THE DERIVE - NEWSLETTER #81

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THE BULLETIN OF THE

DERIVE

USER GROUP

+ CAS-TI

C o n t e n t s:

March 2011

This is a great website: *Images of Mathematicians on postage stamps*

http://jeff560.tripod.com/stamps.html

with not only the images but much information about mathematicians from all over the world

Download two Computer Algebra Systems free of charge:

Octave and Xoctave: http://www.gnu.org/software/octave/

XCAS:

Of Aide for free software XCAS , aid written by **Renée de GRAEVE** Université J Fourier GRENOBLE FRANCE. This AIDE is enormous. I have not the courage for translate it for you. You may find it on the site

www-fourier.ujf-grenoble.fr/~parisse/giac_fr.html

Thanks to Robert Setif for this information

A wonderful book about **Strange Attractors** can be downloaded free of charge (together with programs) from

http://sprott.physics.wisc.edu/sa.htm

http://sprott.physics.wisc.edu/fractals.htm

I plan an extended DNL-paper on this fascinating topic. "*Create your own Strange Attractor!*"
(See two graphs on page 43.)

This is a website on Curves (in German)

http://www.blaesius-anne.de/arbeit_kurven/kurvenG1a.html

SOME RELATED PUBLICATIONS FROM PREVIOUS FORMATEX ACTIVITIES -Book:
"*Research, Reflections and Innovations in Integrating ICT in Education*":

http://www.formatex.org/micte2009/volume1.htm

http://www.formatex.org/micte2009/volume2.htm

http://www.formatex.org/micte2009/volume3.htm

Zwei sehr schöne Bücher:

Rainer Müller, *Klassische Mechanik*, De Gruyter 2010, ISBN 978 3 11 025002 2

Georg Glaeser, *Wie aus der Zahl ein Zebra wurde*, ein mathematisches Fotoshooting, Spektrum 2010, ISBN 978 3 8274 2502 7

For TI-Nspire novices:

Steve Oullette, *TI-Nspire for Dummies*, Wiley Publishing 2009, ISBN 978 0 470 37934 9

Written by a math teacher, *TI-Nspire For Dummies* helps students and educators alike take full advantage of this outstanding tool for teaching and learning math.

Liebe DUG-Mitglieder,

und wieder war ich nicht rechtzeitig mit dem DNL! Wir haben in den 50 Seiten zwar „nur" drei längere Artikel, aber diese haben in einem regen und langen Austausch von emails eine bemerkenswerte Eigendyamik entwickelt. Viele Fragen, Ideen, Änderungen und Verbesserungsvorschläge gingen hin und her. Wir hoffen, dass sich die Bemühungen für die Leser der Beiträge lohnen.

Duncan Mc Dougall zeigt interessante Zusammenhänge in Polynomen mit ganzen Koeffizienten und rationalen Extremwerten. Piotr Trebisz nutzt die ganze „power" eines CAS für die Berechnung und Darstellung von Schneckenhäusern. Robert Setif stellte mir ein nettes Problem, das ich dann doch in seinem Sinn gelöst haben dürfte.

Dietmar Oertel hat zwei inhaltsschwere Papiere geschickt, die demnächst im DNL zu finden sein werden. Roger Folsom stellt im User Forum eine Frage an die DERIVE User, ein langer Brief von ihm steht für das nächste User Forum bereit. Auch mit David Sjöstrand hat es einen ausgedehnten – möglicherweise noch nicht beendeten – Briefwechsel bezüglich des „Beweises" aus dem DNL#80 gegeben (Seite 40).

Ich möchte Euch besonders auf die Internet Links hinweisen. Zwei mächtige CAS sind frei verfügbar. Das Buch von Julien C. Sprott über „Strange Attractors" ist eine Schatzkiste für "Fraktalisten". Die deutschen Bücher auf der Info-seite sind ein wahrer Genuss für Auge und Hirn.

Leider gibt es dieses Mal nur wenig zum TI-Nspire. Version 3 kann beeits von der TI-homepage bezogen werden. Die neuen Möglichkeiten wollen wir im nächsten DNL besprechen.

Ich wünsche Euch eine schöne Zeit und verbleibe mit den besten Grüßen

Dear DUG Members,

and again I was not in time with the DNL! We have 50 pages and „only" three more extended articles. But these developped a remarkable momentum from an intense exchange of emails. Many questions, ideas, changes and proposals for improvements went to and fro. We hope that our efforts will be appreciated by the readers.

Duncan Mc Dougall shows interesting relationships in polynomials with integer coefficients and rational turning points. Piotr Trebisz uses the full „power" of a CAS for the calculation and presentation of snail shells. Robert Setif presents a nice problem, which I could solve in his sense.

Dietmar Oertel sent two very contentful papers which can be found in the next newsletters. Roger Folsom presents a request in the User Forum, a long letter is ready for publication in the next User Forum. I had an extended – possibly not yet finished – email exchange with David Sjöstrand concerning the „Proof" from DNL#80 (page 40).

I'd like to point especially to the internet links. Two powerful CAS can be downloaded free of charge. Julien C. Sprott's book about "Strange Attractors" is a treasure box for "fractalists". The German books presented on the info page are a true pleasure for eyes and brain.

Unfortunately we have little stuff for the TI-Nspire this time. Version 3 can be downloaded from the TI-website. We will talk about the new features in the next DNL.

I wish you a good time and remain with my best regards,

Download all *DNL*-DERIVE- and TI-files from

http://www.austromath.at/dug/

The *DERIVE-NEWSLETTER* is the Bulletin of the *DERIVE* & CAS-*TI User Group*. It is published at least four times a year with a contents of 40 pages minimum. The goals of the *DNL* are to enable the exchange of experiences made with *DERIVE*, *TI*-CAS and other CAS as well to create a group to discuss the possibilities of new methodical and didactical manners in teaching mathematics.

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Contributions:
Please send all contributions to the Editor. Non-English speakers are encouraged to write their contributions in English to reinforce the international touch of the *DNL*. It must be said, though, that non-English articles will be warmly welcomed nonetheless. Your contributions will be edited but not assessed. By submitting articles the author gives his consent for reprinting it in the *DNL*. The more contributions you will send, the more lively and richer in contents the *DERIVE* & CAS-*TI Newsletter* will be.

Next issue: June 2011

Preview: Contributions waiting to be published

Some simulations of Random Experiments, J. Böhm, AUT, Lorenz Kopp, GER
Wonderful World of Pedal Curves, J. Böhm
Tools for 3D-Problems, P. Lüke-Rosendahl, GER
Financial Mathematics 4, M. R. Phillips
Hill-Encription, J. Böhm
Simulating a Graphing Calculator in *DERIVE*, J. Böhm
Henon & Co, J. Böhm
Do you know this? Cabri & CAS on PC and Handheld, W. Wegscheider, AUT
An Interesting Problem with a Triangle, Steiner Point, P. Lüke-Rosendahl, GER
Overcoming Branch & Bound by Simulation, J. Böhm, AUT
Graphics World, Currency Change, P. Charland, CAN
Cubics, Quartics – Interesting features, T. Koller & J. Böhm
Logos of Companies as an Inspiration for Math Teaching
Exciting Surfaces in the FAZ / Pierre Charland´s Graphics Gallery
BooleanPlots.mth, P. Schofield, UK
Old traditional examples for a CAS – what´s new? J. Böhm, AUT
Truth Tables on the TI, M. R. Phillips
Where oh Where is It? (GPS with CAS), C. & P. Leinbach, USA
Embroidery Patterns, H. Ludwig, GER
Mandelbrot and Newton with *DERIVE*, Roman Hašek, CZ & Rob Gough, UK
A Conics-Explorer, J. Böhm, AUT
Tutorials for the NSpireCAS, G. Herweyers, BEL
Some Projects with Students, R. Schröder, GER
Dirac Algebra, Clifford Algebra, D. R. Lunsford, USA
Treating Differential Equations (M. Beaudin, G. Piccard, Ch. Trottier)
Structured Combinatorics, D. Oertel, GER
A new approach to Taylor Series, D. Oertel, GER
Statistics with TI-Nspire, G. Herweyers, BEL
Cesar Multiplication, G. Schödl, AUT
Find your very own Strange Attractor, J. Böhm, AUT

and others

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I invite you browsing in higher mathematics journals. "emis" opens a spectacular view to the work of many mathematicians free of charge. You can find a rich collection of websites hosting mathematics journals from universities and mathematical societies from all over the world.

The Electronic Library of Mathematics

http://www.emis.de/elibm/journals/index.html

Acta Mathematica Academiae Paedagogicae Nyíregháziensis

Acta Mathematica Universitatis Comeniana,

Acta Universitatis Apulensis, Alba Julia, Romania

An International Scientific Journal of Sapientia University of Transylania, Romania

Analele Stiintifice ale Universitatii Ovidius Constanta, Romania

Annales Academiæ Scientiarum Fennicæ, Helsinki

Annales Mathematicae et Informaticae, Esterházy Károly College Eger

Applied Mathematics E-Notes, Taiwan

Archivum Mathematicum, Faculty of Science of Masaryk University Brno

Advances in Geometry

Algebra Montpellier Announcements

Algebraic and Geometric Topology (see also: http://msp.warwick.ac.uk/gtp/)

Balkan Journal of Geometry and Its Applications (Balkan Society of Geometers)

Beiträge zur Algebra und Geometrie / Contributions to Algebra and Geometry

Bulletin of the Venezuelan Mathematical Association

Sociedad de Estadística e Investigación Operativa

Bulletin of the Belgian Mathematical Society Simon Stevin

Bulletin of the Malaysian Mathematical Sciences Society

Bulletin of the Malaysian Mathematical Sciences Society

Bulletin, Classe des Sciences Mathématiques et Naturelles, Sciences mathématiques

Divulgaciones Matemáticas, University of Zulia, Venezuela

Documenta Mathematica – Journal der deutschen Mathematikervereinigung

Electronic Journal of Probability, University of Washington

The Electronic Journal of the Argentine Society for Informatics and Operations Research

The Electronic Journal of Combinatorics, University of Pennsylvania

The Electronic Journal of Differential Equations, Southwest Texas State University

The Electronic Journal of Linear Algebra, Israel

Electronic Journal of Qualitative Theory of Differential Equations, Univ. of Szeged, Hungary

Electronic Transactions on Numerical Analysis, University of Kent, OH, USA

Ensaios Matematicas, Brazilian Mathematical Society

Hindawi Open Access Journals, A bundle of various journals

International Journal of Open Problems in Computer Science and Mathematics,

INTEGERS - Electronic Journal of Combinatorial Number Theory, Univ. of West-Georgia

Journal of Applied Analysis, Lemgo, Germany

Journal of Complex Analysis, Lemgo, Germany

Electronic Journal for History of Probability and Statistics,

Journal for Geometry and Graphics, Lemgo, Germany

Journal of Complex Analysis, Brown University, USA

Journal of Inequalities in Pure and Applied Mathematics, Victoria University of Technology

Journal of Integer Sequences, University of Waterloo, Canada

Journal de Théorie des Nombres de Bordeaux

Living Reviews in Relativity, Max Planck Institute

Lobachevskii Journal of Mathematics, Kazan State University

Matematicki Vesnik, Mathematical Society of Serbia

Mathematica Bohemica

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Diophantine Polynomials (Part 1)

Duncan E. McDougall, duncanemcdougall@hotmail.com

(1)

What is a Diophantine Polynomial? It is a polynomial of degree 2, 3 or 4 which is factorable in the set of integers and whose derivative is factorable in the set of rational numbers. Why we want to discuss them is to facilitate curve sketching.

The polynomials which we are about to examine can be used for both the grade 11 and the calculus student, because the intercepts are easy to find, and the *y*-values for the maxima and minima are shared among the families of curves. For example, we can ask a grade 11 student to sketch $y = x^3 + x^2 - 16x - 16$ by finding both the *x* and *y* intercepts. We can use the very same polynomial for the calculus 12 student who can not only find the intercepts easily, but can more readily find the *x*-values for both maxima and minima because the derivative is easy to factor.

My belief is that students should learn a complicated algorithm in simple progressive steps using straightforward numbers. Diophantus worked with integers and rational numbers only. Pedagogically, Diophantus was really on to something because he created methods which involved a lot of processing and sequencing while focusing on whole numbers. The distractions I refer to in curve sketching are complex and irrational numbers. It is difficult enough to learn some five to eight steps gathering enough data in order to effectively and accurately sketch a cubic, quartic, or quintic polynomial and/or a rational expression which may involve a diagonal asymptote without difficult-to-work-with numbers. If the student has the burden (when first learning the process) of working with irrational or complex numbers, along with concentrating on behavior of the curve and concavity, then he/she might simply declare

(2)

"whatever" and drop the task. If the numbers are whole or integral (Diophantus), then his/her focus remains where it should be- on the algorithm. The task of the educator is to demonstrate algorithms in such a way that the student can master the process in sequence. The solution is to stick to the Diophantine process and to model examples that involve process and sequencing without getting tangled up with irrational numbers. To some readers this may be a no-brainer, but it is not as simple as it sounds to find cubics, or quartics with single integral roots whose derivatives have single rational roots. Finding these involved testing hundreds of polynomials using DERIVE, as I was determined to find easy-to-calculate polynomials, which would facilitate graphing curves like $y = x^3 + 11x^2 + 24x$ without worrying about irrational and complex numbers. There was another challenge of course, and that was to keep the constant of the polynomial relatively small so that working without a calculator would not be arduous.

Another aspect of this approach with whole numbers is that when the student knows that the numbers are designed to work, learning of the method or algorithm remains the priority. The student also knows that there is something wrong if the numbers do not work. It's kind of a security blanket for the beginner, but it eliminates doubt that so often takes away confidence in ability and performance. Later on, after mastering the technique, the student gains confidence through the ease of this, and therefore he can tackle problems with both irrational and complex numbers.

It is my objective to propose families of cubics and quartics which are factorable in the integers and whose derivatives are factorable in the set of rational numbers. I will also

(3)

propose methods using DERIVE by which you can construct your own polynomials. We'll start with the very basic table of linear and quadratic polynomials then lead up to the cubics and quartics. I will end the paper with a brief discussion of the quintic, which should have worked but didn't.

The following diagram (Table 1) contains all the various linear and quadratic forms along with the general set of cubics.

Using Table I

For example, take a cubic of the form $\left(x+a^2\right)\left(x+2ab\right)\left(x+b^2\right)$ whose derivative has rational roots. Choosing any integers $a=1$ and $b=3$ for example, our new polynomial is $\left(x+1\right)\left(x+6\right)\left(x+9\right)$ with roots $-1,-6,$ and -9. The differential form is $3x^2+32x+69$ whose roots are -3 and $-\frac{23}{3}$.

(4)

Table I

Family Function	Roots	Derivative	Roots	Conditions on coefficients and constants to have integral roots
a	none	0	none	not applicable
$a\,x$	$x=0$	a	none	not applicable
$a\,x+b$	$x=\frac{-b}{a}$	a	none	not applicable
$(x+a)^2$	$x=-a$	$2\,(x+a)$	$x=-a$	none
$x^2+a\,x$	$x=0,-a$	$2x+a$	$x=-\frac{a}{2}$	a must be even
$x^2+x(a+b)+a\,b=$ $=(x+a)(x+b)$	$x=-a,-b$	$2x+a+b$	$x=\frac{-a-b}{2}$	a and b are both odd or both even
$acx^2+x(ad+bc)+bd=$ $=(ax+b)(cx+d)$	$x=-\frac{b}{a},-\frac{d}{c}$	$2acx+ad+bc$	$x=\frac{-ad-bc}{2ac}$	$a\neq 0,\ c\neq 0$ $ad+bc$ must either equal ac or be an even multiple of it
$(x+a)^3$	$x=-a$	$3(x+a)^2$	$x=-a$	none

Family Function	Roots	Derivative	Roots	Conditions on coeffs & consts to have integral roots
$(x+a)^2(x+b)$	$x=-a, -b$	$2(x+a)(3x+2b+a)$	$x=-a, \frac{-2b-a}{3}$	$2b+a$ is a multiple of 3
$x(x+a)(x+b)$	$x=0, -a, -b$	$3x^2+2x(a+b)+ab$	$x=\frac{-(a+b)\pm\sqrt{a^2-ab+b^2}}{3}$	a^2-ab+b^2 equals zero or a perfect square
$(x+a^2)(x+2ab)(x+b^2)$	$x=-a^2, -2ab, -b^2$	$3x^2+$ $+x(2a^2+4ab+2b^2)+$ $+ab(2a^2+ab+2b^2)$	$x=-ab$, $\frac{-2a^2-ab-2b^2}{3}$	$2a^2-ab+2b^2$ must be a multiple of 3
$(x+1)(x-a)(x+a)$	$x=-1, a, -a$	$3x^2-2x-a^2$	$x=\frac{1\pm\sqrt{1+3a^2}}{3}$	$4+3a^2$ must be a perfect square ($a =$ $=0,1,4,15,...$)

(5)

The polynomials in Table II consist of the particular numerical families with single roots. These are the ones that are ready to use in your classroom today.

As we observe the families in Table II (page 9), it is hard not to notice the pattern 8, 15, 21, 30, 35, and 36. It is a quadratic arithmetic sequence, whose elements (except for a couple) all work as families of curves.

In terms of methods for single roots let us begin by entering the form $x(x+a)(x+b)$ into DERIVE. This guarantees a factorable form. Press C for Calculus and differentiate. The resulting form is put in function form as DECLARE. Now we can either guess values and hope that our quadratic is factorable, or fix a value for "*a*", and then guess values for *"b"* until the quadratic is factorable. The question is, do we have anything to guide our guessing? In fact, we do. Visually, the values of *"x"* for maxima and minima will occur between the first and last *x*-intercepts. Hence, if we were to choose 0 and 8 as two of our first and last roots, we would know that the third one must come between them. It's just a question of leaving enough room between the roots so that the critical points can occur as integers and/or rational numbers. Algebraically, we enter $x(x-a)(x-8)$ into DERIVE, and then differentiate giving $3x^2+2x(a-8)+8a$. Since

(6)

we have a quadratic; the discriminant B^2-4AC must equal a perfect square in order to be factorable. Using the command DECLARE, we set $f(a)=4a^2-32a+256=4(a^2-8a+64)$ and evaluate (or use the TI83 where second function gives "TABLE" and we search it for perfect squares). Both 3 and 5 come up quickly implying both $x(x-3)(x-8)$ and $x(x-5)(x-8)$ have derivatives whose roots are rational.

