THE DERIVE - NEWSLETTER #86

ISSN 1990-7079

THE BULLETIN OF THE

DERIVE®

USER GROUP

+ CAS-TI

Contents:

1 Letter of the Editor

2 Editorial - Preview

Guido Herweyers
3 Statistics with TI-Nspire (2)

Michel Beaudin and Gilles Picard
19 Solving 2nd Order ODEs
Two Non-analytical Methods Revisited

Roland Schröder
32 Points in a Circle

Nelson Urrego
36 Basic Steps of Shor's Quantum Algorithm

42 From Dietmar Oertel's Taylor Series Gallery

43 User Forum

June 2012

There will be two interesting panel discussions at TIME 2012:

DERIVE is Still Special

Organizers: Michel Beaudin, Josef Böhm, José Luis Galan, Carl Leinbach

Abstract

At the TIME 2006 conference in Dresden, several DERIVE program users were shocked to hear that DERIVE would be discontinued and replaced by the new product TI-Nspire CAS. In 2007, a transition period started where users asked TI developers to implement some interesting features of DERIVE. TI-Nspire CAS, with two platforms and a wider spectrum – due mainly to numerous integrated applications–, could not satisfy all wishes but tried to address some of them: one particular example is found in the latest operating system of Nspire CAS (OS 3.2) where 3D parametric plotting has been added with David Parker on board. But there are still special and unique features of DERIVE that cannot be found in Nspire CAS.

In this panel discussion, we would like participants attending this session to talk about what they still find special in DERIVE: to do so, live examples in various fields of mathematics will be shown to the audience using DERIVE. One goal of this discussion is to provide constructive ideas to Texas Instruments developers: proclaimed at its beginning as the "true successor of DERIVE", Nspire CAS has all the potential needed to satisfy many requests of users.

Topics for discussion include, but are not limited to: Subexpressions, Showing Step Simplification, 2D implicit plotting, precision digits, cubic equations, piecewise continuous functions, Expand/Collect for trig expressions, creating random numbers without the randseed, plotting polyhedrons in 3D, iteration, Fourier series, polynomial systems, Gröbner bases. Those who would like to present one's favourite DERIVE special which has not been implemented in TI-Nspire so far are welcome to join.

I would like to ask you about your DERIVE SPECIAL which you are still missing in TI-NspireCAS. We know that the Nspire-developers are eager to implement as many DERIVE features as possible – but this needs time. Much has been done since the first presentation of TI-Nspire in Dresden 2006 but the DERIVIANs are still waiting for some improvements.

We will publish your wishes together with the TIME 2012 results in our newsletter.

This is a second panel discussion (part of the abstract):

Assessment and mathematical placement exams with technology

An effective yield of the use of technology in mathematics education can only be expected if the tools are also applied accordingly in the exam situation. Discussing this topic we have to keep in mind on the one hand, the process oriented assessment as an integral part of the learning process, and on the other hand, product oriented mathematical placement exams and final exams as a proof of authority for the students.

Your contributions would be very welcome, Josef.

Liebe DUG-Mitglieder,

nun ist es mir doch tatsächlich gelungen, den DNL#86 noch vor der TIME 2012 fertig zu stellen. Die revidierte Fassung des DNL#29 muss da für noch etwas warten.

Im Rahmen der TIME 2012 wird es zwei interessante Podiumsdiskussionen geben: Wir werden über DERIVE-features sprechen, die wir in der aktuellen TI-NspireCAS-Version noch vermissen. Die zweite Diskussion betrifft Probleme und Chancen von Technologie unterstützten Prüfungen. Hinweise auf die Diskussionsinhalte finden Sie auf der Informationsseite. Sie sind herzlich eingeladen, Ihre Meinung oder Beiträge zu beiden Themen auch im Rahmen des DNL abzugeben. Im Herbst werde ich über die Ergebnisse der Diskussionen berichten.

Nun einige Hinweise zum Inhalt dieses DNL: Guido Herweyers' schönes Statistikskript wird weiter geführt. Michel und Gilles haben in Málaga einen spannenden Vortrag zur numerischen Lösung von Differentialgleichungen 2. Ordnung gehalten. Da sie sich auf die Behandlung mit dem V200 beschränkten, habe ich versucht, ihre Ideen mit DERIVE bzw. mit TI-NspireCAS aufzugreifen.

Eine wiederum sehr gelungene schülergerechte Untersuchung wird von Roland Schröder beigesteuert, während der Artikel unseres kolumbianischen Freunds, Nelson Urrego, wohl schon etwas weiter von der Schulmathematik entfernt ist. Zu seinem Thema habe ich noch ein paar sehr brauchbare websites gefunden.

Dietmar Oertel hat mir eine ganze Kollektion von DERIVE-Grafiken geschickt, die er im Zusammenhang mit seinen Untersuchungen mit Taylorreihen in stundenlangen Sitzungen erzeugt hat. Einen kleinen Blick in seine Galerie können Sie auf Seite 42 werfen.

Ich wünsche einen schönen Sommer und freue mich auf den DNL#87.

Dear DUG Members,

yes, I was successful in finishing this DNL before start of TIME 2012. The revised version on DNL#29 must wait a little while.

In the frame of TIME 2012 we will have two interesting panel discussions: we will speak about DERIVE-features which we have been missing in the NspireCAS versions until now. The second discussion will focus on the problems and chances of technology supported assessments. Abstracts of both discussions can be found on the information page. You are invited to contribute or to comment both subjects in our DNL. I will report about the results of the discussions an publish your possible reactions in the fall-issue of the DNL.

I have some remarks on the contents of this Newsletter: You can find the second part of Guido Herweyers' excellent statistics paper. Michel and Gilles gave an exciting lecture on the numerical solution of 2nd order ODEs in Málaga. As they showed the V200-treatment only I tried to follow their ideas with DERIVE and TI-NspireCAS.

Roland Schröder contributed again a very fine student centred investigation. Our friend from Colombia, Nelson Urrego sent a paper which has some distance from school mathematics. I found a couple of interesting websites referring to the contents of Nelson's article.

Dietmar Oertel sent a collection of DERIVE graphs produced by him in connection with his investigations of Taylor series. I believe his words when writing that he needed hours and hours plotting the families of curves. You may take a tiny look on his gallery on page 42.

Finally I'd like to wish you a fine summer and am looking forward to presenting DNL#87 in three months

Viele Grüße, kindest regards

The *DERIVE-NEWSLETTER* is the Bulletin of the *DERIVE* & CAS-*TI User Group*. It is published at least four times a year with a content of 40 pages minimum. The goals of the *DNL* are to enable the exchange of experiences made with *DERIVE*, *TI*-CAS and other CAS as well to create a group to discuss the possibilities of new methodical and didactical manners in teaching mathematics.

Editor: Mag. Josef Böhm
D´Lust 1, A-3042 Würmla, Austria
Phone: ++43-(0)660 3136365
e-mail: nojo.boehm@pgv.at

Contributions:
Please send all contributions to the Editor. Non-English speakers are encouraged to write their contributions in English to reinforce the international touch of the *DNL*. It must be said, though, that non-English articles will be warmly welcomed nonetheless. Your contributions will be edited but not assessed. By submitting articles the author gives his consent for reprinting it in the *DNL*. The more contributions you will send, the more lively and richer in contents the *DERIVE* & CAS-*TI Newsletter* will be.

Next issue: September 2012

Preview: Contributions waiting to be published

Some simulations of Random Experiments, J. Böhm, AUT, Lorenz Kopp, GER
Wonderful World of Pedal Curves, J. Böhm, AUT
Tools for 3D-Problems, P. Lüke-Rosendahl, GER
Hill-Encription, J. Böhm, AUT
Simulating a Graphing Calculator in *DERIVE*, J. Böhm, AUT
Do you know this? Cabri & CAS on PC and Handheld, W. Wegscheider, AUT
An Interesting Problem with a Triangle, Steiner Point, P. Lüke-Rosendahl, GER
Overcoming Branch & Bound by Simulation, J. Böhm, AUT
Graphics World, Currency Change, P. Charland, CAN
Cubics, Quartics – Interesting features, T. Koller & J. Böhm, AUT
Logos of Companies as an Inspiration for Math Teaching
Exciting Surfaces in the FAZ / Pierre Charland´s Graphics Gallery
BooleanPlots.mth, P. Schofield, UK
Old traditional examples for a CAS – what´s new? J. Böhm, AUT
Truth Tables on the TI, M. R. Phillips, USA
Where oh Where is It? (GPS with CAS), C. & P. Leinbach, USA
Embroidery Patterns, H. Ludwig, GER
Mandelbrot and Newton with *DERIVE*, Roman Hašek, CZK
Tutorials for the NSpireCAS, G. Herweyers, BEL
Some Projects with Students, R. Schröder, GER
Structures in the Set of Prime Numbers, D. Oertel, GER
Dirac Algebra, Clifford Algebra, D. R. Lunsford, USA
Laplace Transforms, ODEs and CAS, G. Piccard & Ch. Trottier, CAN
A New Approach to Taylor Series, D. Oertel, GER
Cesar Multiplication, G. Schödl, AUT
Henon & Co; Find your very own Strange Attractor, J. Böhm, AUT
Rational Hooks, J. Lechner, AUT
Mathematical Model for Snail Shells (4), P. Trebisz, GER
Simulation of Dynamic Systems with various Tools, J. Böhm, AUT

and others

Impressum:
Medieninhaber: *DERIVE* User Group, A-3042 Würmla, D´Lust 1, AUSTRIA
Richtung: Fachzeitschrift
Herausgeber: Mag. Josef Böhm

Statistics with TI-Nspire 3.1 (Part 2)

Visualising and Simulating Dynamically with TI-Nspire 3.1

Guido Herweyers, KHBO Campus Oostende
guido.herweyers@khbo.be

(3) Quantitative ungrouped data

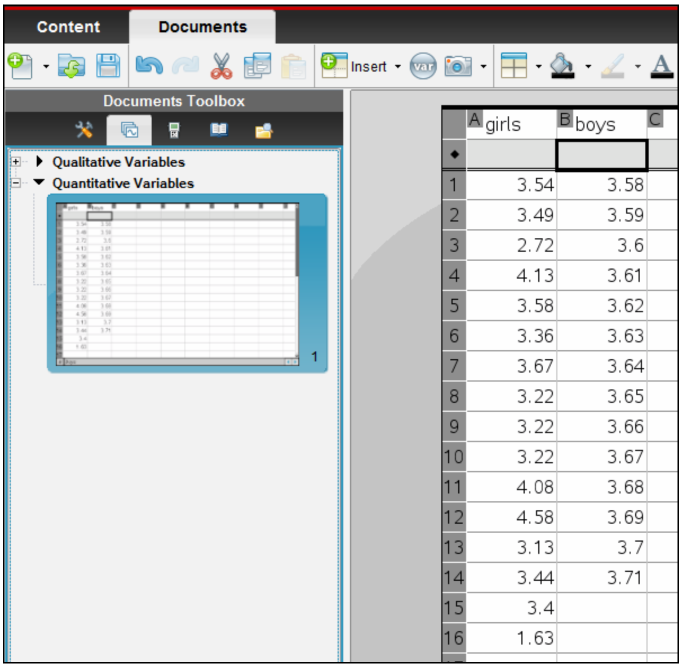
Example 4:

Given are the following birth weights of 16 girls and 14 boys (in kg). Data from an Excel data stock can be imported directly into a TI-Nspire spreadsheet page by Copy and Paste. For some countries is one obstacle to overcome: the decimal comma must be changed to a decimal point first.
(From German: Extras > Optionen > International > …)

Girls	Boys
3,54	3,58
3,49	3,59
2,72	3,6
4,13	3,61
3,58	3,62
3,36	3,63
3,67	3,64
3,22	3,65
3,22	3,66
3,22	3,67
4,08	3,68
4,58	3,69
3,13	3,7
3,44	3,71
3,4	
1,63	

Girls	Boys
3.54	3.58
3.49	3.59
2.72	3.6
4.13	3.61
3.58	3.62
3.36	3.63
3.67	3.64
3.22	3.65
3.22	3.66
3.22	3.67
4.08	3.68
4.58	3.69
3.13	3.7
3.44	3.71
3.4	
1.63	

→ Start a new problem within the same file (in the document menu **Insert Problem**), change the problem name **Problem 2** to **Quantitative Variables**, and insert a **Lists & Spreadsheet** page.

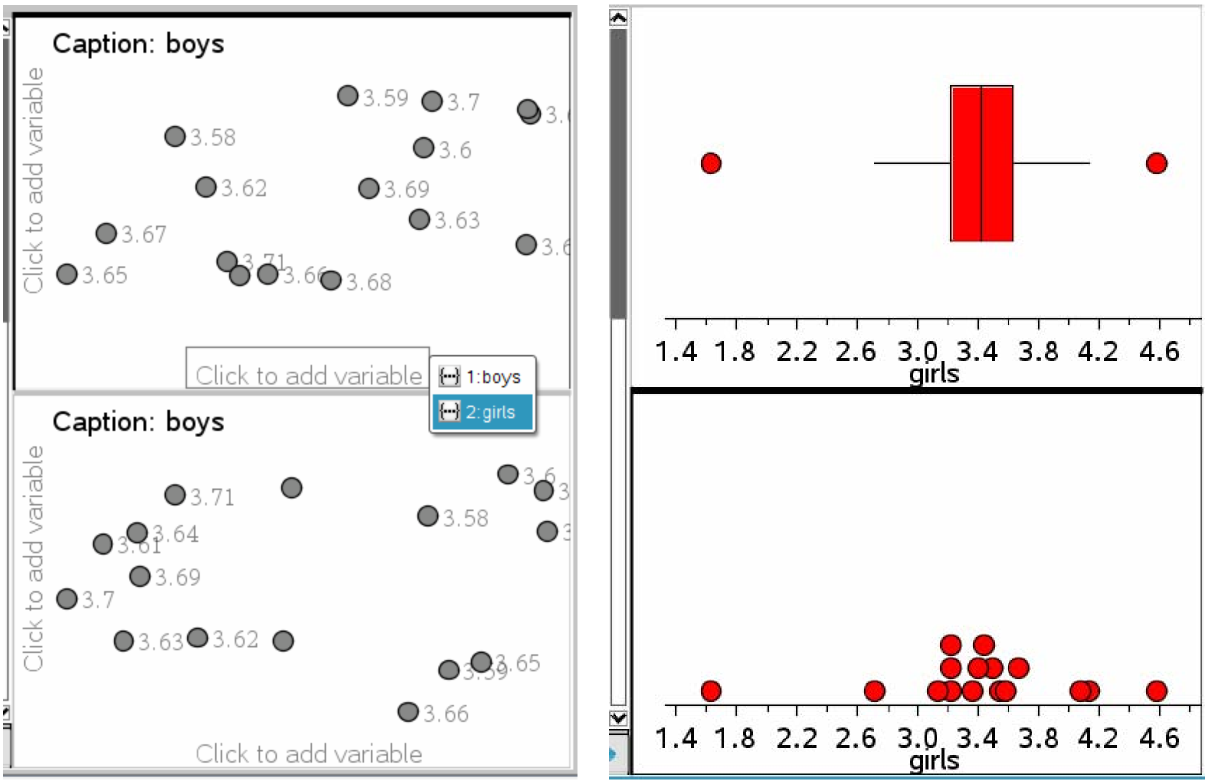


Copy the data into columns *A* and *B*. (Take care to highlight the according region of cells = 16 × 2 before pasting.)

Name the columns as girls and boys.

Split the screen in three windows as shown below and insert two **Data & Statistics** applications into the empty windows.

→ Click in both windows on the bottom for adding the variable girls. By default appears a Dot Plot in both windows. Change the diagram in the upper window in a Box Plot (right mouse click = rmcl).

