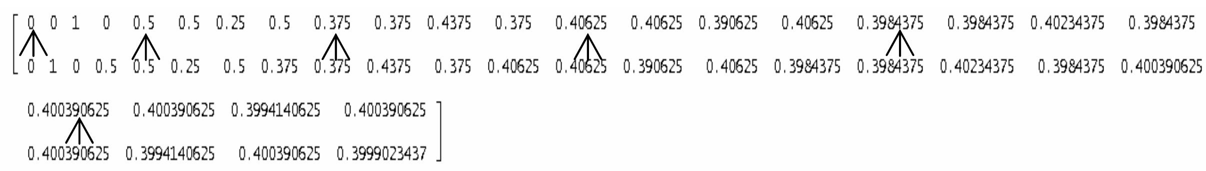
Base Case: Look at the terms of the sequence printed out above

General Case: Assume the result holds for all $k < n$. Then

$$P_{n,1} = \frac{1}{2}\left(P_{n-3,1} + P_{n-2,1}\right) = \frac{1}{2}\left(P_{n-4,2} + P_{n-2,1}\right) \quad (1)$$

$$P_{n-1,2} = \frac{1}{2}\left(P_{n-4,2} + P_{n-3,2}\right) = \frac{1}{2}\left(P_{n-4,2} + P_{n-2,1}\right) \quad (2)$$

Where $P_{n,1}$ designates the n-th term in the first sequence and $P_{n,2}$ the same term in the second sequence. The second equality in both (1) and (2) are a result of the induction hypothesis.



Observation 2: $P_{4n,1} = P_{4n,2}$ for all $n = 0,1,2,3, \ldots$

Proof: At the moment, it seems like the definition of the sequence is not going to get us to an obvious proof of this conjecture.

Let's see if something pops out by looking at the sequence in its rational number presentation. So let's turn to DERIVE:

```
        pts(n, m := 1/2, pt) :=
          Prog
            pt := [0, 0; 0, 1; 1, 0]
            k := 4
#1:         Loop
              If k > n
                 RETURN pt
              pt := APPEND(pt, [m·(pt↓(k - 3) + pt↓(k - 2))])
              k :+ 1

#2:     pts(20)'
```

#3:

$$\begin{bmatrix} 0 & 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{4} & \frac{1}{2} & \frac{3}{8} & \frac{3}{8} & \frac{7}{16} & \frac{3}{8} & \frac{13}{32} & \frac{13}{32} & \frac{25}{64} & \frac{13}{32} & \frac{51}{128} & \frac{51}{128} & \frac{103}{256} & \frac{51}{128} \\ 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{4} & \frac{1}{2} & \frac{3}{8} & \frac{3}{8} & \frac{7}{16} & \frac{3}{8} & \frac{13}{32} & \frac{13}{32} & \frac{25}{64} & \frac{13}{32} & \frac{51}{128} & \frac{51}{128} & \frac{103}{256} & \frac{51}{128} & \frac{205}{512} \end{bmatrix}$$

Observation 3: $P_{4i-1,2} = P_{4i,2} = P_{4i+2,2}$ for all $i = 1,2,3, \ldots$

Proof: Assume that the result holds for all $k < i$

$$P_{4i-1,2} = \frac{1}{2}\left(P_{4i-4,2} + P_{4i-3,2}\right) = \frac{1}{2}\left(P_{4(i-1),2} + P_{4i-3,2}\right) = \frac{1}{2}\left(P_{4(i-1)+2,2} + P_{4i-3,2}\right) =$$

$$= \frac{1}{2}\left(P_{4i-2,2} + P_{4i-3,2}\right) = P_{4i,2}$$

by definition of the recursive sequence.

The next to last equality was a result of the induction hypothesis.

Finally, $P_{4i+2,2} = \frac{1}{2}\left(P_{4i-2,2} + P_{4i,2}\right) = P_{4i,2}$ by the sequence definition and the first part of this proof.

If we combine Observation 1 and Observation 3 we have the proof for Observation 2. Thus, the part of the "Geometric Reasoning" that states that the limit of the sequence of points lies on the line $y = x$ is indeed correct.

But:

What is the value of the limit?

Finding the Limit of $\{P_{4i,1}\}$

$$\begin{bmatrix} 0 & 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{4} & \frac{1}{2} & \frac{3}{8} & \frac{3}{8} & \frac{7}{16} & \frac{3}{8} & \frac{13}{32} & \frac{13}{32} & \frac{25}{64} & \frac{13}{32} & \frac{51}{128} & \frac{51}{128} & \frac{103}{256} & \frac{51}{128} & \frac{205}{512} & \frac{205}{512} & \frac{409}{1024} & \frac{205}{512} \\ 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{4} & \frac{1}{2} & \frac{3}{8} & \frac{3}{8} & \frac{7}{16} & \frac{3}{8} & \frac{13}{32} & \frac{13}{32} & \frac{25}{64} & \frac{13}{32} & \frac{51}{128} & \frac{51}{128} & \frac{103}{256} & \frac{51}{128} & \frac{205}{512} & \frac{205}{512} & \frac{409}{1024} & \frac{205}{512} & \frac{819}{2048} \end{bmatrix}$$

Finding this limit and then invoking observation 3 and one more observation, we can easily use a classic ε, δ proof to show that the limit of this subsequence is the limit of the entire sequence.

Looking at the sequence of first coordinates, we see that the even terms for $i \geq 2$ (remember I call the first term P_0) have successive powers of two in the denominator . Here is a DERIVE program and its result to look at this sequence:

```
Even_Terms(n, ev, pt) :=
  Prog
    pt := [0, 0; 0, 1; 1, 0]
    ev := [0, 1]
    k := 4
    Loop
      If k > n
         RETURN ev
      pt := APPEND(pt, [1/2·(pt↓(k - 3) + pt↓(k - 2))])
      If MOD(k, 2) = 1
         ev := APPEND(ev, [pt↓k↓1])
      k :+ 1

Even_Terms(35)
```

$$\left[0,\ 1,\ \frac{1}{2},\ \frac{1}{4},\ \frac{3}{8},\ \frac{7}{16},\ \frac{13}{32},\ \frac{25}{64},\ \frac{51}{128},\ \frac{103}{256},\ \frac{205}{512},\ \frac{409}{1024},\ \frac{819}{2048},\ \frac{1639}{4096},\ \frac{3277}{8192},\ \frac{6553}{16384},\ \frac{13107}{32768},\ \frac{26215}{65536}\right]$$

Observation 4: $P_{2i,1} = P_{2(i-1),1} + \dfrac{(-1)^{\left[\frac{i}{2}\right]}}{2^{i-1}}$ for all $i = 1,2,3,\ldots$ and $\left[\dfrac{i}{2}\right]$ denotes the floor function.

Proof: Once again, we will assume that the result holds for all $k < i$.

We go back to the basic definition for the basic definition sequence, P_n, and work from there.

$$P_{2i,1} = \frac{1}{2}\left(P_{2i-3,1} + P_{2i-1,1}\right) = \frac{1}{2}\left(P_{2i-3,1} + P_{2(i-1),1}\right)$$

This argument is laden with notation and not terribly instructive, so let's give only an overview of how it goes:

Break the attack into two cases: i even and i odd i.e. $2i$ a multiple of 4 and not a multiple of 4. It is really the first case that we want, but need to prove it for all even terms. Basically, Observations 1 and 3 get the $P_{2i-3,1}$ term above to a previous multiple of 4 and then we work back up. The arithmetic gets messy and the exponents are a little hard to handle, but it eventually all works out. Note that the sign change always takes place at the multiples of 4. As was mentioned: Observations 1 & 3 are the keys.

Observation 5: $P_{4i,1} = P_{4(i-1),1} + \dfrac{(-1)^{i-1}}{2\cdot 4^i}$ for $i = 1, 2, 3, \ldots$,and thus,

$$P_{4i,1} = \sum_{k=1}^{i}\frac{(-1)^{k-1}}{2\cdot 4^k} = \frac{1}{2}\sum_{k=1}^{i}\frac{(-1)^{k-1}}{4^k}.$$

Proof: This is just a matter of extracting the terms from Observation 4.

Observation 6: $\lim\limits_{n\to\infty} P_{4n,1} = \dfrac{2}{5}$ and, thus, $\lim\limits_{n\to\infty} P_{4n,2} = \dfrac{2}{5}$.

Proof: We turn this one over to DERIVE:

$$\frac{1}{2}\cdot\sum_{k=0}^{\infty}\frac{(-1)^k}{4^k} = \frac{2}{5}$$

Finally, we need only show that the sequences of first and second coordinates converge. We show that they are Cauchy Sequences of Real Numbers and use the fact that the Real Numbers are a complete metric space, i.e. all Cauchy Sequences converge.

Observation 7: The sequences $\{P_{n,1}\}$ and $\{P_{n,2}\}$ are Cauchy Sequences.

Proof: Let $\varepsilon > 0$, Observations1, 3, and 4 have shown that for any two adjacent terms in the interval from $4i$ to $4(i+1)$ the absolute value of the differences are: $0, \frac{1}{2^i}, \frac{1}{2^i}, \frac{1}{2^{i+1}}$, respectively. Take the largest of these differences, $\frac{1}{2^i}$, and say that $\left|P_{4\left[\frac{n}{4}\right],1} - \frac{2}{5}\right| < \frac{\varepsilon}{4}$.

Now, choose N such that for $n > N$, $\frac{1}{2^{\left[\frac{n}{4}\right]}} < \frac{\varepsilon}{4}$ and $\left|P_{4\left[\frac{n}{4}\right],1} - \frac{2}{5}\right| < \frac{\varepsilon}{4}$ then if $m, n > N$ we have

$$\left|P_{n,1} - P_{m,1}\right| = \left|\left(P_{n,1} - P_{4\left[\frac{n}{4}\right],1}\right) - \left(P_{m,1} - P_{4\left[\frac{m}{4}\right],1}\right) + \left(P_{4\left[\frac{n}{4}\right],1} - \frac{2}{5}\right) - \left(P_{4\left[\frac{m}{4}\right],1} - \frac{2}{5}\right)\right| \le \varepsilon$$

Thus, $P_{n,1}$ is a Cauchy Sequence and hence converges to the same limit as $P_{4i,1}$.

The sequence $P_{4i,2}$ is just one term ahead of $P_{4i,1}$ and, thus, also converges to $\frac{2}{5}$.

Carl Is Finally Finished!

How Josef Attacked The Challenge

My first approach:

This was the function for creating the visualisation, giving a sequence of points:

```
P(n) :=
  If n = 0
     [0, 0]
     If n = 1
        [0, 1]
        If n = 2
           [1, 0]
           1/2·P(n - 3) + 1/2·P(n - 2)

VECTOR(P(n), n, 0, 20)'
```

$$\begin{bmatrix} 0 & 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{4} & \frac{1}{2} & \frac{3}{8} & \frac{3}{8} & \frac{7}{16} & \frac{3}{8} & \frac{13}{32} & \frac{13}{32} & \frac{25}{64} & \frac{13}{32} & \frac{51}{128} & \frac{51}{128} & \frac{103}{256} & \frac{51}{128} & \frac{205}{512} \\ 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{4} & \frac{1}{2} & \frac{3}{8} & \frac{3}{8} & \frac{7}{16} & \frac{3}{8} & \frac{13}{32} & \frac{13}{32} & \frac{25}{64} & \frac{13}{32} & \frac{51}{128} & \frac{51}{128} & \frac{103}{256} & \frac{51}{128} & \frac{205}{512} & \frac{205}{512} \end{bmatrix}$$

I inspected the (sorted) numerators of the 1st components (the x-values) in order to find the pattern:

Because of the recursive nature of the definition it needs a long calculation time finding the list of the first 100 numerators! The next function works iterative and is much faster:

```
pts(n, m := 1/2, pt) :=
  Prog
    pt := [0, 0; 0, 1; 1, 0]
    k := 4
    Loop
      If k > n
         RETURN pt
      pt := APPEND(pt, [m·(pt↓(k - 3) + pt↓(k - 2))])
      k :+ 1
```

```
SORT(VECTOR(NUMERATOR(k), k, (pts(60))↓↓1))
[0, 0, 0, 1, 1, 1, 1, 1, 3, 3, 3, 7, 13, 13, 13, 25, 51, 51, 51, 103, 205, 205, 205, 409, 819, 819,
 819, 1639, 3277, 3277, 3277, 6553, 13107, 13107, 13107, 26215, 52429, 52429, 52429, 104857, 209715,
 209715, 209715, 419431, 838861, 838861, 838861, 1677721, 3355443, 3355443, 3355443, 6710887,
 13421773, 13421773, 13421773, 26843545, 53687091, 53687091, 53687091, 107374183]
```

which is (for the first 60 fractions) without counting repeated appearances:

[0, 1, 3, 7, 13, 25, 51, 103, 205, 409, 819, 1639, 3277, 6553, 13107, 26215, 52429, 104857, 209715, 419431, 838861, 1677721, ….]

Starting with 7 we have always a package of 4 values containing the first and then three times the next value. I investigated the sequence of values from above starting with $n = 4$ which gives element 7:

n=4	7 =	$2^2 + 3$	
n=5	13 =	$2 \cdot 7 - 1 =$	$2^3+2 \cdot 3-1$
n=6	25 =	$2 \cdot 13 - 1 =$	$2^4+2^2 \cdot 3-3$
n=7	51 =	$2 \cdot 25+1 =$	$2^5+2^3 \cdot 3-2 \cdot 3+1$
n=8	103 =	$2 \cdot 51+1 =$	$2^6+2^4 \cdot 3-2^2 \cdot 3+3$
n=9	205 =	$2 \cdot 103-1 =$	$2^7+2^5 \cdot 3-2^3 \cdot 3+2 \cdot 3-1$
n=10	409 =	$2 \cdot 205-1 =$	$2^8+2^6 \cdot 3-2^4 \cdot 3+2^2 \cdot 3-3$
n=11	819 =	$2 \cdot 409+1 =$	$2^9+2^7 \cdot 3-2^5 \cdot 3+2^3 \cdot 3-2 \cdot 3+1$
n=12	1639 =	$2 \cdot 819+1 =$	$2^{10}+2^8 \cdot 3-2^6 \cdot 3+2^4 \cdot 3-2^2 \cdot 3+3$

…

…

The elements of the sequence formed by the first row of $P(n)$ from above are the numerators divided by 2^n.

I start with the elements with n = 4, 8, 12, … and try finding a general formula for the numerators.

This was for me the real "funny part" of the problem!

#1: $2^{4\cdot i-2} + 3\cdot\sum_{k=0}^{i-1} 2^{4\cdot k} - 3\cdot 2^{2}\cdot\sum_{k=0}^{i-2} 2^{4\cdot k}$

#2: $\dfrac{2^{4\cdot i+1}}{5} + \dfrac{3}{5}$

#3: $\mathrm{VECTOR}\left(\dfrac{\dfrac{2^{4\cdot i+1}}{5} + \dfrac{3}{5}}{2^{4\cdot i}},\ i,\ 0,\ 10\right)$

#4: $\left[1, \dfrac{7}{16}, \dfrac{103}{256}, \dfrac{1639}{4096}, \dfrac{26215}{65536}, \dfrac{419431}{1048576}, \dfrac{6710887}{16777216}, \dfrac{107374183}{268435456}, \dfrac{1717986919}{4294967296}, \dfrac{27487790695}{68719476736}, \dfrac{439804651111}{1099511627776}\right]$

#5: $\dfrac{\dfrac{2^{4\cdot i+1}}{5} + \dfrac{3}{5}}{2^{4\cdot i}} = \dfrac{3\cdot 2^{-4\cdot i}}{5} + \dfrac{2}{5}$

#6: $\lim_{i\to\infty}\left(\dfrac{3\cdot 2^{-4\cdot i}}{5} + \dfrac{2}{5}\right) = \dfrac{2}{5}$

Derive simplifies expression #1 to a nice formula. Applying VECTOR I can check the correctness of expression #5 and in the last step the limit of the partial sequence is given – what can easily be calculated without a CAS, of course.

I repeat the procedure for elements with n = 5, 9, 13, … and end again with the limit $\frac{2}{5}$.

#7: $2^{4\cdot i-1} + 3\cdot 2\cdot\sum_{k=0}^{i-1} 2^{4\cdot k} - 3\cdot 2^{3}\cdot\sum_{k=0}^{i-2} 2^{4\cdot k} - 1$

I can proceed in a similar way for the remaining elements of the sequence.