#1: $x\cdot(x-a)\cdot(x-8)$

#2: $\frac{d}{dx}(x\cdot(x-a)\cdot(x-8))$

#3: $3\cdot x^2-2\cdot x\cdot(a+8)+8\cdot a$

#4: FACTOR($3\cdot x^2-2\cdot x\cdot(a+8)+8\cdot a$, Rational, x)

#5: $3\cdot x^2-2\cdot x\cdot(a+8)+8\cdot a$

You may like to continue working in DERIVE; then use the TABLE-command in connection with SELECT in order to find (more) values for *a* leading to a perfect square, Josef:

#6: SOLVE($3\cdot x^2-2\cdot x\cdot(a+8)+8\cdot a$, x)

#7: $x=-\frac{\sqrt{(a^2-8\cdot a+64)}-a-8}{3}\vee x=\frac{\sqrt{(a^2-8\cdot a+64)}+a+8}{3}$

#8: SELECT(FLOOR($\sqrt{(v_2)}$) = $\sqrt{(v_2)}$, v, TABLE($a^2-8\cdot a+64$, a, -100, 100))

#9: $\begin{bmatrix} -7 & 169 \\ 0 & 64 \\ 3 & 49 \\ 5 & 49 \\ 8 & 64 \\ 15 & 169 \end{bmatrix}$

#10: $\frac{d}{dx}(x\cdot(x+7)\cdot(x-8)) = 3\cdot x^2-2\cdot x-56$

#11: SOLVE($3\cdot x^2-2\cdot x-56$, x) = $\left(x=\frac{14}{3}\vee x=-4\right)$

I don't pretend to have all the families necessarily, but applying translations to any given family will yield many polynomials. The following is a small sample arrived at by adding a constant to all the terms:

given family	$x(x-3)(x-8)$
add 1	$(x+1)(x-2)(x-7)$
add 2	$(x+2)(x-1)(x-6)$
add 3	$(x+3)x(x-5)$
add 4	$(x+4)(x+1)(x-4)$
add 5	$(x+5)(x+2)(x-3)$
add 6	$(x+6)(x+3)(x-2)$
add 7	$(x+7)(x+4)(x-1)$
add 8	$(x+8)(x+5)(x)$, etc.

Interestingly enough, the entire above (all members of the famly) shares a maximum height of $\frac{400}{27}=\frac{5}{3}\cdot\frac{4}{3}\cdot\frac{20}{3}$,, and minimum low of -36, and the difference between their corresponding x-coordinates is exactly $\frac{14}{3}$. (The students can prove this, Josef)

A linear relationship exists between these values and those found in the quartics. We shall explore this after exploring quartic family of curves.

(7)

Table II

Family Function	Roots	Derivative	Roots	Transformation
$x(x+3)(x+8)$ x^3+11x^2+24x	0, -3, -8	$(3x+4)(x+6)$ $3x^2+22x+24$	$-\frac{4}{3},-6$	$(x\pm k)(x\pm 3a\pm k)(x\pm 8a\pm k)$
$x(x+5)(x+8)$ x^3+13x^2+40x	0, -5, -8	$(3x+20)(x+2)$ $3x^2+26x+40$	$-\frac{20}{3},-2$	$(x\pm k)(x\pm 5a\pm k)(x\pm 8a\pm k)$
$x(x+7)(x+15)$ x^3+22x^2+105x	0, -7, -15	$(3x+35)(x+3)$ $3x^2+44x+105$	$-\frac{35}{3},-3$	$(x\pm k)(x\pm 7a\pm k)(x\pm 15a\pm k)$
$x(x+8)(x+15)$ x^3+23x^2+120x	0, -8, -15	$(3x+10)(x+12)$ $3x^2+46x+120$	$-\frac{10}{3},-12$	$(x\pm k)(x\pm 8a\pm k)(x\pm 15a\pm k)$
$x(x+5)(x+21)$ x^3+26x^2+105x	0, -5, -21	$(3x+7)(x+15)$ $3x^2+52x+105$	$-\frac{7}{3},-15$	$(x\pm k)(x\pm 5a\pm k)(x\pm 21a\pm k)$
$x(x+16)(x+21)$ x^3+37x^2+336x	0, -16, -21	$(3x+56)(x+6)$ $3x^2+74x+336$	$-\frac{56}{3},-6$	$(x\pm k)(x\pm 16a\pm k)(x\pm 21a\pm k)$
$x(x+26-a)(x+26)$ $x^3+x^2(52-a)+26x(26-a)$	0, a - 26, - 26	$3x^2+52x+26a-a^2$	not rational	
$x(x+14)(x+30)$ x^3+44x^2+420x	0, -14, -30	$(3x+70)(x+6)$ $3x^2+88x+420$	$-\frac{70}{3},-6$	$(x\pm k)(x\pm 14a\pm k)(x\pm 30a\pm k)$
$x(x+16)(x+30)$ x^3+46x^2+480x	0, -16, -30	$(3x+20)(x+24)$ $3x^2+92x+480$	$-\frac{20}{3},-24$	$(x\pm k)(x\pm 16a\pm k)(x\pm 30a\pm k)$
$x(x+33-a)(x+33)$ $x^3+x^2(66-a)+33x(33-a)$	0, a - 33, -33	$3x^2+2x(66-a)+$ $+1089-33a$	not rational	
$x(x+11)(x+35)$ x^3+46x^2+385x	0, -11, -35	$(3x+77)(x+5)$ $3x^2+92x+385$	$-\frac{77}{3},-5$	$(x\pm k)(x\pm 11a\pm k)(x\pm 35a\pm k)$
$x(x+24)(x+35)$ x^3+59x^2+840x	0, -24, -35	$(3x+28)(x+30)$ $3x^2+118x+840$	$-\frac{28}{3},-30$	$(x\pm k)(x\pm 24a\pm k)(x\pm 35a\pm k)$
$x(x+36-a)(x+36)$ $x^3+x^2(72-a)+36x(36-a)$	0, a - 36, -36	$3x^2+2x(72-a)+$ $+1296-36a$	not rational	

(8)

Having fully explored the cubic, the quartic family of curves presented quite a challenge because there would be three roots, (other than zero), to find. Visually, I opted for a span of 7, (one less than the 8 for cubics) entered $x(x-a)(x-b)(x-7)$ into DERIVE, fixed $a=3$ (only because it had worked with the cubic), took the derivative and evaluated *"b"* from 1 to 7 hoping some value *"b"* would work. The derived form was $4x^3+x^2(-3b-30)+x(20b+42)-21b$ and by declaring *"f"* as the function I simply tested values for *"b"* and systematically factored (pressing F). To my great delight $b=4$ worked, giving $2(x-6)(x-1)(2x-7)$. Observing 3 and 4 together, I acted on a hunch that Pythagorean Triples might work. So following in the footsteps of Diophantus, I tried triplets beginning with odd numbers and even numbers, and they worked beautifully. An added bonus were those triplets with consecutive legs such as 20 - 21 - 29 and 119 - 120 - 169, etc., which also worked wonderfully. The patterns appear in Table III.

The numerical families of the form $x(x+a)(x+b)(x+c)$ appear in Table IV. The numerical families for the form $x^2(x+a)(x+b)$ appear in Table V.

Editor's comment: How to use Table II for obtaining a bundle of cubics giving extremal values with rational *x*-coordinates:

```
#1:   p(x, a, k) := (x + k)·(x + 3·a + k)·(x + 8·a + k)
```

Two examples using Table II

$$\#2:\quad \text{VECTOR}([p(x, 2, k)], k, -3, 3) = \begin{bmatrix} (x+3)\cdot(x-3)\cdot(x+13) \\ (x-2)\cdot(x+4)\cdot(x+14) \\ (x-1)\cdot(x+5)\cdot(x+15) \\ x\cdot(x+6)\cdot(x+16) \\ (x+1)\cdot(x+7)\cdot(x+17) \\ (x+2)\cdot(x+8)\cdot(x+18) \\ (x+3)\cdot(x+9)\cdot(x+19) \end{bmatrix}$$

$$\#3:\quad \text{SOLVE}\left(\frac{d}{dx}(x\cdot(x+6)\cdot(x+16)), x\right) = \left(x = -\frac{8}{3} \vee x = -12\right)$$

$$\#4:\quad \text{VECTOR}([p(x, a, -2)], a, -3, 3) = \begin{bmatrix} (x-2)\cdot(x-11)\cdot(x-26) \\ (x-2)\cdot(x-8)\cdot(x-18) \\ (x-2)\cdot(x-5)\cdot(x-10) \\ (x-2)^3 \\ (x+1)\cdot(x-2)\cdot(x+6) \\ (x-2)\cdot(x+4)\cdot(x+14) \\ (x-2)\cdot(x+7)\cdot(x+22) \end{bmatrix}$$

$$\#5:\quad \text{SOLVE}\left(\frac{d}{dx}((x-2)\cdot(x+4)\cdot(x+14)), x\right) = \left(x = -\frac{2}{3} \vee x = -10\right)$$

(9)

Table III

Family Function	Roots	Derivative	Roots	Conditions for integral roots
$(x+a)^4$	$-a$	$4(x+a)^3$	$-a$	no restrictions
$(x+a)^3(x+b)$	$-a\ \ -b$	$(x+a)^2(4x+a+3b)$	$-a, \frac{-3b-a}{4}$	$3b+a$ must be a multiple of 4 for $b=1$, $a=1,5,9,\ldots,4k-3$ for $b=2$, $a=2,6,10,\ldots,4k-2$ for $b=3$, $a=3,7,11,\ldots,4k-1$ for $b=4$, $a=4,8,12,\ldots,4k$
$(x+a)^2(x+b)^2$	$-a,\ -b$	$2(x+a)(x+b)(2x+a+b)$	$-a,-b,\frac{-a-b}{2}$	a and b must both be even or both odd
$(x+a)^2(x+b)(x+c)$	$-a,\ -b,\ -c$	$(x+a)(4x^2+x(3b+3c+2a)+2bc+ab+ac)$	$\frac{-3b-3c-2a\pm\sqrt{4a^2+9b^2+9c^2-4ab-4ac-14bc}}{8}$	$4a^2+9b^2+9c^2-4ab-4ac-14bc$ must be a perfect square
The product of x, $x+2n+1, x+2n^2+2n$ and $x+2n^2+4n+1$	$0,-2n-1,$ $-2n^2-2n,$ $-2n^2-4n-1$	$2(x+n)(x+2n^2+3n+1)(2x+2n^2+4n+1)$	$-n,-2n^2-3n-1,\frac{-2n^2-4n-1}{2}$	no conditions
The product of x, $x+2n, x+n^2-1$ and $x+n^2+2n-1$	$0,-2n,$ $1-n^2,$ $-n^2-2n+1$	$2(x+n-1)(x+n^2+n)(2x+n^2+2n-1)$	$1-n,-n^2-n,\frac{-n^2-2n+1}{2}$	no conditions

(10)

Table IV

Family Function	Roots	Derivative	Roots	Transfomation
Odd Pythagorean Triplets				
$x(x+3)(x+4)(x+7)$	0, -3, -4, -7	$2(x+1)(x+6)(2x+7)$	$-1,-6,-\frac{7}{2}$	$(x\pm k)(x\pm 3a\pm k)(x\pm 4a\pm k)(x\pm 7a\pm k)$
$x(x+5)(x+12)(x+17)$	0, -5, -12, -17	$2(x+2)(x+15)(2x+17)$	$-2,-15,-\frac{17}{2}$	$(x\pm k)(x\pm 5a\pm k)(x\pm 12a\pm k)(x\pm 17a\pm k)$
$x(x+7)(x+24)(x+31)$	0, -7, -24, -31	$2(x+3)(x+28)(2x+31)$	$-3,-28,-\frac{31}{2}$	$(x\pm k)(x\pm 7a\pm k)(x\pm 24a\pm k)(x\pm 31a\pm k)$
Even Pythagorean Triplets				etc
$x(x+3)(x+4)(x+7)$	0, -3, -4, -7	$2(x+1)(x+6)(2x+7)$	$-1,-6,-\frac{7}{2}$	$(x\pm k)(x\pm 3a\pm k)(x\pm 4a\pm k)(x\pm 7a\pm k)$
$x(x+6)(x+8)(x+14)$	0, -6, -8, -14	$4(x+2)(x+7)(x+12)$	$-2,-7,-12$	$(x\pm k)(x\pm 6a\pm k)(x\pm 8a\pm k)(x\pm 14a\pm k)$
$x(x+8)(x+15)(x+23)$	0, -8, -15, -23	$2(x+3)(x+20)(x+23)$	$-3,-20,-\frac{23}{2}$	$(x\pm k)(x\pm 8a\pm k)(x\pm 15a\pm k)(x\pm 23a\pm k)$
Consecutive-Leg Triplets				etc
$x(x+3)(x+4)(x+7)$	0, -3, -4, -7	$2(x+1)(x+6)(2x+7)$	$-1,-6,-\frac{7}{2}$	$(x\pm k)(x\pm 3a\pm k)(x\pm 4a\pm k)(x\pm 7a\pm k)$
$x(x+20)(x+21)(x+41)$	0, -20, -21, -41	$2(x+6)(x+35)(2x+41)$	$-6,-35,-\frac{41}{2}$	$(x\pm k)(x\pm 20a\pm k)(x\pm 21a\pm k)(x\pm 41a\pm k)$
$x(x+119)(x+120)(x+239)$	0, -119, -120, -239	$2(x+35)(x+204)(2x+239)$	$-35,-204,-\frac{239}{2}$	$(x\pm k)(x\pm 119a\pm k)(x\pm 120a\pm k)(x\pm 239a\pm k)$ etc

(11)

Table V

Family Function	Roots	Derivative	Roots	Transformation
$x^2(x+5)(x-7)$	0, -5, 7	$2x(x-5)(2x+7)$	$0, 5, -\frac{7}{2}$	$(x\pm k)^2(x+5a\pm k)(x-7a\pm k)$
$x^2(x+5)(x+2)$	0, -5, -2,	$x(x+5)(4x+5)$	$0, -4, -\frac{5}{4}$	$(x\pm k)^2(x+5a\pm k)(x+2a\pm k)$
$x^2(x+5)(x+9)$	0, -5, -9	$2x(x+3)(2x+15)$	$0, -3, -\frac{15}{2}$	$(x\pm k)^2(x+5a\pm k)(x+9a\pm k)$
$x^2(x+7)(x+10)$	0, -7, -10	$x(x+4)(4x+35)$	$0, -4, -\frac{35}{4}$	$(x\pm k)^2(x+7a\pm k)(x+10a\pm k)$
$x^2(x+9)(x+14)$	0, -9, -14	$x(x+12)(4x+21)$	$0, -12, -\frac{21}{4}$	$(x\pm k)^2(x+9a\pm k)(x+14a\pm k)$

(12)

The Quintic

In terms of multiple roots the quintic lends itself nicely to easy-to-work-with numbers which are small in quantity. However, for quintics of the form $x(x-a)(x-b)(x-c)(x-d)$, the derivative has no rational roots primarily because of Fermat's Last Theorem whereby there are no integral values for which $x^4 + y^4 = z^4$. Having run the computer through thousands of number combinations (just to be sure), no derivative with rational roots could be found. Our Table VI contains multiple roots only. Table VII contains the numerical families for the forms $x^3(x+a)(x+b)$ and $x^2(x+a)^2(x+b)$.

With all the patterns that do work, it was too tempting not to try to make a linear link among the cubic, quartic, and quintic forms. Let us examine the following facts:

	Smallest Root	Largest Root	Range	Sum of Roots
I. Cubic Form $x(x-5)(x-8)$	0	8	8	13
Derivative $(3x-20)(x-2)$	2	$\frac{20}{3}$	$\frac{14}{3}$	$\frac{26}{3}=\frac{520}{60}$
II. Quartic Form $x(x-3)(x-4)(x-7)$	0	7	7	14
Derivative $2(x-1)(2x-7)(x-6)$	1	6	5	$\frac{21}{2}=\frac{630}{60}$
III. Quintic Form $x(x-2)(x-3)(x-4)(x-6)$	0	6	6	15
Derivative $5x^4-60x^3+240x^2-360x+144$	0.616036…	5.383963…	4.767926	12

(13)

Table VI

Family Function	Roots	Derivative	Roots	Conditions for integral roots
$(x+a)^5$	$-a$	$5(x+a)^4$	$-a$	none
$(x+a)^4(x+b)$	$-a,\ -b$	$(x+a)^3(5x+a+4b)$	$-a, \frac{-a-4b}{5}$	$a+4b$ must be a multiple of 5
$(x+a)^3(x+b)(x+c)$	$-a,\ -b,\ -c$	$(x+a)^2(5x^2+2x(a+2b+2c)+ab+ac+3bc)$	$-a$, $\frac{-(a+2b+2c)\pm\sqrt{\Delta}}{5}$ $\Delta=a^2+4b^2+4c^2-ab-ac-7bc$	$a^2+4b^2+4c^2-ab-ac-7bc$ must be zero or a perfect square
$(x+a)^3(x+b)^2$	$-a,\ -b$	$(x+a)^2(x+b)(5x+2a+3b)$	$-a, -b, \frac{-2a-3b}{5}$	$2a+3b$ must be a multiple of 5
$(x+a)^2(x+b)^2(x+c)$	$-a,\ -b,\ -c$	$(x+a)(x+b)(5x^2+x(3a+3b+4c)+ab+2ac+2bc)$	$-a, -b, \frac{-(3a+3b+4c)\pm\sqrt{\Delta}}{10}$ $\Delta=9a^2+9b^2+16c^2-2ab-16ac-16bc$	Conditions: Δ must be a perfect square
$x^2(x+a)(x+b)(x+c)$	$0,\ -a,\ -b,\ -c$	$x(5x^3+x^2(4a+4b+4c)+x(3ab+3ac+3bc)+2abc)$	no rational roots	not applicable
$x^2(x+a)(x+b)(x+c)(x+d)$	$0,\ -a,\ -b,\ -c,\ -d$	$5x^4+4x^3(a+b+c+d)+3x^2(ab+ac+ad+bc+bd+cd)+$ $+2x(abc+abd+acd+bcd)+abcd$	no rational roots	not applicable

(14)

Table VII

Family Function	Roots	Derivative	Roots	Transformation
$x^3(x+3)(x+4)$ $x^5+7x^4+12x^3$	0, -3, -4	$x^2(x+2)(5x+18)$ $5x^4+28x^3+36x^2$	$0,-2,-\frac{18}{5}$	$(x\pm k)^3(x+3a\pm k)(x+4a\pm k)$
$x^3(x+3)(x+11)$ $x^5+14x^4+33x^3$	0, -3, -11,	$x(x+9)(5x+18)$ $5x^4+56x^3+99x^2$	$0,-9,-\frac{11}{5}$	$(x\pm k)^3(x+3a\pm k)(x+11a\pm k)$
$x^3(x+4)(x+7)$ $x^5+11x^4+28x^3$	0, -4, -7	$x^2(x+6)(5x+14)$ $5x^4+44x^3+84x^2$	$0,-6,-\frac{14}{5}$	$(x\pm k)^3(x+4a\pm k)(x+7a\pm k)$
$x^3(x+5)(x+12)$ $x^5+17x^4+60x^3$	0, -5, -12	$x^2(x+10)(5x+18)$ $5x^4+68x^3+180x^2$	$0,-10,-\frac{18}{5}$	$(x\pm k)^3(x+5a\pm k)(x+12a\pm k)$
$x^2(x-3)^2(x-1)$ $x^5-7x^4+15x^3-9x^2$	0, 3, 1	$x(x-3)(x-2)(5x-3)$ $5x^4-28x^3+45x^2-18x$	$0,3,2,\frac{3}{5}$	$(x\pm k)^2(x-3a\pm k)^2(x-a\pm k)$
$x^2(x-3)^2(x-2)$ $x^5-8x^4+21x^3-18x^2$	0, 3, 2	$x(x-3)(x-1)(5x-12)$ $5x^4-32x^3+63x^2-36x$	$0,3,1,\frac{12}{5}$	$(x\pm k)^2(x-3a\pm k)^2(x-2a\pm k)$
$x^2(x-3)^2(x-7)$ $x^5-13x^4+51x^3-63x^2$	0, 3, 7	$x(x-6)(x-3)(5x-7)$ $5x^4-52x^3+153x^2-126x$	$0,6,3,\frac{7}{5}$	$(x\pm k)^2(x-3a\pm k)^2(x-7a\pm k)$
$x^2(x-3)^2(x+4)$ $x^5-2x^4-15x^3+36x^2$	0, 3, -4	$x(x-3)(x+3)(5x-8)$ $5x^4-8x^3-45x^2+72x$	$0,3,-3,\frac{8}{5}$	$(x\pm k)^2(x-3a\pm k)^2(x+4a\pm k)$

(15)

We realize very quickly that we can come close to rational roots, but cannot obtain them as our constant term could have to be a multiple of 5 in order to be factorable, which is impossible in this situation.