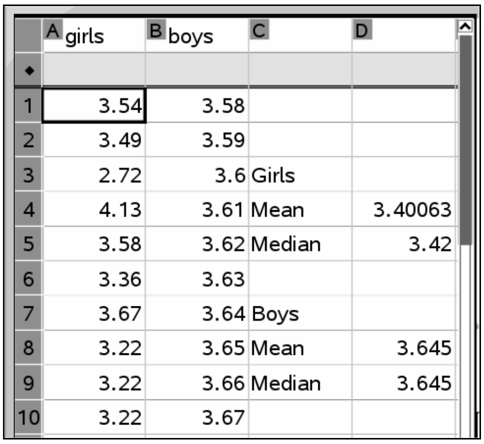


→ Enter "Girls" in cell *C3* of the spreadsheet (under quotes), then "Mean" in cell *C4* and "Median" in cell *C5*.

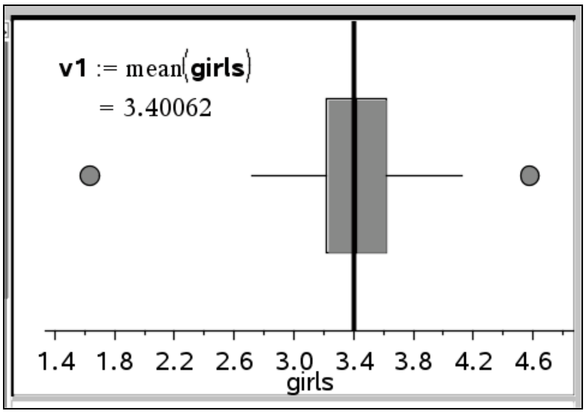
Type **=mean(girls)** in cell *D4* followed by pressing the (ENTER)-key; the mean appears.

Type **=median(girls)** in cell *D5* followed by pressing the (ENTER)-key; the median appears.

You can do the same for the boys' data in cells *C7* to *D9*.



	A girls	B boys	C	D
◆				
1	3.54	3.58		
2	3.49	3.59		
3	2.72	3.6	Girls	
4	4.13	3.61	Mean	3.40063
5	3.58	3.62	Median	3.42
6	3.36	3.63		
7	3.67	3.64	Boys	
8	3.22	3.65	Mean	3.645
9	3.22	3.66	Median	3.645
10	3.22	3.67		

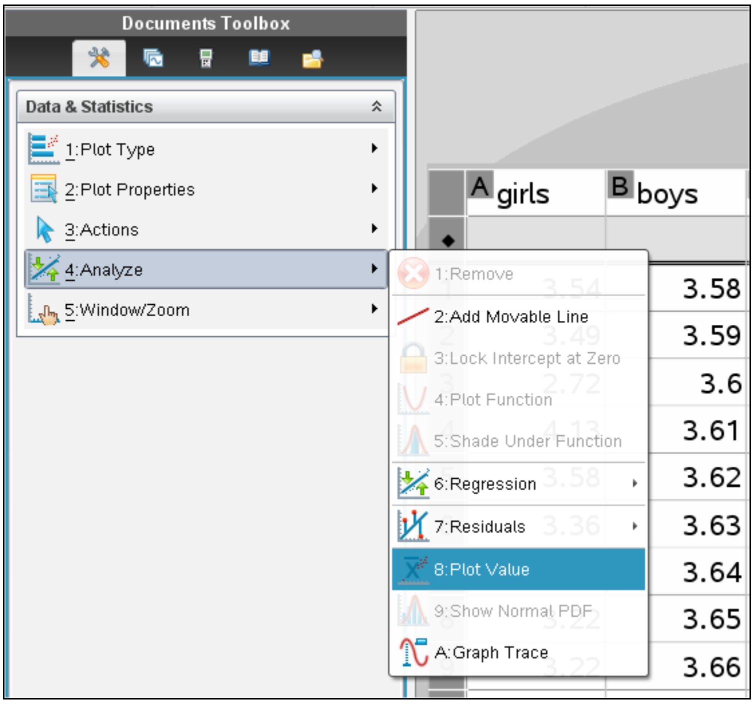


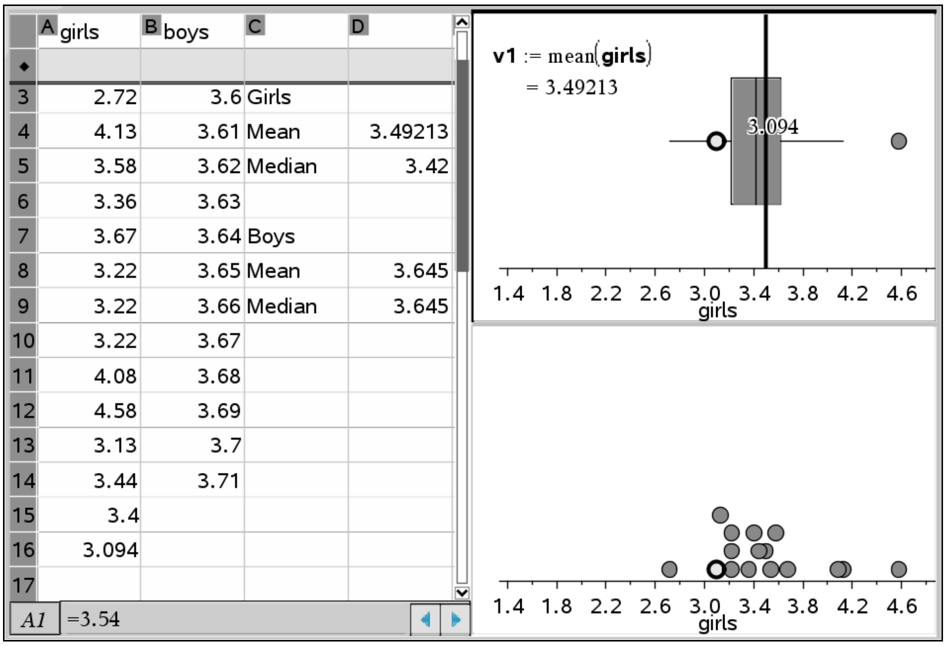
→ Click into the Box Plot window and then choose first the **Analyze** menu in the **Documents Toolbox** and then the option **Plot Value**.

Fill in for **v1:= mean(girls)**. A vertical line in the position of the mean will appear (right above).

Now you can grip one outlier (left mouse click on it) and move it along. You can observe how the data changes together with a change of the mean. Investigate the influence of the outlier on mean and median.

The data are altered accordingly in the spreadsheet and dot plot, too.



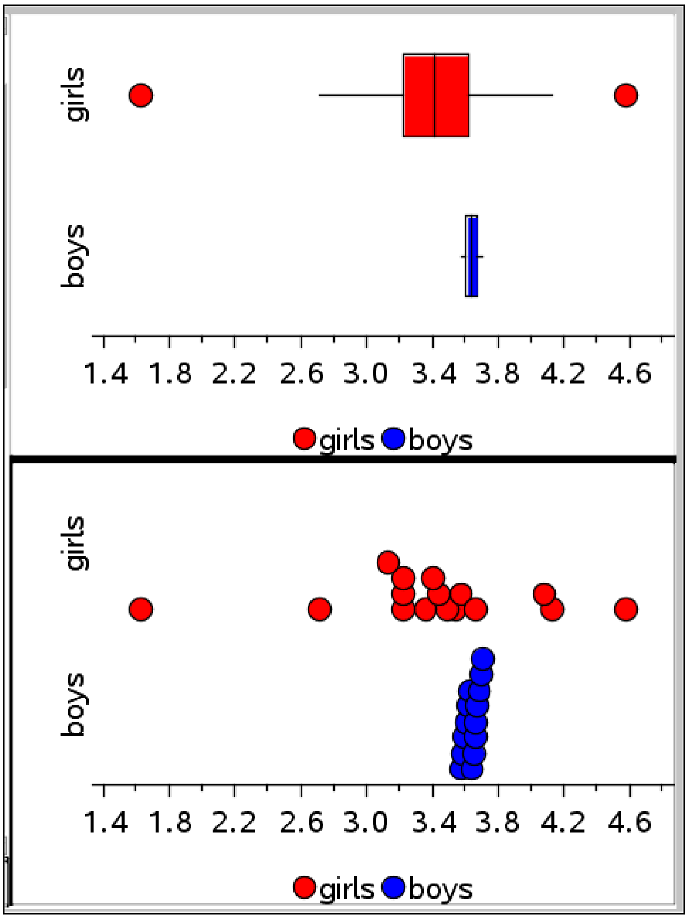


	A girls	B boys	C	D
◆				
3	2.72	3.6	Girls	
4	4.13	3.61	Mean	3.49213
5	3.58	3.62	Median	3.42
6	3.36	3.63		
7	3.67	3.64	Boys	
8	3.22	3.65	Mean	3.645
9	3.22	3.66	Median	3.645
10	3.22	3.67		
11	4.08	3.68		
12	4.58	3.69		
13	3.13	3.7		
14	3.44	3.71		
15	3.4			
16	3.094			
17				

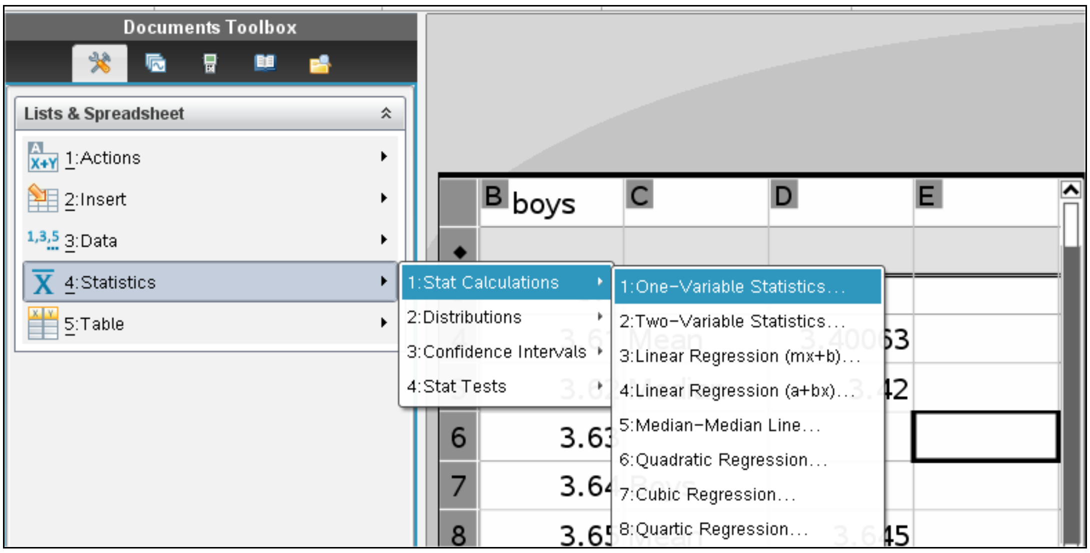
Hint: It is easy to come back to the original data by pressing the **Undo**-key: 

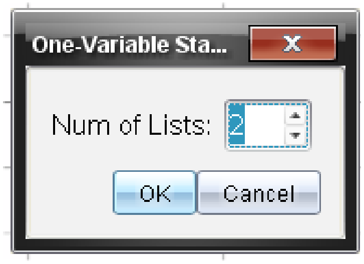
Fortunately it is possible to forbid (to lock) altering data. Open a **Calculator page** and type **lock(girls)**. Now try again moving the outlier. Command **unlock(girls)** gives the data free again.

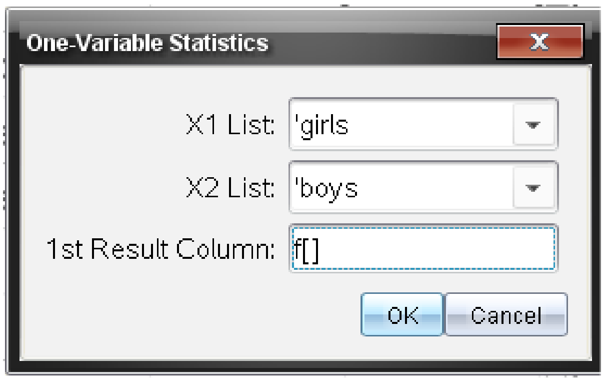
→ Restore the original data. Click on the vertical mean line and press (Del). Make a right mouse click on the girls below the horizontal axis to achieve **Add X variable** and then click on the offered variable boys. Do the same in the Box Plot window. Adapt the colours. Compare the data and their representations of girls and boys for these samples.

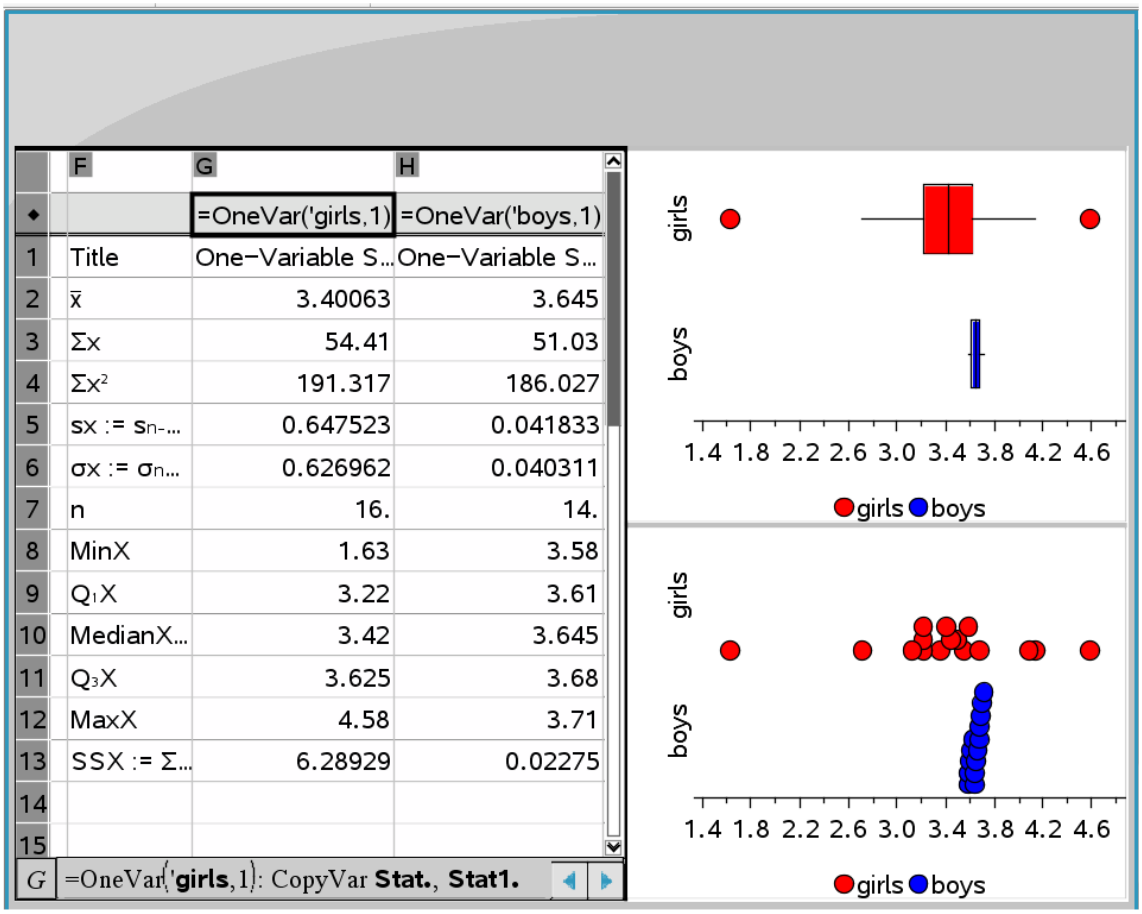


→ Now we will find all statistics of the given data. Click in one cell of column *D* in the spreadsheet, choose the **Statistics** menu, continue with **Stat Calculations** and choose option **One-Variable Statistics**.