For n = 6, 10, 14, …

$2^{4\cdot i} + 3\cdot 2^{2}\cdot\sum_{k=0}^{i-1} 2^{4\cdot k} - 3\cdot\sum_{k=0}^{i-1} 2^{4\cdot k}$

And finally for n = 3, 7, 11, 15, …

$$\text{VECTOR}\left(\frac{2}{5}-\frac{2^{-4\cdot i-3}}{5}, i, 0, 10\right)=\left[\frac{3}{8}, \frac{51}{128}, \frac{819}{2048}, \frac{13107}{32768}, \frac{209715}{524288}, \frac{3355443}{8388608}, \frac{53687091}{134217728}, \frac{858993459}{2147483648}, \frac{13743895347}{34359738368}, \frac{219902325555}{549755813888}, \frac{3518437208883}{8796093022208}\right]$$

$$\lim_{i\to\infty}\left(\frac{2}{5}-\frac{2^{-4\cdot i-3}}{5}\right)=\frac{2}{5}$$

All partial sequences tend to the same limit, so the limit is $\frac{2}{5}$.

Without doubt the CAS was a very strong support for my calculations and checking the results, but …

I must admit that I was not really satisfied with my PROOF because I could not show that the pattern of the numerators and of the fractions as a whole will remain until infinity.

Inspired by Carl´s PROOF and by the fact that only natural numbers are involved I was quite sure that a proof by induction must be the right "recipe".

I used my formulae – which I had derived in the previous attempt –for generating a list of all fractions appearing in the sequence:

$$\#4:\quad p(k) := \left[\frac{2^{-k}}{5}+\frac{2}{5}, \frac{2}{5}-\frac{3\cdot 2^{-k-1}}{5}, \frac{2}{5}-\frac{2^{-k-2}}{5}, \frac{3\cdot 2^{-k-3}}{5}+\frac{2}{5}\right]$$

$$\#5:\quad \text{VECTOR}(p(k), k, 1, 21, 4) = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{3}{8} & \frac{7}{16} \\ \frac{13}{32} & \frac{25}{64} & \frac{51}{128} & \frac{103}{256} \\ \frac{205}{512} & \frac{409}{1024} & \frac{819}{2048} & \frac{1639}{4096} \\ \frac{3277}{8192} & \frac{6553}{16384} & \frac{13107}{32768} & \frac{26215}{65536} \\ \frac{52429}{131072} & \frac{104857}{262144} & \frac{209715}{524288} & \frac{419431}{1048576} \\ \frac{838861}{2097152} & \frac{1677721}{4194304} & \frac{3355443}{8388608} & \frac{6710887}{16777216} \end{bmatrix}$$

We have a table of the first 24 different fractions appearing in the sequence.

Then I started from the very beginning:

I came back to the original sequences of the 1st and 2nd components. My consideration was that both components are created in the same way, then I could stick to only one of them and I chose the x-coordinate. Function `pts(n)` returns the first n first coordinates of the sequence of points.

`pts(41)'`

$$\left[\begin{matrix} 0 & 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{4} & \frac{1}{2} & \frac{3}{8} & \frac{3}{8} & \frac{7}{16} & \frac{3}{8} & \frac{13}{32} & \frac{13}{32} & \frac{25}{64} & \frac{13}{32} & \frac{51}{128} & \frac{51}{128} & \frac{103}{256} & \frac{51}{128} \\ 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{4} & \frac{1}{2} & \frac{3}{8} & \frac{3}{8} & \frac{7}{16} & \frac{3}{8} & \frac{13}{32} & \frac{13}{32} & \frac{25}{64} & \frac{13}{32} & \frac{51}{128} & \frac{51}{128} & \frac{103}{256} & \frac{51}{128} & \frac{205}{512} \end{matrix}\right.$$

$$\begin{matrix} \frac{205}{512} & \frac{205}{512} & \frac{409}{1024} & \frac{205}{512} & \frac{819}{2048} & \frac{819}{2048} & \frac{1639}{4096} & \frac{819}{2048} & \frac{3277}{8192} & \frac{3277}{8192} & \frac{6553}{16384} & \frac{3277}{8192} & \frac{13107}{32768} & \frac{13107}{32768} \\ \frac{205}{512} & \frac{409}{1024} & \frac{205}{512} & \frac{819}{2048} & \frac{819}{2048} & \frac{1639}{4096} & \frac{819}{2048} & \frac{3277}{8192} & \frac{3277}{8192} & \frac{6553}{16384} & \frac{3277}{8192} & \frac{13107}{32768} & \frac{13107}{32768} & \frac{26215}{65536} \end{matrix}$$

$$\left.\begin{matrix} \frac{26215}{65536} & \frac{13107}{32768} & \frac{52429}{131072} & \frac{52429}{131072} & \frac{104857}{262144} & \frac{52429}{131072} & \frac{209715}{524288} \\ \frac{13107}{32768} & \frac{52429}{131072} & \frac{52429}{131072} & \frac{104857}{262144} & \frac{52429}{131072} & \frac{209715}{524288} & \frac{209715}{524288} \end{matrix}\right]$$

I prepared another tool: I wanted to address each single element of the sequence, used the formulae p(k) from above and took in account the fact that it is better to consider packages of eight elements in a row instead of only four.

```
      el(n) :=
        If n ≤ 4
           [0, 0, 1, 0]↓n
           If MOD(n, 8) = 5 ∨ MOD(n, 8) = 6 ∨ MOD(n, 8) = 0
              (p(4·FLOOR((n + 3)/8) - 3))↓1
#6:           If MOD(n, 8) = 1 ∨ MOD(n, 8) = 2 ∨ MOD(n, 8) = 4
                 (p(4·FLOOR((n + 3)/8) - 3))↓3
                 If MOD(n, 8) = 7
                    (p(4·FLOOR((n + 3)/8) - 3))↓2
                    (p(4·FLOOR((n + 3)/8) - 3))↓4

#7:   VECTOR(el(k), k, 41)
```

#8: $$\left[0, 0, 1, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{2}, \frac{3}{8}, \frac{3}{8}, \frac{7}{16}, \frac{3}{8}, \frac{13}{32}, \frac{13}{32}, \frac{25}{64}, \frac{13}{32}, \frac{51}{128}, \frac{51}{128}, \frac{103}{256}, \frac{51}{128}, \frac{205}{512}, \frac{205}{512}, \frac{409}{1024}, \frac{205}{512}, \frac{819}{2048}, \frac{819}{2048}, \frac{1639}{4096}, \frac{819}{2048}, \frac{3277}{8192}, \frac{3277}{8192}, \frac{6553}{16384}, \frac{3277}{8192}, \frac{13107}{32768}, \frac{13107}{32768}, \frac{26215}{65536}, \frac{13107}{32768}, \frac{52429}{131072}, \frac{52429}{131072}, \frac{104857}{262144}, \frac{52429}{131072}, \frac{209715}{524288}\right]$$

Please compare with the first row of simplified `pts(41)'` from above.

Generalization of the problem

For keeping the procedure more general I introduce the matrix **ini** which is the matrix defined by points #2 and #3; the first point is the origin and $m = \frac{1}{2}$ by default (m can be changed).

```
        ptss(n, ini, pt, m := 1/2) :=
          Prog
            pt := APPEND([[0, 0]], ini)
            k := 4
#9:         Loop
              If k > n
                 RETURN pt
              pt := APPEND(pt, [m·(pt↓(k - 3) + pt↓(k - 2))])
              k :+ 1
```

As the first and second coordinates are following the same rule, it is sufficient to investigate only one of them. I am choosing the *x*-coordinates.

#10: $\text{ptss}\left(41, \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix}\right)_1$

#11: $\left[0, x_1, x_2, \frac{x_1}{2}, \frac{x_2 + x_1}{2}, \frac{2\cdot x_2 + x_1}{4}, \frac{x_2 + 2\cdot x_1}{4}, \frac{4\cdot x_2 + 3\cdot x_1}{8}, \frac{3\cdot(x_2 + x_1)}{8}, \frac{6\cdot x_2 + 7\cdot x_1}{16}, \frac{7\cdot x_2 + 6\cdot x_1}{16}, \frac{12\cdot x_2 + 13\cdot x_1}{32}, \frac{13\cdot(x_2 + x_1)}{32}, \frac{26\cdot x_2 + 25\cdot x_1}{64}, \frac{25\cdot x_2 + 26\cdot x_1}{64}, \frac{52\cdot x_2 + 51\cdot x_1}{128}, \frac{51\cdot(x_2 + x_1)}{128}, \frac{102\cdot x_2 + 103\cdot x_1}{256}, \frac{103\cdot x_2 + 102\cdot x_1}{256}, \frac{204\cdot x_2 + 205\cdot x_1}{512}, \frac{205\cdot(x_2 + x_1)}{512}, \frac{410\cdot x_2 + 409\cdot x_1}{1024}, \frac{409\cdot x_2 + 410\cdot x_1}{1024}, \frac{820\cdot x_2 + 819\cdot x_1}{2048}, \frac{819\cdot(x_2 + x_1)}{2048}, \frac{1638\cdot x_2 + 1639\cdot x_1}{4096},\right.$

…

…

For me it is important to double check the single steps of the procedure:

Substituting [0,1] for $x = [x_1, x_2]$ results in the coefficients of x_2 which is the list of the 1st coordinates of the points:

$\left[0, 0, 1, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{2}, \frac{3}{8}, \frac{3}{8}, \frac{7}{16}, \frac{3}{8}, \frac{13}{32}, \frac{13}{32}, \frac{25}{64}, \frac{13}{32}, \frac{51}{128}, \frac{51}{128}, \frac{103}{256}, \frac{51}{128}, \frac{205}{512}, \frac{205}{512}, \frac{409}{1024}, \frac{205}{512}, \frac{819}{2048}, \frac{819}{2048}, \frac{1639}{4096}, \frac{819}{2048}, \frac{3277}{8192}, \frac{3277}{8192}, \frac{6553}{16384}, \frac{3277}{8192}, \frac{13107}{32768}, \frac{13107}{32768}, \frac{26215}{65536}, \frac{13107}{32768}, \frac{52429}{131072}, \frac{52429}{131072}, \frac{104857}{262144}, \frac{52429}{131072}, \frac{209715}{524288}\right]$

I substitute [1,0] for $x = [x_1, x_2]$ for obtaining the coefficients of x_1 (= 2nd coordinates of the points):

#13: $\left[0, 1, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{2}, \frac{3}{8}, \frac{3}{8}, \frac{7}{16}, \frac{3}{8}, \frac{13}{32}, \frac{13}{32}, \frac{25}{64}, \frac{13}{32}, \frac{51}{128}, \frac{51}{128}, \frac{103}{256}, \frac{51}{128}, \frac{205}{512}, \frac{205}{512}, \frac{409}{1024}, \frac{205}{512}, \frac{819}{2048}, \frac{819}{2048}, \frac{1639}{4096}, \frac{819}{2048}, \frac{3277}{8192}, \frac{3277}{8192}, \frac{6553}{16384}, \frac{3277}{8192}, \frac{13107}{32768}, \frac{13107}{32768}, \frac{26215}{65536}, \frac{13107}{32768}, \frac{52429}{131072}, \frac{52429}{131072}, \frac{104857}{262144}, \frac{52429}{131072}, \frac{209715}{524288}, \frac{209715}{524288}\right]$

```
#14:  x2_p(n) := el(n)

#15:  x1_p(n) := el(n + 1)
```

I split the fractions into their summands try to proof the pattern of the coefficients by induction.

Assume that the rule is valid until element x1_p(n) with mod(n,8) = 0; we would like to find element x1_p(n+1) by 1/2 *(x1_p(n+1-3)+x1_p(n+1-2)) = = 1/2*(x1_p(n-2)+x1_p(n-1)).

Then x1_p(n-2) with mod(n-2,8) = 6 and x1_p(n-1) with mod(n-1,8) = 7 will - hopefully - give x1_p(n+1) with mod(n+1,8) = 1.

First of all I perform a – successful – check again (for n = 40):

$$\#16: \quad x1\_p(40) = \frac{209715}{524288}$$

$$\#17: \quad x1\_p(41) = \frac{1}{2}\cdot(x1\_p(38) + x1\_p(39))$$

$$\#18: \quad \frac{209715}{524288} = \frac{209715}{524288}$$

I copied function `el(n)` because am needing the subexpressions for the different cases of mod(n,8).

```
       el(n) :=
         If n ≤ 4
            [0, 0, 1, 0]↓n
            If MOD(n, 8) = 5 ∨ MOD(n, 8) = 6 ∨ MOD(n, 8) = 0
               (p(4·FLOOR((n + 3)/8) - 3))↓1
#19:           If MOD(n, 8) = 1 ∨ MOD(n, 8) = 2 ∨ MOD(n, 8) = 4
                  (p(4·FLOOR((n + 3)/8) - 3))↓3
                  If MOD(n, 8) = 7
                     (p(4·FLOOR((n + 3)/8) - 3))↓2
                     (p(4·FLOOR((n + 3)/8) - 3))↓4
```

Then x1_p(n-2) (with mod(n,8) = 6):

$$\#20: \quad \text{SUBST}\left(\left(p\left(4\cdot\text{FLOOR}\left(\frac{n+3}{8}\right) - 3\right)\right)_1, n, n-2\right)$$

$$\#21: \quad \frac{2^3\cdot 2^{-4\cdot\text{FLOOR}(n/8 + 1/8)}}{5} + \frac{2}{5}$$

$$\#22: \quad \text{VECTOR}\left(\frac{2^3\cdot 2^{-4\cdot\text{FLOOR}(n/8 + 1/8)}}{5} + \frac{2}{5}, n, 8, 48, 8\right) = \left[\frac{1}{2}, \frac{13}{32}, \frac{205}{512}, \frac{3277}{8192}, \frac{52429}{131072}, \frac{838861}{2097152}\right]$$

Expression #22 are elements #6, 14, 22, 30, …

x1_p(n-1) (with mod(n,8) = 7):

$$\#23: \quad \text{SUBST}\left(\left(p\left(4\cdot\text{FLOOR}\left(\frac{n+3}{8}\right) - 3\right)\right)_2, n, n-1\right)$$

$$\#24: \quad \frac{2}{5} - \frac{2^2\cdot 3\cdot 2^{-4\cdot\text{FLOOR}(n/8 + 1/4)}}{5}$$

$$\#25: \quad \text{VECTOR}\left(\frac{2}{5} - \frac{2^2\cdot 3\cdot 2^{-4\cdot\text{FLOOR}(n/8 + 1/4)}}{5}, n, 8, 48, 8\right) = \left[\frac{1}{4}, \frac{25}{64}, \frac{409}{1024}, \frac{6553}{16384}, \frac{104857}{262144}, \frac{1677721}{4194304}\right]$$

Expression #25 are elements #7, 15, 23, 31, …

This is - should be - the next element in the sequence x1_p(n+1) (with mod(n,8) = 1):

$$\#26:\quad \text{SUBST}\left(\left[p\left(4\cdot\text{FLOOR}\left(\frac{n+3}{8}\right)-3\right)\right]_3,\ n,\ n+1\right)$$

$$\#27:\quad \frac{2}{5}-\frac{2^{1}\cdot 2^{-4\cdot\text{FLOOR}(n/8+1/2)}}{5}$$

$$\#28:\quad \text{VECTOR}\left(\frac{2}{5}-\frac{2^{1}\cdot 2^{-4\cdot\text{FLOOR}(n/8+1/2)}}{5},\ n,\ 8,\ 48,\ 8\right)=\left[\frac{3}{8},\ \frac{51}{128},\ \frac{819}{2048},\ \frac{13107}{32768},\ \frac{209715}{524288},\ \frac{3355443}{8388608}\right]$$

The next check holds:

$$\#29:\quad \frac{1}{2}\cdot\left(\left[\frac{1}{2},\ \frac{13}{32},\ \frac{205}{512},\ \frac{3277}{8192},\ \frac{52429}{131072},\ \frac{838861}{2097152}\right]+\left[\frac{1}{4},\ \frac{25}{64},\ \frac{409}{1024},\ \frac{6553}{16384},\ \frac{104857}{262144},\ \frac{1677721}{4194304}\right]\right)$$

$$\#30:\quad \left[\frac{3}{8},\ \frac{51}{128},\ \frac{819}{2048},\ \frac{13107}{32768},\ \frac{209715}{524288},\ \frac{3355443}{8388608}\right]$$

Now follows the interesting step: 1/2*(#21 + #24) = #27 ??

$$\#31:\quad \frac{1}{2}\cdot\left(\frac{2^{3}\cdot 2^{-4\cdot\text{FLOOR}(n/8+1/8)}}{5}+\frac{2}{5}+\left(\frac{2}{5}-\frac{2^{2}\cdot 3\cdot 2^{-4\cdot\text{FLOOR}(n/8+1/4)}}{5}\right)\right)$$

$$\#32:\quad \frac{2^{2}\cdot 2^{-4\cdot\text{FLOOR}(n/8+1/8)}}{5}-\frac{2^{1}\cdot 3\cdot 2^{-4\cdot\text{FLOOR}(n/8+1/4)}}{5}+\frac{2}{5}$$

DERIVE **does not** simplify further because it has no information about the nature of n. But we have: n is divisible by 8 (mod(n,8) = 0).