<u>Summary:</u>

If nothing else, the reader now has a list (not complete) of cubic and quartic polynomials with multiple or single integral roots whose derivatives have multiple or single rational roots. The quintic avails itself to multiple but not to single roots.

I would never have attempted all this work without the user-friendly program DERIVE, as I was able to test many polynomials in seconds and quickly find derivative and corresponding factored forms. It goes without saying that the same Diophantine process can be applied to rational forms, making life a little easier for the curve sketcher.

Part 2 of the Diophantine Polynomials will follow.

Additional Editor's Comments

I enjoyed this contribution very much. As you might know from earlier articles I liked to use the computer as a training partner for the students providing random generated problems from various fields of the curriculum.

Duncan mentioned that he used the TABLE function of the TI-83 for discovering certain expressions as perfect squares. I felt inspired to use the *TI-Nspire* for producing random generated polynomial functions. Take the quartics presented in Table IV (the Pythagorean Triplets).

First of all we can – preferable together with the students – derive the condition(s) for having rational roots of the first derivative. (These are the first three lines in the Calculator Application of *NspireCAS* below.) Then let's discuss how to produce Pythagorean Triplets – and finally use the spread sheet application to create random Pythagorean Triplets.

if $x = m^2 - n^2,\ y = 2mn,\ z = m^2 + n^2$
then $x^2 + y^2 = z^2$

This is shown in the next screen shot.

$g(x,a,b,c):=x\cdot(x+a)\cdot(x+b)\cdot(x+a+b)$	*Done*
$\frac{d}{dx}(g(x,a,b,c))$	$4\cdot x^3+6\cdot(a+b)\cdot x^2+(2\cdot a^2+6\cdot a\cdot b+2\cdot b^2)\cdot x+a\cdot(a+b)\cdot b$
$\text{zeros}\left(\frac{d}{dx}(g(x,a,b,c)),x\right)$	$\left\{\frac{\sqrt{a^2+b^2}-a-b}{2},\frac{-(\sqrt{a^2+b^2}+a+b)}{2},\frac{-(a+b)}{2}\right\}$
$\text{zeros}\left(\frac{d}{dx}(g(x,60,-32,28)),x\right)$	$\{-48,-14,20\}$
$\text{zeros}\left(\frac{d}{dx}(g(x,35,12,47)),x\right)$	$\left\{-42,\frac{-47}{2},-5\right\}$
$\text{zeros}\left(\frac{d}{dx}(g(x,8,-6,2)),x\right)$	$\{-6,-1,4\}$

In columns A and B are defined random generated integer numbers between -10 and +10 (without 0). Cell A1 and B1 contain the same formula which must be copied down – say 20 times.
In columns C, D and E are the values for the constants *a*, *b*, and *a* + *b* of the polynomial *g*.

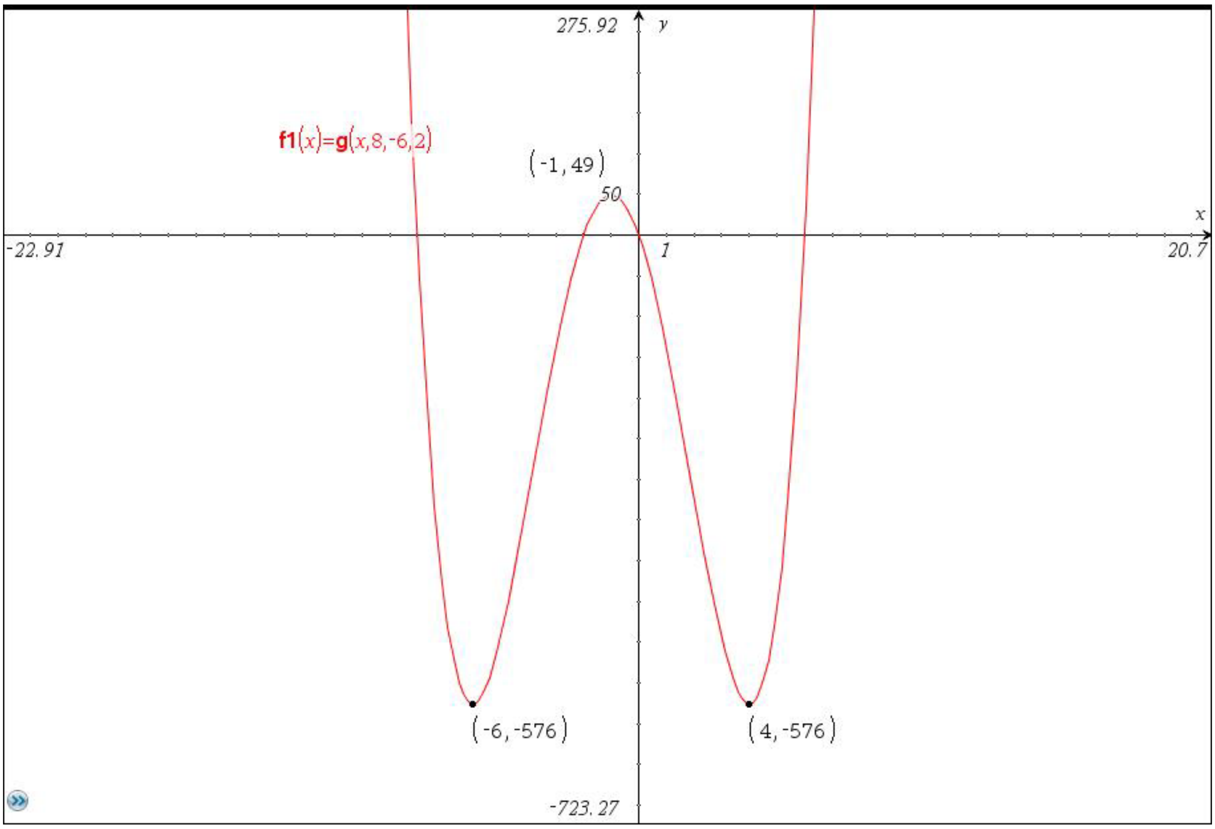
As you can see we obtain pretty much nice triplets of numbers.

	A m	B n	C a	D b	E	F	G	H	I	J
◆			=abs(a[]^2−b[]^2)	=2*a[]*b[]	=c[]+d[]					
1	8	-2	60	-32	28					
2	-3	-9	72	54	126					
3	1	6	35	12	47					
4	-2	-1	3	4	7					
5	6	1	35	12	47					
6	-10	8	36	-160	-124					
7	-3	-4	7	24	31					
8	1	5	24	10	34					
9	-1	7	48	-14	34					
10	4	7	33	56	89					
11	1	7	48	14	62					
12	-7	-6	13	84	97					
13	10	3	91	60	151					
14	-4	-9	65	72	137					
15	10	-5	75	-100	-25					
16	-1	3	8	-6	2					
17	7	-10	51	-140	-89					
18	-7	-5	24	70	94					
19	1	7	48	14	62					

A1 $=\text{randint}(1,10)\cdot(-1)^{\text{randint}(1,2)}$

The next three lines of the calculator present a test, if the roots of the first derivative are really rational numbers. I took the values of rows 1, 5, and 16.

Let the students confirm the calculation by plotting one or more function graphs and investigating the turning points:



The last column of Duncan´s Table IV gives advice how to create many polynomials from one “mother polynomial” applying easy to perform transformations.

I add a small program (function) how to generate random quartics of this type having fixed the values *a* and *b* of the “mother function”.

quartic(8,-6) produces “children of” $x\,(x + 8)\,(x – 6)\,(x + 2)$.

$$\frac{d}{dx}(g(x,a,b,c))$$
$$4\cdot x^3+6\cdot(a+b)\cdot x^2+(2\cdot a^2+6\cdot a\cdot b+2\cdot b^2)\cdot x+a\cdot(a+b)\cdot b$$

$$\text{zeros}\left(\frac{d}{dx}(g(x,a,b,c)),x\right)$$
$$\left\{\frac{\sqrt{a^2+b^2}-a-b}{2},\frac{-(\sqrt{a^2+b^2}+a+b)}{2},\frac{-(a+b)}{2}\right\}$$

$$\text{zeros}\left(\frac{d}{dx}(g(x,60,-32,28)),x\right) \quad \{-48,-14,20\}$$

$$\text{zeros}\left(\frac{d}{dx}(g(x,35,12,47)),x\right) \quad \left\{-42,\frac{-47}{2},-5\right\}$$

$$\text{zeros}\left(\frac{d}{dx}(g(x,8,-6,2)),x\right) \quad \{-6,-1,4\}$$

$$quartic(8,-6) \quad x^4-24\cdot x^3+16\cdot x^2+1536\cdot x-5120$$

$$quartic(8,-6) \quad x^4+4\cdot x^3-444\cdot x^2-896\cdot x+3520$$

$$quartic(8,-6) \quad x^4-20\cdot x^3-50\cdot x^2+1500\cdot x-3591$$

$$\text{zeros}\left(\frac{d}{dx}((x^4-20\cdot x^3-50\cdot x^2+1500\cdot x-3591)\cdot 1),x\right) \quad \{-5,5,15\}$$

$$\text{zeros}(x^4-20\cdot x^3-50\cdot x^2+1500\cdot x-3591,x) \quad \{-9,3,7,19\}$$

11/99

```
"quartic" stored successfully
Define quartic(u,v)=
Func
Local a,k
a:=randInt(-5,5)·(-1)^randInt(1,2)
k:=randInt(-5,5)·(-1)^randInt(1,2)
expand((x+k)·(x+u·a+k)·(x+v·a+k)·(x+(u+v)·a+k))
EndFunc
```

Don’t forget defining a randseed when using the function another time. Otherwise you might find the same polynomials. You can extend this function by implementing a random generated “mother”, too.

Add more types of quartics and …

It should be no problem transferring this procedure to the Voyage 200 and/or the good old TI-92.

This would not be the “*DERIVE Newsletter*” if I wouldn’t add a *DERIVE* comment, too. Duncan wrote in his paper that he used *DERIVE* to find many connections between the various types of polynomials and to quickly derive and factorize.

Let me use *DERIVE* to create problems (again “members of certain families”) more automated and to find representants of the families more systematically.

I start with Table I, next to the last example.

The first expressions confirm Duncan’s entries in his Table I

#1: $f(x, a, b) := (x + a^2)\cdot(x + 2\cdot a\cdot b)\cdot(x + b^2)$

#2: $g(x, a, b) := \frac{d}{dx} f(x, a, b)$

#3: $g(x, a, b) := 3\cdot x^2 + x\cdot(2\cdot a^2 + 4\cdot a\cdot b + 2\cdot b^2) + a\cdot b\cdot(2\cdot a^2 + a\cdot b + 2\cdot b^2)$

#4: SOLUTIONS$(2\cdot a^2 + a\cdot b + 2\cdot b^2 - 3\cdot k = 0, b)$

#5: SOLUTIONS$(g(x, a, b) = 0, x)$

#6: $\left[- a\cdot b, - \frac{2\cdot a^2 + a\cdot b + 2\cdot b^2}{3}\right]$

I don't want to experiment and then be lucky to have $2a^2 + ab + 2b^2$ divisible by 3. I have a powerful CAS at my disposal and this shall do the work – but I have to give correct and accurate orders …

#7: SOLUTIONS$\left(- \frac{2\cdot a^2 + a\cdot b + 2\cdot b^2}{3} = k, b\right) = \left[- \frac{\sqrt{3}\cdot\sqrt{(- 5\cdot a^2 - 8\cdot k)} + a}{4}, \frac{\sqrt{3}\cdot\sqrt{(- 5\cdot a^2 - 8\cdot k)} - a}{4}\right]$

#8: $(\sqrt{3}\cdot\sqrt{(- 5\cdot a^2 - 8\cdot k)})^2 = - 3\cdot(5\cdot a^2 + 8\cdot k)$

#9: SOLUTIONS$(- 3\cdot(5\cdot a^2 + 8\cdot k) = 0, a) = \left[\frac{2\cdot\sqrt{10}\cdot\sqrt{(-k)}}{5}, - \frac{2\cdot\sqrt{10}\cdot\sqrt{(-k)}}{5}\right]$

#10: $a = \frac{2\cdot\sqrt{10}\cdot\sqrt{(-k)}}{5}$

#11: $b =$ SUBST$\left(- \frac{\sqrt{3}\cdot\sqrt{(- 5\cdot a^2 - 8\cdot k)} + a}{4}, a, \frac{2\cdot\sqrt{10}\cdot\sqrt{(-k)}}{5}\right) = \left(b = - \frac{\sqrt{10}\cdot\sqrt{(-k)}}{10}\right)$

Expressions #7 - #11 show how to find *a* and *b* dependet on the parameter *k* - which must not necessarily be an integer as we will discover soon.

#12: $f\left(x, \frac{2\cdot\sqrt{10}\cdot\sqrt{(-k)}}{5}, - \frac{\sqrt{10}\cdot\sqrt{(-k)}}{10}\right) = \frac{(5\cdot x + 4\cdot k)\cdot(5\cdot x - 8\cdot k)\cdot(10\cdot x - k)}{250}$

#13: cub1(k) := EXPAND$((5\cdot x + 4\cdot k)\cdot(5\cdot x - 8\cdot k)\cdot(10\cdot x - k))$

cub1(k) is the recipe to create a cubic of requested form. Take any *k*, e.g. *k* = 7:

#14: cub1(7) $= 250\cdot x^3 - 1575\cdot x^2 - 14700\cdot x + 10976$

#15: FACTOR(cub1(7)) $= (5\cdot x + 28)\cdot(5\cdot x - 56)\cdot(10\cdot x - 7)$

You may ask the students: "Where are *a* and *b*?"

We solve the polynomial and its first derivative as well hoping to obtain only rational roots:

#16: $\text{SOLUTIONS}(\text{cub1}(7) = 0, x) = \left[-\frac{28}{5}, \frac{56}{5}, \frac{7}{10}\right]$

#17: $\text{SOLUTIONS}\left(\frac{d}{dx}(250\cdot x^3 - 1575\cdot x^2 - 14700\cdot x + 10976) = 0, x\right) = \left[7, -\frac{14}{5}\right]$

#18: $\text{cub1}\left(-\frac{2}{3}\right) = 250\cdot x^3 + 150\cdot x^2 - \frac{400\cdot x}{3} - \frac{256}{27}$

#19: $\text{SOLUTIONS}\left(\text{cub1}\left(-\frac{2}{3}\right) = 0, x\right) = \left[-\frac{1}{15}, \frac{8}{15}, -\frac{16}{15}\right]$

#20: $\text{SOLUTIONS}\left(\frac{d}{dx}\text{cub1}\left(-\frac{2}{3}\right) = 0, x\right) = \left[-\frac{2}{3}, \frac{4}{15}\right]$

Luckily it worked. We can ask ourselves, how to achieve integer roots for the given function and the derivative as well? Inspecting #10 and #11 from above, we conclude that *k* must be a multiple of 10 (divided or not by 250). Ok, let me try:

#21: $\frac{\text{cub1}(-30)}{250} = x^3 + 27\cdot x^2 - 1080\cdot x - 3456$

#22: $\text{SOLUTIONS}\left(\frac{\text{cub1}(-30)}{250} = 0, x\right) = [-3, 24, -48]$

#23: $\text{SOLUTIONS}\left(\frac{d}{dx}\frac{\text{cub1}(-30)}{250} = 0, x\right) = [12, -30]$

#24: $\frac{\text{cub1}(110)}{250} = x^3 - 99\cdot x^2 - 14520\cdot x + 170368$

#25: $\text{FACTOR}(x^3 - 99\cdot x^2 - 14520\cdot x + 170368) = (x - 11)\cdot(x + 88)\cdot(x - 176)$

#26: $\text{SOLUTIONS}\left(\frac{d}{dx}(x^3 - 99\cdot x^2 - 14520\cdot x + 170368) = 0, x\right) = [-44, 110]$

Finally I am treating the family of family functions appearing in Table II.

#27: $\frac{d}{dx}(x\cdot(x + a)\cdot(x + b)) = 3\cdot x^2 + x\cdot(2\cdot a + 2\cdot b) + a\cdot b$

#28: $\text{SOLUTIONS}(3\cdot x^2 + x\cdot(2\cdot a + 2\cdot b) + a\cdot b = 0, x)$

#29: $\left[-\frac{\sqrt{(a^2 - a\cdot b + b^2)} + a + b}{3}, \frac{\sqrt{(a^2 - a\cdot b + b^2)} - a - b}{3}\right]$

I asked myself how to find suitable “partners“ which might be able to found a “family“ to use also Duncan’s very charming notation.

Then it would be necessary to have a function which presents possible partners to one given root, resulting in integer roots for the function and rational roots for the derivative.

#30: $\text{partner}(a, l, u) := \left(\text{SELECT}\left(v_2 = \text{FLOOR}(v_2), v, \text{VECTOR}\left(\left[b, \sqrt{(a^2 - a\cdot b + b^2)}\right], b, l, u\right)\right)\right)\downarrow\downarrow 1$

#31: partner(3, -100, 100) = [-5, 0, 3, 8]

The VECTOR-command produces pairs of b with $l \le b \le u$ and the root – with a given a. The SELECT command filters all pairs with an integer root and collects only the first elements of the pairs – these are possible parters to found a family.