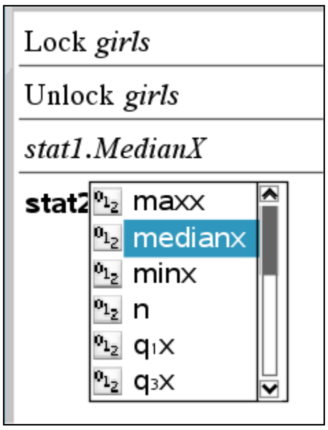




	F	G	H
◆		=OneVar('girls,1)	=OneVar('boys,1)
1	Title	One−Variable S...	One−Variable S...
2	x̄	3.40063	3.645
3	Σx	54.41	51.03
4	Σx²	191.317	186.027
5	sx := sn-...	0.647523	0.041833
6	σx := σn...	0.626962	0.040311
7	n	16.	14.
8	MinX	1.63	3.58
9	Q₁X	3.22	3.61
10	MedianX...	3.42	3.645
11	Q₃X	3.625	3.68
12	MaxX	4.58	3.71
13	SSX := Σ...	6.28929	0.02275
14			
15			

G =OneVar('girls,1): CopyVar Stat., Stat1.

→ Change again the data (by dragging) and observe the changes of the statistics.



→ Open a **Calculator App** and type **stat1**. and select one statistic of the appearing list, followed by (ENTER). Do the same for **stat2.** in order to get the statistics of the second list (the boys).

stat1.results delivers a matrix giving all statistics for the girls' data. TI-Nspire stores automatically all statistics calculations in stat1, stat2, stat3, …

stat1.MedianX	3.42
stat2.MedianX	3.645

stat1.results

"Title"	"One−Variable Statistics"
"$\bar{x}$"	3.40063
"Σx"	54.41
"Σx^2"	191.317
"$sx := s_{n-1}x$"	0.647523
"$\sigma x := \sigma_n x$"	0.626962
"n"	16.
"MinX"	1.63
"Q_1X"	3.22
"MedianX"	3.42
"Q_3X"	3.625
"MaxX"	4.58
"$SSX := \Sigma(x-\bar{x})^2$"	6.28929

Statistics calculations can also be performed in a Calculator App via the Statistics menu.

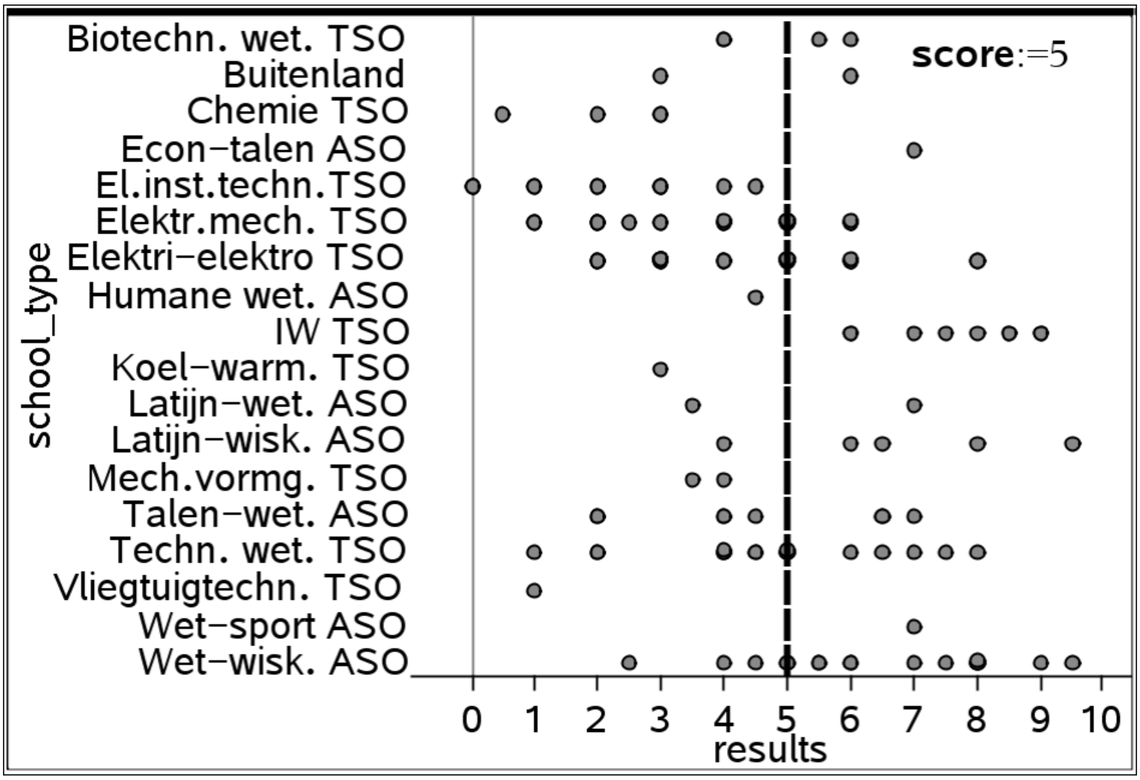
Example 5:

In this example ungrouped quantitative data are split according categories. The population is the data set of the first year bachelor students at the KHBO Campus in Oostende in the academic year 2008/09 (137 students). It follows an overview of the results of a mathematics test in the first semester (maximum score 10 points) together with the type of secondary school attended by the students. There are two lists: results and school_type.

	A results	B school_type
◆		
1	3	Chemie TSO
2	5	Wet−wisk. ASO
3	5.5	Wet−wisk. ASO
4	5	Wet−wisk. ASO
5	6	Latijn−wisk. ASO
6	7	Latijn−wet. ASO
7	2	Techn. wet. TSO
8	2	Chemie TSO
9	7	Wet−sport ASO
10	4	Wet−wisk. ASO
11	4.5	Humane wet. ASO
12	4.5	Techn. wet. TSO
13	7	Talen−wet. ASO
14	4.5	Wet−wisk. ASO
15	4	Biotechn. wet. TSO
B	*school_type*	

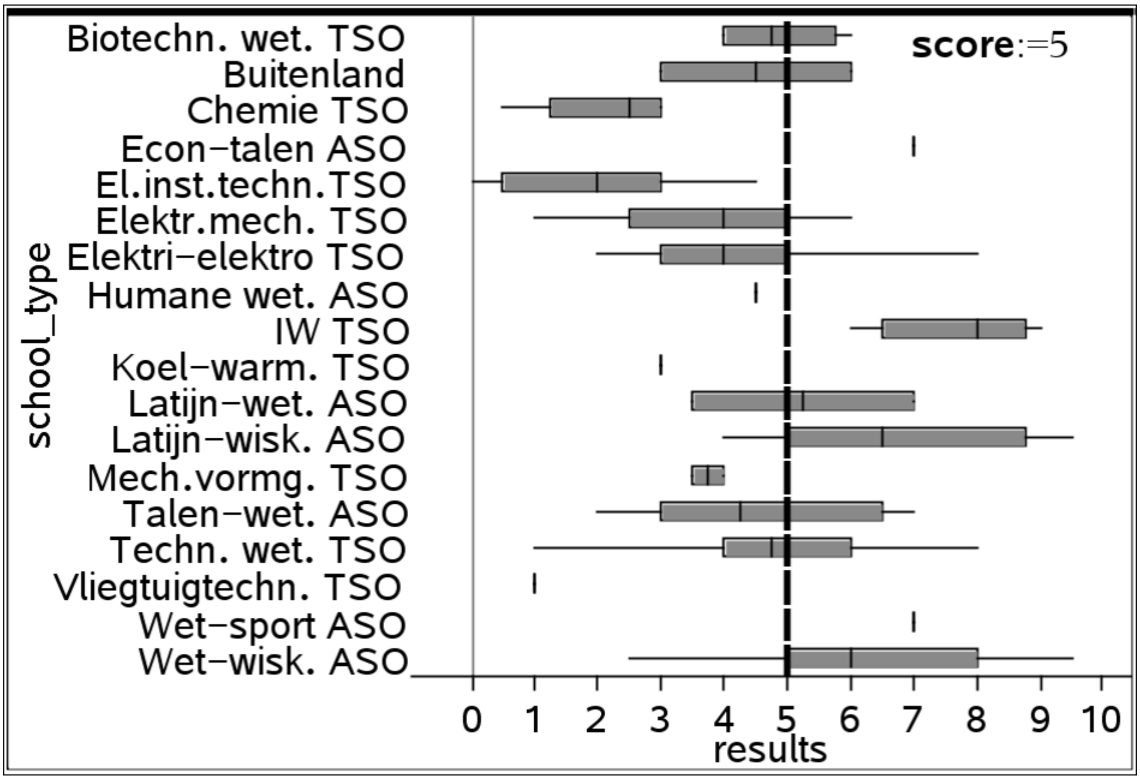
Insert a new **Data & Statistics** page.

Add variable results for the *x*-axis and variable school_type for the *y*-axis. This gives a Dot Plot of the test results split according to the various school types. Now add a vertical line for score 5 (again in **Documents Toolbox > Analyze > Plot value** and change the variable name **v1** to score).

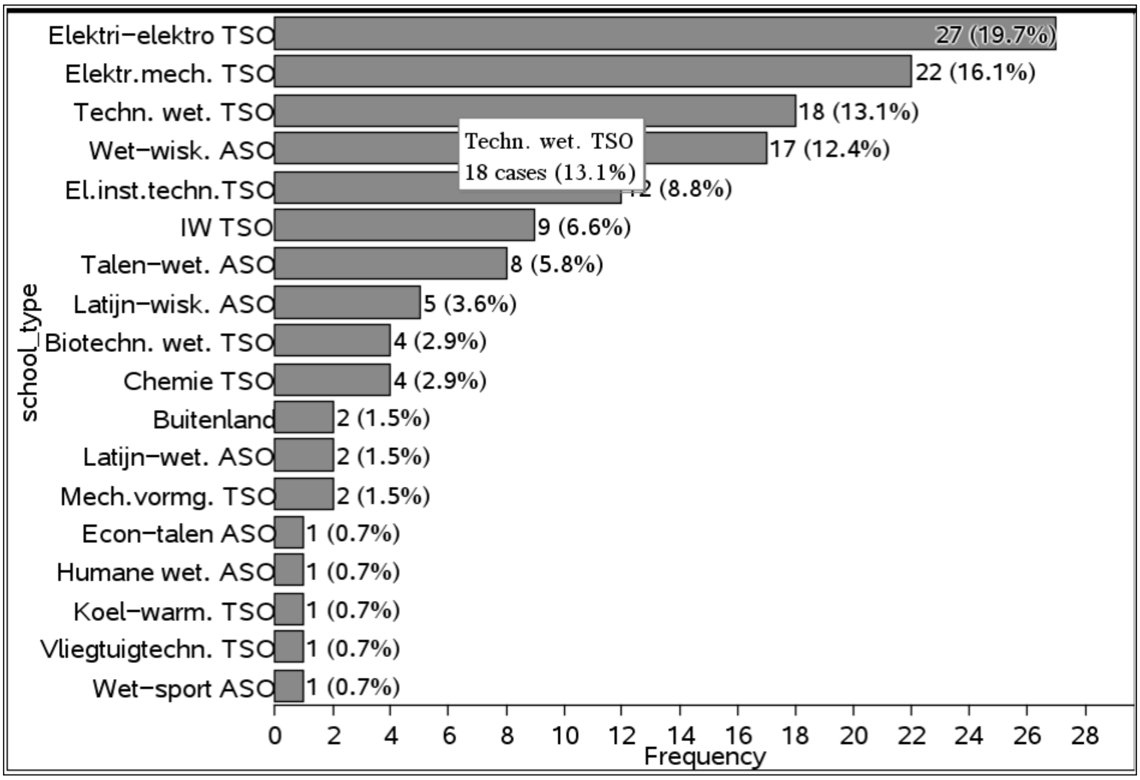


The dot plot gives a clear impression of the individual results and which secondary school types reached low or high scores.

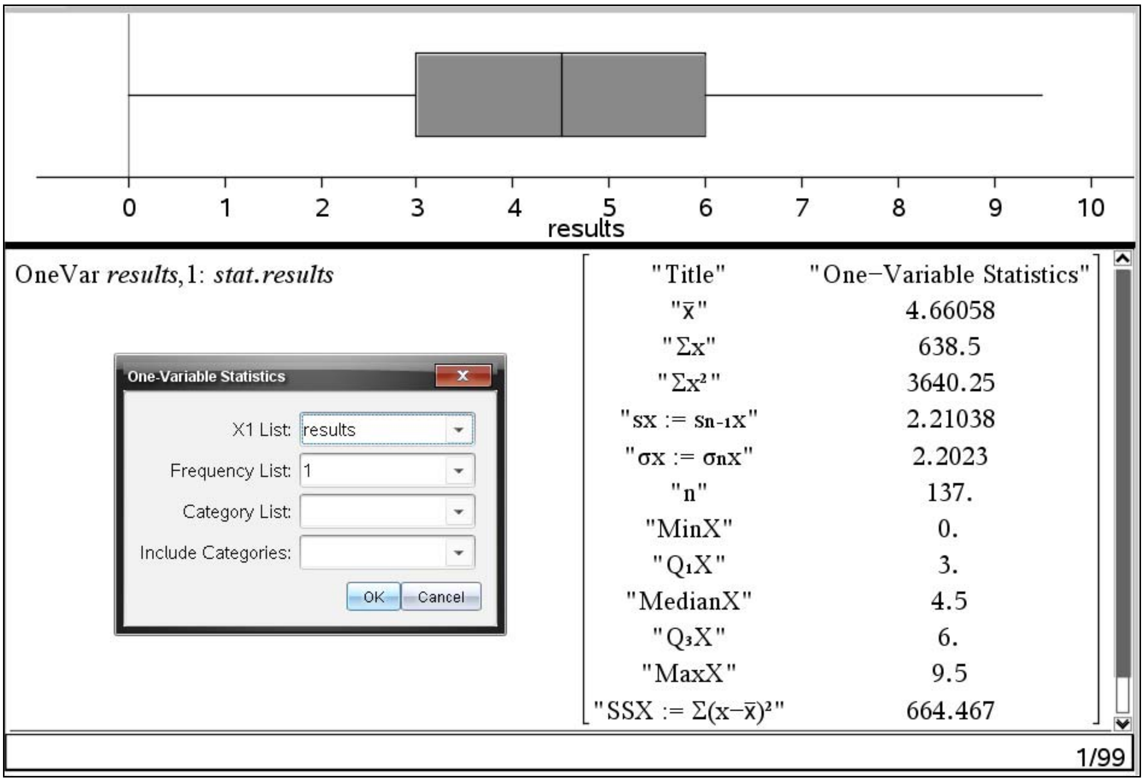
→ Change the Dot Plot into a Box Plot. We can read off that there are six types with a median greater than score 5 and 12 types with a median less 5.



→ Remove the variable results from the *x*-axis (rmcl on results).



→ Insert a new page and split it horizontally into a **Data & Statistics App** and a **Calculator App**. Collect all data in one single Box Plot and in the statistics of the 137 data records.