We know that: for all n with mod(n,8) = 0: floor(n/8 + 1/8) = floor(n/8 + 1/4) = n/8, so we can proceed:

$$\#33:\quad \frac{2^{2}\cdot 2^{-4\cdot(n/8)}}{5}-\frac{2^{1}\cdot 3\cdot 2^{-4\cdot(n/8)}}{5}+\frac{2}{5}$$

$$\#34:\quad \frac{2}{5}-\frac{2^{(2-n)/2}}{5}$$

We can do this with the CAS, too. Let´s take in account that n is divisible by 8. I substitute n by 8n_ and try again simplifying the expression.

$$\frac{1}{2}\cdot\left(\frac{2^{3}\cdot 2^{-4\cdot FLOOR(n/8+1/8)}}{5}+\frac{2}{5}+\left(\frac{2}{5}-\frac{2^{2}\cdot 3\cdot 2^{-4\cdot FLOOR(n/8+1/4)}}{5}\right)\right)$$

$$\frac{1}{2}\cdot\left(\frac{2^{3}\cdot 2^{-4\cdot FLOOR(8\cdot n\_/8+1/8)}}{5}+\frac{2}{5}+\left(\frac{2}{5}-\frac{2^{2}\cdot 3\cdot 2^{-4\cdot FLOOR(8\cdot n\_/8+1/4)}}{5}\right)\right)$$

$$\frac{2^{2}\cdot 2^{-4\cdot FLOOR(n\_+1/8)}}{5}-\frac{2^{1}\cdot 3\cdot 2^{-4\cdot FLOOR(n\_+1/4)}}{5}+\frac{2}{5}$$

n_ :∈ Integer (0, ∞)

$$\frac{2}{5}-\frac{2^{1-4\cdot n\_}}{5}$$

$$\frac{2}{5}-\frac{2^{1-4\cdot(n/8)}}{5}$$

$$\frac{2}{5}-\frac{2^{(2-n)/2}}{5}$$

I "simplify" expression #27 in the same way:

$$\#27:\quad \frac{2}{5}-\frac{2^{1}\cdot 2^{-4\cdot FLOOR(n/8+1/2)}}{5}$$

$$\#35:\quad \frac{2}{5}-\frac{2^{1}\cdot 2^{-4\cdot FLOOR(n/8+1/2)}}{5}$$

$$\#36:\quad \frac{2}{5}-\frac{2^{1}\cdot 2^{-4\cdot(n/8)}}{5}$$

$$\#37:\quad \frac{2}{5}-\frac{2^{(2-n)/2}}{5}$$

We can repeat the procedure for all cases and proof show the identities of
x1_p(n+1) = 1/2·(x1_p(n–2)+x1_p(n–1)) for all positions of n within a package of 8 in a row.

It is obvious that for the second part of the x-value = x2_p(n) the proof will also hold.

What we also can see is the fact that the full x-value will be $\frac{2}{5}x_1+\frac{2}{5}x_2+f1(n)x_1+f2(n)x_2$ where $f1$ and $f2$ are functions with 2^n in the denominator. The same is happening with the y-values.

Calculating the limits, the functions are tending to 0 and the limit of the sequence of points with $[x_0,y_0]$ = [0,0] will end in $\left[\frac{2}{5}(x_1+x_2),\frac{2}{5}(y_1+y_2)\right]$.

See an example: Initial points are [0,0], [5,-4] and [11,9].

If my idea holds then the sequence should end in $\left[\frac{2}{5}(5+11),\frac{2}{5}(-4+9)\right]=[6.4,2]$.

#47: $\text{ptss}\left(100, \begin{bmatrix} 5 & -4 \\ 11 & 9 \end{bmatrix}\right)_{[95, \ldots, 100]}$

#48: $\begin{bmatrix} 6.4 & 2 \\ 6.4 & 2 \\ 6.4 & 2 \\ 6.4 & 2 \\ 6.4 & 2 \\ 6.4 & 2 \end{bmatrix}$

I introduce a more general function including variable `pt` (= a matrix) for the initial points:

```
#49:  ptsG(n, pt, m := 1/2) :=
        Prog
          k := 4
          Loop
            If k > n
               RETURN pt
            pt := APPEND(pt, [m·(pt↓(k - 3) + pt↓(k - 2))])
            k :+ 1
```

#50: $\text{ptsG}\left(100, \begin{bmatrix} 0 & 0 \\ 5 & -4 \\ 11 & 9 \end{bmatrix}\right)_{100}$

#51: $\left[\frac{3602879701896389}{562949953421312}, \frac{140737488355327}{70368744177664}\right]$

#52: [6.4, 2]

Initial points are [-3, 5], [5, –4] and [11, 9]. What is the convergence point now, if there is one?

#53: $\text{ptsG}\left(100, \begin{bmatrix} -3 & 5 \\ 5 & -4 \\ 11 & 9 \end{bmatrix}\right)_{100}$

#54: $\left[\frac{204069358115225}{35184372088832}, \frac{1688849860263931}{562949953421312}\right]$

#55: [5.8, 3]

Can you find out the rule?

$$\left[0.4 \cdot x_3 + 0.4 \cdot x_2 + 0.2 \cdot x_1 , 0.4 \cdot y_3 + 0.4 \cdot y_2 + 0.2 \cdot y_1\right]$$

Proof this!

What happens if *m* ≠ 1/2? Conjectures? Proofs?

So What Was The Role Of The CAS?

❶ Although the DNL #22 Article said that it was finding a limit, it really only gave our intuition a "nudge".

❷ To really know that (0.4, 0.4) is the limit, a proof was required. The CAS can not construct a proof. There is no button to push.

❸ This is where a "partnership" develops. The student, and instructor, have to understand what it is that the CAS and other technologies can do to help with the reasoning process.

❹ Visualization is a powerful aid. Sometimes it takes the form of graphical displays, other times it may be just to generate a large number of terms or examples, or, as it this case it was to give a display that made certain patterns "stick out."

❺ As instructors, we need to "let a thousand flowers bloom", i.e. let our students try their own strategies and exercise the limits of the CAS and other technologies. Our role is to gently critique and offer guidance through suggestions. In this case, a real strategy did not emerge until it became clear that the denominators were powers of 2. Everything else emerged from this very simple observation.

Postlude

Here's What Rüdiger Saw

The original article was from DERIVE Newsletter #22. Rüdiger Baumann sent a short note for the DERIVE Newsletter pointing to the fact that little generalization leads to Edward Sawada's "Misguided Missile" contribution (also from DNL#22).

Rüdiger recommended the ITERATES-procedure because the recursive procedure is too slow.

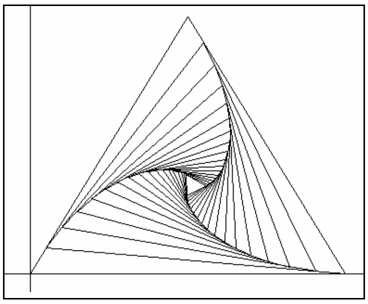
#85: pts_baum(r, s, ini, n) := ITERATES([b, c, r·a + s·b], [a, b, c], ini, n)

This is the "Leinbach-Brubaker Sequence":

$$\#86:\quad \text{pts\_baum}\left(0.5,\ 0.5,\ \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix},\ 50\right)$$

And this is Edward´s missile:

$$\#87:\quad \text{pts\_baum}\left(0.9,\ 0.1,\ \begin{bmatrix} 0 & 0 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \\ 1 & 0 \end{bmatrix},\ 50\right)$$

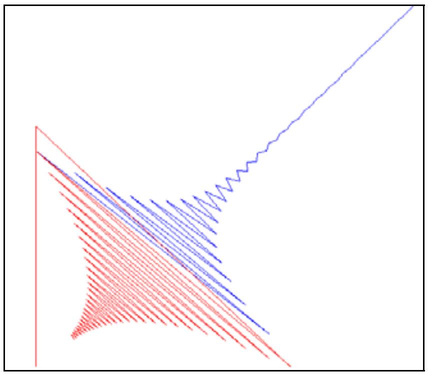


Playing with the parameters in Rüdiger’s function leads to interesting patterns (limits?)

#89: $\text{pts\_baum}\left(0.15,\ 0.9,\ \begin{bmatrix} 0 & 0 \\ 0 & 2 \\ 2 & 0 \end{bmatrix},\ 50\right)$

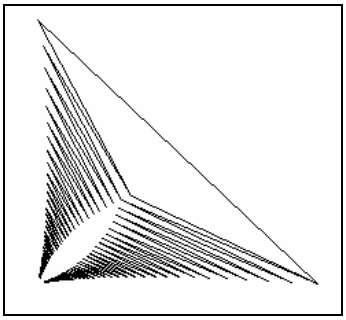
#90: $\text{pts\_baum}\left(0.05,\ 0.9,\ \begin{bmatrix} 0 & 0 \\ 0 & 2 \\ 2 & 0 \end{bmatrix},\ 50\right)$

#89 red and #90 blue



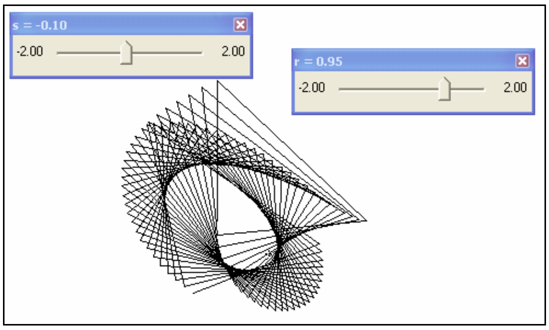
Let´s produce a TWIN

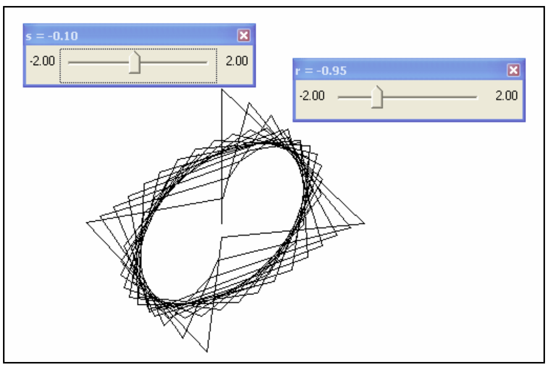
$$\left[\text{pts\_baum}\left(0.02,\ 0.9,\ \begin{bmatrix} 0 & 3 \\ 1 & 1 \\ 3 & 0 \end{bmatrix},\ 80\right),\ \text{pts\_baum}\left(0.02,\ 0.9,\ \begin{bmatrix} 3 & 0 \\ 0 & 3 \\ 1 & 1 \end{bmatrix},\ 80\right)\right]$$



Again working with my “beloved” sliders:

Why not trying to introduce sliders for the parameters r and s and investigate their influence on the sequence of points?





Are this really conics?

Try a proof!!

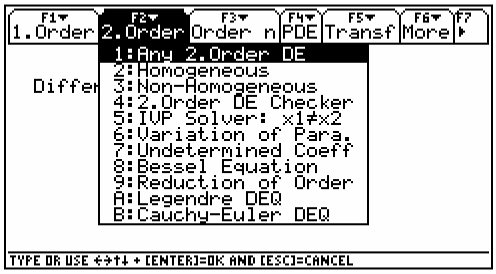
If you find other (better?) proofs for the presented problems then please send them.

Carl and Josef

Differential Equations Made Easy (2)

Review and Description of some features, Josef Böhm

In DNL#74 I presented Nils Hahnfeld's tool *DEQME* and was busy with Menu F1 dealing with 1st order ODEs. In this article I'd like to proceed to Menu F2 which offers treating 2nd order differential equations.



I'll try to address all Menu options from 1: through B: using the occasion to demonstrate parallel how to apply *DERIVE* for solving the problems and comparing with the solutions given by other CASs.
I don´t hesitate to admit that I learned – again after many years – about specific DEs – and I liked to do this. Most of the examples are from a textbook *Differential Equations*[1].

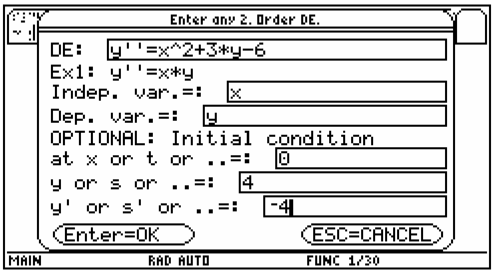
I´ll start with 1: Any 2.Order DE and observe the reaction of *DEQME*. My first package of differential equations is:

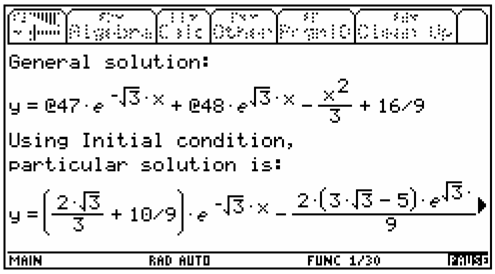
(1) $y'' = x^2 + 3y - 6;\ \ y(0) = 4,\ y'(0) = -4$

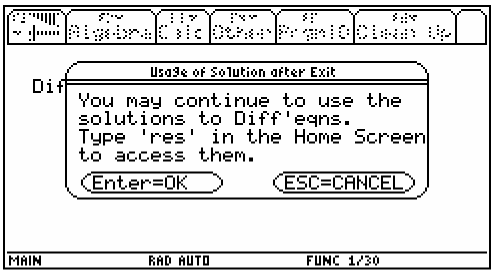
(2) $y'' + 4y = 12x;\ \ y(0) = 5,\ y'(0) = 7$

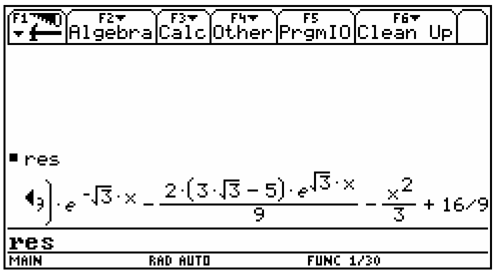
(3) $y'' + y = 2\sin(x)\cdot\sin(2x)$

Example (1)









In order to make comparing the tools easier I will do that example for example. I will am starting with *DERIVE* and proceed with *WIRIS, wxMaxima*, *MuPAD*, and with *TI-Nspire*, too, of course.

DERIVE's Online Help informs about the syntax for solving 2nd order ODEs:

DSOLVE2(p, q, r, x, c1, c2) simplifies to an explicit general solution of the linear second order ordinary differential equation

$$y'' + p(x)\cdot y' + q(x)\cdot y = r(x)$$

DSOLVE2_BV(p, q, r, x, x0, y0, x2, y2) is similar to DSOLVE2, but simplifies to a specific solution that satisfies the boundary conditions y=y0 at x=x0 and y=y2 at x=x2.

DSOLVE2_IV(p, q, r, x, x0, y0, v0) is similar toDSOLVE2_BV, but simplifies to a specific solution that satisfies the initial conditions y=y0 and y'=v0 at x=x0.

#1: $\text{DSOLVE2\_IV}(0, -3, 6 - x^2, x, 0, 4, -4)$

#2: $$e^{\sqrt{3}\cdot x}\cdot\left(\frac{26}{9} - \frac{2\cdot\sqrt{3}}{3}\right) + e^{-\sqrt{3}\cdot x}\cdot\left(\frac{2\cdot\sqrt{3}}{3} + \frac{26}{9}\right) + \frac{3\cdot x^2 - 16}{9}$$

Now let *WIRIS* try the job:

prob1 $:=\text{solve}(y''(x)=x^2+3\cdot y(x)-6, y(0)=4, y'(0)=-4);$

$$\text{prob1} \rightarrow \left\{\left\{y(x)=\left(\frac{2\cdot\sqrt{3}}{3}+\frac{10}{9}\right)\cdot e^{-\sqrt{3}\cdot x}+\left(-\frac{2\cdot\sqrt{3}}{3}+\frac{10}{9}\right)\cdot e^{\sqrt{3}\cdot x}-\frac{x^2}{3}+\frac{16}{9}\right\}\right\}$$

I applied a "trick" introducing prob1 in order to avoid presenting the equation together with the result in one line which would have been difficult to print it here in a reasonable size.