#31: partner(3, -100, 100) = [-5, 0, 3, 8]

#32: $\text{VECTOR}\left(\text{SOLUTIONS}\left(\frac{d}{dx}(x\cdot(x + 3)\cdot(x + p)) = 0, x\right), p, [-5, 0, 3, 8]\right)$

#33: $\begin{bmatrix} 3 & -\frac{5}{3} \\ 0 & -2 \\ -1 & -3 \\ -6 & -\frac{4}{3} \end{bmatrix}$

There are only 4 possible partners between -100 and +100. (There are the same four between -10000 and + 10000 – after 55 seconds calculation time!). The roots of the derivate for all partners are given in #33. Let's have another try. Possibly 7 would be more attractive for partners?

#38: partner(7, -100, 100) = [-33, -8, 0, 7, 15, 40]

#39: $\text{VECTOR}\left(\text{SOLUTIONS}\left(\frac{d}{dx}(x\cdot(x + 7)\cdot(x + p)) = 0, x\right), p, [-33, -8, 0, 7, 15, 40]\right)$

#40: $\begin{bmatrix} 21 & -\frac{11}{3} \\ -4 & \frac{14}{3} \\ 0 & -\frac{14}{3} \\ -7 & -\frac{7}{3} \\ -3 & -\frac{35}{3} \\ -28 & -\frac{10}{3} \end{bmatrix}$

My Conclusion: I felt very inspired by Duncan's contribution. I find it useful mainly because of two reasons: it makes math teacher's daily work easier and it presents a lot of problems which could (and should) discussed with the students, e.g. filling in the Tables' last columns which are "Conditions for integral or rational roots" and the possible "Transformations" or including considerations about the 2nd derivative, … And this is much more than only a tool for generating bundles of "beautiful" investigations of polynomial functions, Josef.
See also Duncan's mail on page 38.

Ein Mathematisches Modell für Schneckenhäuser

A Mathematical Model for Snail Shells

von Piotr Trebisz
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Wahrscheinlich ist schon jedem DERIVE-Nutzer der spiralförmige Torus aufgefallen der auf der Verpackung von Derive abgebildet ist. Dieses geometrische Objekt hat mich dazu inspiriert, einen Beitrag über Schneckenhäuser zu schreiben, die auf logarithmischen Spiralen basieren.

Schneckenhäuser sind ein sehr schönes Beispiel dafür, dass logarithmische Spiralen ein häufig wiederkehrendes Muster in der Natur sind. Logarithmische Spiralen sind selbstähnlich, das heißt, dass eine kleine Spirale genauso aussieht wie eine große Spirale. Das ermöglicht Schnecken ihre Häuser kontinuierlich weiterzubauen, ohne dass sich deren Geometrie ändert.

Ich werde im Laufe der folgenden Artikelreihe ein mathematisches Modell für solche Schneckenhäuser vorstellen. Da die von mir vorgestellten Schneckenhäuser einige besondere Eigenschaften haben, die so in der Literatur noch nirgendwo beschrieben wurden, nehme ich mir die Freiheit heraus, sie nach mir zu benennen: *Die Trebisz-Spiralen*.

Beginnen wir bei der Konstruktion unserer Schneckenhäuser mit einer einfachen Variante, bei der uns als Basis eine planare logarithmische Spirale dient, ähnlich wie bei einer Nautilus-Schale. Ich nenne sie „*Trebisz-Spirale – Typ 1*"

It is very likely that all DERIVE users have noticed the spiral formed torus on the package and the manual. This geometric object inspired me to write a paper on logarithmic based snail shells.

Snail shells are a very beautiful example for the fact that logarithmic spirals are a frequently recurring pattern in nature. Logarithmic spirals are self similar, i.e. a small spiral looks like a large one. And this enables snails building their shells continuously without changing their geometry.

Over the following series of contributions I will introduce a mathematic model for such snail shells. As these snail shells will show some special properties, which I didn't find elsewhere in literature, I take the freedom to call them *Trebisz Spirals*.

We will start constructing our snail shells with an easy model, taking a plane logarithmic spiral as base, very similar to a shell of a nautilus. I call it "*Trebisz-Spiral – Type 1*".

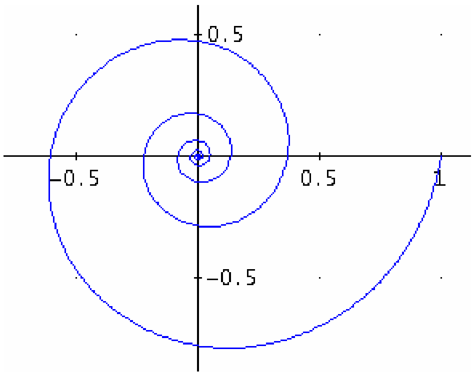
Als Gerüst für das Schneckenhaus dient eine planare logarithmische Spirale die nach ihrem Radius bzw. ihrer Länge parametrisiert ist. Der Parameter „*t*" steht hierbei für den Radius bzw. die Länge der Spirale, während „*b*" die Basis des Logarithmus ist.

A plane logarithmic spiral which is parameterized according its radius and its length, respectively is serving as the shell's scaffold. Parameter "*t*" is standing for the radius and the length of the spiral, respectively, and "*b*" is the base of the logarithm.

```
#1:   [CaseMode := Sensitive, InputMode := Word]

#2:   [b :∊ Real (1, ∞), m :∊ Real [0, ∞), t :∊ Real [0, ∞), s :∊ Real [−π, +π]]

#3:   [w :∊ Real, n :∊ Integer (0, ∞), k :∊ Integer]

#4:   SNAIL(b, t) := [0, t·COS(2·π·LOG(t, b)), t·SIN(2·π·LOG(t, b))]
```

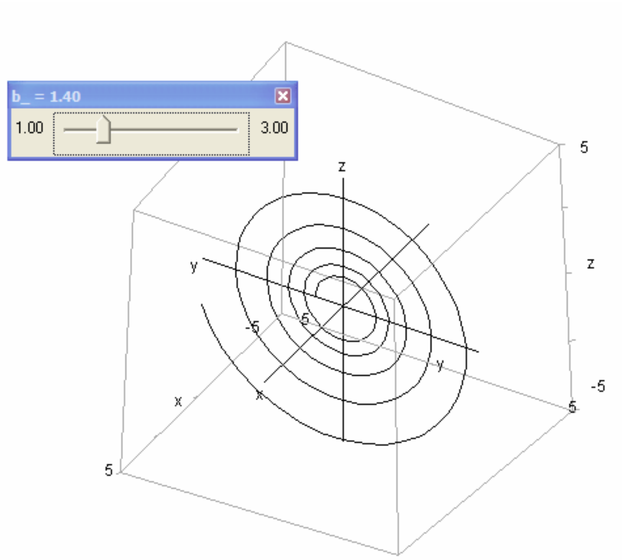
```
Example:

SNAIL(e, t) SUB [2,3]

with
```

$$0 \le t \le 1$$



The frequent reader of the DNL knows my devotion to sliders. We can animate the base spirals in the 3D-plot window:

```
SNAIL(b_)
```



See the respective plot parameters to for presenting 5 rotations:

	Minimum	Maximum	Number of Panels
s:	0	1	1
t:	0	5	200

Josef

Dass diese Spirale nach ihrem Radius parametrisiert ist, lässt sich leicht überprüfen, indem man sich anschaut, wie weit ein Punkt auf dieser Spirale vom Zentrum entfernt ist in Abhängigkeit von „*t*“.

It is easy to check that the radius is the parameter of this spiral by inspecting the distance of a point on the spiral to the centre dependent on “*t*”.

```
#5:    |SNAIL(b, t)| = t
```

Dass die Spirale auch nach Ihrer Länge parametrisiert ist, lässt sich dadurch zeigen, dass die Tangente an jedem Punkt der Spirale eine konstante Länge hat

The fact that this spiral has also its length as parameter can be shown by demonstrating that the tangent on each spiral point has constant length.

$$\#6: \quad \left|\frac{d}{dt}\,\mathrm{SNAIL}(b,\ t)\right| = \frac{\sqrt{\mathrm{LN}(b)^2 + 4\cdot\pi^2}}{\mathrm{LN}(b)}$$

Wie man sieht, ist die Spirale auch nach der Länge parametrisiert, aber multipliziert mit einem konstanten Maßstabs-Faktor.

Um den Mantel des Schneckenhauses zu konstruieren benötigen wir ein lokales mitdrehendes Koordinatensystem aus 2 Vektoren die senkrecht zueinander und zur Tangente der Spirale stehen.

Dies gelingt uns, indem wir die 2. Ableitung der Spirale nach „*t*“ bestimmen. Dabei machen wir uns zu Nutze, dass die 2. Ableitung einer Kurve, die nach ihrer Länge parametrisiert ist, einen Vektor ergibt, der in der Krümmungsebene der Kurve liegt und der senkrecht zu ihrer Tangente steht.

As one can see the spiral is also parameterized wrt its length multiplied by a constant scaling factor.

For constructing the surface of the snail shell we need a local co-rotating system of coordinates consisting of two vectors which are perpendicular to each other and perpendicular wrt the tangent of the spiral.

The 2nd derivative of the spiral wrt “*t*” can help. With its length as parameter the 2nd derivative of the curve results in a vector lying in the osculation plane of the curve perpendicular to its tangent.

$$\#7: \quad \left(\frac{d}{dt}\,\mathrm{SNAIL}(b,\ t)\right) \cdot \left(\frac{d}{dt}\right)^2 \mathrm{SNAIL}(b,\ t) = 0$$

Das Kreuzprodukt aus Tangente und 2. Ableitung liefert uns einen weiteren Vektor der senkrecht zur Tangente und zur 2. Ableitung steht. Dieser Vektor soll unsere lokale mitdrehende X-Achse sein. Und die lokale Y-Achse können wir direkt aus der 2. Ableitung nach „t“ bestimmen. Die lokale mitdrehende Z-Achse entspricht der Tangente der Spirale.

The cross product from tangent and 2nd derivative delivers another vector which is perpendicular to both tangent and 2nd derivative. This vector shall form our local co-rotating X-axis. The local Y-axis is given directly by the 2nd derivative wrt “t”. the local co-rotating Z-axis is the spiral tangent.

$$\#8: \quad X\_Achse(b, t) := SIGN\left(\left(\frac{d}{dt} SNAIL(b, t)\right) \times \left(\frac{d}{dt}\right)^2 SNAIL(b, t)\right)$$

$$\#9: \quad Y\_Achse(b, t) := - SIGN\left(\left(\frac{d}{dt}\right)^2 SNAIL(b, t)\right)$$

$$\#10: \quad Z\_Achse := SIGN\left(\frac{d}{dt} SNAIL(b, t)\right)$$

Es lässt sich zeigen dass dieses lokale mitdrehende Koordinatensystem die selbe Händigkeit besitzt wie das globale Koordinatensystem, in dem die Spirale definiert ist.

We can show that this local co-rotating system of coordinates has the same direction as the global system where the spiral is defined.

$$\#11: \quad X\_Achse(b, t) \times Y\_Achse(b, t) = Z\_Achse$$

Die Mantelfläche soll einfachheitshalber einen kreisförmigen Querschnitt haben. Der Radius der Mantelfläche soll linear mit dem Radius bzw. mit der Länge der Spirale wachsen, „*m*“ ist der Linearfaktor.

To make it easy we will have a circle as cross section of the surface. The radius of the surface shall increase linearly according to the radius – and the length – of the spiral with "*m*" as linear-factor.

$$\#12: \quad Mantel(b, m, t, s) := m \cdot t \cdot (COS(s) \cdot X\_Achse(b, t) + SIN(s) \cdot Y\_Achse(b, t))$$

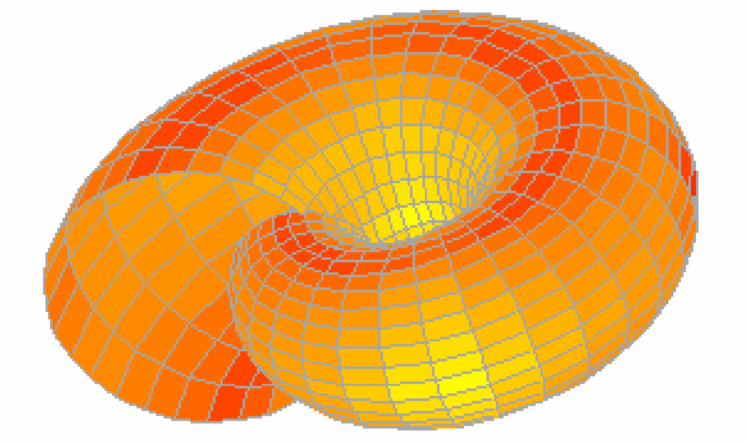
Und schon ist das Schneckenhaus fertig. Es ist die Summe aus Spirale und Mantel.

Here we go, the snail shell is ready. It is the sum of spiral and surface.

$$\#13: \quad SpiralTorusYZ\_R(b, m, t, s) := Mantel(b, m, t, s) + SNAIL(b, t)$$

$$\#14: \quad SpiralTorusYZ\_R\left(0.5, \frac{2}{3}, t, s\right)$$

$0 \le s \le 2\pi$, $0 \le t \le 1$; 40 panels each



Will man das Schneckenhaus nicht nach dem Radius, sondern nach Zahl der Umdrehungen parametrisieren, dann ersetzt man „*t*“ durch „*b^t*“. Der Radius wächst dann nicht mehr linear zu „*t*“ sondern zu b^t.

When we don't take the radius as parameter but the number of rotations then we have to substitute "*t*" by "*b^t*". Then the radius does not increase linear wrt "*t*" but to b^t.

#15: $\left|\mathrm{SNAIL}(b,\ b^t)\right| = b^t$

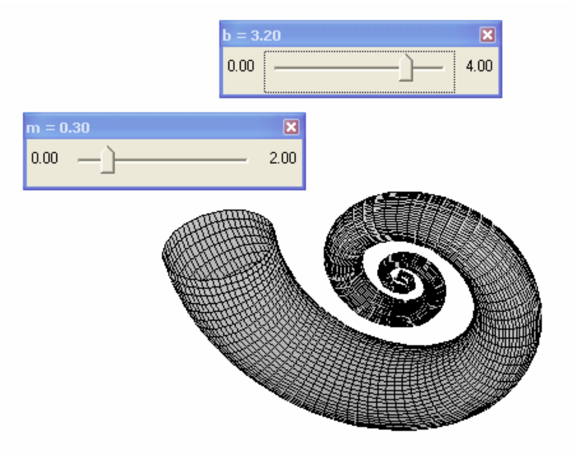
#16: $\mathrm{SpiralTorusYZ\_U}(b,\ m,\ t,\ s) := \mathrm{Mantel}(b,\ m,\ b^t,\ s) + \mathrm{SNAIL}(b,\ b^t)$

Here again the slider bars give some useful insight about the importance of the parameters b and m.

I introduce the sliders in order to animate the "linear-factor m" and the base of the spiral as well.

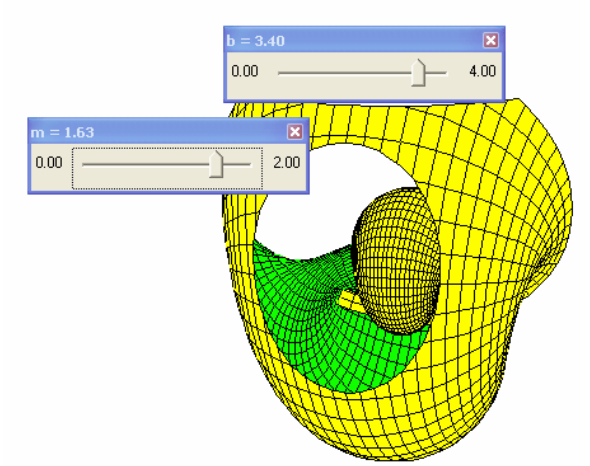
See the snail shell with t = 2 and two combinations of b and m.

Josef



SpiralTorusYZ_R(b, m, t, s)

$0 \le t \le 1$



SpiralTorusYZ_U(b, m, t, s)

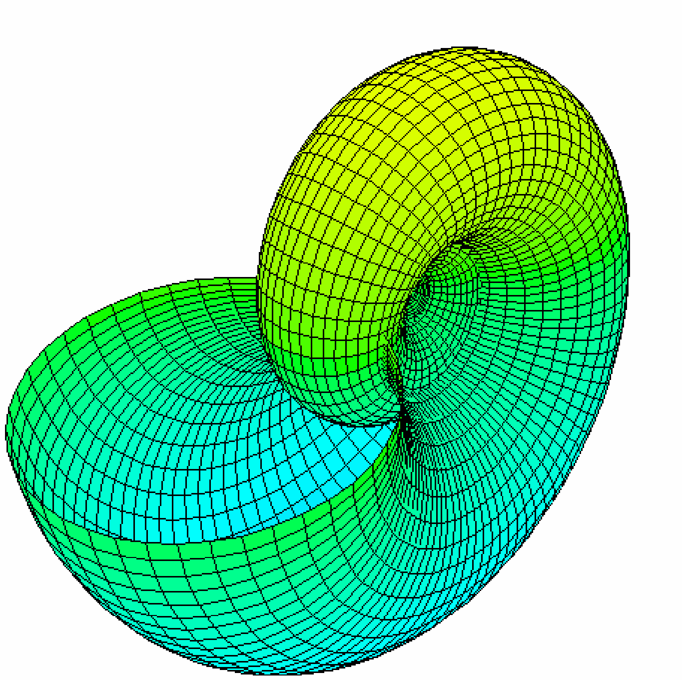
$-1 \le t \le 1{,}2$

Es gibt leider ein Problem mit dem Linear-Faktor „*m*". Wird er zu groß gewählt, dann durchdringt sich die Mantelfläche bei jeder Umdrehung selbst. Beispiel: $b = e$ und $m = 2/3$.

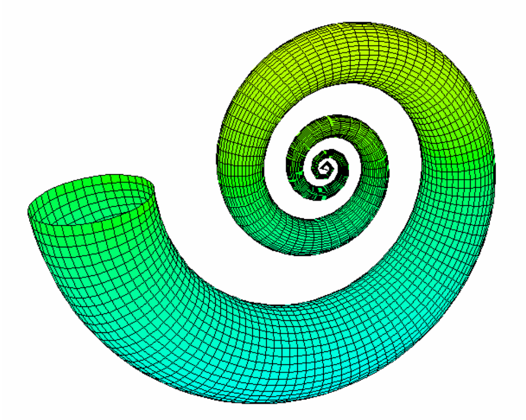
#17: $\mathrm{SpiralTorusYZ\_R}\left(e,\ \frac{2}{3}\right)$

$0 \le t \le 1$ and $-\pi \le s \le \pi$

Unfortunately we are facing a problem with the linear-factor "*m*". Choosing *m* too big then the surface will interpenetrate at each rotation, e.g. $b = e$ and $m = 2/3$.