Example 6:

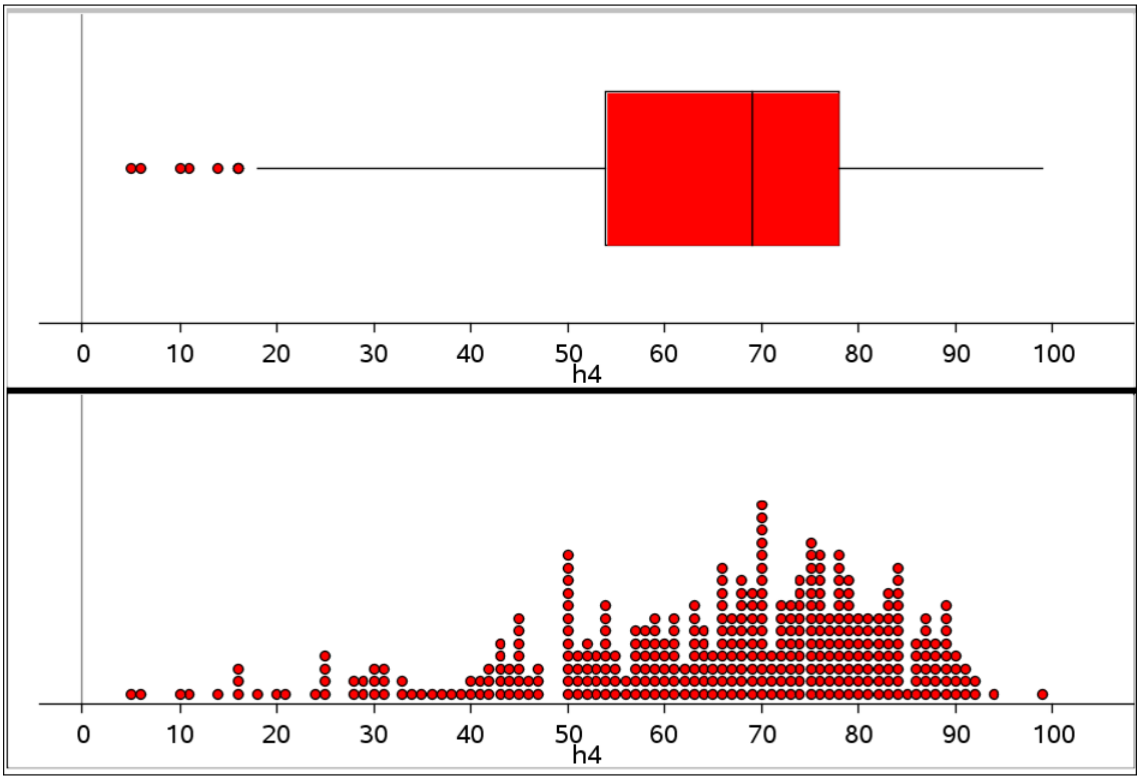
During summer holidays the Belgian Coast is watched by life guards. Many students feel attracted by this adventurous holiday job. Apprenticeship for life guard is not to be underestimated. In addition to an extended theoretical course a couple of difficult swimming tests must be passed.

The training for life guards is organized annually. At first the students must pass the theoretical exam. The course exists of seven main parts which are all assessed separately – each of them with 100 points maximum. For passing the theoretical part at least 50% of each main part must be reached. The students who have passed the first part can go on with the practical part of the exam (swimming tests, knowledge about knots, first aid in case of accidents). The results of the theoretical exam are published on a website annually [4].

There were 367 participants for the course in 2010-2011. Each of them got a number between 1 and 367. The results are given sorted wrt the numbers in the TI-Nspire-file.

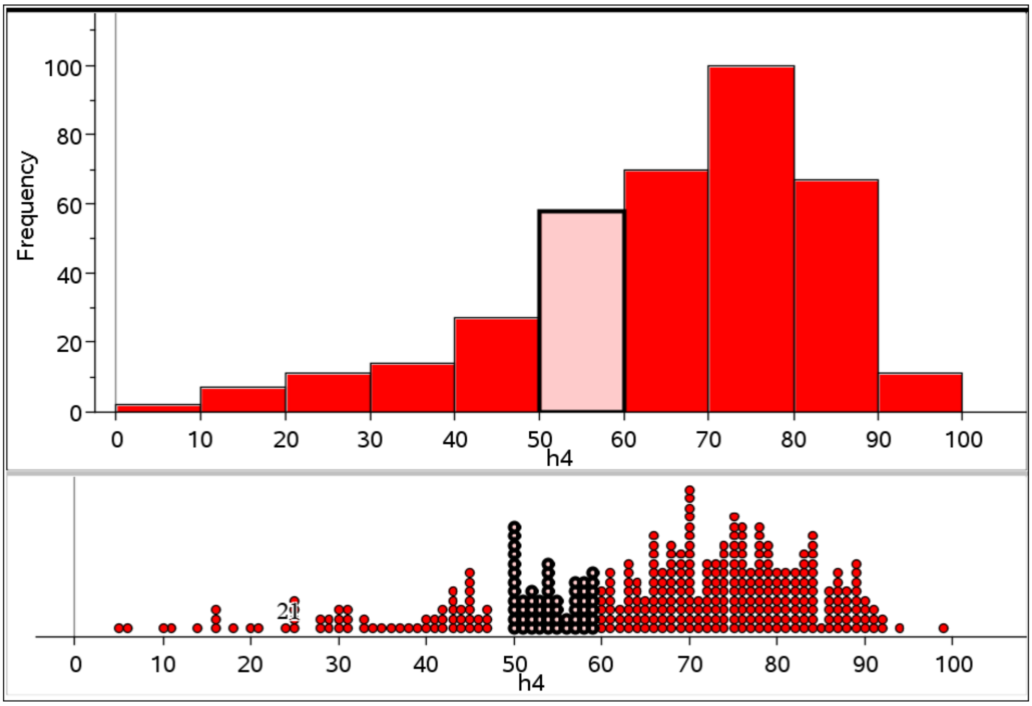
→ Insert a **Calculator** page and lock the data with the command
lock h1,h2,h3,h4,h5,h6,h7

→ Insert a new page with two **Data & Statistics** applications. Produce a Dot Plot and a Box Plot as well of the results of main part 4 (given in variable h4).



→ Change the Box Plot to a Histogram, enlarge the histogram window (by dragging the horizontal borderline). Experiment with the class width (by dragging a vertical side of any rectangle). One can set the class width manually, too (rmcl on the histogram, **Option 5: Bin Settings**). Move the cursor over the histogram in order to read off the frequencies. You can represent the relative frequencies making again a right mouse click on the histogram and choosing **Scale > Percent**.

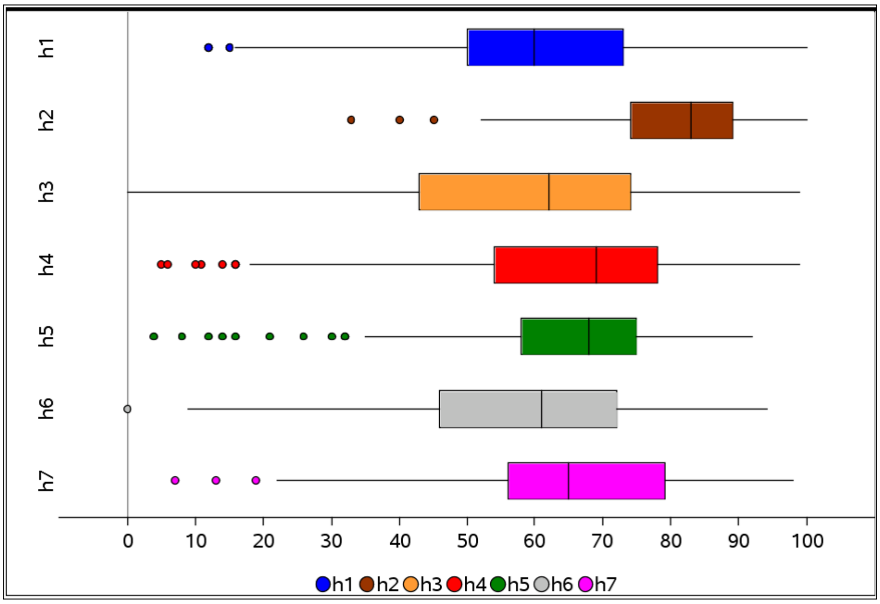
When selecting one class the corresponding data are marked in the dot plot. Clicking in the free space outside of the diagram cancels the selection.



→ The class frequencies can be calculated in a Calculator page using the command **countIf**. You will obtain the frequency list for the classes [0,10[, [10,20[, …, [100,110[with the command line **seq(countIf(h4, 10k ≤ ? ≤ 20(k+1), k, 0, 10)**. The list can also be generated in a column of the spreadsheet by typing **=seq(countIf(h4, 10k ≤ ? ≤ 20(k+1), k, 0, 10)** into the grey cell in the second row of the column.

Lock *h1,h2,h3,h4,h5,h6,h7*	*Done*
countIf(*h4*,20≤?<30)	11
seq(countIf(*h4*,10·*k*≤?<10·(*k*+1)),*k*,0,10)	{2,7,11,14,27,58,70,100,67,11,0}

→ Compare the results for all main parts in one window by their box plots and discuss the differences.



→ Call the statistics of the results of first three main parts of the exam. Find the connections to the respective box plots.

OneVar 3,*h1*,*h2*,*h3*: *stat.results*

"Title"	"One−Variable Statistics"	"_____"	"_____"
"$\bar{x}$"	59.9019	81.0198	57.8392
"Σx"	21984.	28681.	21227.
"Σx²"	1.43359E6	2.36305E6	1.38082E6
"sx := s_{n-1}x"	17.8566	10.5536	20.4505
"σx := σ_nx"	17.8323	10.5387	20.4226
"n"	367.	354.	367.
"MinX"	12.	33.	0.
"Q_1X"	50.	74.	43.
"MedianX"	60.	82.	62.
"Q_3X"	73.	89.	74.
"MaxX"	100.	100.	99.
"SSX := $\Sigma(x-\bar{x})^2$"	116702.	39316.9	153070.

Lists h2 and h7 contain empty cells for the students who didn't participate in the exams of these main parts.

Empty cells of a list (or of a column in a speadsheet table) are marked by an underscore (_) or by the word **void**.

dim(*h2*)	367
mean(*h2*)	$\frac{28681}{354}$
$\frac{\text{sum}(h2)}{367}$	$\frac{28681}{367}$
isVoid(*h2*)	
{false,false,false,false,false,false,false,false,false,false,false,false,false,false▸	
countIf(isVoid(*h2*),?=false)	354
countIf(isVoid(*h2*),?=true)	13
h2	
{90,97,78,83,90,85,85,92,68,90,86,63,93,78,97,95,88,79,89,93,95,79,66,89▸	

There are 13 empty cells in list h2.

(4) Quantitative grouped data

Example 7:

Given are the frequencies for main part 4 results for the classes as shown in the table as follows:

[0,10[	[10,20[	[20,30[	[30,40[	[40,50[	[50,60[	[60,70[	[70,80[	[80,90[	[90,100[
2	7	11	14	27	58	70	100	67	11

→ Create a histogram based on this frequency table. Find the relative frequencies and the cumulative frequencies as well. Draw the relative frequency graph.

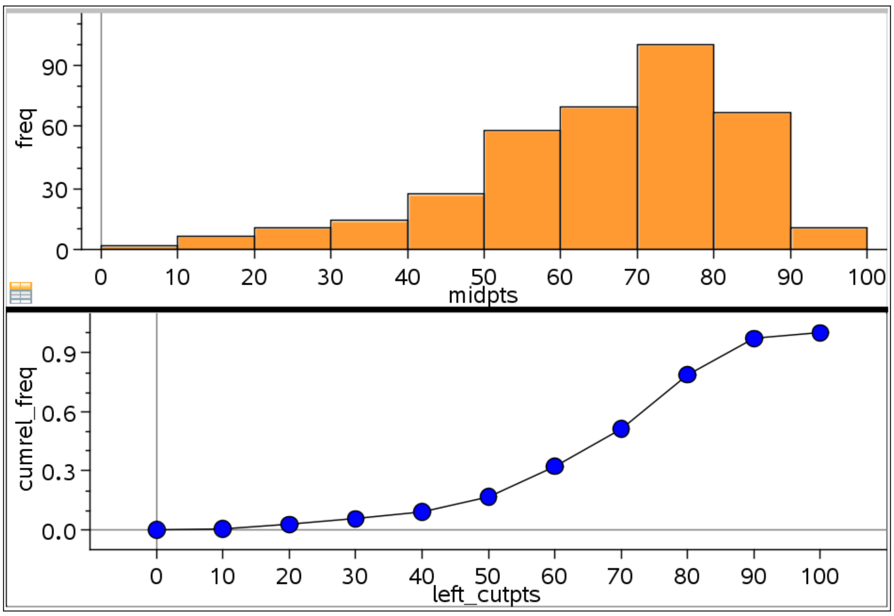
	A left_cutpts	B midpts	C freq	D rel_freq	E cum_rel_freq
◆	=seq(10*k,k,0,10)		=seq(countif('h4,10*k≤?<1	=round(freq/(sum(freq)),3)	
1	0	5	2	0.005	0
2	10	15	7	0.019	0.005
3	20	25	11	0.03	0.024
4	30	35	14	0.038	0.054
5	40	45	27	0.074	0.092
6	50	55	58	0.158	0.166
7	60	65	70	0.191	0.324
8	70	75	100	0.272	0.515
9	80	85	67	0.183	0.787
10	90	95	11	0.03	0.97
11	100				1.

The second column contains the list class of the midpoints of the class intervals. This list is easy done: click in cell *B1* and type

= (a1 + a2)/2

followed by (ENTER) (start a formula with an equation sign like in MS-Excel). Now you can copy down this formula until cell *B10*. For doing this move the cursor to the right bottom corner of cell *B1* until a plus-sign is appearing. Keep the left mouse key pressed and move the cursor down to *B10* to spread the formula.

Column *E* is filled with the cumulate frequencies: write 0 into cell *E1* and the formula = **e1+d1** in cell *E2*. Copy down the cell contents until cell *E10*.



(5) Scatter diagrams and regression

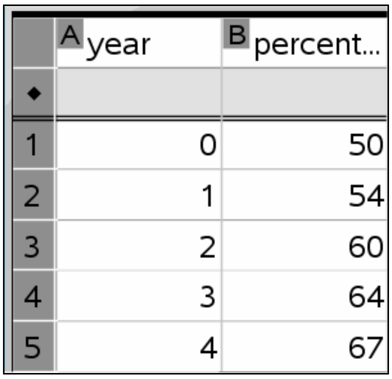
Example 8:

Given is the table containing the percentage of Belgian homes having internet access (source NIS):

Year	2005	2006	2007	2008	2009
Percentage	50	54	60	64	67

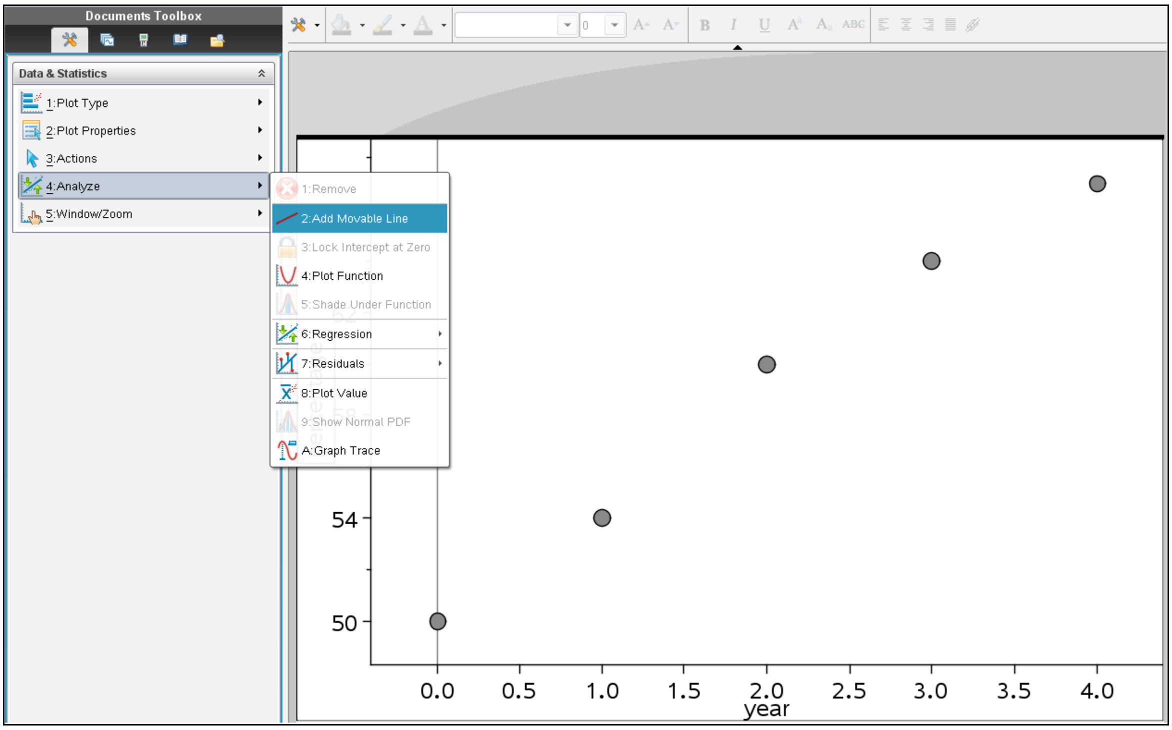
→ Represent the given data in a scatter diagram.