Next in the row is *wxMaxima*:

```
(%i1) eqn_1: 'diff(y,x,2) = x^2+3*y-6;
```

(%o1) $$\frac{d^2}{d\,x^2}\,y = 3\,y + x^2 - 6$$

```
(%i2) ode2(%o1, y, x);
```

(%o2) $$y = \%k1\,\%e^{\sqrt{3}\,x} + \%k2\,\%e^{-\sqrt{3}\,x} - \frac{3\,x^2 - 16}{9}$$

```
(%i17) ic2(%o2, x=0, y=4, 'diff(y,x)=-4);
```

(%o17) $$y = -\frac{(6\sqrt{3} - 10)\,\%e^{\sqrt{3}\,x}}{9} + \frac{(6\sqrt{3} + 10)\,\%e^{-\sqrt{3}\,x}}{9} - \frac{3\,x^2 - 16}{9}$$

What about *MuPAD*?

```
ivp:=ode({y''(x)=x^2+3*y(x)-6,y(0)=4,y'(0)=-4},y(x))
```

$$\text{ode}\left(\left\{y(0) = 4,\ y'(0) = -4,\ -3\cdot y(x) - x^2 + \frac{\partial^2}{\partial x^2}\,y(x) + 6\right\},\ y(x)\right)$$

```
solve(ivp)
```

$$\left\{e^{-\sqrt{3}\cdot x}\cdot\left(\frac{2\cdot\sqrt{3}}{3}+\frac{10}{9}\right) - e^{\sqrt{3}\cdot x}\cdot\left(\frac{2\cdot\sqrt{3}}{3}-\frac{10}{9}\right) - \frac{x^2}{3} + \frac{16}{9}\right\}$$

Last but not least see the *TI-Nspire*:

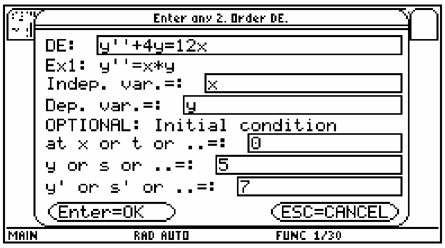
$\text{deSolve}\left(y''=x^2+3\cdot y-6 \text{ and } y(0)=4 \text{ and } y'(0)=-4, x, y\right)$

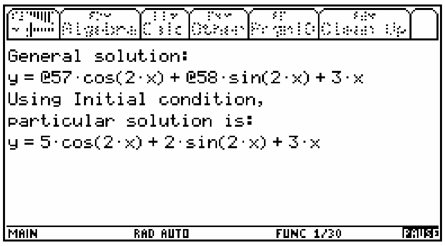
$$y=\left(\frac{2\cdot\sqrt{3}}{3}+\frac{10}{9}\right)\cdot e^{-\sqrt{3}\cdot x}-\frac{2\cdot\left(3\cdot\sqrt{3}-5\right)\cdot e^{\sqrt{3}\cdot x}}{9}-\frac{x^2}{3}+\frac{16}{9}$$

TI-Nspire works like the TI-92PLUS and TI-Voyage 200. Please wait, *DEQME* is able to do a lot more than only solving this kind of DEs assisted by a nice form for entering the data.

All systems behave the same so far. I can assure that things will change!

Example (2)





DERIVE:

$$\text{DSOLVE2\_IV}(0,\ 4,\ 12\cdot x,\ x,\ 0,\ 5,\ 7) = 4\cdot\text{SIN}(x)\cdot\text{COS}(x) - 10\cdot\text{SIN}(x)^2 + 3\cdot x + 5$$

WIRIS:

prob2 := solve(y"(x)+4y(x)=12x,y(0)=5,y'(0)=7);

$$\text{prob2} \to \left\{\left\{y(x)=6\cdot x\cdot\sin(2\cdot x)^2+\frac{\sin(2\cdot x)}{2}+6\cdot x\cdot\cos(2\cdot x)^2+5\cdot\cos(2\cdot x)\right\}\right\}$$

$$\text{prob2} := \text{simplify}\left(6\cdot x\cdot\sin(2\cdot x)^2+\frac{\sin(2\cdot x)}{2}+6\cdot x\cdot\cos(2\cdot x)^2+5\cdot\cos(2\cdot x)\right);$$

$$\text{prob2} \to 5\cdot\cos(2\cdot x)+\frac{\sin(2\cdot x)}{2}+6\cdot x$$

WxMaxima:

(%i20) eqn_2: 'diff(y,x,2)+4*y=12*x;

(%o20) $\frac{d^2}{d\,x^2}\,y + 4\,y = 12\,x$

(%i21) ode2(%o20,y,x);

(%o21) $y = \%k1\,\sin(2\,x) + \%k2\,\cos(2\,x) + 3\,x$

(%i24) ic2(%o21,x=0,y=5,'diff(y,x)=7);

(%o24) $y = 2\,\sin(2\,x) + 5\,\cos(2\,x) + 3\,x$

MuPAD provides the same result and so do the TIs!

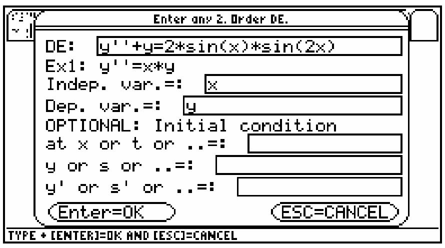
WIRIS needs an extra simplification – it does not simplify $6x\sin^2(2x) + 6x\cos^2(2x)$ to $6x$. The *WIRIS* result seems to be wrong! I am using *WIRIS* itself to check the result – and the right side gives $24x$ instead of $12x$.

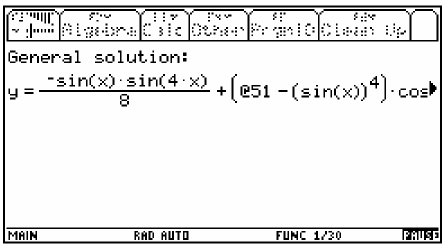
$$y(x) := 5\cdot\cos(2\cdot x)+\frac{\sin(2\cdot x)}{2}+6\cdot x \to x\mapsto 5\cdot\cos(2\cdot x)+\frac{\sin(2\cdot x)}{2}+6\cdot x$$

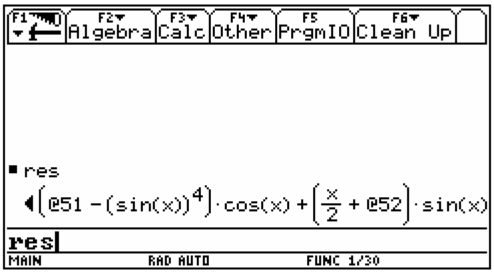
$$y''(x)+4\cdot y(x) \to 24\cdot x$$

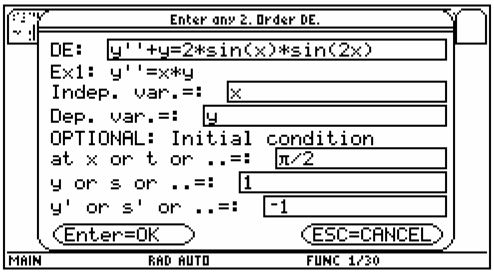
I could have used also *DEQME*'s 2nd order DE Checker to check the presented (correct) solutions!

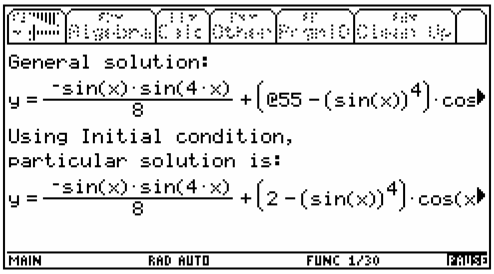
Example (3) (including an additional IVP)**:**

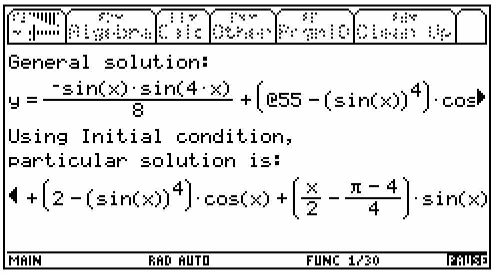












Working with *DERIVE* is interesting:

$$\mathrm{DSOLVE2\_IV}\left(0,\ 1,\ 2\cdot\mathrm{SIN}(x)\cdot\mathrm{SIN}(2\cdot x),\ x,\ \frac{\pi}{2},\ 1,\ -1\right)$$

$$-\frac{\mathrm{COS}(x)\cdot\mathrm{COS}(4\cdot x)}{8}-\frac{\mathrm{SIN}(x)\cdot\mathrm{SIN}(4\cdot x)}{8}+\frac{\mathrm{COS}(x)\cdot\mathrm{COS}(2\cdot x)}{2}+\frac{13\cdot\mathrm{COS}(x)}{8}+\left(\frac{x}{2}-\frac{\pi}{4}+1\right)\cdot\mathrm{SIN}(x)$$

$$\frac{\sqrt{(4\cdot x^2+1)}\cdot\mathrm{COS}(\mathrm{ATAN}(2\cdot x)-x)}{4}-\frac{\sqrt{(4\cdot\pi^2-32\cdot\pi+233)}\cdot\mathrm{SIN}\left(\mathrm{ATAN}\left(\frac{13}{2\cdot(\pi-4)}\right)-x\right)}{8}+\frac{\mathrm{COS}(3\cdot x)}{8}$$

[Trigonometry := Expand, Trigpower := Sines]

$$\mathrm{COS}(x)\cdot\left(2-\frac{\mathrm{SIN}(x)^2}{2}\right)+\left(\frac{x}{2}-\frac{\pi}{4}+1\right)\cdot\mathrm{SIN}(x)$$

There are three different appearances of the result depending on the Trig Mode Settings. You need some skills (or intuiton and luck – just try) to find the appropriate settings in order to obtain a compact form of the solution. Plotting all solutions on the same axes and checking by substituting the solution into the given equation shows the identity of the expressions.

$$y(x) :=-\frac{\mathrm{SIN}(x)\cdot\mathrm{SIN}(4\cdot x)}{8}+(2-\mathrm{SIN}(x)^4)\cdot\mathrm{COS}(x)+\left(\frac{x}{2}-\frac{\pi-4}{4}\right)\cdot\mathrm{SIN}(x)$$

$$y''(x)+y(x)-2\cdot\mathrm{SIN}(x)\cdot\mathrm{SIN}(2\cdot x)=0$$

WIRIS presents its solution in a very “extended” form. Plotting and applying the appropriate Trig Mode in *DERIVE* confirms the identity with the other results.

$$\mathrm{solve}(y''(x)+y(x)=2\sin(x)\cdot\sin(2x),y(\pi/2)=1,y'(\pi/2)=-1)$$

$$\rightarrow\left\{\left[y(x)=-\frac{\sin(x)\cdot\sin(-2\cdot x)}{4}-\frac{\sin(x)\cdot\sin(4\cdot x)}{8}-\frac{\sin(x)\cdot\sin(2\cdot x)}{4}+\left(\frac{x}{2}-\frac{\pi}{4}+1\right)\cdot\sin(x)\right.\right.$$

$$y(x):=-\frac{\sin(x)\cdot\sin(-2\cdot x)}{4}-\frac{\sin(x)\cdot\sin(4\cdot x)}{8}-\frac{\sin(x)\cdot\sin(2\cdot x)}{4}+\left(\frac{x}{2}-\frac{\pi}{4}+1\right)\cdot\sin(x)-\frac{\cos(4\cdot x)\cdot\cos(x)}{8}+\frac{\cos(x)\cdot\cos(2\cdot x)}{2}+\frac{13\cdot\cos(x)}{8};$$

This was the first step. Simplifying this result once more delivers the result as above (*DERIVE*):

$$\text{simplify}\left(-\frac{\sin(x)\cdot\sin(-2\cdot x)}{4}-\frac{\sin(x)\cdot\sin(4\cdot x)}{8}-\sin(\ldots\right.$$

$$\rightarrow\left(-\frac{\sin(x)^2}{2}+2\right)\cdot\cos(x)+\left(\frac{x}{2}-\frac{\pi}{4}+1\right)\cdot\sin(x)$$

WxMaxima (correct result):

$$y=\frac{\cos(3\,x)+4\,x\sin(x)-\cos(x)}{8}-\frac{(\%pi-4)\sin(x)}{4}+2\cos(x)$$

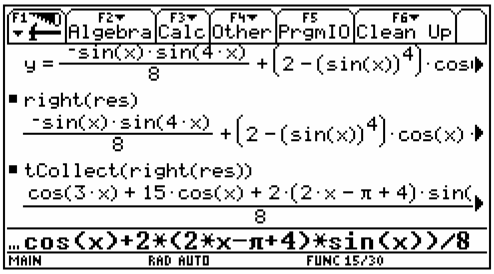
I believe that you are not surprised now that *MuPad* also delivers another form of the result – which proved to be wrong (satisfies the IVs, but does not satisfy the DE).

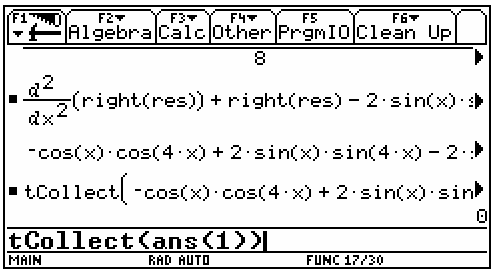
solve(ivp3)

$$\left\{\frac{23\cdot\cos(x)}{16}+\sin(x)+\frac{\cos(3\cdot x)}{32}+\frac{\cos(5\cdot x)}{32}-\frac{\pi\cdot\sin(x)}{4}+\frac{x\cdot\sin(x)}{2}\right\}$$

Finally I apply tCollect on the *DEQME*-output and receive a new appearance (right).

Then I substitute into the given equation and calculate the difference of the left side of the equation and the expected right side and hope that the result would be 0.





Both next problems read originally as follows:

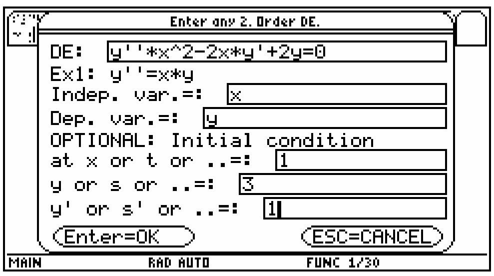
Verify that y_1 and y_2 are solutions of the DE. Then find a particular solution of the form $y = c_1\,y_1 + c_2\,y_2$ that satisfies the given initial conditions.

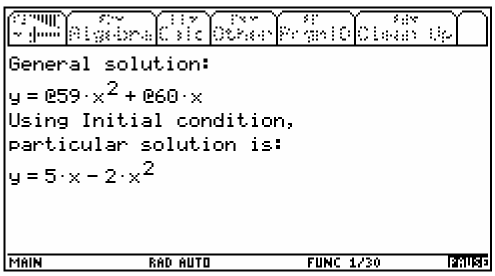
(4) $x^2y''-2x\,y'+2y=0;\;\; y_1=x,\, y_2=x^2;\, y(1)=3,\, y'(1)=1$

(5) $x^2y''+2x\,y'-6y=0;\;\; y_1=x^2,\, y_2=x^{-3};\, y(2)=10,\, y'(2)=15$

Example (4)

I try solving the DE and leave the check of the solutions y_1 and y_2 for later.





Again we don't encounter any problem using *DEQME*!