Wird „*m*“ zu klein gewählt, dann klafft bei jeder Umdrehung eine hässliche Lücke zwischen der Mantelfläche. Beispiel $b = e$ und $m = 1/4$.



$$\#18: \quad \text{SpiralTorusYZ\_R}\left(e,\ \frac{1}{4}\right)$$

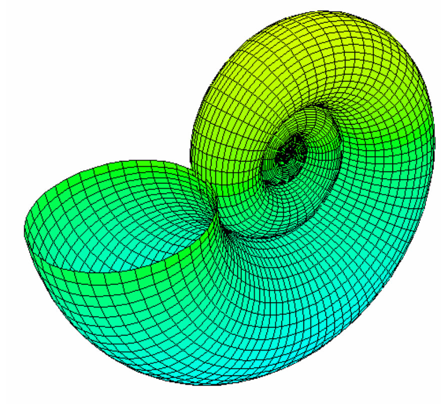
Choosing "*m*" too small, then we have an ugly gap between the surface, e.g. $b = e$ and $m = 1/4$

Wie kann man zur Basis „*b*“ das richtige „*m*“ so bestimmen dass sich die Mantelfläche nach jeder Umdrehung exakt berührt, ohne sich selbst zu durchdringen? $b = e$ und $m = 0.4593761774$.

How to determine the right “*m*“ for given base “*b*“ that the surface is osculating exactly after each full rotation without interpenetrating? $b = e$ and $m = 0.4593761774$.

$$\#19: \quad \text{SpiralTorusYZ\_R}\left(e,\ \frac{2071448}{4509263}\right)$$

$$\#20: \quad \text{APPROX}\left(\frac{2071448}{4509263}\right) = 0.4593761774$$



Dazu müssen wir uns den Schnitt des Schneckenhauses durch die XY-Ebene anschauen. Die Z-Koordinate von #16 (vereinfacht) muss also 0 sein. Wir lösen die entsprechende Gleichung nach „*t*“ auf:

We have to investigate the section of the snail shell with the XY-plane. Hence the Z-coordinate of simplified #16 must equal zero. We solve the respective equation for "*t*":

$$\#21: \quad b^t \cdot \left(\left(\frac{2\cdot\pi\cdot m\cdot \text{SIN}(s)}{\sqrt{(\text{LN}(b)^2 + 4\cdot\pi^2)}} + 1\right)\cdot \text{SIN}(2\cdot\pi\cdot t) - \frac{m\cdot \text{LN}(b)\cdot \text{SIN}(s)\cdot \text{COS}(2\cdot\pi\cdot t)}{\sqrt{(\text{LN}(b)^2 + 4\cdot\pi^2)}}\right) = 0$$

$$\#22: \quad t = \frac{\text{ATAN}\left(\dfrac{m\cdot \text{LN}(b)\cdot \text{SIN}(s)}{\sqrt{(\text{LN}(b)^2 + 4\cdot\pi^2)} + 2\cdot\pi\cdot m\cdot \text{SIN}(s)}\right)}{2\cdot\pi}$$

Wir können die gefundene Lösung verallgemeinern, denn das Schneckenhaus schneidet die XY-Ebene nicht nur einmal, sondern bei jeder halben Umdrehung. Wenn wir also zu der Lösung ein beliebiges ganzzahlige Vielfaches von 1/2 dazu addieren, dann erhalten wir wieder eine gültige Lösung. „*k*“ ist eine beliebige ganze Zahl.

We can generalize the found solution because the snail shell does not intersect the XY-plane only once but at every half rotation. We obtain another valid solution by adding any integer multiple of ½. ”*k*” is an arbitrary integer.

#22:
$$t = \frac{\mathrm{ATAN}\left(\dfrac{m\cdot \mathrm{LN}(b)\cdot \mathrm{SIN}(s)}{\sqrt{\mathrm{LN}(b)^2 + 4\cdot\pi^2} + 2\cdot\pi\cdot m\cdot \mathrm{SIN}(s)}\right)}{2\cdot\pi} + \frac{k}{2}$$

Für die Lösung des Selbstdurchdringungsprolems können wir aber nur jene Lösungen für „*t*“ gebrauchen, die sich in einer ganzen Umdrehung unterscheiden, deshalb schränken wir die Lösungen für „*t*“ wie folgt ein:

For solving the interpenetrating problem we can use only such solutions for “*t*” which differ in a full rotation. We set a constraint for „*t*“ as follows:

#23:
$$t = \frac{\mathrm{ATAN}\left(\dfrac{m\cdot \mathrm{LN}(b)\cdot \mathrm{SIN}(s)}{\sqrt{\mathrm{LN}(b)^2 + 4\cdot\pi^2} + 2\cdot\pi\cdot m\cdot \mathrm{SIN}(s)}\right)}{2\cdot\pi} + k$$

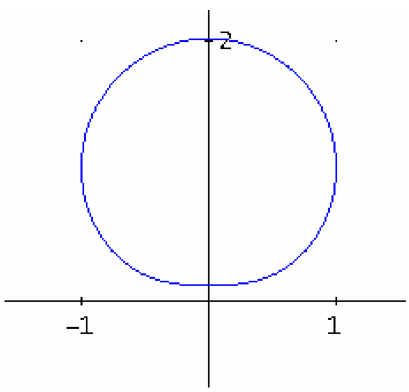
Setzt man diese Lösung für „*t*“ in die Schneckenhaus-Formel ein, wird die Z-Koordinate 0 und man erhält den XY-Schnitt der Fläche, wo man sehen kann, ob sie sich selbst durchdringt oder nicht. Schauen wir uns mal ein paar Schnittkurven für $k = 0$ an:

Substituting this solution for "*t*" in in the formula for the snail house the Z-coordinate will become 0 and one obtains the XY-intersection of the surface. This section shows whether the surface is interpenetrating or not.
We will look at some intersection curves for $k = 0$:

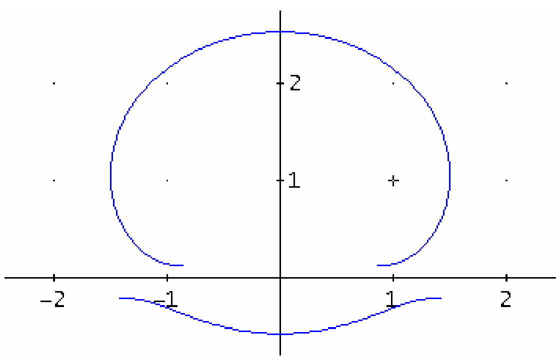
#24:
$$\left(\mathrm{SUBST}\left(\mathrm{SpiralTorusYZ\_U}(b, m, t, s),\ t,\ \frac{\mathrm{ATAN}\left(\dfrac{m\cdot \mathrm{LN}(b)\cdot \mathrm{SIN}(s)}{\sqrt{\mathrm{LN}(b)^2 + 4\cdot\pi^2} + 2\cdot\pi\cdot m\cdot \mathrm{SIN}(s)}\right)}{2\cdot\pi}\right)\right)_{[1,\ 2]}$$

#25:
$$\left[m\cdot b^{\mathrm{ATAN}(m\cdot \mathrm{LN}(b)\cdot \mathrm{SIN}(s)/(\sqrt{\mathrm{LN}(b)^2 + 4\cdot\pi^2} + 2\cdot\pi\cdot m\cdot \mathrm{SIN}(s)))/(2\cdot\pi)}\cdot \mathrm{COS}(s),\ b^{\mathrm{ATAN}(m\cdot \mathrm{LN}(b)\cdot \mathrm{SIN}(s)/(\sqrt{\mathrm{LN}(b)^2 + 4\cdot\pi^2} + 2\cdot\pi\cdot m\cdot \mathrm{SIN}(s)))/(2\cdot\pi)}\cdot \frac{\mathrm{SIGN}(\sqrt{\mathrm{LN}(b)^2 + 4\cdot\pi^2} + 2\cdot\pi\cdot m\cdot \mathrm{SIN}(s))\cdot\sqrt{\sqrt{\mathrm{LN}(b)^2 + 4\cdot\pi^2}\cdot(m^2\cdot \mathrm{SIN}(s)^2 + 1) + 4\cdot\pi\cdot m\cdot \mathrm{SIN}(s)}}{(\mathrm{LN}(b)^2 + 4\cdot\pi^2)^{1/4}}\right]$$

$b = e$ and $m = 1$



$b = e$ and $m = 3/2$



Hoppla!!! Warum ist denn im 2. Beispiel die Schnittkurve unterbrochen? Der Grund dafür liegt in der Lösung, die wir für „*t*“ bestimmt haben. Formal gesehen ist die Lösung vollkommen in Ordnung. Wenn man sie zur Probe in die Z-Koordinate des Schneckenhauses einsetzt, dann wird diese 0.

Nur hat die Lösung einen gravierenden Schönheitsfehler! Bei bestimmten Kombinationen von „*b*“ und „*m*“ zerfällt die Schnittkurve in zwei Teile, die zwar genau zusammenpassen, sich aber auch in genau einer halben Umdrehung unterscheiden! Das erkennt man daran, dass sich das zweite Stück der Schnittkurve auf der anderen Seite der X-Achse befindet. Dieses Kurvenstück gehört zum Schnitt des Schneckenhauses mit der XY-Ebene nach der nächsten halben Umdrehung!

Offensichtlich gibt es einen bestimmen Wertebereich von „*s*“, wo es in der Y-Koordinate zu einem Vorzeichenwechsel kommt. Und das wiederum führt zu einem „Umdrehungswechsel“ in der Schnittkurve! Dieser „Umdrehungswechsel“ zeigt sich sehr schön, wenn man sich die Schnittkurven für zwei Lösungen anschaut, die sich in einer halben Umdrehung unterscheiden.

Oops!! Why is there an interruption in the 2nd example? The reason lies in our solution for "*t*". Formally the solution is absolutely ok. The Z-coordinate is becoming 0 by substituting this solution, but …

… it is a solution with a serious flaw. Certain combinations of “*b*“ and “*m*“ result in a decomposing of the intersection curve in two parts which – fitting exactly to each other – differ in exact one half rotation! This can be recognized by the fact that the second part of the intersection curve is lying on the other side of the X-axis. This part of the curve is part of the intersection of the shell with the XY-plane after the next half rotation!

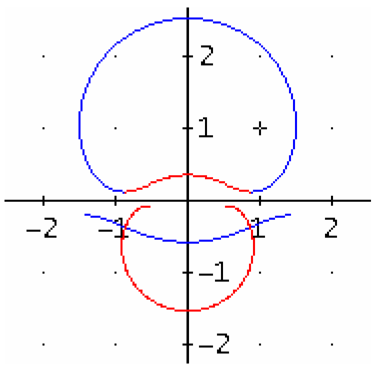
Obviously there is a certain domain of "*s*" which results in a sign change of the Y-coordinate. And this leads subsequently to a change of the rotation in the intersection curve! This “rotation change” appears very clear by observing the intersection curves for two solutions which differ in a half rotation.

$$t1 = \frac{\mathrm{ATAN}\left(\dfrac{m\cdot \mathrm{LN}(b)\cdot \mathrm{SIN}(s)}{\sqrt{(\mathrm{LN}(b)^2 + 4\cdot\pi^2)} + 2\cdot\pi\cdot m\cdot \mathrm{SIN}(s)}\right)}{2\cdot\pi}$$

und/and

$$t2 = \frac{\mathrm{ATAN}\left(\dfrac{m\cdot \mathrm{LN}(b)\cdot \mathrm{SIN}(s)}{\sqrt{(\mathrm{LN}(b)^2 + 4\cdot\pi^2)} + 2\cdot\pi\cdot m\cdot \mathrm{SIN}(s)}\right)}{2\cdot\pi} - \frac{1}{2}$$

b = *e* and *m* = 3/2



Wie man sieht, fügen sich die Teile der zwei Schnittkurven passgenau zusammen. Gibt es eine Möglichkeit die Lösung zu reparieren?

Es gibt tatsächlich einen Weg, dazu müssen wir uns genauer anschauen für welche Wertebereiche von „*s*“ der Vorzeichenwechsel stattfindet. Gucken wir uns dazu die 4 Terme genauer an, aus denen die Y-Koordinate der Schnittkurve zusammengesetzt ist.

As one can see the two parts of the two intersection curves are matching. Is there a possibility to repair the solution?

Yes, indeed, there is a chance. We must investigate very accurate which domain of "*s*" is causing the sign change. Let's have a thorough look at the four expressions which build the Y-coordinate of the intersection curve.

$$\#28:\quad b^{\text{ATAN}(m\cdot \text{LN}(b)\cdot \text{SIN}(s)/(\sqrt{\text{LN}(b)^2 + 4\cdot\pi^2} + 2\cdot\pi\cdot m\cdot \text{SIN}(s)))/(2\cdot\pi)}$$

$$\#29:\quad \sqrt{\sqrt{\text{LN}(b)^2 + 4\cdot\pi^2}\cdot(m^2\cdot \text{SIN}(s)^2 + 1) + 4\cdot\pi\cdot m\cdot \text{SIN}(s)}$$

$$\#30:\quad (\text{LN}(b)^2 + 4\cdot\pi^2)^{1/4}$$

Diese 3 Terme haben alle immer ein positives Vorzeichen, hier finden wir die Antwort also nicht. Was ist mit dem 4. Term?

The first three expressions are always positive, we will not find the answer here. What's about the fourth expression?

$$\#31:\quad \text{SIGN}(\sqrt{\text{LN}(b)^2 + 4\cdot\pi^2} + 2\cdot\pi\cdot m\cdot \text{SIN}(s))$$

In der Tat! Das ist ein SIGNUM-Term der sein Vorzeichen für einen bestimmten Wertebreich von „*s*" ändert wenn folgende Bedingung erfüllt ist:

Indeed, this is an expression containing the sign-function. It changes its sign for a certain domain of „*s*" under the following condition:

$$\#32:\quad \sqrt{\text{LN}(b)^2 + 4\cdot\pi^2} < 2\cdot\pi\cdot m$$

Nun können wir die Lösung für „*t*" so reparieren, dass die Schnittkurve immer geschlossen ist und es nicht zu einem „Umdrehungswechsel" kommt.

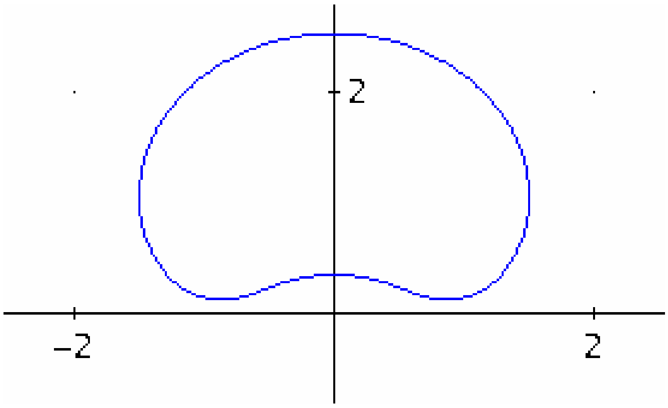
Now we are able to adapt the solution for "*t*" in order to obtain always a closed intersection curve without any "rotation change".

$$\#33:\quad t = \frac{\text{SIGN}(\sqrt{\text{LN}(b)^2 + 4\cdot\pi^2} + 2\cdot\pi\cdot m\cdot \text{SIN}(s)) - 1}{4} + \frac{\text{ATAN}\left(\frac{m\cdot \text{LN}(b)\cdot \text{SIN}(s)}{\sqrt{\text{LN}(b)^2 + 4\cdot\pi^2} + 2\cdot\pi\cdot m\cdot \text{SIN}(s)}\right)}{2\cdot\pi} + k$$

$$\#34:\quad t = \frac{\text{SIGN}(\sqrt{\text{LN}(b)^2 + 4\cdot\pi^2} + 2\cdot\pi\cdot m\cdot \text{SIN}(s)) + 4\cdot k - 1}{4} + \frac{\text{ATAN}\left(\frac{m\cdot \text{LN}(b)\cdot \text{SIN}(s)}{\sqrt{\text{LN}(b)^2 + 4\cdot\pi^2} + 2\cdot\pi\cdot m\cdot \text{SIN}(s)}\right)}{2\cdot\pi}$$

Was habe ich gemacht? Ich habe die Lösung so erweitert, dass sie immer dann um eine halbe Umdrehung zurückgesetzt wird, wenn der Vorzeichenwechsel stattfindet. Dadurch kompensieren sich beide Effekte und wir erhalten eine wunderschöne geschlossene Schnittkurve für genau eine Umdrehung! Schauen wir uns nochmal die problematische Schnittkurve von vorhin an, wenn wir die reparierte Lösung einsetzen ($b = e$, $m = 3/2$).

What was the trick? I extended the solution in such a way that it will be set back by a half rotation when a sign change is occuring. Hence both effects are compensating each other and we get a wonderful closed intersection curve for exact one rotation!
Let's have one more look at the problematic intersection curve from above applying the adapted solution ($b = e$, $m = 3/2$).



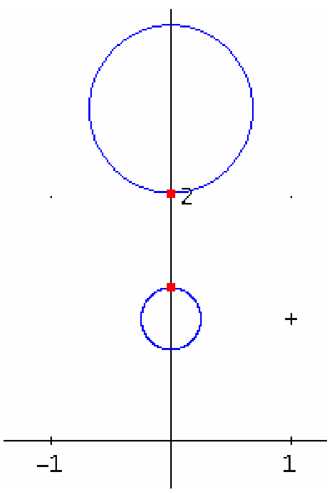
Nachdem es uns endlich gelungen ist die Schnittkurve des Schneckenhauses mit der XY-Ebene pro Umdrehung zu berechnen, können wir uns daran machen das Selbstdurchdringungsproblem zu lösen. Dazu müssen wir den Parameter „*m*“ so bestimmen dass sich die Schnittkurven von zwei aufeinander folgenden Umdrehungen exakt berühren ohne sich zu schneiden. Schauen wir uns die Schnitte für $k = 0$ und $k = 1$ an.

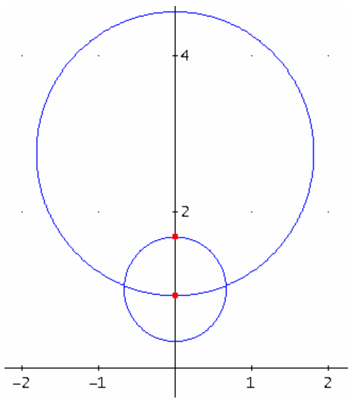
Substituiere für „*t*“ aus #33 (für $k = 0$ und dann für $k = 1$) in SpiralTorusYZ_U und setze dann die entsprechenden Wertepaare für „*b*“ und „*m*“ ein. (Substituiere wie in #24 um #34 und #35 zu erhalten – hier aus Platzgründen nicht angegeben.) Das ergibt die folgenden Schnitte.

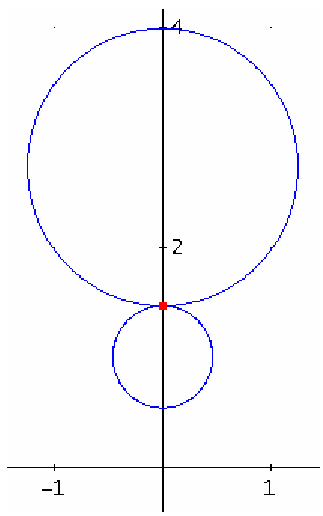
Having succeeded calculating the intersection curve of the shell with the XY-plane per rotation we can start solving the interpenetration problem. We have to determine the parameter “*m*“ in such a way that the intersection curves of two consecutive rotations are osculating without intersecting. Let’s inspect the intersections for $k = 0$ and $k = 1$.