Open a new **Lists & Spreadsheet** page and fill in the data (year = 0 corresponds with year = 2005).

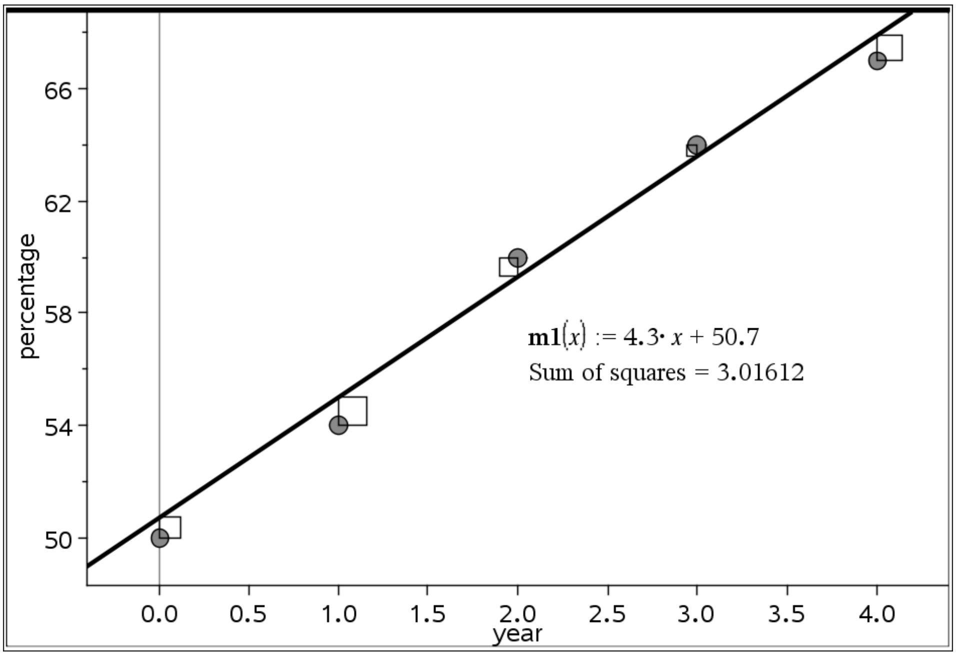


Insert a **Data & Statistics** page. Set year on the horizontal axis and percentage on the vertical one.

Add a Moveable Line via the **Analyze** menu.

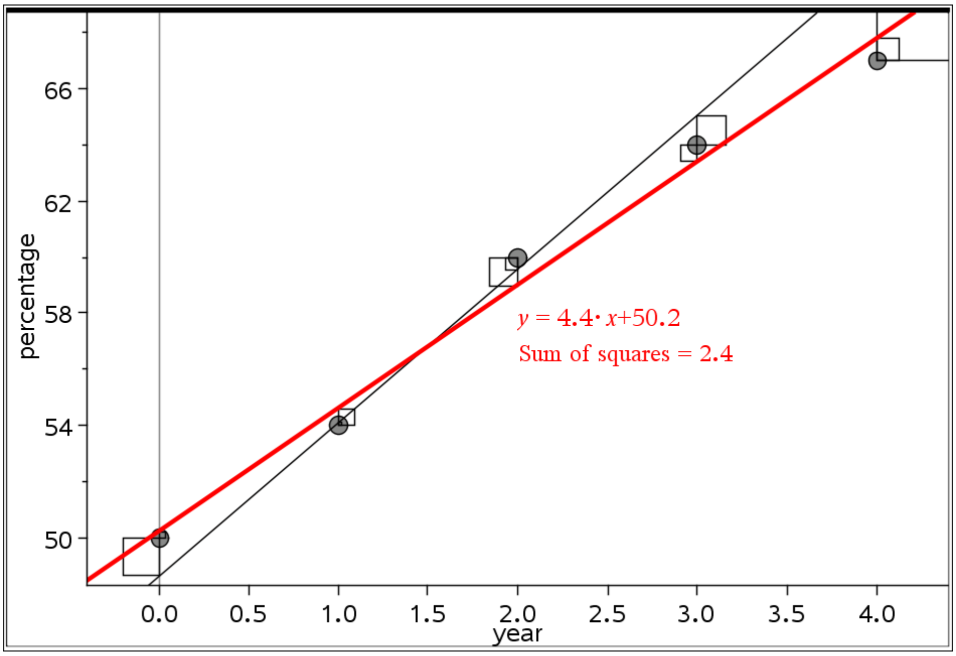


Continue in the **Analyze** menu with option **Residuals** > **1: Show Residual Squares**.

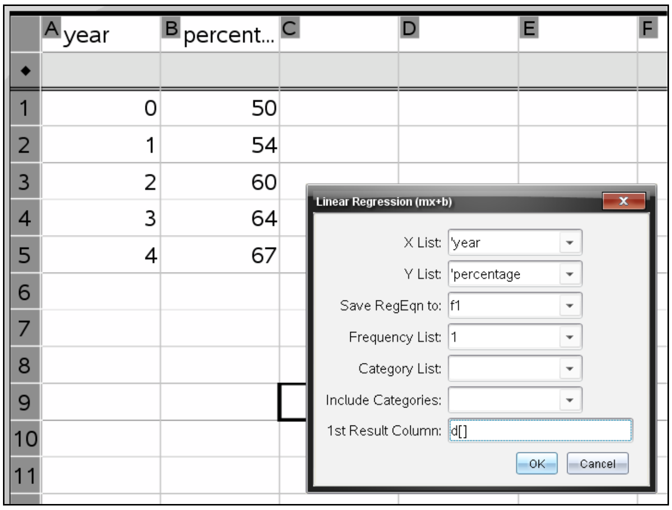


Move the line (You can move the line by gripping it around the midpoint of the visible segment – the cursor changes into a fourfold arrow – and you can rotate the line by gripping at the ends of the segment – the cursor changes in a circle formed arrow). Play around and try to find the minimum value of the **Sum of Squares** by experimenting with the position of the line.

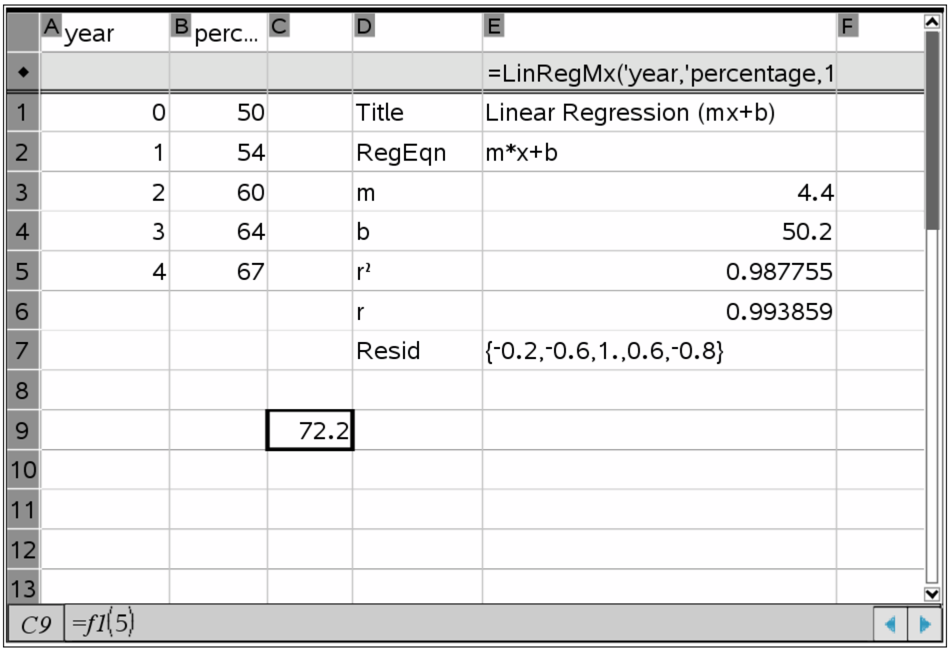
Finally compare your "best straight line" and its corresponding sum of squares with TI-Nspire's results. Use the menu **Analyze** > **Regression** > **Show Linear (mx + b)** followed by **Analyze > Residuals > Show Residual Squares**.



Return to the **Data & Statistics** page (by pressing Ctrl + ← or via the page sorter). Click into one cell of column *C* and choose menu **Statistics > Stat Calculations > Linear Regression (mx + b)** …



The coefficient of determination $r^2 = 98{,}8\%$ is excellent. The equation of the regression line is stored as function $f1$. Estimate the percentage for 2010 in cell *F9* applying the formula **=f1(5)**.



Example 9:

The graph of a time series is a scatter diagram with data of a quantitative variable on the vertical axis as function of time on the horizontal axis. The points are connected by segments. In this way the trend of development of the variable can be studied.

The tables given below show the development of the numbers of weddings and divorces in Belgium in the years 1995 to 2009.

Aantal huwelijken per gemeente, 1995 - 2009

Refnis	ADMINISTRATIEVE EENHEDEN	1995	1996	1997	1998	1999	2000	2001	2002	2003	2004	2005	2006	2007	2008	2009
Totaal		51.402	50.552	47.759	44.393	44.171	45.123	42.110	40.434	41.777	43.296	43.141	44.813	45.561	45.613	43.303

Bron (verplichte vermelding): Algemene Directie Statistiek en Economische Informatie - Thematische Directie Samenleving.

Aantal echtscheidingen per gemeente, 1995 - 2009

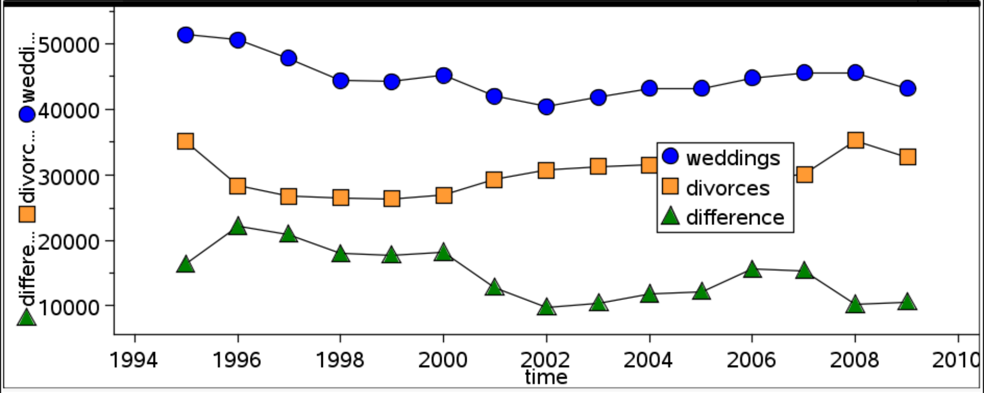
Refnis	ADMINISTRATIEVE EENHEDEN	1995	1996	1997	1998	1999	2000	2001	2002	2003	2004	2005	2006	2007	2008	2009
Totaal		34.983	28.402	26.748	26.503	26.423	27.002	29.314	30.628	31.355	31.405	30.840	29.189	30.081	35.366	32.606

Bron (verplichte vermelding): Algemene Directie Statistiek en Economische Informatie - Thematische Directie Samenleving.

→ Represent both time series and additionally represent the differences of the data.

	A time	B weddings	C divorces	D difference	E	F	G	H
◆				='weddings–'divorces				
1	1995	51402	34983	16419				
2	1996	50552	28402	22150				
3	1997	47759	26748	21011				
4	1998	44393	26503	17890				
5	1999	44171	26423	17748				

A2:A15 =a1+1



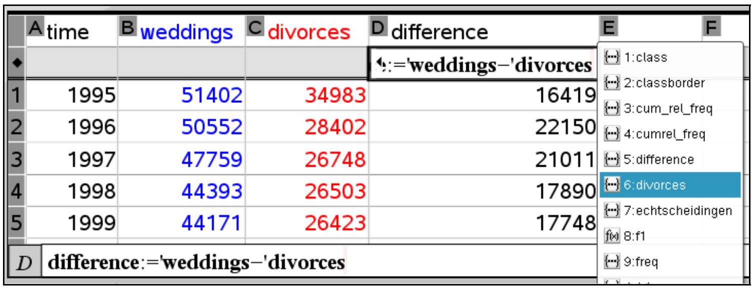
After typing the name difference for column *D* automatically appears **difference:=** when clicking in the grey formula cell below the caption of the column.

Accomplish the formula with

difference:=weddings – divorces

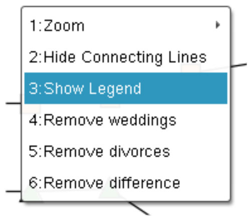
Hints:

The names of the variables are offered in a list via the [var] key in document menu. Then it is easy to insert a name into a formula.



You need **Add Y Variable** to present more than only one time series.

The legend does not appear automatically.



Solving Second Order ODEs, Two Non-analytical Methods Revisited

Michel BEAUDIN and Gilles PICARD

Ecole de technologie supérieure

1100, Notre-Dame street West, Montreal (Quebec) Canada

michel.beaudin@etsmtl.ca

gilles.picard@etsmtl.ca

ABSTRACT

Mathematics can still be taught without using a CAS and this is probably the case in most schools and universities. Although CAS and technology are often used by instructors to demonstrate or illustrate mathematical concepts, they are rarely used by students. When we consider our mathematics curriculum, "Differential Equations" is one course that we firmly believe can and should benefit from the use of CAS. In this talk, we will report how our ODE course has evolved, as our engineering students have access to technology (Voyage 200 symbolic calculator) in the classroom at all times. This talk will show examples of what students still do by hand and what CAS allows us to do now to enrich the learning experience.

We will consider the series solutions of a second order equation with variable coefficients and numerical solution of the first order equivalent system. Many textbooks do not show the relation between these two subjects. With a CAS on every desk, we can ask students to compare results obtained with both methods. Of course, technology is a must to support this. As teachers, we still want our students to be able to do some specific computations manually. For example, they have to find the recurrence formula by hand for the coefficients of the series solution. However, we also want students to be able to compute, with some accuracy, the value of the solution at a certain point using partial sums or even graph this approximate solution. Then, converting the same equation into a first order system, they can plot the numerical generated curve obtained by the built-in RK method in the Voyage 200 or create a table of values for the approximate solution.