The next page shows the DE solved by *DERIVE*, *WIRIS, wxMaxima*, and *MuPad*:

$$\text{DSOLVE2\_IV}\left(-\frac{2}{x}, \frac{2}{x^2}, 0, x, 1, 3, 1\right) = 5\cdot x - 2\cdot x^2$$

WIRIS:

Example (4)

solve$(x^2\cdot y''(x)-2x\cdot y'(x)+2y(x)=0, y(1)=3, y'(1)=1) \rightarrow \{\{y(x)=-2\cdot e^{2\cdot \ln(x)}+5\cdot x\}\}$

simplify$(y(x)=-2\cdot e^{2\cdot\ln(x)}+5\cdot x) \rightarrow y(x)=-2\cdot x^2+5\cdot x$

WIRIS needs again an extra simplification for $e^{\ln x} = x$. Then see *wxMaxima* …

```
eqn_4:'x^2*'diff(y,x,2)-2*x*'diff(y,x,1)+2*y=0;
```

$$x^2\left(\frac{d^2}{d\,x^2}y\right) - 2\,x\left(\frac{d}{d\,x}y\right) + 2\,y = 0$$

```
ode2(%o11,y,x);
```

$$y = \%k1\,x^2 + \%k2\,x$$

```
ic2(%o12,x=1,y=3,'diff(y,x)=1);
```

$$y = 5\,x - 2\,x^2$$

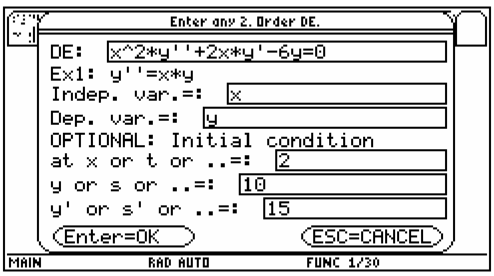
… followed by *MuPAD* (compact form of the procedure performed on page 26):

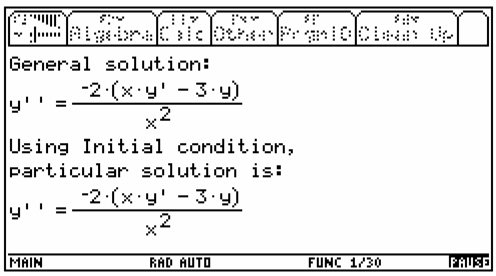
```
solve(ode({x^2*y''(x)-2*x*y'(x)+2*y(x)=0,y(1)=3,y'(1)=1},y(x)))
```

$\{5\cdot x - 2\cdot x^2\}$

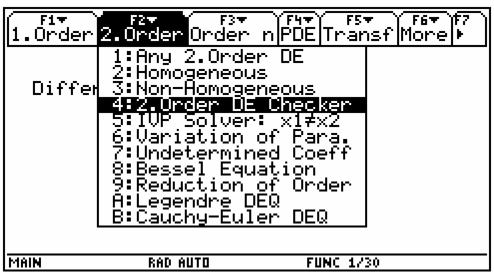
This runs pretty well, let´s try the next example which looks very similar.

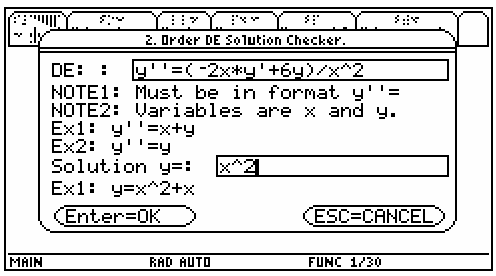
Example (5)

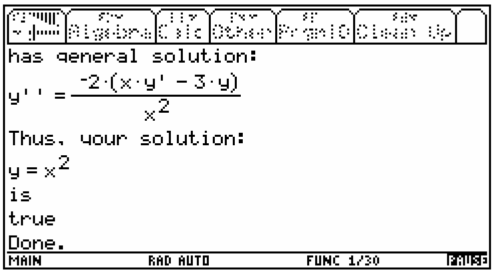


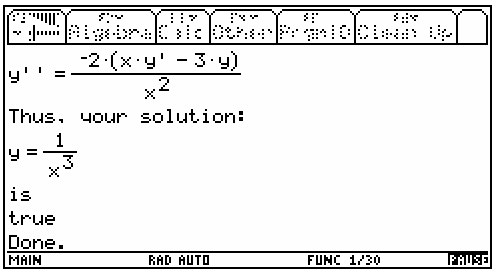


Interestingly this DE cannot be solved although it looks very similar to Example (4). Maybe that it doesn´t have any solution? I try the 2. Order DE Checker offered in Option 4 for y_1 and y_2:









y_1 and y_2 are solutions – as expected. So, we have solved the problem given in the textbook. I leave this DE for a later treatment and will look how DERIVE and other Computeralgebra Systems are performing. Doubtless I am starting again with DERIVE:

$$\#2:\quad \text{DSOLVE2\_IV}\left(\frac{2}{x}, -\frac{2}{x^2}, 0, x, 2, 10, 15\right) = \text{inapplicable}$$

I was not very much surprised about #2. The TI's algorithms are very close related with the *DERIVE* algorithms. (David Stoutemyer implemented the DE-package in *DERIVE* and the TIs as well.)

I checked the validity of y_1 and y_2 as solutions with *DERIVE*. Any linear combination of y_1 and y_2 should also form a solution, hence:

$$\text{sol}(x) := c1\cdot x^2 + c2\cdot x^{-3}$$

$$x^2\cdot\text{sol}''(x) + 2\cdot x\cdot\text{sol}'(x) - 6\cdot\text{sol}(x) = 0$$

So we are very curious what other systems will answer. Our next candidate is *WIRIS*:

Example (5)

$\text{prob5} := \text{solve}(x^2\cdot y''(x) + 2x\cdot y'(x) - 6\cdot y(x) = 0, y(2) = 10, y'(2) = 15);$

$\text{prob5} \longrightarrow \{\{y(x) = -16.\cdot e^{-3.\cdot\ln(x)} + 3.\cdot e^{2.\cdot\ln(x)}\}\}$

$\text{simplify}(y(x) = -16\cdot e^{-3\cdot\ln(x)} + 3\cdot e^{2\cdot\ln(x)}) \longrightarrow y(x) = \dfrac{3\cdot x^5 - 16}{x^3}$

We obtain for $x > 0$ $\;y(x) = -16x^{-3} + 3x^2$. Then I try a compact form with *wxMaxima*.

```
ic2(ode2(x^2*'diff(y,x,2)+2*x*'diff(y,x,1)-6*y=0,y,x),x=2,y=10,'diff(y,x)=15);
```

$$y = 3\,x^2 - \frac{16}{x^3}$$

MuPad delivers the same solution without any problems. I don´t know why *DERIVE* refuses solving this equation, do you know?

I asked Albert Rich – one of the fathers of *DERIVE* – and I received his answer:

Hello Josef,

Thanks for your inquiry. Since David Stoutemyer wrote the DSOLVE packages for ***Derive***, he would be the best one to help resolve the deficiency you found.

Aloha,
Albert

So my next mail was sent to David and he also answered very soon:

Hello Josef!

Karen and I are fine. We too took a trip to Tanzania this fall. Fantastic!

DSOLVE2_IV(...) doesn't search for the pattern for Euler-type ODEs.

However, often more than one method is applicable to a given ODE. One of the other implemented methods solved your first example, but not the second.

However, it would clearly be a good idea to add Euler-type equations to the list of patterns to try.

-- best regards, david

Working out this review I remembered (supported by my old text books, of course) that DEs of this type - $x^2 \cdot y''(x) + p \cdot x \cdot y'(x) + q \cdot y(x) = r(x)$ - are *Euler Equations* – and fortunately enough, Nils has provided one option for solving this type of equations – even stepwise. *DEQME* in its recent version solves only the homogenous DE but we can use his tool for the inhomogeneous form, too.

For the very few among you, who – like myself – have forgotten the algorithm solving an *Euler DE,* I'll solve another example by hands and then check my calculations!

Example (6)

Find the general solution of $x^2 y'' - 3x\,y' + 8y = x^2 + 2x$ for $x > 0$.
If $y(x)$ is a solution for $x > 0$ then $y(-x)$ is a solution for $x < 0$.

The "trick" is applying the substitution $x = e^s$. Then $u(s) = y(e^s) = y(x) = u(\log x)$.

The following is a useful application of the chain rule – for the students.

$x = e^s$ and $y(x) = y(e^s) = u(s)$

$u'(s) = y'(e^s) \cdot e^s = y' \cdot x$

$u''(s) = y''(e^s) \cdot e^s \cdot e^s + y'(e^s) \cdot e^s = y'' \cdot e^{2s} + y' \cdot e^s = y'' \cdot x^2 + y' \cdot x$

hence

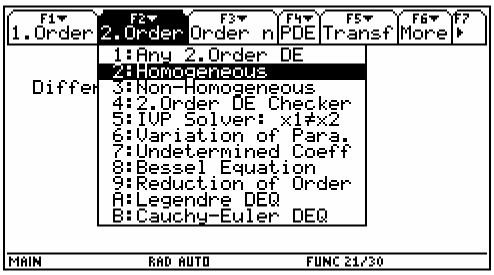
$y''(x) \cdot x^2 = u''(s) - u'(s)$ and $y'(x) \cdot x = u'(s)$

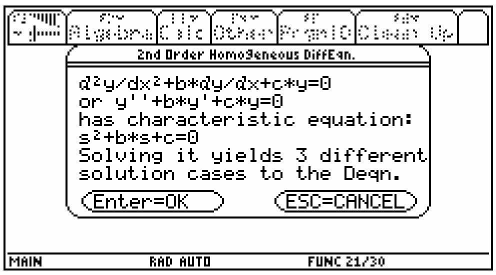
The given differential equation reads now (in variables s und u):

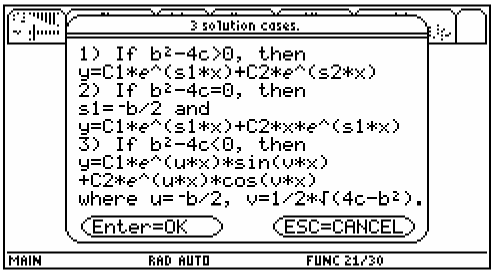
$u'' - u' - 3u' + 8u = e^{2s} + 2e^s$

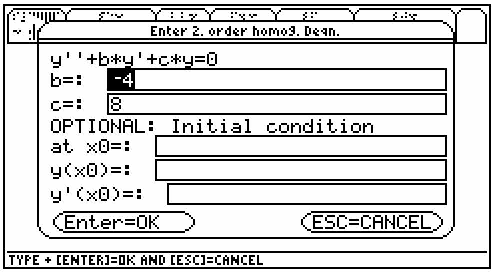
$u'' - 4u' + 8u = e^{2s} + 2e^s$

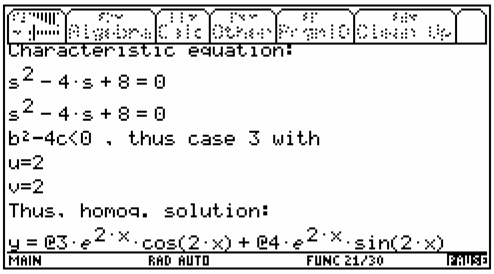
This is an nonhomogeneous ODE of 2nd order with constant coefficients. At first we have to find the general solution of the respective homogeneous DE. We ask *DEQME*:







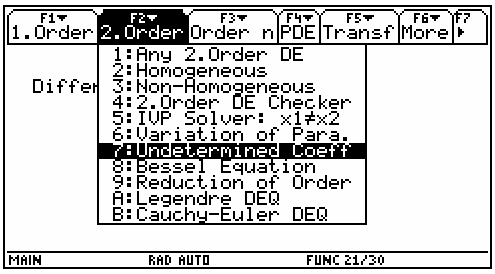


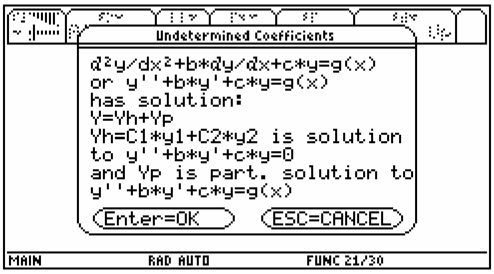


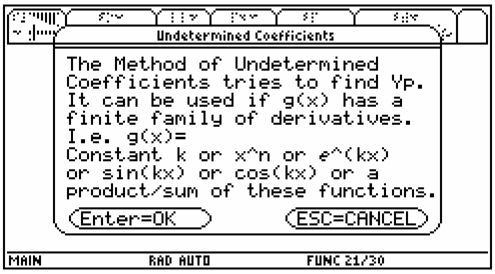
As we are restricted to *y* and *x* in this option, we have to rewrite the solution as

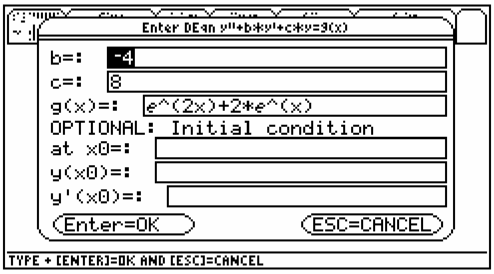
$$u(s) = c_1 \cdot e^{2s}\cos(2s) + c_2 \cdot e^{2s}\sin(2s).$$

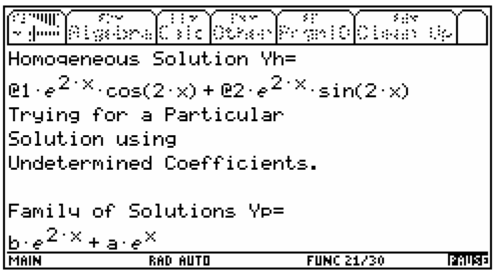
We need a particular solution to accomplish the solution for the nonhomogeneous equation. The method of undetermined coefficients seems to be appropriate. Again we ask *DEQME* for information and support.

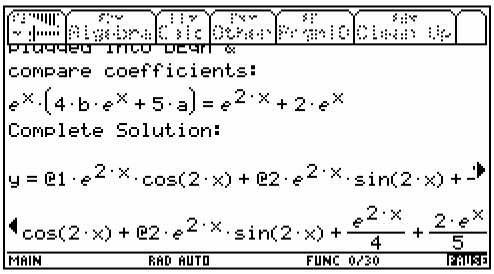












Now we have solved the (*u*,*s*)-DE which must be transformed back to the *y*(*x*)-function:

$x = e^s \leftrightarrow s = \log x$

(*u*,*s*): $\quad u(s) = c_1 \cdot e^{2s}\cos(2s) + c_2 \cdot e^{2s}\sin(2s) + \dfrac{e^{2s}}{4} + \dfrac{2e^s}{5}$

(*y*, *x*): $\quad y(x) = c_1 \cdot x^2\cos(2\log x) + c_2 \cdot x^2\sin(2\log x) + \dfrac{x^2}{4} + \dfrac{2x}{5}.$

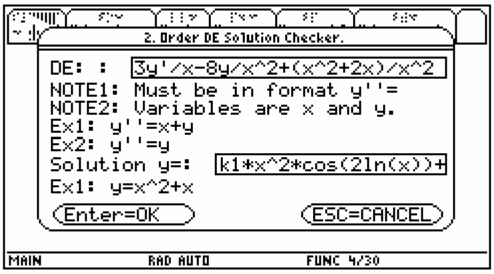
I regret having to admit that *DERIVE* is unable to solve this equation, so we will try with *wxMaxima*:

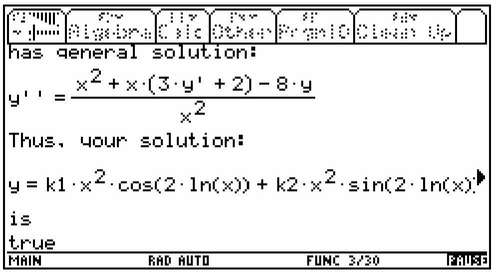
```
ode2(x^2*'diff(y,x,2)-3*x*'diff(y,x,1)+8*y=x^2+2*x,y,x);
```

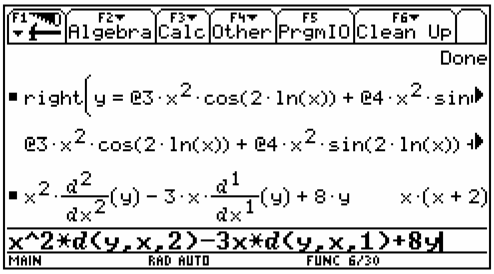
$$y=\frac{\left(5\,x^2+8\,x\right)\sin(2\log(x))^2+\left(5\,x^2+8\,x\right)\cos(2\log(x))^2}{20}+x^2(\%k1\sin(2\log(x))+\%k2\cos(2\log(x)))$$

I used the occasion to demonstrate Options 2 and 3 of the F2-menu.

We can apply the *DEQME* built-in DE Checker (Option 4) or do it on our own in the Home Screen (see below) for checking the solution.







As you might guess I was not really satisfied about *DERIVE*'s inability solving one or the other *Euler Equation.* So look at this:

(7) $x^2y''-3x\,y'+8y=x^2+2x$

(8) $x^2y''+2x\,y'-6y=0;\ y(2)=10,\ y'(2)=15$

(9) $x^2y''+2x\,y'-6y=30x;\ y(2)=10,\ y(6)=15$

(10) $x^2y''+2x\,y'-6y=30x;\ y(2)=10,\ y'(6)=15$

(11) $x^2y''+2x\,y'-6y=30x;\ y'(2)=10,\ y'(6)=15$

All of them cannot be solved using DSOLVE2, DSOLVE2_IV and DSOLVE2_BV respectively.
My EULER-functions do a better job.