Substitute for “*t*” from #33 (with $k = 0$ and then with $k = 1$) in SpiralTorusYZ_U and then substitute again the respective values for *b* and *m*. (Substitute like in #24 to obtain #34 and #35, which are both omitted here to save space.) This gives the following intersections:

$b = e,\ m = ¼$ | $b = e,\ m = 3/2$ | $b = e,\ m = 0.4593761774$







Wie man sieht durchdringen sich die beiden Schnitte entlang der Y-Achse auf gegenüber liegenden Seiten der Schnittkurven. Die X-Koordinate muss also 0 sein. Wir lösen nach „*s*“.

As you can see the both intersections interpenetrate along the Y-axis at opposite sides of the curves. Hence, the X-coordinate must become 0. We solve for "*s*".

#37:
$$\text{SOLUTIONS}\left(m\cdot b^{\left(\text{SIGN}\left(\sqrt{\text{LN}(b)^2+4\cdot\pi^2}+2\cdot\pi\cdot m\cdot\text{SIN}(s)\right)-1\right)/4+\text{ATAN}\left(m\cdot\text{LN}(b)\cdot\text{SIN}(s)/\left(\sqrt{\text{LN}(b)^2+4\cdot\pi^2}+2\cdot\pi\cdot m\cdot\text{SIN}(s)\right)\right)/(2\cdot\pi)}\cdot\text{COS}(s)=0,\ s\right)=\left[\frac{\pi}{2},\ -\frac{\pi}{2},\ \frac{3\cdot\pi}{2}\right]$$

#38:
$$\text{SOLUTIONS}\left(m\cdot b^{\left(\text{SIGN}\left(\sqrt{\text{LN}(b)^2+4\cdot\pi^2}+2\cdot\pi\cdot m\cdot\text{SIN}(s)\right)+3\right)/4+\text{ATAN}\left(m\cdot\text{LN}(b)\cdot\text{SIN}(s)/\left(\sqrt{\text{LN}(b)^2+4\cdot\pi^2}+2\cdot\pi\cdot m\cdot\text{SIN}(s)\right)\right)/(2\cdot\pi)}\cdot\text{COS}(s)=0,\ s\right)=\left[\frac{\pi}{2},\ -\frac{\pi}{2},\ \frac{3\cdot\pi}{2}\right]$$

Die Lösungen für *s* sind: / The solutions for *s* are:

$$k=0\wedge s=\frac{\pi}{2}\qquad\text{and}\qquad k=1\wedge s=-\frac{\pi}{2}$$

Die beiden Lösungen werden eingesetzt (in #35 und #36).

Both solutions are substituted (in #35 and #36).

#39:
$$\left[0,\ b^{\text{ATAN}\left(m\cdot\text{LN}(b)/\left(\sqrt{\text{LN}(b)^2+4\cdot\pi^2}+2\cdot\pi\cdot m\right)\right)/(2\cdot\pi)}\cdot\frac{\sqrt{(m^2+1)\cdot\sqrt{\text{LN}(b)^2+4\cdot\pi^2}+4\cdot\pi\cdot m}}{\left(\text{LN}(b)^2+4\cdot\pi^2\right)^{1/4}}\right]$$

#40:
$$\left[0,\ b^{\left(3+\text{SIGN}\left(\sqrt{\text{LN}(b)^2+4\cdot\pi^2}-2\cdot\pi\cdot m\right)\right)/4-\text{ATAN}\left(m\cdot\text{LN}(b)/\left(\sqrt{\text{LN}(b)^2+4\cdot\pi^2}-2\cdot\pi\cdot m\right)\right)/(2\cdot\pi)}\cdot\frac{\sqrt{(m^2+1)\cdot\sqrt{\text{LN}(b)^2+4\cdot\pi^2}-4\cdot\pi\cdot m}}{\left(\text{LN}(b)^2+4\cdot\pi^2\right)^{1/4}}\right]$$

Wenn sich die Mantelfläche nur berühren, aber nicht durchdringen soll, müssen beide Y-Koordinaten gleich sein. Folgende Gleichung muss also nach „*m*" gelöst werden:

zweite Komponenten von #39 = zweite. Komponente von #40.

If the surface shall only osculate (and not interpenetrate) then both Y-coordinates must be equal. So we have to solve the following equation for "m":

second component of #39 = second component of #40.

#41:
$$b^{\left(3+\text{SIGN}\left(\sqrt{\text{LN}(b)^2+4\cdot\pi^2}-2\cdot\pi\cdot m\right)\right)/4-\text{ATAN}\left(m\cdot\text{LN}(b)/\left(\sqrt{\text{LN}(b)^2+4\cdot\pi^2}-2\cdot\pi\cdot m\right)\right)/(2\cdot\pi)}\cdot\frac{\sqrt{(m^2+1)\cdot\sqrt{\text{LN}(b)^2+4\cdot\pi^2}-4\cdot\pi\cdot m}}{\left(\text{LN}(b)^2+4\cdot\pi^2\right)^{1/4}}=b^{\text{ATAN}\left(m\cdot\text{LN}(b)/\left(\sqrt{\text{LN}(b)^2+4\cdot\pi^2}+2\cdot\pi\cdot m\right)\right)/(2\cdot\pi)}\cdot\frac{\sqrt{(m^2+1)\cdot\sqrt{\text{LN}(b)^2+4\cdot\pi^2}+4\cdot\pi\cdot m}}{\left(\text{LN}(b)^2+4\cdot\pi^2\right)^{1/4}}$$

Nebenbei kann man aus dieser Gleichung ablesen, welcher Umdrehungszahl eine Windung "*w*" entspricht, also der Umdrehungszahl nach der sich die Mantelfläche selber berührt.

Additionally one can read off from this equation the rotation number which is corresponding to one coil "w" i.e. the rotation number when the surface is osculating itself.

#42:
$$w = \frac{3 + \mathrm{SIGN}(\sqrt{\mathrm{LN}(b)^2 + 4\cdot\pi^2} - 2\cdot\pi\cdot m)}{4} - \frac{\mathrm{ATAN}\left(\frac{m\cdot\mathrm{LN}(b)}{\sqrt{\mathrm{LN}(b)^2 + 4\cdot\pi^2} - 2\cdot\pi\cdot m}\right) + \mathrm{ATAN}\left(\frac{m\cdot\mathrm{LN}(b)}{\sqrt{\mathrm{LN}(b)^2 + 4\cdot\pi^2} + 2\cdot\pi\cdot m}\right)}{2\cdot\pi}$$

Damit erhalten wir eine 3. Art von Schneckenhaus, das nach der Anzahl der Windungen parametrisiert ist.

Doing so we obtain another kind of snail shell which is parameterized according to the number of coils.

#43:
$$\mathrm{SpiralTorusYZ\_W}(b, m, t, s) := \mathrm{SpiralTorusYZ\_U}\left(b, m, t\cdot\left(\frac{3 + \mathrm{SIGN}(\sqrt{\mathrm{LN}(b)^2 + 4\cdot\pi^2} - 2\cdot\pi\cdot m)}{4} - \frac{\mathrm{ATAN}\left(\frac{m\cdot\mathrm{LN}(b)}{\sqrt{\mathrm{LN}(b)^2 + 4\cdot\pi^2} - 2\cdot\pi\cdot m}\right) + \mathrm{ATAN}\left(\frac{m\cdot\mathrm{LN}(b)}{\sqrt{\mathrm{LN}(b)^2 + 4\cdot\pi^2} + 2\cdot\pi\cdot m}\right)}{2\cdot\pi}\right), s\right)$$

Die Gleichung (#41) kann vereinfacht werden

Equation (#41) can be simplified.

#44:
$$b^{(3 + \mathrm{SIGN}(\sqrt{\mathrm{LN}(b)^2 + 4\cdot\pi^2} - 2\cdot\pi\cdot m))/2 - \mathrm{ATAN}(m\cdot\mathrm{LN}(b)/(\sqrt{\mathrm{LN}(b)^2 + 4\cdot\pi^2} - 2\cdot\pi\cdot m))/\pi}\cdot((m^2 + 1)\cdot\sqrt{\mathrm{LN}(b)^2 + 4\cdot\pi^2} - 4\cdot\pi\cdot m) = b^{\mathrm{ATAN}(m\cdot\mathrm{LN}(b)/(\sqrt{\mathrm{LN}(b)^2 + 4\cdot\pi^2} + 2\cdot\pi\cdot m))/\pi}\cdot((m^2 + 1)\cdot\sqrt{\mathrm{LN}(b)^2 + 4\cdot\pi^2} + 4\cdot\pi\cdot m)$$

Die Gleichung ist leider höchst nichtlinear und läßt sich nicht analytisch nach „*m*“ lösen, deshalb müssen wir das Newton-Verfahren anwenden.

This equation is not linear at all and cannot be solved for “m“ analytically. So we have to apply the Newton-Raphson method.

#45:
$$f(b, m) := b^{(3 + \mathrm{SIGN}(\sqrt{\mathrm{LN}(b)^2 + 4\cdot\pi^2} - 2\cdot\pi\cdot m))/2 - \mathrm{ATAN}(m\cdot\mathrm{LN}(b)/(\sqrt{\mathrm{LN}(b)^2 + 4\cdot\pi^2} - 2\cdot\pi\cdot m))/\pi}\cdot((m^2 + 1)\cdot\sqrt{\mathrm{LN}(b)^2 + 4\cdot\pi^2} - 4\cdot\pi\cdot m) - b^{\mathrm{ATAN}(m\cdot\mathrm{LN}(b)/(\sqrt{\mathrm{LN}(b)^2 + 4\cdot\pi^2} + 2\cdot\pi\cdot m))/\pi}\cdot((m^2 + 1)\cdot\sqrt{\mathrm{LN}(b)^2 + 4\cdot\pi^2} + 4\cdot\pi\cdot m)$$

Gesucht ist ein Wert für „*m*“ für den gilt f(*b*,*m*) = 0. Für das Newton-Verfahren wird die 1. Ableitung von f(*b*,*m*) nach „*m*“ benötigt. g(*b*,*m*) ist die 1. Ableitung von f(*b*,*m*).

We would like to find a value for "*m*" with f(*b*,*m*) = 0. For the Newton-algorithm we need the first derivative of f(*b*,*m*) wrt "*m*". g(*b*,*m*) is the first derivative of f(*b*,*m*).

$$\#46: \quad g(b, m) := \frac{d}{dm} f(b, m)$$

Leider ist *DERIVE* nicht selbstständig in der Lage die 1. Ableitung von f(*b*,*m*) nach „*m*“ zu bestimmen. Jeder Versuch führt zu einer Fehlermeldung wegen zu wenig Speicher.

Wenn wir aber *DERIVE* exakt vorgeben wie es die 1. Ableitung bestimmen soll, ist die Software doch dazu in der Lage. Dazu zerlegen wir f(*b*,*m*) in 4 Teilfunktionen …

Unfortunately *DERIVE* is unable to find this first derivative. Each attempt leads to an error message: Memory Exhausted.

If we give some hints how to find the first derivative then the software is able to differentiate f(*b*,*m*). We decompose f(*b*,*m*) into 4 partial functions …

(I omit printing this – from the didactical point of view – interesting part of Piotr´s file in order to save space. I invite you to follow his steps in expressions #47 - #55 of his dfw-file. #55 is the final result for g(*m*,*b*). See also my comment at the end of this article, Josef)

$$\#55: \quad g(b, m) := b^{(3 + SIGN(\sqrt{LN(b)^2 + 4\cdot\pi^2} - 2\cdot\pi\cdot m))/2 - ATAN(m\cdot LN(b)/(\sqrt{LN(b)^2 + 4\cdot\pi^2} - 2\cdot\pi\cdot m))/\pi} \cdot \left(2\cdot m\cdot\sqrt{LN(b)^2 + 4\cdot\pi^2} - \frac{LN(b)^2}{\pi} - 4\cdot\pi\right) - b^{ATAN(m\cdot LN(b)/(\sqrt{LN(b)^2 + 4\cdot\pi^2} + 2\cdot\pi\cdot m))/\pi} \cdot \left(2\cdot m\cdot\sqrt{LN(b)^2 + 4\cdot\pi^2} + \frac{LN(b)^2}{\pi} + 4\cdot\pi\right)$$

Den Newton-Algorithmus kann man als kleines *DERIVE*-Programm realisieren.

The Newton-Raphson algorithm is realised in form of a short *DERIVE* program.

```
       h(b, m, n) :=
         Prog
           δm :∈ Real
#56:       WRITE(APPEND("m := ", APPROX(STRING(m), n)))
           δm := m - APPROX(f(b, m), 2·n)/APPROX(g(b, m), 2·n)
           RETURN APPROX(δm, n)

       M_YZ(b, n) :=
         Prog
#57:       V :∈ Vector
           V := ITERATES(h(b, m, n), m, 0)
           RETURN V↓DIM(V)
```

M_YZ(b, n) berechnet das passende "*m*" zur Basis "*b*" auf "*n*" Stellen genau. Und da wir auch die Windungszahl berechnet haben, können wir nun das Schneckenhaus auch nach der Windungszahl parametrisieren. Das findet sich in SpiralFlaecheYZ_W.

M_YZ(b, n) calculates the matching "m" fort he base "b" accurate to "n" decimal places. And as we have calculated the coil number, too, we are able to parameterize the snail house according to this number, giving SpiralFlaecheYZ_W.

Now we can find the answer for my question on page 23:

"How to obtain this very special m?"

$$\#63:\quad \text{M\_YZ}(e,\ 10) = \frac{2071448}{4509263}$$

$$\#64:\quad \text{M\_YZ}(e,\ 10) = 0.4593761774$$

Here I came across a strange - bug? Preparing the publication of this article I reproduced all calculation steps and I failed simplifying (and approximating as well) expression #63. I asked Piotr and he answered that he is aware of this fact, but there will occur nor problems in the respective mth-file. So I stored his dfw-file as an mth-file - loosing all graphs and comments – and surprisingly enough, now it worked – as you can see above.

What happened here? I switched back to the dfw-file and opened the function g(m,b) via Author > Function Definition. I saw the function for a moment and then there was in place of the function the message "Memory Full" in the Edit Window!! I tried calculating the first derivative of f(m,b) – no success, Memory Exhausted.

When I did the same in the mth-file, the same message appeared in the Function Definition Window, but when simplifying M_YZ which is based on f(m,b) and the predefined g(m,b) DERIVE continues cooperating with the user … These are expressions #63 and #64 above. Don't ask my why. Hopefully Albert Rich is reading this remark and maybe that he will provide some explanation for this strange behaviour.

If one of our members knows the reason or if he has some conjectures then please let us know, Josef (and Piotr, of course)

I continue with Piotr's closing lines.

Ich bevorzuge es immer auf $n + 2$ Stellen genau zu rechnen. Eine Extra Vor-Komma-Stelle und eine Extra Nach-Komma-Stelle für den Rundungsfehler

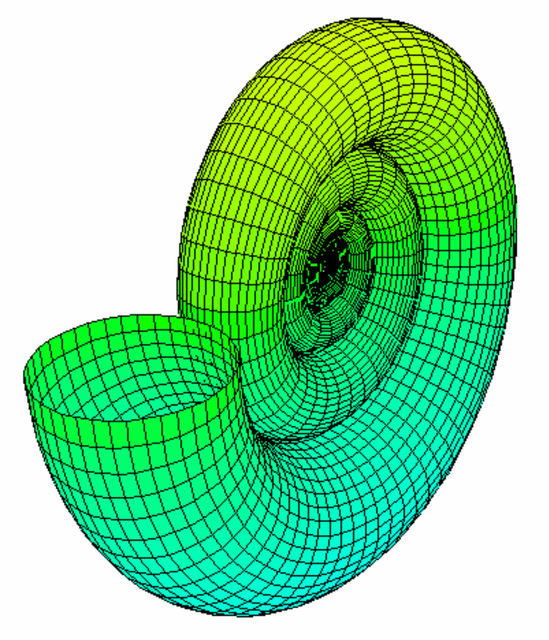
I prefer calculating for $n + 2$ decimal places. One extra digit in front of the decimal point and one extra digit at the end of the number to consider a possible rounding error.

```
#58:  SpiralFlaecheYZ_R(b, n, t, s) := SpiralTorusYZ_R(b, M_YZ(b, n), t, s)

#59:  SpiralFlaecheYZ_U(b, n, t, s) := SpiralTorusYZ_U(b, M_YZ(b, n), t, s)

#60:  SpiralFlaecheYZ_W(b, n, t, s) := SpiralTorusYZ_W(b, M_YZ(b, n), t, s)
```

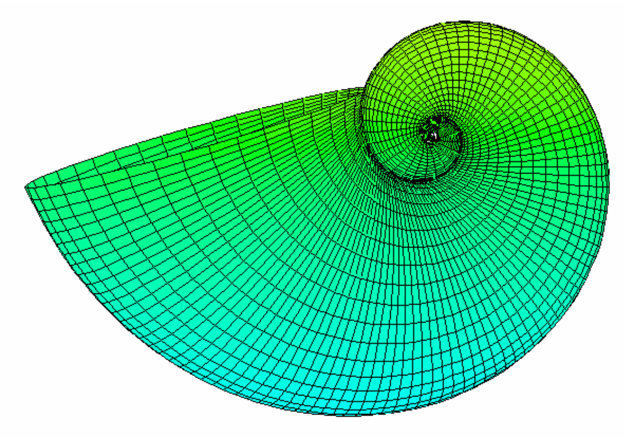
Wir schließen mit ein paar hübschen Schneckenhäuser für die Basen 2 und 6.

We close presenting some nice snail shells for bases 2 and 6.