1. Introduction

Let us say a word about the idea behind this paper. Once someone has decided to use computer algebra in his teaching, he has to decide for what particular topics he will use it and when it is appropriate to do so. We know that computer algebra helps a lot in calculus and differential equations, but not so much in mathematical analysis, for example. Many colleagues or instructors continue to think that computer algebra is mainly useful for doing applications, not also for teaching mathematical concepts. In this paper, we want to show that the Voyage 200 symbolic calculator can be used in a very original way to solve numerically second order linear differential equations. We will explore the precision of results obtained with the "Power series" method and compare with values from a "Runge-Kutta" algorithm implemented on the Voyage 200. We will be using 2 out of 6 graphing windows of the device: the Diff Equations and the Sequence graphing modes. We will present an approach where pencil and paper techniques are still required but where technology completes the analysis.

2. Power series solutions using technology: more can be done

At ETS, the Voyage 200 calculator has been used since September 1999 but questions about solving ODEs using power series are usually formulated in the same way, year after year. Let's look at the classic approach for this topic. Students are given a differential equation of the following type:

$$p(x)y''+q(x)y'+r(x)y=0, \quad y(0)=y_0, y'(0)=v_0.$$

In fact, the initial conditions are sometimes defined elsewhere than at 0, at $x=x_0$ for example, but a simple change of variables ($v=x-x_0$) will revert the problem to this general case. The functions $\frac{q(x)}{p(x)}$ and $\frac{r(x)}{p(x)}$ are supposed to be analytic at 0 and $p(0) \neq 0$ (said otherwise, 0 is an ordinary point of the differential equation). Then students are told, without proof, that there exists a general solution of the problem by mean of power series:

$$y(x)=\sum_{n=0}^{\infty} c_n x^n \quad \text{with} \quad c_0=y_0 \text{ and } c_1=v_0.$$

These series converge for $|x|<R$ where R is at least equal to the distance between 0 and the nearest singularities (if we suppose that p, q and r have no common factors, the nearest singularity is a complex point a which is the closest to 0 and satisfying $p(a) = 0$). Students have to find a recurrence formula in order to compute recursively the coefficients of the series. This is done by hand. Having found many coefficients, they can use them to get an approximation value of $y(\alpha)$ with α between $-R$ and R. The problem is that we never ask them how accurate is this approximation! In fact, they usually only have to estimate $y(\alpha)$ using no more than the 5 first non vanishing terms… This paper wants to reverse this approach. We will show how this subject can be enriched with the aid of technology. Moreover, because Voyage 200 "desolve" command does not possess, like big CAS running on computers, a "power series method", this "disadvantage" can become an advantage. A good mix of paper and pencil techniques and, also, a use of RK numerical method for first order systems can be used and give a nice solution of the problem. Let us be more concise, using an example.

3. Concrete examples

If we want to show what we are doing exactly in the classroom, the best way will be to give concrete examples. Let us take the following problems:

$$eq1: \left(x^2+4\right)y''+3xy'-4y=0, \quad y(0)=4,\, y'(0)=1.$$

$$eq2: (2x-9)y''-y'-4xy=0, \quad y(0)=8,\, y'(0)=-2.$$

For each of them, students are asked to find a series solutions centered at 0 and use it to estimate the value of $y(\alpha)$ where α belongs to the interval of convergence. In fact, for $eq1$, we note that there are singularities located at $x = \pm\, 2i$, so the series $\sum_{n=0}^{\infty} c_n x^n \left(c_0=4, c_1=1\right)$ will certainly converge for $|x|<2$. We will take $\alpha = 1$ in this case. For $eq2$, the only singularity is at 9/2, so the series $\sum_{n=0}^{\infty} c_n x^n \left(c_0=8, c_1=-2\right)$ will converge for $|x|<9/2$. We will take $\alpha = 2$ in this case. One can estimate $y(\alpha)$ by using a partial sum of the form $\sum_{k=0}^{n} c_k$ for different values of n.

An important question is: how many terms are needed in order to be confident for the value of $y(\alpha)$? What if we only use the first 5 non vanishing terms of the series? At ETS, every student in the classroom has his own Voyage 200 on his desk. They know that the pencil and paper ability required to find the recurrence formula is still important. After recalling some properties of power series — namely that term by term differentiation is allowed within the interval of convergence —, they will set $y=\sum_{n\in\mathbb{Z}} c_n x^n$ instead $y=\sum_{n=0}^{\infty} c_n x^n$ of because we can simply decide that $c_n=0$ when $n < 0$. This way, y' and y'' will keep the same indices, we can even omit these indices recalling that all summations are on $n\in\mathbb{Z}$ (we first saw this approach in [1] in the 80's). Substituting $y=\sum c_n x^n$ into the differential equation and, collecting similar terms, leads to the recurrence formula (because if $\sum_{n=0}^{\infty} a_n x^n=0$ then $a_n=0\ \ \forall n$). Once it is obtained, they can compute some additional coefficients and, then, estimate $y(\alpha)$. Our students will manually do the substitutions and find the recurrence formula. For $eq1$, this would lead to:

Solve $\left(x^2+4\right)y''+3xy'-4y=0$

Using $y=\sum c_n x^n \qquad y'=\sum n c_n x^{n-1} \qquad y''=\sum n(n-1)c_n x^{n-2}$

$$x^2\sum n(n-1)c_n x^{n-2}+4\sum n(n-1)c_n x^{n-2}+3x\sum n c_n x^{n-1}-4\sum c_n x^n=0$$

$$\sum n(n-1)c_n x^n+\sum 4n(n-1)c_n x^{n-2}+\sum 3n c_n x^n+\sum -4c_n x^n=0$$

Making the change $n\to n+2$ in the second summation so that they are all in terms of x^n

$$\sum n(n-1)c_n x^n + \sum 4(n+2)(n+1)c_{n+2}x^n + \sum 3nc_n x^n + \sum -4c_n x^n = 0$$

$$\sum \left[n(n-1)c_n + 4(n+2)(n+1)c_{n+2} + (3n-4)c_n \right] x^n = 0$$

Thus $(n^2 - n + 3n - 4)c_n + 4(n+2)(n+1)c_{n+2} = 0$

and the recurrence formula will be $c_{n+2} = \dfrac{-\left(n^2 + 2n - 4\right)}{4(n+2)(n+1)} c_n$

Since for $eq1$, $y(0) = 4$ and $y'(0) = 1$, using $n = 0,1,2\ldots$ in this formula gives us the coefficients and the solution $y(x) = 4 + x + 2x^2 + \dfrac{1}{24}x^3 - \dfrac{1}{6}x^4 + \ldots$

Using a partial sum with 5 terms gives us an approximation of the solution at $x = 1$, $y(1) \approx \dfrac{55}{8} = 6.875$

But what can be said about the precision of such an estimate? The process of calculating more terms, manually, to get more precision is quite tedious. Here is where technology can help. We show students that their Voyage 200 has a "sequence" graphing mode that can be used in order to generate as many coefficients as they need. Furthermore, the adaptive built-in RK method can find numerically the value of $y(\alpha)$ if the user has transformed the second order ODE into a first order system. So, when comes time to teach power series solutions to our students, we can do a better job if we decide to make use of technology.

We summarize in the following table the changes that have occurred for both of us for the past few years. We moved from questions a), b) and c) on the left to questions a), b), c) and d) on the right.

<table>
<tr><th>No technology involved</th><th colspan="2">Voyage 200 involved</th></tr>
<tr><td>a) Use a power series representation and find the recurrence formula.</td><td colspan="2">a) Use a power series representation and find the recurrence formula.</td></tr>
<tr><td>b) Find the interval of convergence of the series.</td><td colspan="2">b) Find the interval of convergence of the series.</td></tr>
<tr><td rowspan="2">c) Estimate the value of $y(\alpha)$ by using the 5 first non vanishing terms.</td><td>c) Find the value of $y(\alpha)$ by using as much terms as you need.</td><td>Define the partial sum $\sum_{k=0}^{n} c_k \alpha^k \to s(n)$ (α fixed) and compute the value of $y(\alpha)$ with 2 "stable" decimals.</td></tr>
<tr><td>d) When the value of $y(\alpha)$ has become "stable", make a second check using RK method.</td><td>By converting the second order ODE into a first order system and using the DE graphing mode window.</td></tr>
</table>

Figure 1 Table showing the way questions are given to the students for power series solutions of ODEs

Here are summarized the results to some of the former questions for both problems:

$eq1$ ($\alpha = 1$)	$eq2$ ($\alpha = 2$)
a) $c_0 = 4, c_1 = 1,$ $c_{n+2} = -\dfrac{\left(n^2+2n-4\right)}{4(n+1)(n+2)}c_n \quad (n \ge 0).$	a) $c_0 = 8, c_1 = -2,$ $c_{n+1} = \dfrac{\left(2n^2-3n\right)c_n - 4c_{n-2}}{9n(n+1)} \quad (n \ge 1).$
c) $y(1) = 6.88$ (first 5 non vanishing terms). when using more terms, the answer seems to stabilize to 6.89.	c) $y(2) = 0.14$ (first 5 non vanishing terms). when using more terms, the answer seems to stabilize to 0.84.

Figure 2 Results for our two ODEs

Figure 2 indicates that if we only use the 5 first non vanishing terms, the value of $y(1)$ will be a good estimate in the case of $eq1$ while the value of $y(2)$ will be very bad in the case of $eq2$. With technology, this can be easily corrected. Because of the poor estimate (0.14) obtained in $eq2$, we will show how Voyage 200 can help for this equation.

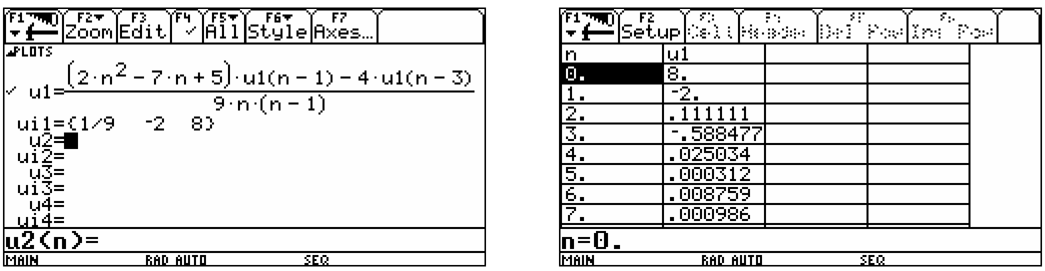
First, the Y Editor of the Voyage 200, in sequence graphing mode, requires a sequence defined in the form of "$u1(n)$ =" (see figure 3 a). This means that the students have to rewrite the recurrence formula using the change $n \to n-1$, thus obtaining

$$c_n = \frac{\left(2(n-1)^2 - 3(n-1)\right)c_{n-1} - 4c_{n-3}}{9(n-1)n} = \frac{\left(2n^2 - 7n + 5\right)c_{n-1} - 4c_{n-3}}{9n(n-1)}.$$

In Voyage 200, initial values should be given in reverse order (see figure 3 a) and, in this particular case where the recurrence formula is of order 3, we need to compute by hand the value of c_2.

Recalling that $c_n = 0$ when $n < 0$, we find

$$c_2 = \frac{\left(2\cdot 2^2 - 7\cdot 2 + 5\right)\cdot(-2) - 4\cdot(0)}{9\cdot 2\cdot 1} = \frac{2}{18} = \frac{1}{9}.$$



(a) **(b)**

Figure 3 Voyage 200 Y Editor in "sequence" graphic mode (a) and (b) showing a table of values.

The coefficients of the series solution are now available through this special function $u1(n)$, as can be seen in figure 3 b. Partial sums for finding a stable value for $y(2)$ are now computed:

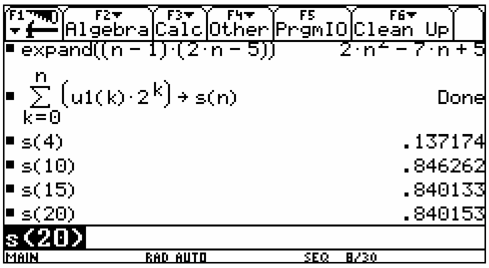
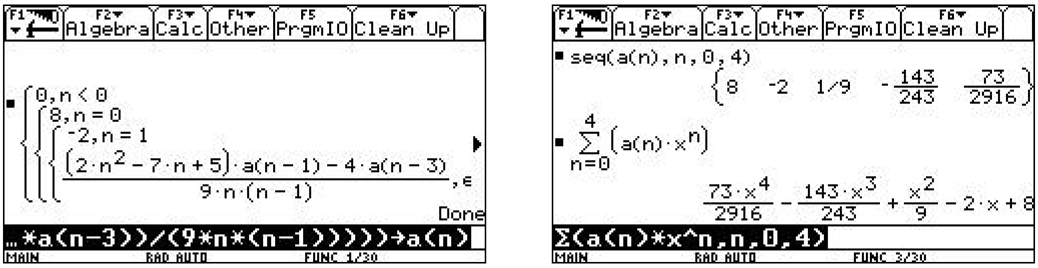


Figure 4 Function definition and evaluation

Figure 4 shows us that an appropriate 2 decimal value for $y(2)$ should be 0.84. Using only the first 5 non vanishing terms (that is $s(4)$ because $c_0, c_1, \ldots, c_4$ are all different from 0), we would conclude that $y(2) = 0.137174...$

Another way of generating a partial sum for the series solution, without using the graphic environment, is to use the "when(, ,)" command which is equivalent to a "if then else " instruction. Let's create a conditional function $a(n)$ giving us, for $n = 0, 1, 2, 3\cdots$ the desired coefficients. For the above example, the function must give us a value of 0 if $n < 0$, 8 if $n = 0$, -2 if $n = 1$ and $\dfrac{\left(2n^2 - 7n + 5\right)a_{n-1} - 4a_{n-3}}{9n(n-1)}$ if $n = 2, 3, 4, 5\cdots$. Figure 5 a below shows such a function created using four cascading when commands.



(a) **(b)**

Figure 5 A function to generate the coefficients, and the partial sum of the series solution.

Looking at figure 5 b, we see one advantage of this approach. The coefficients are in exact mode instead of floating point values as in figure 3 b. On the other hand this approach is much slower if we need to calculate more precision since evaluating $a(n)$ requires recursion back to 0 for each coefficient instead of relying on previous value already calculated as with our previous $u1(n)$ function. But the user will get the same values as those obtained using the "sequence" graphic mode (see figure 6).

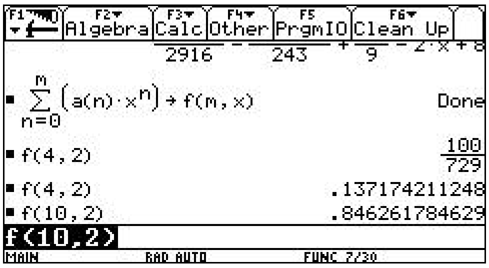
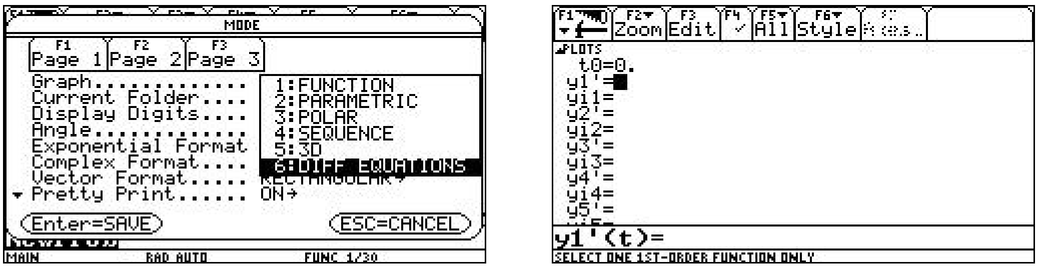


Figure 6 Estimating $y(2)$ with the $a(n)$ function.