Examples (7) – (11)

$$\text{DSOLVE2}\left(-\frac{3}{x},\ \frac{8}{x^2},\ \frac{x^2+2\cdot x}{x^2}\right)=\text{inapplicable}$$

but now

$$\text{EULER2}\left(-\frac{3}{x},\ \frac{8}{x^2},\ \frac{x^2+2\cdot x}{x^2}\right)=c1\cdot x^2\cdot\text{COS}(2\cdot\text{LN}(x))+c2\cdot x^2\cdot\text{SIN}(2\cdot\text{LN}(x))+\frac{x^2}{4}+\frac{2\cdot x}{5}$$

$$\text{EULER2\_IV}\left(\frac{2}{x}, -\frac{6}{x^2}, 0, x, 2, 10, 15\right) = 3\cdot x^2 - \frac{16}{x^3}$$

Parameter #9 in EULER2_BV (= k) = 0 (= default) for y0=y(x0) and y1 = y(x1)
= 1 for y0 = y(x0) and y1 = y'(x1)
= 2 for y0 = y'(x0) and y1 = y'(x1)

$$\text{EULER2\_BV}\left(\frac{2}{x}, -\frac{6}{x^2}, \frac{30}{x}, x, 2, 10, 6, 15\right) = \frac{145\cdot x^2}{88} - \frac{15\cdot x}{2} + \frac{1620}{11\cdot x^3}$$

$$\text{EULER2\_BV}\left(\frac{2}{x}, -\frac{6}{x^2}, \frac{30}{x}, x, 2, 10, 6, 15, 1\right) = \frac{310\cdot x^2}{163} - \frac{15\cdot x}{2} + \frac{22680}{163\cdot x^3}$$

$$\text{EULER2\_BV}\left(\frac{2}{x}, -\frac{6}{x^2}, \frac{30}{x}, x, 2, 10, 6, 15, 2\right) = \frac{1805\cdot x^2}{968} - \frac{15\cdot x}{2} - \frac{6480}{121\cdot x^3}$$

one check:

$$f(x) := \frac{1805\cdot x^2}{968} - \frac{15\cdot x}{2} - \frac{6480}{121\cdot x^3}$$

$$x^2\cdot f''(x) + 2\cdot x\cdot f'(x) - 6\cdot f(x) = 30\cdot x$$

We will ask *DEQME* to solve example (8) after having recognized that (8) is of *Euler* type on page 43 investigating Option B: Cauchy-Euler DEQ.

wxMaxima confirms the solution of boundary value problem (9):

```
bc2(ode2(x^2*'diff(y,x,2)+2*x*'diff(y,x)-6*y=30*x,y,x),x=2,y=10,x=6,y=15);
```

$$y = \frac{145\, x^2}{88} - \frac{15\, x}{2} + \frac{1620}{11\, x^3}$$

muPAD says that I am right with my solutions for (9), (10) and (11), thank you very much, indeed.

```
solve(ode({x^2*y''(x)+2*x*y'(x)-6*y(x)=30*x,y(2)=10,y(6)=15},y(x)))
```

$$\left\{\frac{145\cdot x^2}{88} - \frac{15\cdot x}{2} + \frac{1620}{11\cdot x^3}\right\}$$

```
solve(ode({x^2*y''(x)+2*x*y'(x)-6*y(x)=30*x,y(2)=10,y'(6)=15},y(x)))
```

$$\left\{\frac{310\cdot x^2}{163} - \frac{15\cdot x}{2} + \frac{22680}{163\cdot x^3}\right\}$$

```
solve(ode({x^2*y''(x)+2*x*y'(x)-6*y(x)=30*x,y'(2)=10,y'(6)=15},y(x)))
```

$$\left\{\frac{1805\cdot x^2}{968} - \frac{15\cdot x}{2} - \frac{6480}{121\cdot x^3}\right\}$$

These are two of my "home made" EULER-functions for DERIVE:

```
EULER2(p, q, r, x, c1, c2, ans) :=
  Prog
    ans := DSOLVE2(p·x - 1, q·x^2, SUBST(r·x^2, x, e^x), x, c1, c2)
    If ¬ STRING?(ans)
       LIM(ans, x, LN(x))
       "inapplicable"

EULER2_IV(p, q, r, x, x0, y0, v0, ans, v, s) :=
  Prog
    ans := DSOLVE2(p·x - 1, q·x^2, SUBST(r·x^2, x, e^x), x, c1, c2)
    If ¬ STRING?(ans)
       Prog
         v := ∂(ans, x)
         s := (SOLUTIONS([LIM(ans, x, x0) = y0, LIM(v, x, x0) = v0], [c1, c2]))↓1
         SUBST(ans, [c1, c2], s)
       "inapplicable"
```

I used these functions to solve problems (9) to (11).

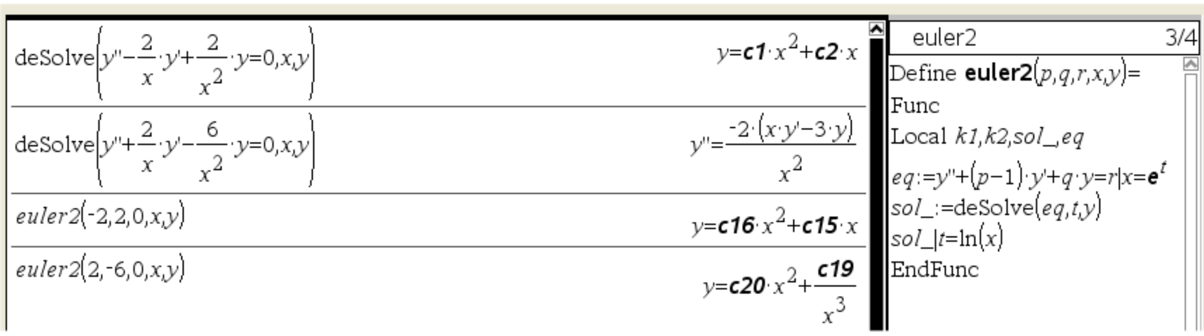
I don´t copy the Boundary Value function `EULER2_BV` here to save space. Its syntax is as follows:

EULER2_BV(p, q, r, x, x1, y1, x2, y2, k)

with $k = 0$ (by default): given are $(x_1, y_1 = y(x_1))$ and $(x_2, y_2 = y(x_2))$
with $k = 1$: given are $(x_1, y_1 = y(x_1))$ and $(x_2, y_2 = y'(x_2))$ and
with $k = 2$: given are $(x_1, y_1 = y'(x_1))$ and $(x_2, y_2 = y'(x_2))$.

WIRIS cannot solve nonhomogeneous *Euler* DEs.

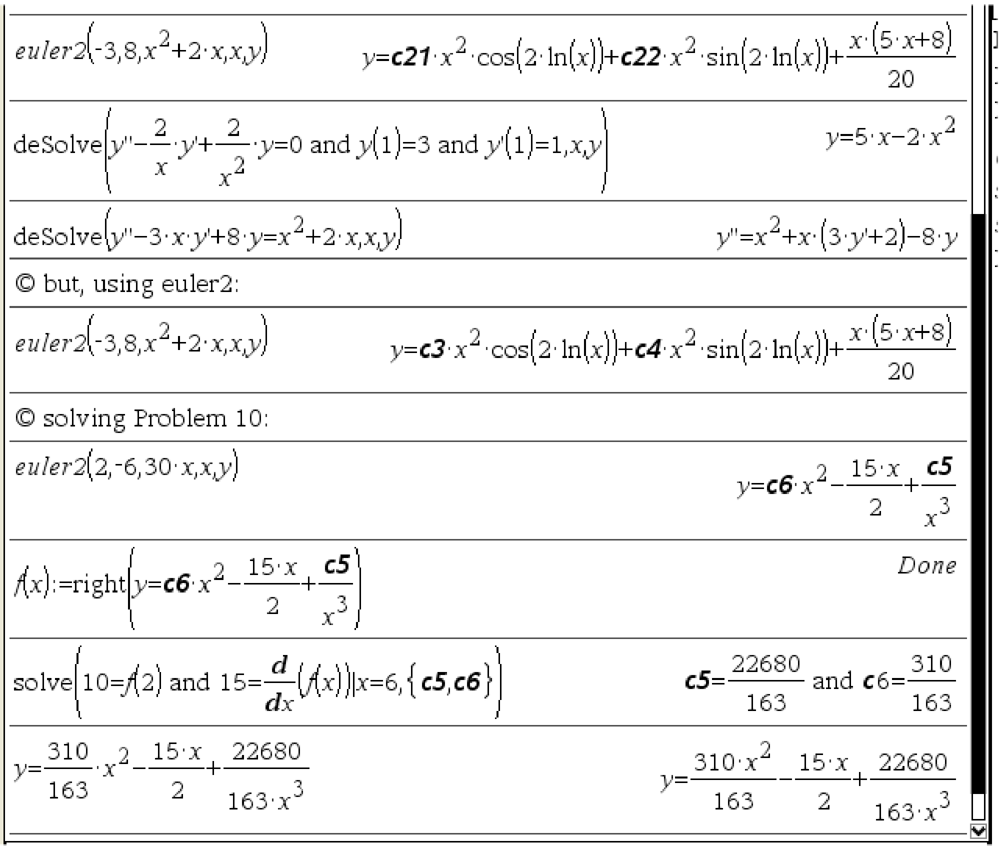
This is function *euler2* for the *TI-Nspire*:



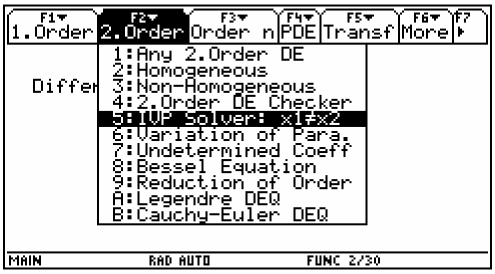
Nspire has the same problems solving *Euler* DEs as the TI-92 PLUS and the Voyage 200 as well. Please read the respective communication with Albert Rich and David Stoutemyer at the end of this article.

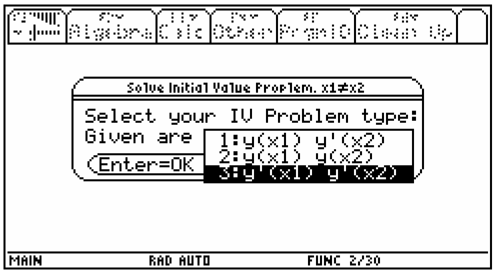
Shame on me, I was not able to produce the IVP and BVP-functions for *TI-Nspire* because I din´t find an easy way to solve the simultaneous equations for the constants ***cn*** (@n on the TI-handhelds can be reset by pressing 8:Clear Home.).

The screen shot on the next page shows some examples worked with *TI-Nspire* applying *euler2*.



DEQME supports initial value and boundary value problems as well in Option 5:



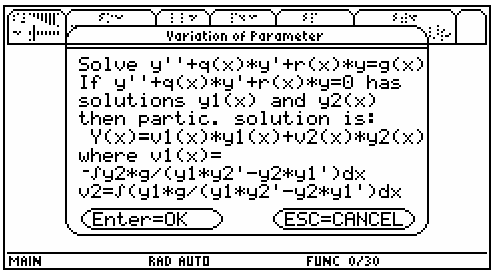


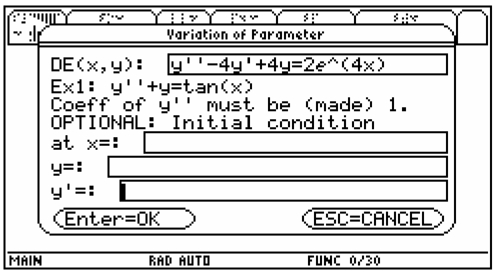
It works pretty well for all 2nd order ODEs which can be solved by the TI's DE-Solver.

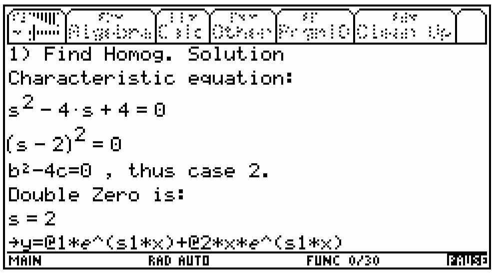
Proceeding investigating menu F2 I reach Option 6: Variation of Parameters. *DEQME* provides stepwise demonstrating this standard algorithm. I'll show two examples:

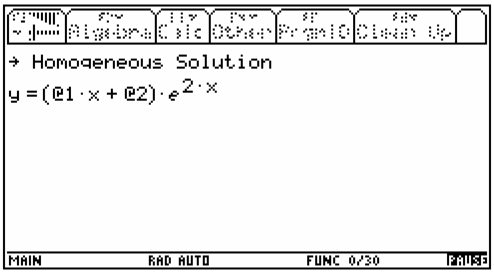
(12) $y'' - 4y' + 4y = 2e^{4x}$

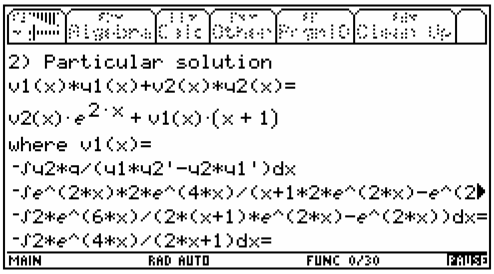
(13) $y'' - 9y = \sinh(2x);\ y(1) = 1,\ y'(1) = -0.5$

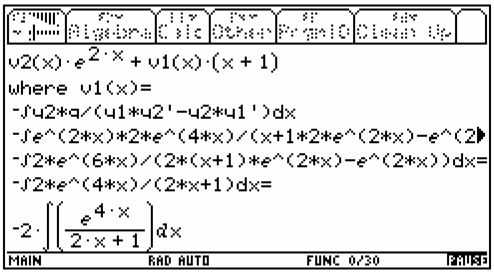


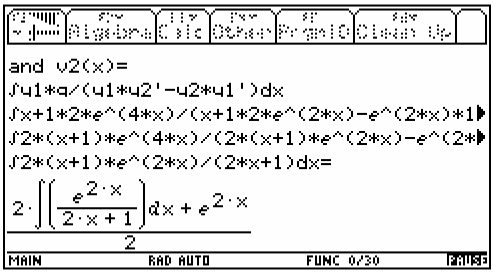


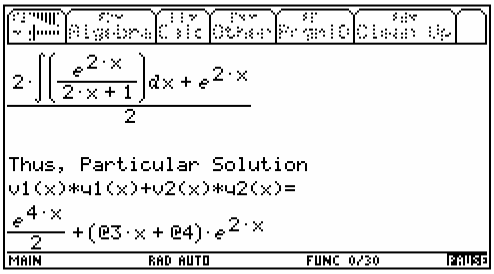












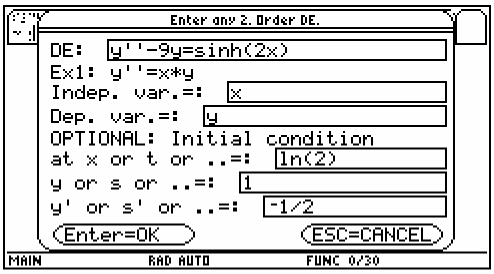
Maybe that students are happy with this, notify the result and proceed to the next example. I am not happy, because I can see that both integrals are not calculated. So this method fails! At the other hand there is a result, which says that the internal DE Solver finds the particular solution! What is my conclusion? What should the students conclude? There must be another method which should be more appropriate. Try in this case the Method of Undetermined Constants!

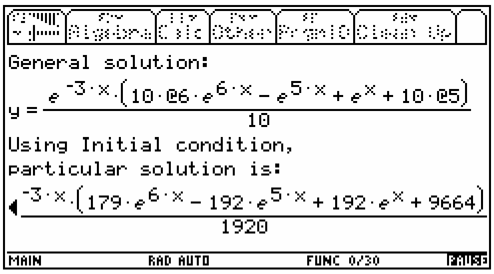
DERIVE returns - without stepwise simplification:

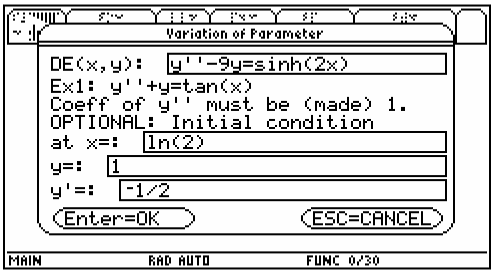
$$\text{DSOLVE2}(-4,\ 4,\ 2\cdot e^{4\cdot x}) = \frac{e^{4\cdot x}}{2} + e^{2\cdot x}\cdot(c2\cdot x + c1)$$

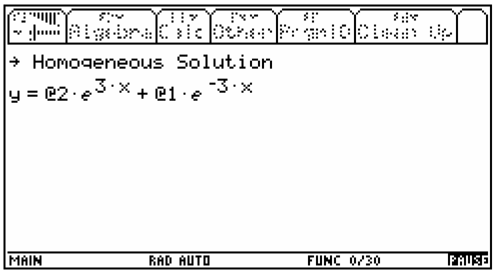
I recommend applying *DERIVE´*s Stepwise Simplification of this Solving procedure, it is an experience following the 30 “steps”. You will learn a lot about *DERIVE´*s “thinking”.