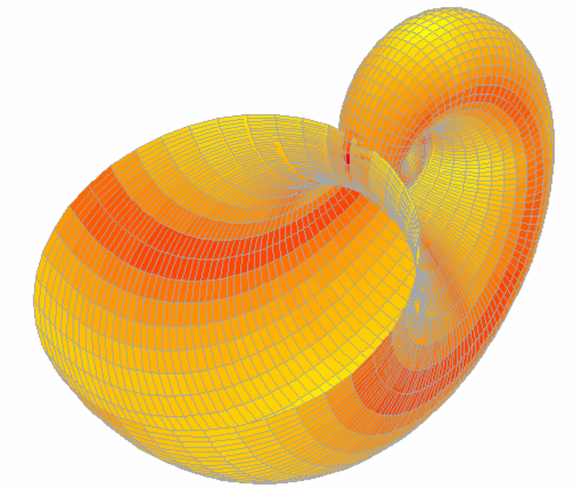
Base 2



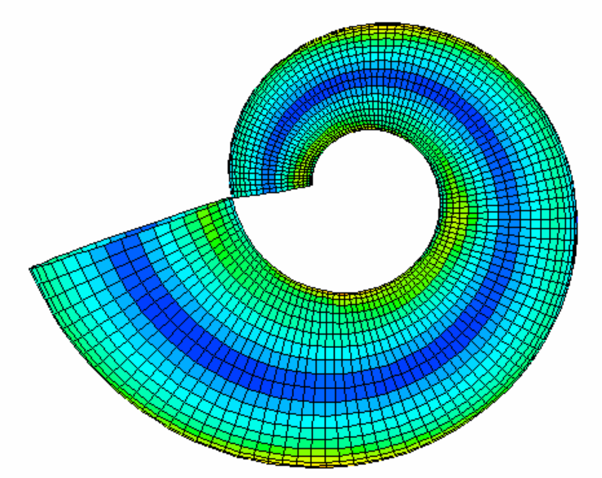
Base 6



Base 6



Base *e* with exactly one coil



In Snail_shells.dfw finden sich alle Details. Die Zusammenfassung aller wichtigen Funktionen für das Plotten der Schneckenhäuser und für weiteres Experimentieren bildet die mth-Datei Logarithmic_Spiral_Surfaces.mth.

The respective dfw-file is called Snail_shells.dfw. All important functions for plotting the shells and experimenting with various parameters are collected in an mth-file, called Logarithmic_Spiral_Surfaces.mth.

If there are any serious mistakes in the English translation – I am quite sure that there are some – then please don´t blame Piotr Trebisz. I received the German version and tried to produce the English translation.

Josef

As you can see in the following screen shots, MuPad and *MATHEMATICA* can both derive function f(b,m). The derivative is much bulkier than Piotr's result. I plotted f(*e*,*m*) and g(*e*,*m*) in MuPad and compared with the DERIVE plot. It looks pretty the same, so the results should be identical?

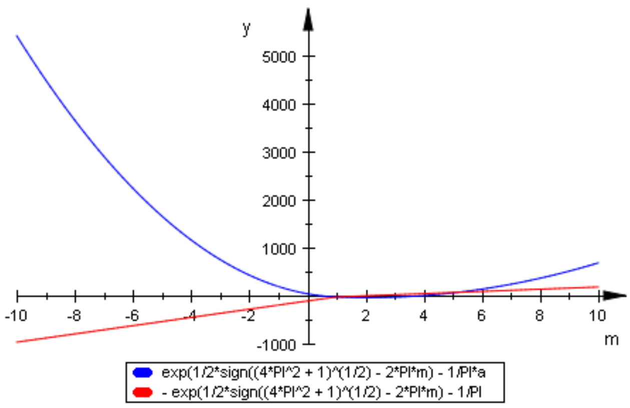
```
f(b,m):=b^((3 + sign(sqrt(ln(b)^2 + 4*PI^2) - 2*PI*m))/2 - arctan(m*ln(b)/(sqrt
-b^(arctan(m*ln(b)/(sqrt(ln(b)^2 + 4*PI^2) + 2*PI*m))/PI)*((m^2 + 1)*sqrt(ln(b)
```

$$b^{\frac{\operatorname{sign}\left(\sqrt{\ln(b)^2+4\cdot\pi^2}-2\cdot\pi\cdot m\right)}{2}-\frac{\arctan\left(\frac{m\cdot\ln(b)}{\sqrt{\ln(b)^2+4\cdot\pi^2}-2\cdot\pi\cdot m}\right)}{\pi}+\frac{3}{2}}\cdot\left(\sqrt{\ln(b)^2+4\cdot\pi^2}\cdot(m^2+1)-4\cdot\pi\cdot m\right)-b^{\frac{\arctan\left(\frac{\ldots}{\sqrt{\ln\ldots}}\right)}{\ldots}}$$

```
g(b,m):=diff(f(b,m),m)
```

$$-b^{\frac{\operatorname{sign}\left(\sqrt{\ln(b)^2+4\cdot\pi^2}-2\cdot\pi\cdot m\right)}{2}-\frac{\arctan\left(\frac{m\cdot\ln(b)}{\sqrt{\ln(b)^2+4\cdot\pi^2}-2\cdot\pi\cdot m}\right)}{\pi}+\frac{3}{2}}\cdot\left(4\cdot\pi-2\cdot m\cdot\sqrt{\ln(b)^2+4\cdot\pi^2}\right)-b^{\frac{\arctan\left(\frac{m\cdot\ldots}{\sqrt{\ln(b)^2+\ldots}}\right)}{\pi}}$$

```
plotfunc2d(ff(exp(1),m),gg(exp(1),m),m=-10..+10)
```



MuPad above and *MATHEMATICA* below

In[3]:= **D[f[b, m], m]**

Out[3]=

$$-\left(2\sqrt{\log^2(b)+4\pi^2}\,m+4\pi\right)b^{\frac{\tan^{-1}\left(\frac{m\log(b)}{2\pi m+\sqrt{\log^2(b)+4\pi^2}}\right)}{\pi}}-$$

$$\frac{\log(b)\left(4\pi m+(m^2+1)\sqrt{\log^2(b)+4\pi^2}\right)\left(\frac{\log(b)}{2\pi m+\sqrt{\log^2(b)+4\pi^2}}-\frac{2m\pi\log(b)}{\left(2\pi m+\sqrt{\log^2(b)+4\pi^2}\right)^2}\right)b^{\frac{\tan^{-1}\left(\frac{m\log(b)}{2\pi m+\sqrt{\log^2(b)+4\pi^2}}\right)}{\pi}}}{\pi\left(\frac{m^2\log^2(b)}{\left(2\pi m+\sqrt{\log^2(b)+4\pi^2}\right)^2}+1\right)}+$$

$$\left(2m\sqrt{\log^2(b)+4\pi^2}-4\pi\right)b^{\frac{1}{2}\left(\operatorname{sgn}\left(\sqrt{\log^2(b)+4\pi^2}-2\pi\cdot m\right)+3\right)-\frac{\tan^{-1}\left(\frac{m\,\mathrm{LN}(b)}{\sqrt{\log^2(b)+4\pi^2}-2m\pi}\right)}{\pi}}+$$

$$\log(b)\left((m^2+1)\sqrt{\log^2(b)+4\pi^2}-4m\pi\right)\left(-\frac{\frac{\mathrm{LN}(b)}{\sqrt{\log^2(b)+4\pi^2}-2m\pi}+\frac{2m\pi\,\mathrm{LN}(b)}{\left(\sqrt{\log^2(b)+4\pi^2}-2m\pi\right)^2}}{\pi\left(\frac{m^2\,\mathrm{LN}(b)^2}{\left(\sqrt{\log^2(b)+4\pi^2}-2m\pi\right)^2}+1\right)}-\operatorname{sgn}'\left(\sqrt{\log^2(b)+4\pi^2}-2\pi\cdot m\right)\mathrm{CenterDot}^{(0,1)}(\pi,m)\right)$$

$$b^{\frac{1}{2}\left(\operatorname{sgn}\left(\sqrt{\log^2(b)+4\pi^2}-2\pi\cdot m\right)+3\right)-\frac{\tan^{-1}\left(\frac{m\,\mathrm{LN}(b)}{\sqrt{\log^2(b)+4\pi^2}-2m\pi}\right)}{\pi}}$$

Exchange of mails connected with Duncan McDougal´s "Diophantine Polynomials"

Mail from Duncan McDougal as answer on my request concerning the table in (12) on page 13. Duncan had 740/60 as sum of the roots in the last row …
In an earlier mail he had asked: "Just out of curiosity (if I may ask), what are your intentions with this article?"

This was my answer:

I have two intentions:

(1) Finding and exploring patterns is very important in mathematics education and in mathematics in general. Your paper is an excellent example for discovering – and using – patterns.

(2) The second reason is given in my final comments (page 21).

Hi Josef,

You are correct, the sum is 12 and should remain 12. However, the pattern becomes 520, 630, and 740 as an arithmetic sequence and 740/60 reduces to 37/3, a wee bit more than 12. I am still secretly hoping that someone sees a pattern I don't see and will come up with a brilliant discovery on the Quintic. Thus, I am suggesting that although 12 is the probable answer, maybe 37/3 is what it could be??? I acknowledge that I've proven that there is no monic quintic for which the first derivative contains integral and/or rational roots, but the pattern is hard to resist or ignore.

I am still very much interested in this topic and will entertain questions or comments from any interested party. I've only just begun the process for non-monic polynomials and already the quartics look daunting.

Anyway, I am flattered that you like my work enough to use it because that is what it was meant for.

I remain
Yours truly, Duncan McDougall

Danny Ross Lunsford (22 01 2011)

Hello Derivers, the routines provided in the LinearAlgebra.mth file leave something to be desired in practice - has anyone resolutely attacked the problem of diagonalization and determination of eigenspaces of large matrices? For my application large = 8x8. I need a robust and fast diagonalization scheme. (DNL#80).

Hi Derivers, (19 02 2011)

In implementing the Johnson Trotter algorithm one comes to a point where the direction of a mobile integer must be changed. My working area is an (N+2) x 2 matrix with the working permutations in the 1st column and the directions in the 2nd (directions are + or - 1). Eventually one has to change directions for all integers larger than the largest mobile integer - so I have a statement to run over the array

```
MAP(IF(P SUB i SUB 1 > Largest, P SUB i SUB 2 := -P SUB i SUB 2), i, [2,..,N+1])
```

However the sub-array assignment never takes place, even though the statement is executed. What exactly is going on? I know sub-array assignments must be simplified, but that should be happening here. Thanks in advance!
-drl

Hi Derivers, (19 02 2011)

OK see attached - I changed the map statement a bit but still no joy. The SHAPE function is one I made that creates arrays, don't worry about it, the size of the array created is (N+2) x 2. The output is included and you can see that the sign changes are never executed.

(20 02 2011)

It doesn't work - see attached - a function to generate all permutations of 1 to N by accretion was given in Derive Newsletter 41. It creates two nested loops, the inner one with k going from m to 1 in steps of -1, the outer with m going from 2 to N in steps of 1. All the variables are local to the PERMS function. We can make the inner loop into a MAP statement but the syntactically identical function works when the accretion variable t is global, but fails when you make it local to the function - I'm completely baffled

```
#1:   Generate all permutations of N integers by accretion - from DNL41

      PERMS(N, k, m := 2, s := [[]], t) :=
        Loop
          If m > N
             RETURN SORT(s)
             k := m
             t := []
#2:          Loop
               If k = 0 exit
               t := APPEND(t, VECTOR(INSERT(m, v, k), v, s))
               k := -1
             s := t
             m :+ 1

      TPERMS(N, k, m := 2, s := [[1]]) :=
        Loop
          If m > N
             RETURN SORT(s)
#3:       t := []
          MAP(t := APPEND(t, VECTOR(INSERT(m, v, k), v, s)), k, m, 1, -1)
          s := t
          m :+ 1

#4:   t := []

#5:   We've replaced the inner loop over decreasing k by a MAP statement

#6:   t is a global variable which we set to empty
```

#7: TPERMS(3) =

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \\ 2 & 1 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \\ 3 & 2 & 1 \end{bmatrix}$$

```
#8:   It works - now make t a local variable to TPERMS

      TPERMS_(N, k, m := 2, s := [[1]], t) :=
        Loop
          If m > N
             RETURN SORT(s)
#9:       t := []
          MAP(t := APPEND(t, VECTOR(INSERT(m, v, k), v, s)), k, m, 1, -1)
          s := t
          m :+ 1

#10:  TPERMS_(3) = [[3, 2, 1]]

#11:  Doesn't work any more - why??
```

Danny Ross promised an article on Clifford Algebra and Dirac matrices some time ago. I asked him whether his request is in connection with this intended paper for the DNL and he answered.

20 02 2011

Yes, will work out solution to hydrogen fine structure as an example application. See attached for teaser on the algebraic environment. Derive makes it very easy to keep track of the rather complex algebraic operations in the Clifford algebra of Dirac matrices.

On page 48 in DNL#80 I published the email communication with Albert Rich. Unfortunately I did not include the respective website where you can find many details about his fascinating current research. Albert wrote:

Hello Josef,
Yes, feel free to include my 2 January 2011 email in the next DNL. Also please include the website address

http://www.apmaths.uwo.ca/RuleBasedMathematics

As far as my current research, I recommend the following ongoing discussion on the Usenet group sci.math.symbolic

http://groups.google.com/group/sci.math.symbolic/browse_thread/thread/90b4f2e39ee5ad9d?hl=en

in which I am collaborating with Martin to develop powerful new rules for integrating trigonometric expressions. In particular for your next DNL, you might find of interest my last post in which I outline how recurrence equations are transformed into integration rules.

Martin is certainly one of the world's most active and knowledgeable Derive users. Apparently he lives in Germany and is a member of DUG. Do you know his last name?

Aloha,
Albert

Does anybody know, who "Martin" is. I found an email-address which could have been Martin's one: (clicliclic@freenet.de). It would be great if we could find out who is the „secret" DERIVE user? (The email address is not valid any more!)

David Sjöstrand from Sweden wrote:

Dear Josef,
after only having had a short glance at your and Carl´s paper in DNL #80 I couldn´t resist investigating some more sequences with Excel. (I shouldn´t have done this. I have so many things I must do.)
It seems that the sequence P(n)=m*P(n-3) + (1-m)*P(n-2) is convergent for all m and all initial conditions. I attach josef_carl.xls.

David sent a proof for DNL#80, page 22 in his next mail and we started an exciting exchange of mails. The results of this correspondence will form an article in DNL#82, Josef

Notes by R.N. (Roger) Folsom
Monterey, California, USA
rnfolsom@redshift.com.

```
#1:   CaseMode := Sensitive
```

As you will see below, I am trying to formally prove the circumstances under which a particular inequality is true --- and Derive's Simplify or Solve command actually says that it is true. (For now, all I need is sufficient conditions.)

But I have not been able to figure out the Derive6.1 command or commands that will do that.

My sources are Bernhard Kutzler and Vlasta Kolol-Voljc's "Introduction to Derive 6, Advanced Mathematics for Your PC" (hereafter Derive6); and Albert and Joan Rich, Theresa Shelby, and David Stoutemeyer's "Derive User Manual Version 6" (hereafter Derive3).

Derive6's equation solver (page 57), when used to determine whether a relationship is True or False, doesn't apply to what I am trying to do because it doesn't allow conditions to be imposed on the equation's (or, in my case, the inequality's) variables.

Derive6's discussion of the IF tool (on pages 166 and 227) didn't help because I didn't understand page 166 (I don't know any Analytic Geometry), and although I think that I did understand page 227 I couldn't figure out how to apply it to my problem.

Derive3's discussion of the IF tool (pages 309-310), made sense, but again I couldn't figure out how to apply it to my problem.

My understanding is that the IF tool has four fields: a test condition, a "then" clause of results if the test condition is satisfied, an "else" clause of results if the test condition is NOT satisfied, and an "unknown" clause.

Derive3's opening IF example (page 309) is a payroll problem: IF(h <= 40, 10h, 400 + 15(h-40)) where h is hours worked and 10h and 15(h-40) are worker payment amounts. Unfortunately, the problem does not include the unknown clause, which if not included but is needed turns out to be the entire IF(...) tool. An example that does include an unknown clause would have been helpful.

What I want is a tool, or set of tools, that let me state restrictions on my inequality's variables, then state the inequality itself, and returns either "true" or "false", or ? (or the same object) if the restrictions are too weak to determine whether the inequality is true or false.

The r↓n variables below are annual (or some other time period) rates of growth, such as interest rates, inflation and deflation rates, real output growth or decline rates, etc. Rates of growth r↓n can be either positive, zero, or negative.

The number N is the number of time periods being considered. Sometimes "total" rates of growth over N time periods are calculated by summing the N periodic rates of change r↓n (n = 1,2,...,N), which is less accurate than multiplying the N corresponding (1+r↓n) factors. Summing the r↓n can give results that are either larger or smaller than the more accurate multiplication of the (1 + r↓n).

However, a common situation is that over N time periods, more rates of growth r_n are positive than negative. If that difference is sufficiently large, summing the N periodic rates of change r_n gives a lower total rate of growth over N periods than does multiplyng the corresponding $(1 + r_n)$ numbers, so that summation understates the true total rate of change over N periods.

Mathematically, that "common situation" inequality can be written in three basic ways. See statements #2, #3, and #4, below.

SATISFYING THE "COMMON SITUATION" INEQUALITY; A SUFFICIENT CONDITION: All $r_n \geq 0$

If ALL of the r_n are positive or zero ($r_n \geq 0$), the "common situation" inequality holds, because the left side's terms are smaller than the right side's factors. For an example, let N = 3, and see statements #5, #6, #7, and #8. Note that in #7 --- a rearrangement of #2 with N = 3 --- the left side is -1, and ALL of the variables on the right side are positive.
Therefore, a SUFFICIENT condition for the "usual condition" inequality is simply that all all $r_n >= 0$.

$$\#2: \quad \sum_{n=1}^{N} r_n < \prod_{n=1}^{N} (1 + r_n)$$

$$\#3: \quad 0 < \left(\prod_{n=1}^{N} (1 + r_n)\right) - \sum_{n=1}^{N} r_n$$

$$\#4: \quad -1 < \left(\prod_{n=1}^{N} (1 + r_n)\right) - 1 - \sum_{n=1}^{N} r_n$$

For example, if N = 3:

$$\#5: \quad -1 < \left(\prod_{n=1}^{3} (1 + r_n)\right) - 1 - \sum_{n=1}^{3} r_n$$

Statement #5 can be written as

$$\#6: \quad -1 < (r_1 + 1)\cdot(r_2 + 1)\cdot(r_3 + 1) - 1 - (r_3 + r_2 + r_1)$$

In statement #6, multiplying out the $(r_n + 1)$ factors would eliminate the $-1 - (r_3 + r_2 + r_1)$ expression. Doing that (by using Derive's Simplify-Factor command) gives statement #7.

$$\#7: \quad -1 < r_3\cdot(r_2\cdot(r_1 + 1) + r_1) + r_1\cdot r_2$$

or

$$\#8: \quad -1 < r_1\cdot r_2\cdot r_3 + r_2\cdot r_3 + r_1\cdot r_3 + r_1\cdot r_2$$

Note: I have done the same calculations above using N = 5.

Given the sufficient condition that all r↓n ≥ 0, statements #7 and #8 are obviously true because the left side is negative, and every term on the right side is positive.
And since statements #7 and #8 are merely a rearrangement of #5, which is an example of statements #4, #3, and #2, statement #7 demonstrates that if all r↓n ≥ 0, the "usual situation" inequality holds, so that summing N periodic rates of change r↓n (instead of multiplying corresponding 1 + r↓n numbers) understates the true total rate of change over N periods.

HOWEVER, I need to learn how to persuade Derive6.1 to do a formal proof (results either true, false, or unknown) of the preceding paragraph's results, because my next step is to experiment with less restrictive conditions, such as all r↓n >= -1.