The Voyage 200 possesses also a robust adaptive RK method [2], so students can do the same problem a second time, using another method! This is often not seen in a Differential Equations course. The Runge-Kutta methods are usually shown for first order ODE's. To solve a second order equation we need to show students how to transform this equation in a set of first order equations. Of course, once this is done, the CAS calculator will do all the computing though students have to learn how to control the environment.

First, let's look at converting a second order ODE into a system of first order ODE. Every second order equation which can be re-written in the form $y'' = F(x, y, y')$ can be transformed in a canonical way in a system of first order equations. In figure 7 b, we see that the independent variable must be *t* instead of *x*. Furthermore, the dependant variables must be *y*1, *y*2, *y*3,… In fact, this graphic mode is designed to solve system of first order differential equations.



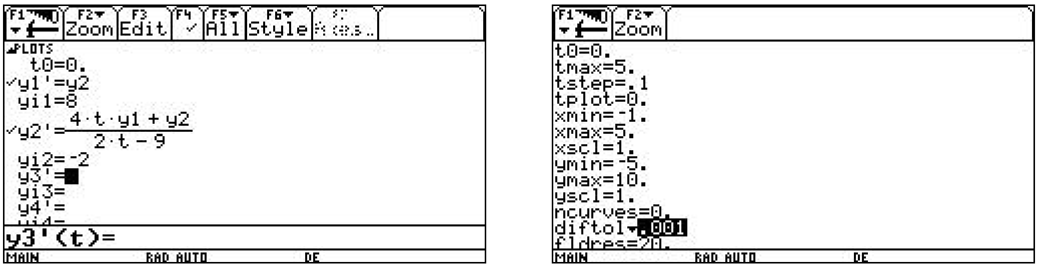
(a) **(b)**

Figure 7 The "differential equations" graphing mode

Recalling that *eq*2 is $eq2: (2x-9)y'' - y' - 4xy = 0, \quad y(0) = 8, y'(0) = -2$, we set $x = t$, then $y = y1$ and $y' = y1' = y2$, we can convert the second order ODE *eq*2 into a first order ODE system *syst_eq*2:

$$syst\_eq2: \begin{cases} y1' = y2 \\ y2' = \dfrac{4ty1 + y2}{2t-9} \end{cases} \qquad \begin{array}{l} y1(0) = 8 \\ y2(0) = -2 \end{array}$$

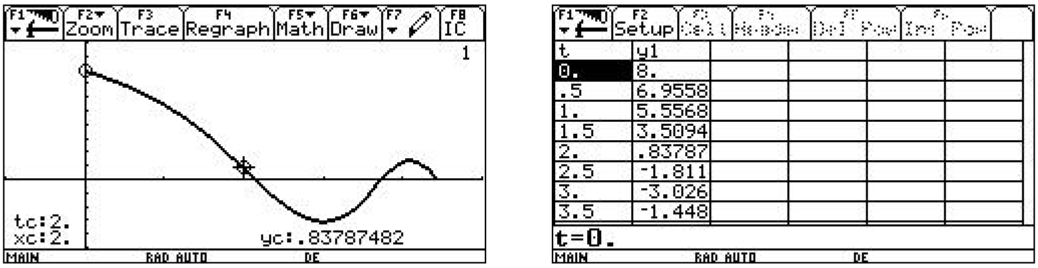
We enter these values in the Voyage 200 DE graphing mode (see figure 8 a). One can notice the particular way the initial conditions are entered ($t0 = 0$ and $yi1 = 8$ for $y1(0) = 8$). In figure 5 we see that we have chosen a graph window with *x* or *t*-axis going from -1 to 5 and the *y*-axis going from –5 to 10.



(a) **(b)**

Figure 8 Preparing to estimate $y(2)$ using RK method for *syst_eq*2

The step for this example will be 0.1 though this being an adaptive step implementation of a Runge-Kutta method the step could be smaller to ensure that the global error will not exceed a value of *diftol* = 0.001 when going from t0 = 0 to tmax = 5. In the F1 menu, one must also set the "Fields" option to "FLDOFF" to prevent the calculator from trying to plot a slope field, option we use when studying with them graphical solution of first order equations. Figure 9 shows the curve computed with this method and a table of values generated.



(a) **(b)**

Figure 9 Solution and table of values obtained using RK method for *syst_eq*2

The value 0.83787 was obtained using a step size of 0.1 with a tolerance of 0.001. Changing the tolerance to 0.0001 would provide $y(2) = 0.83991$. We can be confident with the value 0.84. Note also that the graph "stops" at $x = 4.5$ even though we asked to plot a solution up to $t = 5$! This method being adaptive and wanting to ensure a global tolerance, the calculator isn't capable of calculating a sufficiently small step, thus stopping calculations at the singularity of this equation which is 9/2. Cool!

4. Conclusion

This approach — using a mix of paper and pencil and technology — has many advantages. One among them is the fact that students are able to "discover" their CAS calculator when they have to do this kind of problem using 2 different graphic windows. We have to admit that the RK method is only used to check the results obtained by the power series method: we don't study RK in details (only Euler) but this can be done in a separated numerical analysis course. Another advantage of using the sequence graphing mode is contained in the following remark: students attending this ODE course had followed, earlier, a calculus course where they studied infinite series and convergence.

This is a nice topic where we can talk again of these results. They know that if a series of real numbers $\sum_{n=0}^{\infty} a_n$ converge, then we must have $\lim_{n\to\infty} a_n = 0$. The table of values in figure 3b) shows this but, in order to evaluate a series solution of an ODE at some point, "how long will it take" if we want to be confident with a partial sum? Experimenting with the CAS Voyage 200 gave us a satisfying answer.

Some authors, especially in Applied Differential Equations, will mention that efficient numerical approximation methods and access to computer and more powerful calculators could question the need for power series method in a first course in differential equations, in particular for solutions about a regular point. Some of them will even omit this subject [3]. We still think it's a good idea to link all these mathematic topics together and of course, a partial sum of a series solution (a polynomial) is still a great object to manipulate algebraically.

5. References

[1] Spiegel, Murray R *Applied Differential Equations.* 3rd Edition. Prentice Hall. 1981.

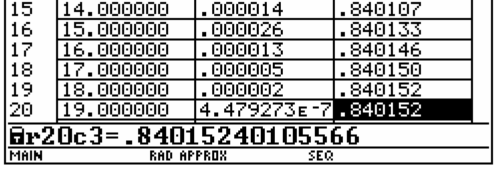
[2] P. Bogacki and L.F. Shampine. *A 3(2) Pair of Runge-Kutta Formulas*. Applied Math. Letters, vol. 2, pp. 1-9, 1989.

[3] Brannan, James R. and Boyce, William E.. *Differential Equations, An Introduction to Modern Methods and applications*. 1st Edition. Wiley. 2007.

Also, in the writing of this article, we took a look at some exams or exercises given by our colleagues to their students. Special thanks to our colleague Chantal Trottier.

When I came home from Málaga and a wonderful travel through Andalusia I studied Michel's and Gilles' paper. I had my own ideas for demonstrating the convergence behaviour of the power series approximation:

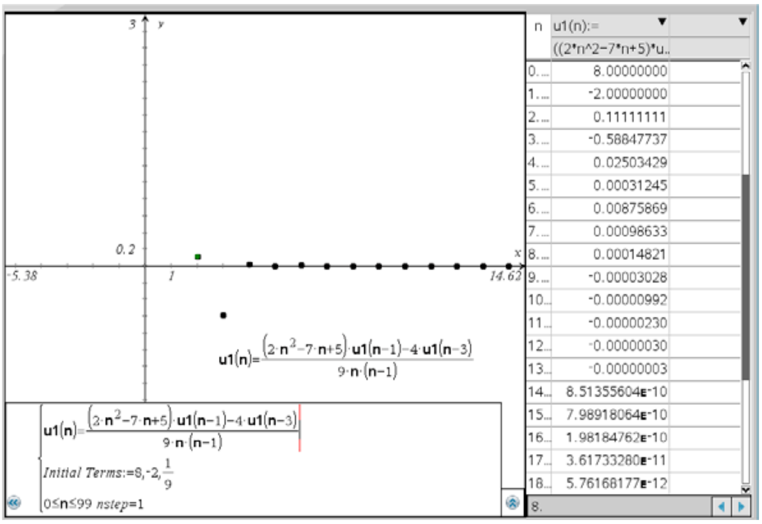
I set up a table in the Voyage 200 DATA-MATRIX Editor:



DATA	n	c[n]*a^n	Σ(c2(n))
	c1	c2	c3
14	13.000000	-.000214	.840093
15	14.000000	.000014	.840107
16	15.000000	.000026	.840133
17	16.000000	.000013	.840146
18	17.000000	.000005	.840150
19	18.000000	.000002	.840152
20	19.000000	4.479273E-7	.840152

This was in 2010. Now we have 2012 and I'd like to transfer the procedure to TI-NspireCAS first and then to DERIVE, too. Josef

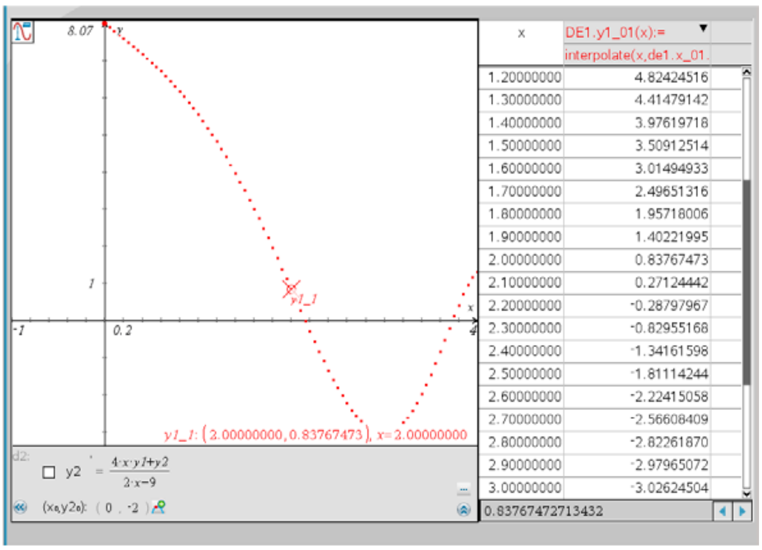
I defined the recurrence equation in the Graphs Application. Take care that the initial elements are given in the right order!



The spreadsheet is the appropriate application for showing the convergence behavior. The value for α can be changed in cell *A1*. The partial sums are given in column *E*.

	A alpha	B n	C coefficients	D summands	E approx_values
◆		=seq(k,k,0,40)			=cumulativesum(summands)
1	2	0	8.00000000	8.00000000	8.00000000
2		1	-2.00000000	-4.0000000...	4.00000000
3		2	0.11111111	0.44444444	4.44444444
4		3	-0.58847737	-4.7078189...	-0.26337449
5		4	0.02503429	0.40054870	0.13717421
6		5	0.00031245	0.00999848	0.14717269
7		6	0.00875869	0.56055592	0.70772861
8		7	0.00098633	0.12624995	0.83397856
9		8	0.00014821	0.03794155	0.87192011
10		9	-0.00003028	-0.0155030...	0.85641711
11		10	-0.00000992	-0.0101553...	0.84626178
12		11	-0.00000230	-0.0047140...	0.84154771
13		12	-0.00000030	-0.0012410...	0.84030664
14		13	-0.00000003	-0.0002140...	0.84009259
15		14	8.51355604E-...	0.00001395	0.84010653
16		15	7.98918064E-...	0.00002618	0.84013271
17		16	1.98184762E-...	0.00001299	0.84014570

The numerical solution of the DE is represented in another Graphs page:



In *DERIVE* I wanted to do a little bit more. I intended to do all the necessary calculations CAS supported:

$$\#1: \quad y(x) := c_n \cdot x^n$$

$$\#2: \quad (2 \cdot x - 9) \cdot y''(x) - y'(x) - 4 \cdot x \cdot y(x) = 0$$

$$\#3: \quad - x^{n-2} \cdot (4 \cdot x^3 + n \cdot x \cdot (3 - 2 \cdot n) + 9 \cdot n \cdot (n - 1)) \cdot c_n = 0$$

$$\#4: \quad 4 \cdot x^{n+1} \cdot c_n + n \cdot x^{n-1} \cdot (3 - 2 \cdot n) \cdot c_n + 9 \cdot n \cdot x^{n-2} \cdot (n - 1) \cdot c_n = 0$$

I omitted the summation symbols for finding the recurrence equation for the coefficients.

I performed substitions in the summands of #4 in order to have a common power x^n in all of them as follows: $n \to n - 1$ in the 1st expression, $n \to n + 1$ in the 2nd one and $n \to n + 2$ in the third one. The rewritten sum is given in #8 where I can factor out x^n:

$$\#5: \quad 4 \cdot x^n \cdot c_{n-1}$$

$$\#6: \quad x^n \cdot (n + 1) \cdot (1 - 2 \cdot n) \cdot c_{n+1}$$

$$\#7: \quad 9 \cdot x^n \cdot (n + 1) \cdot (n + 2) \cdot c_{n+2}$$

$$\#8: \quad 4 \cdot x^n \cdot c_{n-1} + x^n \cdot (n + 1) \cdot (1 - 2 \cdot n) \cdot c_{n+1} + 9 \cdot x^n \cdot (n + 1) \cdot (n + 2) \cdot c_{n+2}$$

#10 gives the recurrence equation – but not in its most comfortable form.

$$\#9: \quad x^n \cdot (9 \cdot (n + 1) \cdot (n + 2) \cdot c_{n+2} + 4 \cdot c_{n-1} + (n + 1) \cdot (1 - 2 \cdot n) \cdot c_{n+1})$$

$$\#10: \quad 9 \cdot (n + 1) \cdot (n + 2) \cdot c_{n+2} + 4 \cdot c_{n-1} + (n + 1) \cdot (1 - 2 \cdot n) \cdot c_{n+1}$$

The greatest index should be *n*.
So I substitute $n \to n - 2$ and obtain #12.

$$\#12: \quad 4 \cdot c_{n-3} + (1 - n) \cdot (2 \cdot n - 5) \cdot c_{n-1} + 9 \cdot n \cdot (n - 1) \cdot c_n$$

Solving this equation for c_n needs a little trick.

$$\#13: \quad \text{SOLVE}(4 \cdot c_{n-3} + (1 - n) \cdot (2 \cdot n - 5) \cdot c_{n-1} + 9 \cdot n \cdot (n - 1) \cdot x\_, \ x\_)$$

$$\#14: \quad x\_ = \frac{4 \cdot c_{n-3} + (1 - n) \cdot (2 \cdot n - 5) \cdot c_{n-1}}{9 \cdot n \cdot (1 - n)}$$

$$\#15: \quad c_n = \frac{4 \cdot c_{n-3} + (1 - n) \cdot (2 \cdot n - 5) \cdot c_{n-1}}{9 \cdot n \cdot (1 - n)}$$

It is not necessary now to calculate c_2 manually! I define the function for the coefficients, cf(n), recursively.

```
        cf(n) :=
          If n < 0
             0
             If n = 0
#16:            8
                If n = 1
                   -2
                   (4·cf(n - 3) + (1 - n)·(2·n - 5)·cf(n - 1))/(9·n·(1 - n))
```

And I show the first 5 coefficients of the power series expansion:

#17: TABLE(cf(n), n, 0, 5)

#18:

0	8
1	-2
2	$\frac{1}{9}$
3	$-\frac{143}{243}$
4	$\frac{73}{2916}$
5	$\frac{41}{131220}$

#19:

0	8
1	-2
2	0.1111111111
3	-0.5884773662
4	0.02503429355
5	0.00031245237

According to the original paper I define a function for the partial sums. Then I am able to calculate the approximations.