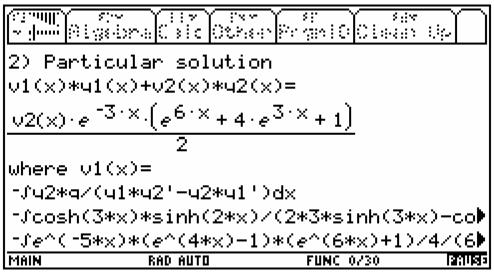
I am trying Parameter Variation once more solving example (13):

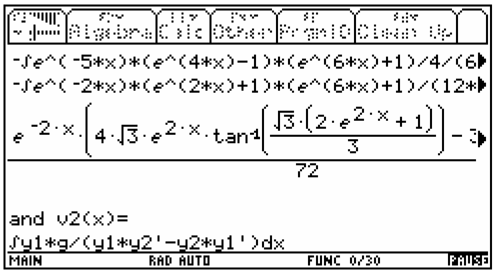


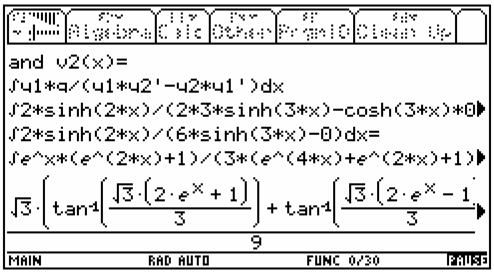


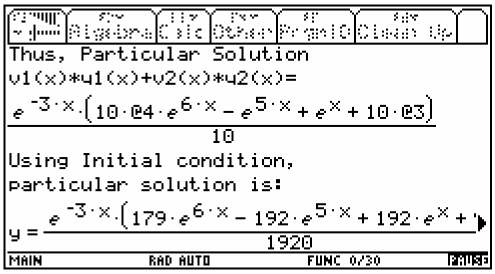












Compare with *DERIVE* followed by *wxMaxima*:

```
DSOLVE2_IV(0, -9, SINH(2·x), x, LN(2), 1, -0.5)
```

$$\frac{179\cdot e^{3\cdot x}}{1920}-\frac{e^{2\cdot x}}{10}+\frac{e^{-2\cdot x}}{10}+\frac{151\cdot e^{-3\cdot x}}{30}$$

```
ic2(ode2('diff(y,x,2)-9*y=sinh(2*x),y,x),x=log(2),y=1,'diff(y,x)=-1/2);
```

$$y=-\frac{\%e^{-2x}\left(\%e^{4x}-1\right)}{10}+\frac{179\,\%e^{3x}}{1920}+\frac{151\,\%e^{-3x}}{30}$$

F2 Option 7: Undetermined Coeff(icients) was treated on page 33.

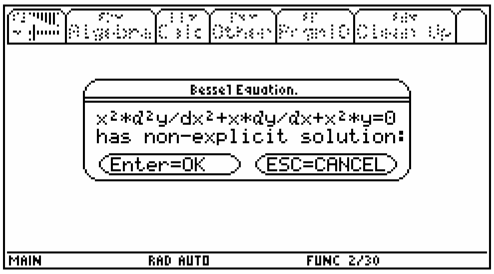
Options 6 and 7 are really nice and students and teachers as well might appreciate this tool. It could be a challenge to implement this "stepwise simplification" for *TI-Nspire* or *DERIVE* or other Computeralgebra systems – as stand alone programs.

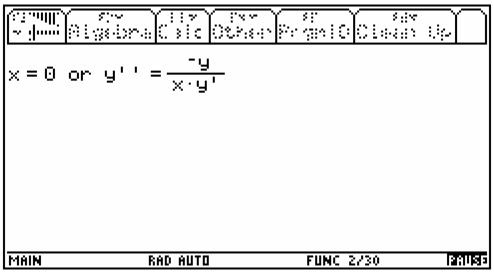
So we can proceed to Option 8: Bessel Equation.

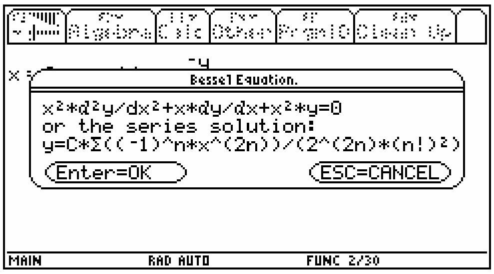
DEQME gives a short explanation of how the *Bessel Equation* looks like. The general form of the *Bessel* DE is:

$$x^2y''+x\,y'+(x^2-m^2)y=0.$$

I learned from the text books that the *Bessel* DE and *Legendre Equation* (Option A) appear when solving Partial Differential Equations (*Laplace Equation*).
Obviously *DEQME* shows only the – most important – case for $m = 0$.







See *MuPad*'s answer:

```
solve(ode(x^2*y''(x)+x*y'(x)+x^2*y(x)=0,y(x)))
Warning: Only Q-solvable exponential solutions will be found! [ode::secondOrder]
```

$\{C2 \cdot J_0(x) + C3 \cdot Y_0(x)\}$

and *wxMaxima*'s answer:

```
(%i30) ode2(x^2*'diff(y,x,2)+x*'diff(y,x)+x^2*y=0,y,x);
(%o30) y = bessel_y(0 , x)%k2 + bessel_j(0 , x)%k1

(%i31) ode2(x^2*'diff(y,x,2)+x*'diff(y,x)+(x^2-4)*y=0,y,x);
(%o31) y = bessel_y(2 , x)%k2 + bessel_j(2 , x)%k1

(%i32) ode2(x^2*'diff(y,x,2)+x*'diff(y,x)+(x^2-3)*y=0,y,x);
```

is $\sqrt{3}$ *an integer? type y or n.*

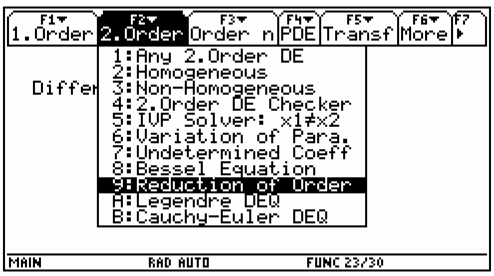
The solution contains the so called *Bessel functions* (see DNLs #18 and #34). *DERIVE* provides a utility file `BesselFunctions.mth`. *Bessel functions* can also be found in `SpecialFunctions.dfw` and `SpecialFunctions.mth` provided in the Users Folder.

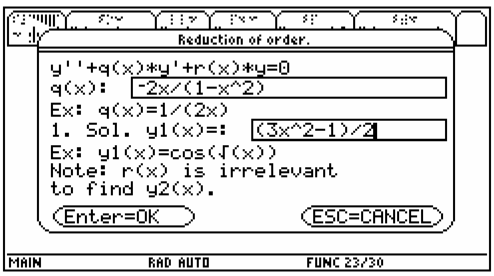
After this short trip into the world of higher mathematics I will return to easier fields.

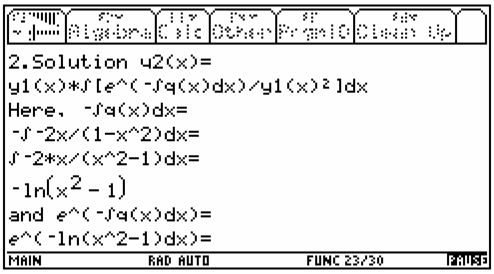
When I found F2 Option 9: Reduction of Order I – again – consulted my text books on ODEs and didn´t find anything matching with this option. I didn´t find one single problem similar to (14) or (15). Given is a 2nd order ODE and one of its solutions. Find the second one.

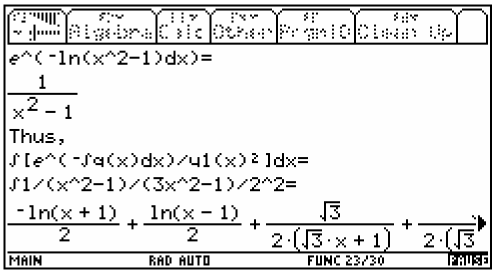
(14) $\quad y''(1-x^2) - 2x\,y' + 6y = 0;\ \ y_1 = \dfrac{3x^2-1}{2}$

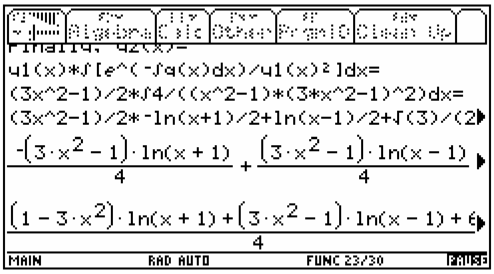
(15) $\quad y'' - 2y' - 3y = 6;\ \ y_1 = e^{3x}$

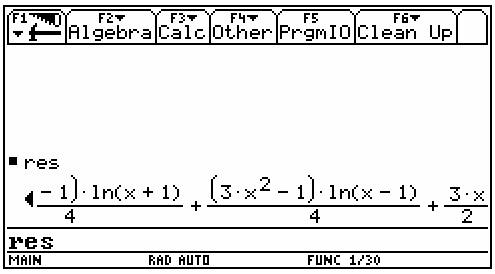










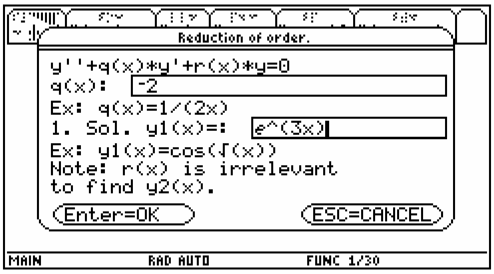


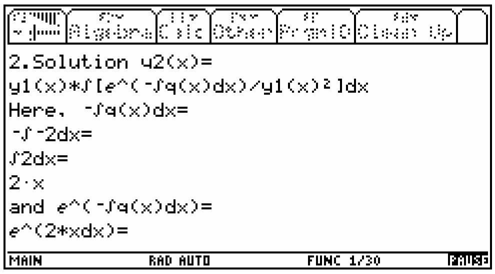
I check the validity of the complete solution – and I can be satisfied. *DEQME* does a good job.

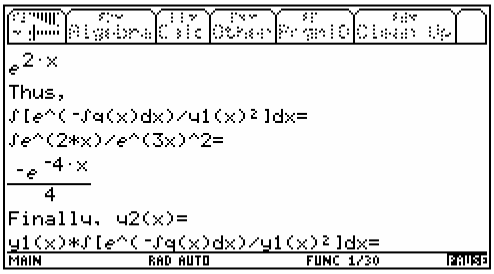
$$loes(x) := c1\cdot\frac{3\cdot x^2-1}{2}+c2\cdot\left(\frac{(3\cdot x^2-1)\cdot LN\left(\frac{x-1}{x+1}\right)}{4}+\frac{3\cdot x}{2}\right)$$

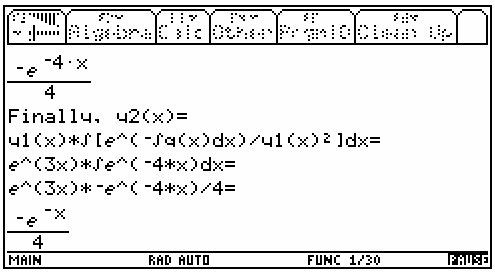
$$loes''(x)\cdot(1-x^2)-2\cdot x\cdot loes'(x)+6\cdot loes(x)=0$$

Example (15)









The complete solution of the homogeneous equation is $c_1 e^{3x} + c_2 e^{-x}$. Check it!

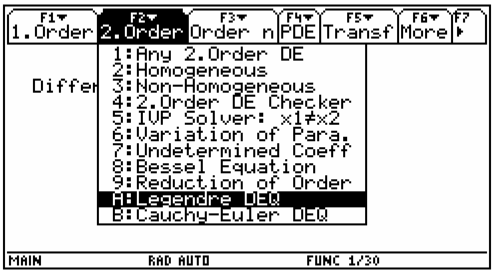
I asked Nils about the relevance of this option because I couldn´t find any related problem in my books. He answered:

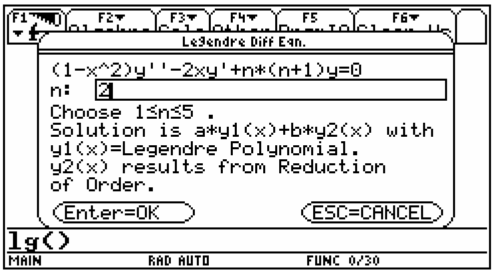
> Yes, "Reduction of order" is described here
>
> http://tutorial.math.lamar.edu/Classes/DE/ReductionofOrder.aspx
>
> Is treated in all good ODE courses.

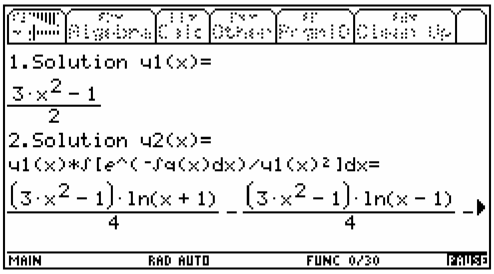
(This is really a fine web site. Try this URL, Josef)

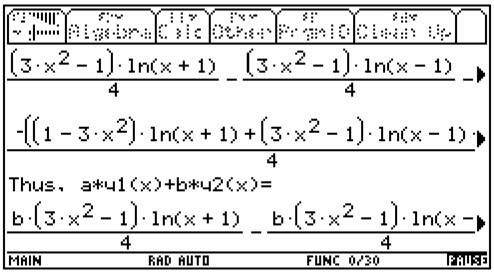
I mentioned the *Legendre Equation* earlier. Option A treats this kind of 2nd order ODE.

(16) *Legendre* DE: $(1-x^2)y''-2x\,y'+n(n+1)=0;\ \ n\in\mathbb{N}$.









Please compare with (14) from above. Now you might find an explanation for my choice of y_1.
The Legendre Polynomials for $n\in\mathbb{N}$ can be produced by the "Formula of Rodrigues":

$$L^{(n)}=\frac{1}{2^n\,n!}\left(\frac{d}{dx}\right)^n(x^2-1)^n.$$

$$\text{VECTOR}\left(\frac{1}{2^k\cdot k!}\cdot\left(\frac{d}{dx}\right)^k(x^2-1)^k,\ k,\ 0,\ 5\right)$$

$$\left[1,\ x,\ \frac{3\cdot x^2-1}{2},\ \frac{x\cdot(5\cdot x^2-3)}{2},\ \frac{35\cdot x^4-30\cdot x^2+3}{8},\ \frac{x\cdot(63\cdot x^4-70\cdot x^2+15)}{8}\right]$$

As calculation times for finding the *Legendre polynomials* increase enourmously on the TIs, Nils restricted for $n \le 5$.

MuPAD:

```
solve(ode((1-x^2)*y''(x)-2*x*y'(x)+2*y(x)=0,y(x)))
```

$$\left\{C6\cdot x+C5\cdot\left(\frac{x\cdot\ln(x-1)}{2}-\frac{x\cdot\ln(x+1)}{2}+1\right)\right\}$$

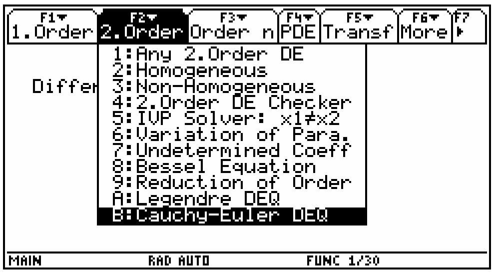
```
solve(ode((1-x^2)*y''(x)-2*x*y'(x)+6*y(x)=0,y(x)))
```

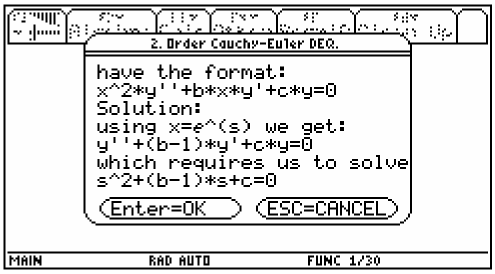
$$\left\{C3\cdot\left(x^2-\frac{1}{3}\right)+C2\cdot\left(\frac{9\cdot x}{4}-\frac{3\cdot\ln(x-1)}{8}+\frac{3\cdot\ln(x+1)}{8}+\frac{9\cdot x^2\cdot\ln(x-1)}{8}-\frac{9\cdot x^2\cdot\ln(x+1)}{8}\right)\right\}$$

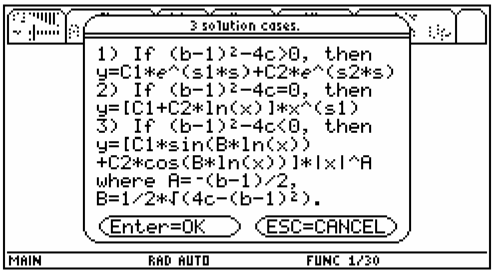
(Nice job for the students: Compare the result with the result from page 40, Problem (14)!)