==

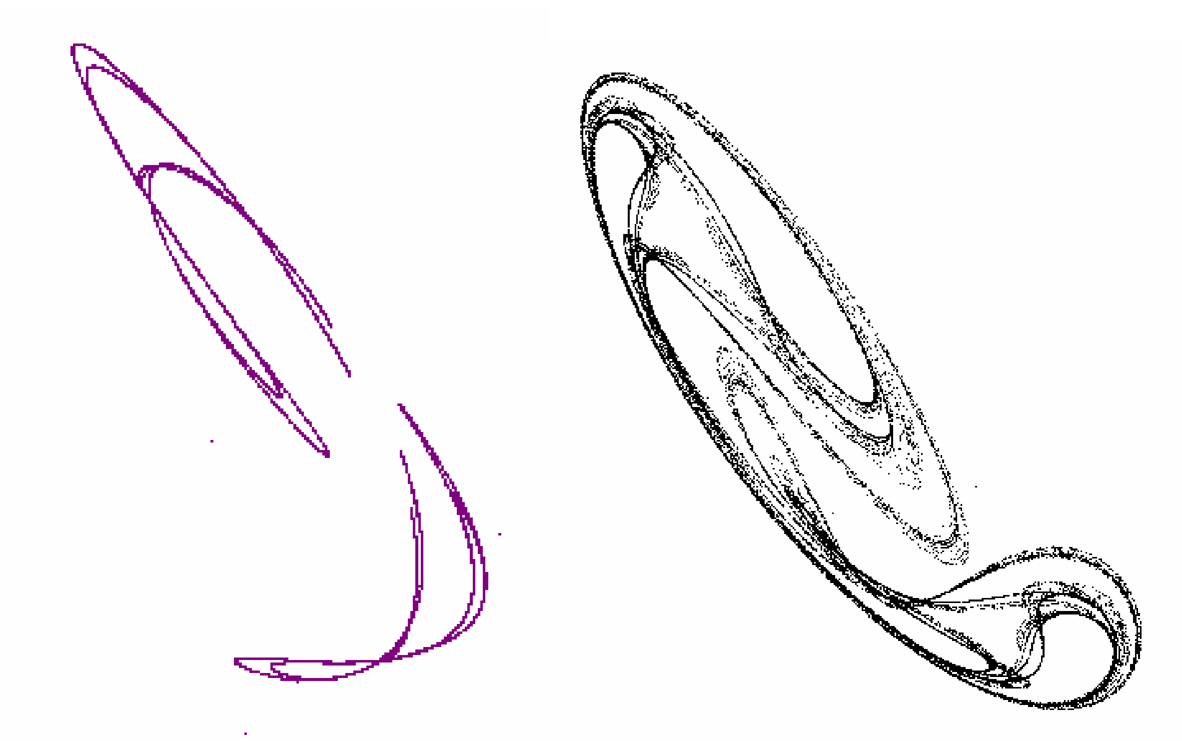
TO DO: Figure out a formal proof --- for any statement #5 through #8 --- that when #10 (below) holds, comes up with "true" rather than the wrong answer of "false."

For example, Simplify-Basic of either IF(#10, #7) or IF(#7, #10) returns the same (written out) IF statement, presumably meaning that Simplify could not determine whether the inequality was true or false, despite the "all r↓n >= 0" variable conditions.

But Solve-Expression of either IF(#10, #7) or IF(#7, #10) concludes that the constrained inequality is false, when I know that it is true (from the analysis that follows #8).

I clearly am not using IF properly. It may not be intended for what I am trying to do. In that case, I need to know what else to do.

These are two Strange Attractors. I am quite sure that I was the first person who saw them on a PC screen. Josef



I received a mail from Robert Setif, a DUG Member from the very first times of its existence:
Dear Josef,
(in bad English)
I am always alive and I spend my time to program often deranged counting with different mathematical softwares (Derive, Mathematica, Maple, Mupad, Xcas, python, pari-gp). Regarding these softwares, do you know others, free preferably?

I send you 3 files which calculate decimals of PI. You can make what you want.of them.I I am unable to ameliorate them on billing(displaying, printing). In fact, the billing of long integers or floats with a lot of decimales by edges(slices) of 3 or 5 or 10 digits.It would be more practical for checking. Could you fix me on this subject?

I should like if there is a function that for instance "time_elapsed(pi_machin(10001)) for calculating the time for that calcul.Thank you very much. I congratulate you for your work.
Best regards.

(in good French)
Je suis toujours en vie et je passe mon temps à programmer des calculs souvent délirants avec différents logiciels mathématiques (Derive, Mathematica, Maple, Mupad, Maxima, Xcas, python, pari-gp). A propos de ces logiciels, en connaissez-vous d'autres, gratuits de préférence?

Je vous envoie 3 fichiers qui calculent des décimales de PI. Vous pouvez en faire ce que vous voulez. Je suis incapable de les améliorer sur l'affichage. En effet, l'affichage de longs entiers ou de flottants avec beaucoup de décimales par tranches de 3 ou 5 ou 10 chiffres serait plus pratique pour la vérification. Pourriez-vous me dépanner à ce sujet?

J'aimerais aussi une fonction comme par exemple "time_elapsed(pi_machin(10001)) qui donnerait le temps nécessaire pour ce calcul. Merci infiniment.
Je vous félicite pour votre travail.Bravo !

These are Robert's files which were attached to his mail :

```
#1:   InputMode := Word

      appin(x, nd) :=
        Prog
          Precision := Approximate
          PrecisionDigits := nd + 5
          NotationDigits := nd
          nit := CEILING(nd/(2·LOG(1/x, 10))) + 2
          DISPLAY(nit)
#2:       t := 1/(2·nit - 1)
          k := 2·nit - 3
          Loop
            t := 1/k - t·x·x
            k := k - 2
            If k < 1 exit
          RETURN t·x
```

#3: $\text{appin}\left(\dfrac{1}{239}, 60\right)$

15

#4: 0.00418407600207472386453821495928545274104806530763195082701961

```
      appi(x, nd) :=
        Prog
          Precision := Approximate
          PrecisionDigits := nd + 5
          NotationDigits := nd
          nit := CEILING(nd/(2·LOG(1/x, 10))) + 2
#5:       t := 1/(2·nit - 1)
          k := 2·nit - 3
          Loop
            t := 1/k - t·x·x
            k := k - 2
            If k < 1 exit
          RETURN t·x
```

#6: $\text{appi}\left(\frac{1}{239},\ 60\right)$

```
#7:   0.00418407600207472386453821495928545274104806530763195082701961
```

#8: $\text{pig(nd)} := 12\cdot\text{appi}\left(\frac{1}{4},\ \text{nd}\right) + 4\cdot\text{appi}\left(\frac{1}{20},\ \text{nd}\right) + 4\cdot\text{appi}\left(\frac{1}{1985},\ \text{nd}\right)$

```
#9:   pig(101)

#10:  3.1415926535897932384626433832795028841971693993751058209749445923078164062862089986 2~
        80348253421170679


#11:  pig(1001)

#12:  3.14159265358979323846264338327950288419716939937510582097494459230781640628620899862~
        80348253421170679821480865132823066470938446095505822317253594081284811174502841027~
        01938521105559644622948954930381964428810975665933446128475648233786783165271201909~
        14564856692346034861045432664821339360726024914127372458700660631558817488152092096~
        28292540917153643678925903600113305305488204665213841469519415116094330572703657595~
        91953092186117381932611793105118548074462379962749567351885752724891227938183011949~
        12983367336244065664308602139494639522473719070217986094370277053921717629317675238~
        46748184676694051320005681271452635608277857713427577896091736371787214684409012249~
        53430146549585371050792279689258923542019956112129021960864034418159813629774771309~
        96051870721134999999837297804995105973173281609631859502445945534690830264252230825~
        33446850352619311881710100031378387528865875332083814206171776691473035982534904287~
        55468731159562863882353787593751957781857780532171226806613001927876611195909216420~
        1989


#13:  pig(2001)
```

The second file was pig.dfw which was pretty the same as pig1.mth above including a comment that Robert had calculated π up to 5000 decimal places and needed approximatey 12 minutes.

I wrote back and asked for some information about this algorithm – and I had some ideas how to answer his request applying string manipulations and the time counter which was presented by René Hugelshofer some time ago. Using the DNL-index I found out that this was in DNL#55.

But read first Robert’s information about this algorithm for calculating π with arbitrary accuracy:

Dear Josef,

Explications about functions appin et appi of my approx of PI.

Of Aide for free software XCAS , aid written by **Renée de GRAEVE** Université J Fourier GRENOBLE FRANCE. This AIDE is enormous. I have not the courage for translate it for you. You may find it on the site

www-fourier.ujf-grenoble.fr/~parisse/giac_fr.html

Bernard Parisse author of XCAS Université J Fourier GRENOBLE FRANCE

Please excuse that shambles of French-English.

Atan(x)=x-x^3/3+x^5/5… PI/4=4 atan(1/5) – atan(1/239)

Pour calculer la somme de n termes de la série on va utiliser la méthode de **Hörner** pour faire le moins possibles de multiplications, on a :

$$\arctan(x)=x(1-x^2(1/3-x^2(1/5-x^2(....-x^2(1/2n-1)))))$$

L'utilisateur doit rentrer la valeur a de x et le nombre n de termes de la série.

```
gregory(a,n):={
local t,k;
t:=1/(2*n-1);
for (k:=2*n-3;k>0;k:=k-2) {
t:=1/k-a^2*t;
}
return (a*t);
};
```

On tape :

```
16*gregory(1/5,18)-4*gregory(1/239,6)
```

On obtient :

```
3.14159265359
```

On tape :

```
evalf(16*gregory(1/5,42)-4*gregory(1/239,20))
```

On obtient avec DIGITS :=60

```
3.14159265358979323846264338327950288419716939937510582097494
```

Soit $R_n(x)$ le reste de la série $\sum_{k=1}^{\infty}(-1)^{k+1}x^{2k-1}/2k-1$: $R_n(x)=\sum_{k=n+1}^{\infty}(-1)^{k+1}x^{2k-1}/2k-1$.

On sait que $|R_n(x)|<|x|^{2n+1}/2n+1$ Pour avoir $|R_n(1/5)|=|R_n(0.2)|<10^{-61}$ il faut que :$2^{2n+1}<(2n+1)10^{2n-60}$ et comme $2^1 0\simeq 10^3$ cela donne si on suppose $2n+1>10$:

$10^{(6n+3)/10}<10^{2n-59}$ soit $593<14n$ soit $n\simeq 42$ On vérifie pour $n=42$ on a $(1/5)^{85}/85<2.56e-62$

On peut aussi écrire si on suppose que $2n+1>10$: $|x|^{2n+1}/2n+1<x^{2n+1}/10<10^{-61}$ donc on va choisir $x^{2n+1}<10^{-60}$ soit $(2n+1)log10(x)<-60$ ou encore $n>((-60)/log10(x)-1)/2$ Pour $x=1/5$ on a $n>42.4202967422$ et comme $2n+1>40$ on peut améliorer la majoration $|x|^{2n+1}/2n+1<x^{2n+1}/40<10^{-61}$ ce qui donne $n>((-60+log10(4))/log10(1/5)-1)/2=41.9896201841$ donc on prend $n=42$ pour $x=1/239$ on a $n>11.66$ donc on prend $n=12$ et on vérifie : $(1/239)^{25}/25=1.38711499837e-61$.

On choisit 62 `Chiffres` pour DIGITS et on tape :

```
evalf(16*gregory(1/5,42)-4*gregory(1/239,12))
```

On obtient :

```
3.1415926535897932384626433832795028841971693993751058209749446
```

On tape :
`evalf(pi`

On obtient :
`3.1415926535897932384626433832795028841971693993751058209749446`

Remarque
Avec cette méthode John Machin calcula 100 décimales de π en 1706.

Then I asked Robert how the result should look like and presented some alternatives. His answer was as follows:

Dear Josef,

It would be like 3.14159 26535 89793 23846 26433

Concerning the calculation time, 3 possibilities:

1. calculate pig(1001) in the background – without showing the result,
2. see only the calculation time as result
3. see the calculation time stored in a global variable

These 3 solutions are very nice, but I can wait patiently... (I was born in 1929 during the crysis!)

Best regards.

Robert

First of all my congratulations to Robert. I really hope that I when I will reach 82 years of life time will also be interested in presenting some 1000 digits of π!!

This is what I found out finally (using René Hugelshofer's time couting trick from DNL#55):

```
#1:   InputMode := Word
```

Global Variables:

```
#2:   [calc_time :=, pi_res :=]

      appi(x, nd) :=
        Prog
          Precision := Approximate
          PrecisionDigits := nd + 5
          NotationDigits := nd
          nit := CEILING(nd/(2·LOG(1/x, 10))) + 2
#3:       t := 1/(2·nit - 1)
          k := 2·nit - 3
          Loop
            t := 1/k - t·x·x
            k := k - 2
            If k < 1 exit
          RETURN t·x
```

#4: $\text{pig(nd)} := 32\cdot\text{appi}\left(\frac{1}{10}, \text{nd}\right) - 4\cdot\text{appi}\left(\frac{1}{239}, \text{nd}\right) - 16\cdot\text{appi}\left(\frac{1}{515}, \text{nd}\right)$

My first attempt which did not satisfy Robert´s expectations:

```
#5:   pig_time(100, 20)
```

#6: $\left[\begin{bmatrix} 3.14159265358979323 8 \\ 46264338327950288419 \\ 71693993751058209749 \\ 44592307816406286208 \\ 99862803482534211706 \\ 7 \end{bmatrix}, 0.05\right]$

Output in columns

```
       pig_t_col(nd, digs, t, t_, time, res, out) :=
         Prog
           t_ := RANDOM(0)
           res := STRING(32·appi(1/10, nd) - 4·appi(1/239, nd) - 16·appi(1/515, nd))
           time := APPROX((RANDOM(0) - t_)/100)
           calc_time := time
           If DIM(res) ≤ digs
              RETURN [res, time]
#7:        out := []
           Loop
             out := APPEND(out, [res↓[1, ..., digs]])
             res := res↓[digs + 1, ..., DIM(res)]
             If DIM(res) ≤ digs
                If t = 1
                   RETURN [[APPEND(out, [res])]`, APPEND("Calc time = ", STRING(time))]
                   RETURN [[APPEND(out, [res])]`]
```

Output in columns to "digs" digits without calculation time,
Calculation time is stored in global variable "calc_time"
which can be given after the calculation (#10)

```
#8:   pig_t_col(501, 40)

      [[ 3.141592653589793238462643383279502884197 ]]
      [[ 7169399375105820974944592307816406286208 ]]
      [[ 9986280348253421170679821480865132823066 ]]
      [[ 4709384460955058223172535940812848111745 ]]
      [[ 0284102701938521105559644622948954930381 ]]
      [[ 9644288109756659334461284756482337867831 ]]
#9:   [[ 6527120190914564856692346034861045432664 ]]
      [[ 8213393607260249141273724587006606315588 ]]
      [[ 1748815209209628292540917153643678925903 ]]
      [[ 6001133053054882046652138414695194151160 ]]
      [[ 9433057270365759591953092186117381932611 ]]
      [[ 7931051185480744623799627495673518857527 ]]
      [[           248912279381830119491 2         ]]

#10:  calc_time = 1.43
```

Output in colums with 1 as third parameter, calculation time is given in the output

```
#11:  pig_t_col(201, 40, 1)

       [[ 3.141592653589793238462643383279502884197 ]                    ]
       [[ 7169399375105820974944592307816406286208 ]                    ]
       [[ 9986280348253421170679821480865132823066 ]                    ]
#12:   [[ 4709384460955058223172535940812848111745 ], Calc time = 0.17 ]
       [[ 0284102701938521105559644622948954930381 ]                    ]
       [[                    96                     ]                    ]
```

Same procedure but output in rows (and "digs" slices) with default digs = 5:

```
      pig_t_row(nd, digs := 5, t, t_, time, res, out) :=
        Prog
          t_ := RANDOM(0)
          res := STRING(32·appi(1/10, nd) - 4·appi(1/239, nd) - 16·appi(1/515, nd))
          time := APPROX((RANDOM(0) - t_)/100)
          calc_time := time
          If DIM(res) ≤ digs
             RETURN [res, time]
#13:      out := "3."
          res := REST(REST(res))
          Loop
            out := APPEND(out, res↓[1, ..., digs], " ")
            res := res↓[digs + 1, ..., DIM(res)]
            If DIM(res) ≤ digs
               If t = 1
                  RETURN APPEND(out, res, ", Calc time = ", STRING(time))
                  RETURN APPEND(out, res)

#14:  pig_t_row(101, 5)

#15:  3.14159 26535 89793 23846 26433 83279 50288 41971 69399 37510 58209 74944 59230 78164

        06286 20899 86280 34825 34211 70679

#16:  calc_time = 0.04

#17:  pig_t_row(101, 5, 1)

#18:  3.14159 26535 89793 23846 26433 83279 50288 41971 69399 37510 58209 74944 59230 78164

        06286 20899 86280 34825 34211 70679, Calc time = 0.05

#19:  pig_t_row(501, 20)

#20:  3.14159265358979323846 26433832795028841971 69399375105820974944 59230781640628620899

        86280348253421170679 82148086513282306647 09384460955058223172 53594081284811174502

        84102701938521105559 64462294895493038196 44288109756659334461 28475648233786783165

        27120190914564856692 34603486104543266482 13393607260249141273 72458700660631558817

        48815209209628292540 91715364367892590360 01133053054882046652 13841469519415116094

        33057270365759591953 09218611738193261179 31051185480744623799 62749567351885752724

        89122793818301194912

#21:  calc_time = 1.46

      pig_time_only(nd, digs, t_, time, res, out) :=
        Prog
          t_ := RANDOM(0)
#22:      pi_res := 32·appi(1/10, nd) - 4·appi(1/239, nd) - 16·appi(1/515, nd)
          time := APPROX((RANDOM(0) - t_)/100)
          RETURN time
```

The last function gives back the calculation time only,
the resulting number is stored as "pi_res"

```
#23:  pig_time_only(501)

#24:  1.47

#25:  pi_res =

        3.14159265358979323846264338327950288419716939937510582097494459230781640628620899~
        86280348253421170679821480865132823066470938446095505822317253594081284811174502841 0~
        27019385211055596446229489549303819644288109756659334461284756482337867831652712019~
        09145648566923460348610454326648213393607260249141273724587006606315588174881520920~
        96282925409171536436789259036001133053054882046652138414695194151160943305727036575~
        95919530921861173819326117931051185480744623799627495673518857527248912279381830119~
        4912

#26:  pig_time_only(2001)

#27:  60.36
```

Robert's answer: Thank you very much for your work. It is great (magnifique en français).

Mathematical Physics Electronic Journal, University of Barcelona

Probability Surveys

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Journal of Analysis and its Applications, Lemgo, Germany

Mathematical Proceedings of the Royal Irish Academy

Memoirs on Differential Equations and Mathematical Physics, Tbilisi

The New York Journal of Mathematics

Novi Sad Journal of Mathematics

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