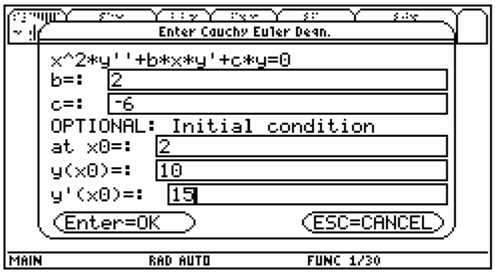
By the way, the *LDE* – algorithm works also for $n = 0$.

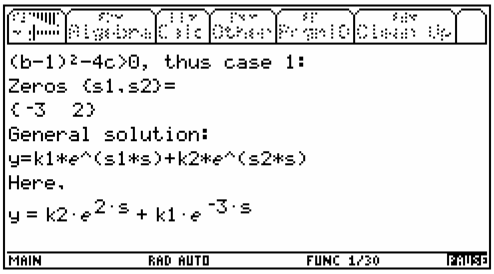
The last option doesn't need much space, because we talked about 2nd order *Euler Equations* and how to treat them on page 32. Option B does this in a very reduced form See example (8) treated with this option:

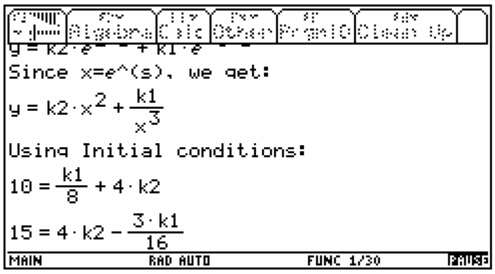


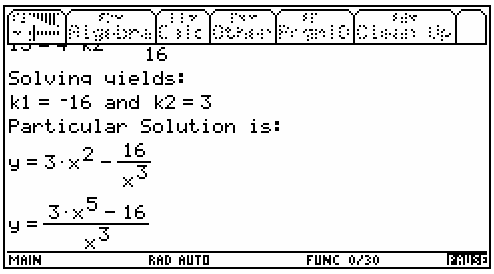












I am looking forward to investigating Menu F3 Order n containing the topics Linar+Const Coeff, 2×Linear System, X' = A ∗ X, X' = A ∗ X + F and Separable DE – and to "remember" a lot about Differential Equations again.

Differential Equations Made Easy for TI-89, 92+ and Voyage 200 is available from
http://www.ti89.com

You can find much information about *Legendre* and *Bessel DEs* at
http://mo.mathematik.uni-stuttgart.de/inhalt/beispiel/beispiel822/
http://en.wikiversity.org/wiki/Legendre_differential_equation
http://www.math.tugraz.at/~berglez/Math_C/Folien_10%28Potenzreihenans%29.pdf

References

[1] C.H.Edwards & D.E.Penney, *Differential Equations*, Prentice Hall 1996
[2] Günter/Kusimin, *Aufgabensammlung zur höheren Mathematik*, VEB Berlin 1964
[3] W.I.Smirnow, *Lehrgang der höheren Mathematik II*, VEB Berlin 1966
[4] E. Weisstein, *CRC Concise Encyclopedia of Mathematics*, CRC Press 1999
[5] *The Calculus Problem Solver*, REA 1985

Differential Equations "stepwise" with DERIVE and TI-Nspire

Josef Böhm, Würmla, Austria

I liked the stepwise simplification in *DEQME* – and Nils wrote that the users of *DEQME* appreciate this feature of *DEQME*, too. On page 39 I mentioned the "challenge" how to create a similar procedure with *DERIVE* or *TI-Nspire* or any other CAS-tool.

Nils and I, we had an extended exchange of emails concerning some features and improvements of the F2 Menu options. So I found some time to face my own challenge. Here are my results for the homogenous 2nd order ODEs with constant coefficients:

The DERIVE Stepwise (without printing the progam):

"Stepwise solution for homogeneous 2nd order ODEs with constant coefficients.

y'' + b y' + c y = 0

hom2_IV(b,c,x0,y0,v0,x,y) with y(x0)=y0 and y'(x0)=v0.

Three examples:

y'' – 4y = 0

```
#1:    hom2_IV(0, -4)

Characteristic equation:

s = -2 ∨ s = 2

Discriminant = 16

Discriminant > 0 → two real solutions: s1 ≠ s2

[2, -2]

General solution: y = c1*e^(s1*x)+c2*e^(s2*x),

hence:

              2·x       - 2·x
#2:    y = c1·e    + c2·e
```

x''(t) – 6 x'(t) + 9 x(t) = 0

```
#3:    hom2_IV(-6, 9, t, x)

Characteristic equation:

s^2 - 6·s + 9 = 0

Discriminant = 0

Discriminant = 0 → one real double solution: s1 = s2

[3]

hence

              3·t         3·t
#4:    x = c1·e    + c2·t·e
```

$y'' - 4y' + 8y = 0,\ y(\pi/2) = 1,\ y'(\pi/2) = -1$

#5: $\text{hom2\_IV}\left(-4,\ 8,\ x,\ y,\ \dfrac{\pi}{2},\ 1,\ -1\right)$

```
Characteristic equation:
s^2 - 4·s + 8 = 0
Discriminant = -16
Discriminant < 0 → two complex solutions: s1 ≠ s2
[2 + 2·i, 2 - 2·i]
General solution: y = c1*e^(RE(s1)*x)*sin(IM(s1)*x)+c2*e^(RE(s2)*x)*cos(IM(s2)*x),
hence:
General solution:
y = e^(2·x)·(c2·COS(2·x) + c1·SIN(2·x))
Solving the Initial Value Problem:
1. equation: y(x0)=y0
c2 = - e^(-π)
2. equation: y(x0)=v0
2·c1 + 2·c2 = e^(-π)
Solutions for c1 and c2 are:
[[3·e^(-π)/2], [- e^(-π)]]
Solution
```

#6: $\begin{bmatrix} \text{General solution} & y = e^{2\cdot x}\cdot(c2\cdot\cos(2\cdot x) + c1\cdot\sin(2\cdot x)) \\ \text{Therefore, the solution is:} & y = e^{2\cdot x - \pi}\cdot\left(\dfrac{3\cdot\sin(2\cdot x)}{2} - \cos(2\cdot x)\right) \end{bmatrix}$

#7: $\text{DSOLVE2\_IV}\left(-4,\ 8,\ 0,\ x,\ \dfrac{\pi}{2},\ 1,\ -1\right) = e^{2\cdot x - \pi}\cdot\left(\dfrac{3\cdot\sin(2\cdot x)}{2} - \cos(2\cdot x)\right)$

The "steps" printed in blue are created by using the DISPLAY-function. We don't have a PAUSE-function like in the TI-89, 92 and Voyage 200 programming language. So I can show the steps in a form of report only.

Unfortunately I cannot enter "*DERIVE*'s interior" in order to make use of *DERIVE*´s stepwise calculation. As I mentioned ealier it is very interesting to apply the *DERIVE* steps on the *DERIVE* functions. Sometimes you might fail. I entered DSOLVE2_IV(-4, 8, 0,x,π/2,1,-1) and was stopped by "Memory exhausted" after three steps ….

I exchanged some mails with José Luis Galan from Spain. He wrote that he and his colleagues are working with this type of functions (including explanatory comments) on their university.

Using the features of the latest version of *TI-Nspire* (there are rumors that release of version 3 will be soon) results also in a nice dialogue-driven commented output of the algorithm.

The Request-, Text-, and Disp-command are of importance. The values for b and c, and for x0, y(x0) and y’(x0) are entered in a dialogue box.

hom2_iv()

DE of form y'' + b*y' + c*y = 0

Enter b,c: 0,−4

hom. DE:

$y''-4\cdot y=0$

Enter Initialvalues x0, y(x0), v0=y'(x0)

ENTER 0 if none

Initialvalues x0,y(x0),y'(x0): 0

Characteristic Equation:

$s^2-4=0$

Discriminant =

16

Discriminant > 0 → two real solutions: s1 ≠ s2

$\{-2,2\}$

General solution is

$y=c2\cdot e^{2\cdot x}+c1\cdot e^{-2\cdot x}$

Done

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hom2_iv()

DE of form y'' + b*y' + c*y = 0

Enter b,c: −4,8

hom. DE:

$y''-4\cdot y'+8\cdot y=0$

Enter Initialvalues x0, y(x0), v0=y'(x0)

ENTER 0 if none

Initialvalues x0,y(x0),y'(x0): pi/2,1,−1

Characteristic Equation:

$s^2-4\cdot s+8=0$

Discriminant =

-16

Discriminant < 0 → two complex solutions: s1 ≠ s2

$\{2+2\cdot i,2-2\cdot i\}$

General solution is

$y=c2\cdot e^{2\cdot x}\cdot\cos(2\cdot x)+c1\cdot e^{2\cdot x}\cdot\sin(2\cdot x)$

Solving the Initial Value Problem:

1. Equation: y(x0) = y0

$1=-c2\cdot e^{\pi}$

2/99

2. Equation: y'(x0)=v0

$-1=-2\cdot c1\cdot e^{\pi}-2\cdot c2\cdot e^{\pi}$

Solutions for c1 and c2 are:

$c1=\frac{3}{2\cdot e^{\pi}}$ and $c2=\frac{-1}{e^{\pi}}$

Special solution is

$y=\frac{3\cdot(e^{x})^{2}\cdot\sin(2\cdot x)}{2\cdot e^{\pi}}-\frac{(e^{x})^{2}\cdot\cos(2\cdot x)}{e^{\pi}}$

Done

The screen shots show the Calculator Application.

This is the first part of the *TI-Nspire* program.

Below is the start of the *DERIVE* function.

Both programs can be downloaded from the DUG-website (contained in mth80.zip).

```
hom2_iv
Define hom2_iv()=
Prgm
Local co,ivs,d,sols,gen_sol,y_1,eq1,eq2,cs,x0,y0,v0
Disp "DE of form y'' + b*y' + c*y = 0"
RequestStr "Enter b,c:",co
co:="{"&co&"}":co:=expr(co)
b:=co[1]: c:=co[2]
Disp "hom. DE: "
Disp y''+b·y'+c·y=0
Disp "Enter Initialvalues x0, y(x0), v0=y'(x0)"
Disp "ENTER 0 if none"
RequestStr "Initialvalues x0,y(x0),y'(x0):",ivs
ivs:="{"&ivs&"}"
ivs:=expr(ivs)
d:=b^2-4·c
sols:=cZeros(s^2+b·s+c,s)
Disp "Characteristic Equation: "
Disp s^2+b·s+c=0
```

```
hom2_IV(b, c, x, y, x0 := ι, y0, v0, d, s_, sols, gen_sol, sp_sol, c1, c2,
  Prog
    d := b^2 - 4·c
    sols := SOLUTIONS(s^2 + b·s + c = 0, s)
    DISPLAY("Characteristic equation:")
    s_ := '(s^2 + b·s + c = 0)
    DISPLAY(s_)
    "DISPLAY(s^2 + b·s + c = 0)"
    DISPLAY("Discriminant" = d)
    If d > 0
       Prog
         DISPLAY("Discriminant > 0 → two real solutions: s1 ≠ s2")
         DISPLAY(sols)
         DISPLAY("General solution: y = c1*e^(s1*x)+c2*e^(s2*x),")
         DISPLAY("hence:")
         gen_sol := y = c1·e^(x·sols↓1) + c2·e^(x·sols↓2)
    If d = 0
       Prog
         DISPLAY("Discriminant = 0 → one real double solution: s1 = s2")
         DISPLAY(sols)
         DISPLAY("General solution: y = c1*e^(s1*x)+c2*x*e^(s2*x),")
         DISPLAY("hence:")
         gen_sol := y = c1·e^(x·sols↓1) + c2·x·e^(x·sols↓1)
```

Dear Noor and Josef,

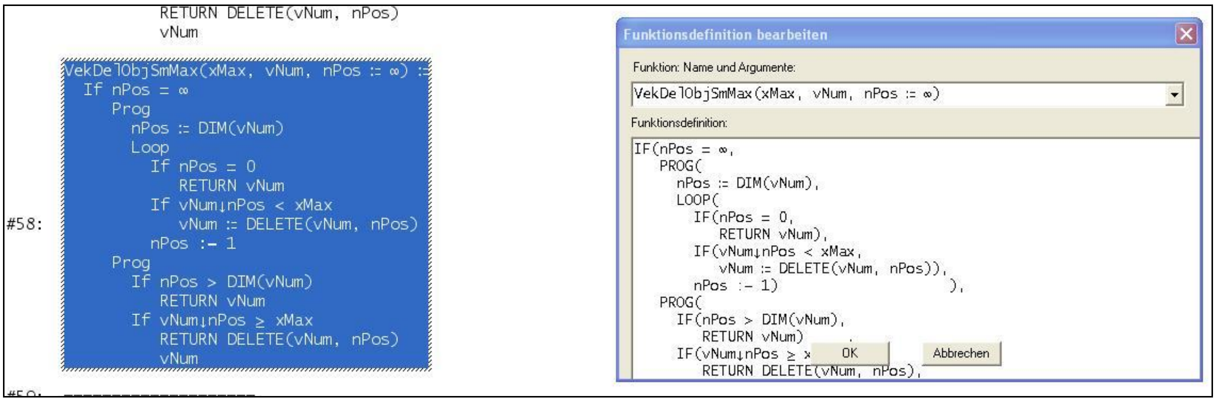
Great to hear from you! From the weather news I have been hearing from Europe, I can assume the picture of your village below is not a current one. :=)

As far as my recent work is concerned, my goal is to pass on the lessons learned in my career to the next generation of CAS developers. To accomplish that, my website promotes using a rule-based approach to implement such systems. As proof-of-concept, the site makes freely the rules required to integrate a large class of expressions, and provides test results favorably comparing Rubi, a rule-based integrator, with the major commercial systems.

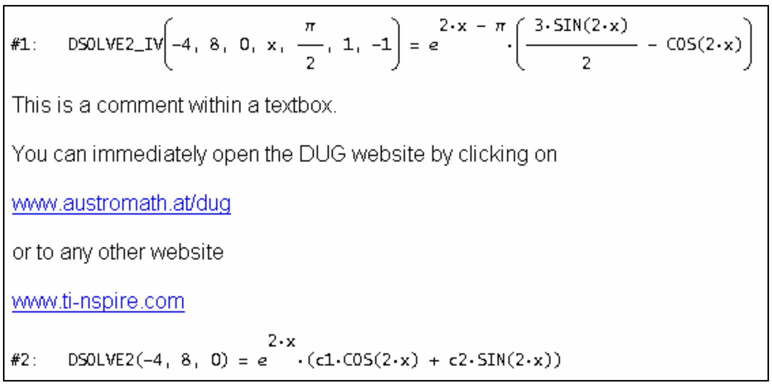
I am heartened to hear the Derive User Group continues to be the virtual home for the community of loyal Derive users around the world. I only hope some entrepreneur will use the knowledge on my website to produce a worthy successor to Derive for the future generations...

Aloha, Albert

A DUG Member from Germany has problems with the program editor window. You can see the Ok and the Abbrechen (= Cancel) button within the written text. Is there anybody facing similar problems? Do you know how to get rid of these nasty buttons?



Do you know this? It is possible to include internet links in a DERIVE-file. Simply write the URL in a text box. It will appear blue and underlined and you can open the URL from within DERIVE. Josef



Hello Derivers, the routines provided in the LinearAlgebra.mth file leave something to be desired in practice - has anyone resolutely attacked the problem of diagonalization and determination of eigenspaces of large matrices? For my application large = 8x8. I need a robust and fast diagonalization scheme.

Danny Ross Lunsford [antimatter33@YAHOO.COM]