#20: $s(\alpha, n) := \sum_{k=0}^{n} cf(k)\cdot\alpha^k$

#21: $s(2, 4) = \frac{100}{729}$

#22: s(2, 4) = 0.1371742112

The table demonstrates the convergence behaviour:

#23: TABLE(s(2, k), k, 0, 10)

#24:

0	8
1	4
2	4.444444444
3	-0.2633744855
4	0.1371742112
5	0.147172687
6	0.7077286095
7	0.8339785601
8	0.8719201101
9	0.8564171052
10	0.8462617846

I wanted to know whether *DERIVE* is able to solve the DE?

#25: $\text{DSOLVE2\_IV}\left(\frac{3\cdot x}{x^2 + 4}, -\frac{4}{x^2 + 4}, 0, x, 8, -2\right)$

#26: inapplicable

Clear *DERIVE*-answer: No, I am not!

But DERIVE has a function available for performing the Taylor series approximation!

(I have to take *t* instead of *y* because *y*(*x*) has been defined above for another purpose; v = y').

#27: $\text{TAYLOR\_ODE2}\left(\dfrac{v + 4\cdot x\cdot t}{2\cdot x - 9}, x, t, v, 0, 8, -2, 5\right)$

#28: $\dfrac{41\cdot x^5}{131220} + \dfrac{73\cdot x^4}{2916} - \dfrac{143\cdot x^3}{243} + \dfrac{x^2}{9} - 2\cdot x + 8$

#29: $\dfrac{4828}{32805}$

#30: 0.1471726870

#29 is the result of substituting 2 in #28, and #30 is its numerical approximation.

#31: $\text{TAYLOR\_ODE2}\left(\dfrac{v + 4\cdot x\cdot t}{2\cdot x - 9}, x, t, v, 0, 8, -2, 10\right)$

#32: $-\dfrac{86064547\cdot x^{10}}{8678218953600} - \dfrac{1751803\cdot x^9}{57854793024} + \dfrac{529297\cdot x^8}{3571283520} + \dfrac{6989\cdot x^7}{7085880} + \dfrac{62063\cdot x^6}{7085880} + \dfrac{41\cdot x^5}{131220} + \dfrac{73\cdot x^4}{2916} -$

#33: $\lim_{x\to 2} \text{TAYLOR\_ODE2}\left(\dfrac{v + 4\cdot x\cdot t}{2\cdot x - 9}, x, t, v, 0, 8, -2, 10\right)$

#34: $\dfrac{57375352024}{67798585575}$

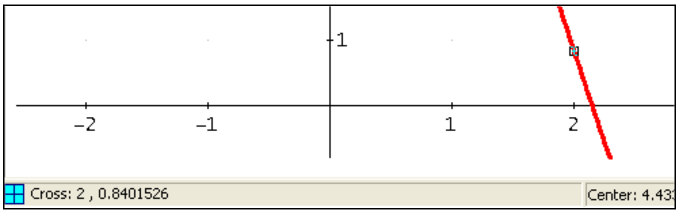
#35: 0.8462617846

#32 is truncated.

Finally I apply *DERIVE*'s implemented RK-method:

#36: [y1 :=, y2 :=]

#37: $\left(\text{RK}\left(\left[y2, \dfrac{4\cdot t\cdot y1 + y2}{2\cdot t - 9}\right], [t, y1, y2], [0, 8, -2], 0.005, 400\right)\right)\downarrow\downarrow[1, 2]$



Tracing the graph of the integral curve gives the value y(2) ≈ 0.84015.

The respective part of the function table is presented below.

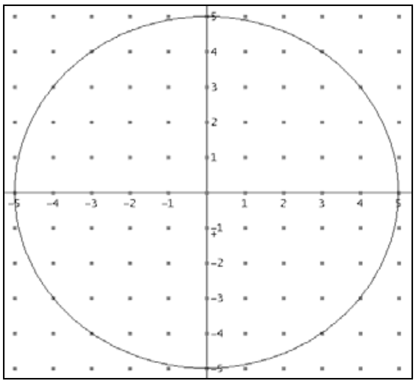
$\left(\text{RK}\left(\left[y2, \dfrac{4\cdot t\cdot y1 + y2}{2\cdot t - 9}\right], [t, y1, y2], [0, 8, -2], 0.005, 405\right)\right)_{[397,\ \ldots,\ 403]}\downarrow\downarrow[1, 2]$

1.98	0.9535140960
1.985	0.9251852125
1.99	0.8968479833
1.995	0.8685034306
2	0.8401525859
2.005	0.8117964894
2.01	0.7834361907

Points in a Circle

Roland Schröder

Grid points in a circle



Given is a circle with its center in (0,0) and its radius *r*. We will count all points with integer coordinates lying within the circle or on its circumference.

What is the relation between the radius of the circle and the number of these points?

We produce a matrix which contains a "1" for each point which is counted and a "0" for all points which are lying outside of the circle.

```
      F(r) :=
        If a^2 + b^2 ≤ r^2
#1:        1
           0

#2:   G(r) := VECTOR(VECTOR(F(r), a, -r, r), b, -r, r)
```

#3: G(4) =

$$\begin{bmatrix} 0&0&0&0&1&0&0&0&0\\ 0&0&1&1&1&1&1&0&0\\ 0&1&1&1&1&1&1&1&0\\ 0&1&1&1&1&1&1&1&0\\ 1&1&1&1&1&1&1&1&1\\ 0&1&1&1&1&1&1&1&0\\ 0&1&1&1&1&1&1&1&0\\ 0&0&1&1&1&1&1&0&0\\ 0&0&0&0&1&0&0&0&0 \end{bmatrix}$$

Nice pattern!!

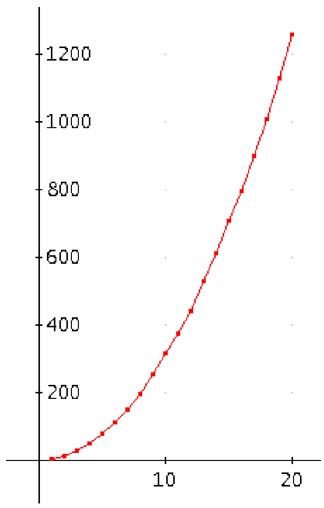
We add all the Ones from *G*(*r*) in order to obtain the number *P*(*r*) of the grid points within the circle with radius *r*.

```
#4:   P(r) := Σ(Σ(G(r)))

#5:   P(4) = 49
```

Next step is to create a value table for the relation *radius r* → *number of points P*(*r*):

```
#6:   VECTOR([r, P(r)], r, 20)
```

What is the equation of the graph corresponding with this relation?



Possible ways to answer might be:

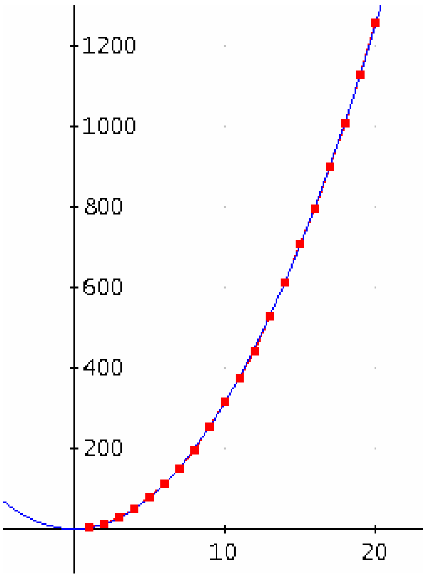
(1) One could assume that it is parabola of form $f(x) = a\,x^2$ and then try to find the apprpriate *a* by installing a slider.

(2) One could apply *DERIVE*'s FIT-function:

$$\text{FIT}\left(\begin{bmatrix} x, & a\cdot x^2 \end{bmatrix}, \text{VECTOR}([r, P(r)], r, 20)\right)$$

$$\frac{1129599\cdot x^2}{361333}$$

$$3.126199378\cdot x^2$$

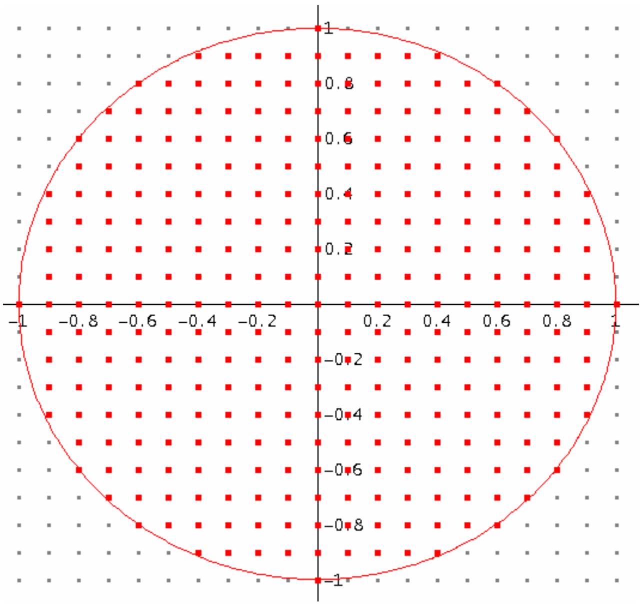


Grid points in the unit circle

We keep the radius constant $r = 1$ and increase the number of points by decreasing the distance between the points, i.e. we refine the grid:

```
#13:   H(x) := VECTOR(VECTOR(F(1), a, -1, 1, x), b, -1, 1, x)
```

For $x = 0.1$ we get:



Counting works now as follows:

```
#15:   count(x) := Σ(Σ(H(x)))

#16:   count(0.1) = 317
```

Each point counted is assigned a small square having the point as its center and the side length *x*. The sum of all areas of all these squares is given by:

```
#17:  area(x) := x^2 ·count(x)
#18:  area(0.1) = 3.17
#19:  area(0.01) = 3.1417
```

The value table for function *area*(*x*) can easily be created, either using the VECTOR command or applying the TABLE function:

$$\text{VECTOR}\left(\left[n,\ \text{area}\left(\frac{0.1}{n}\right)\right],\ n,\ 10\right)$$

1	3.17
2	3.1425
3	3.134444444
4	3.140625
5	3.138
6	3.135833333
7	3.137346938
8	3.13765625
9	3.141358024
10	3.1417

$$\text{TABLE}\left(\left[\frac{0.1}{n},\ \text{area}\left(\frac{0.1}{n}\right)\right],\ n,\ 1,\ 15\right)$$

1	0.1	3.17
2	0.05	3.1425
3	0.03333333333	3.134444444
4	0.025	3.140625
5	0.02	3.138
6	0.01666666666	3.135833333
7	0.01428571428	3.137346938
8	0.0125	3.13765625
9	0.01111111111	3.141358024
10	0.01	3.1417
11	0.009090909090	3.138925619
12	0.008333333333	3.140625
13	0.007692307692	3.140650887
14	0.007142857142	3.139234693
15	0.006666666666	3.141377777

Additional comments of the editor:

I find this a very nice investigation. It could be followed by the best known Monte Carlo method for approxmaing π.

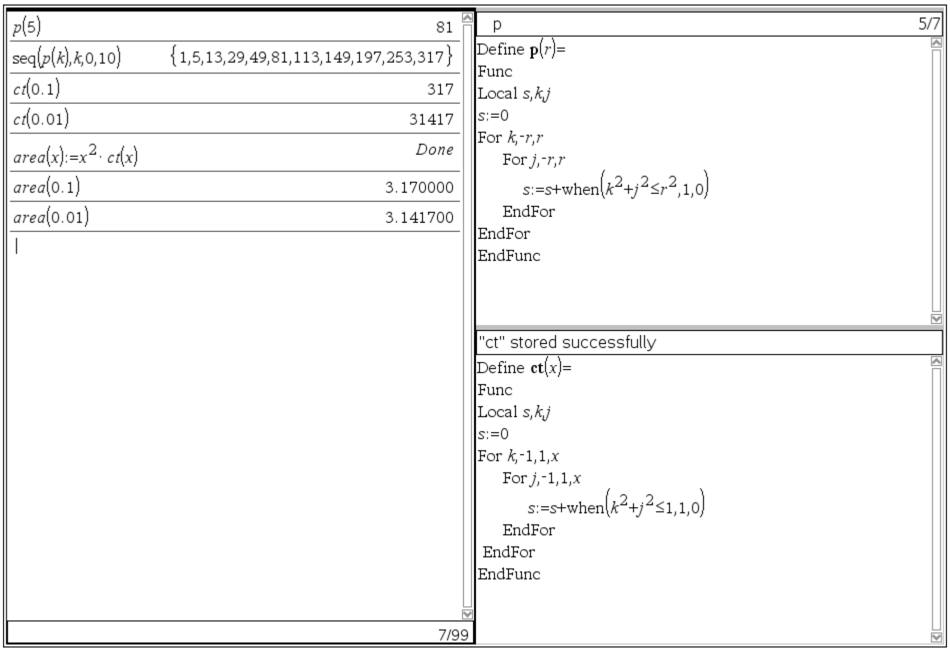
I wanted to reproduce Roland's investigation with TI-NspireCAS and was not able to do it in the same way because I am missing one of my favourite DERIVE commands: VECTOR.

So I developped two short functions *p*(*r*) and *ct*(*x*) and further *area*(*x*).

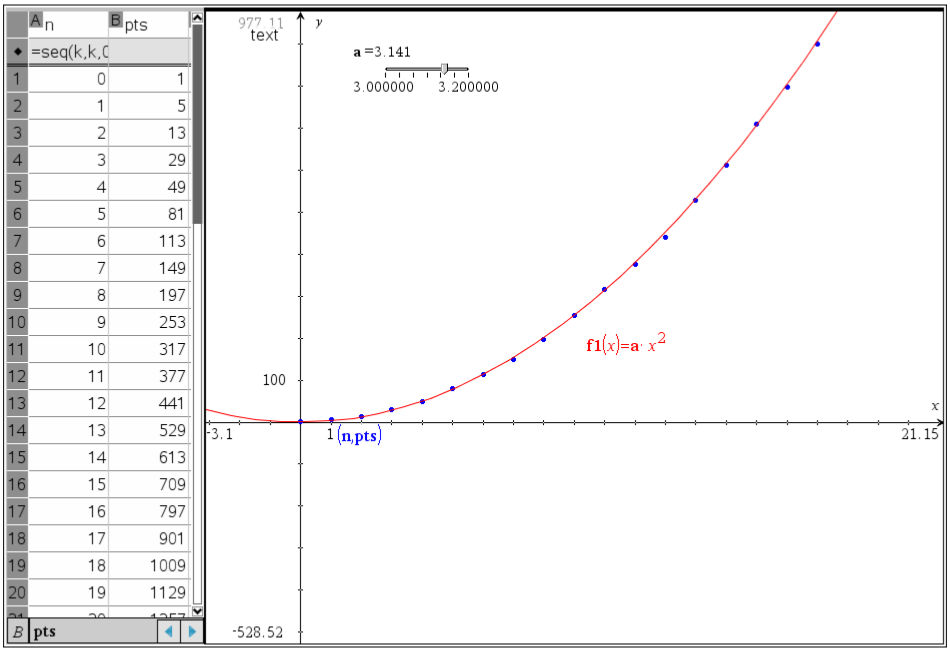
The Lists & Spreadsheet application and the Graphs Application (including a slider bar) were used. Instead of DERIVE's FIT function (which forced the form of the approximating parabola as $a \cdot x^2$ I applied a statistics tool, the quadratic regression.

See the respective screen shots on the next page, Josef

The Calculator App together with the two functions:



Part of the spreadsheet together with the scatter diagram and the approximating parabola followed by the output of the quadratic regression and the list of the approximated areas of the unit circle.



A n	B pts	C distance	D areas	E	F
=seq(k,l					=QuadReg('n,'pts,1):
0	1	_	_	Title	Quadratic Regressi...
1	5	0.100000	3.170000	RegEqn	a*x^2+b*x+c
2	13	0.050000	3.142500	a	3.147892
3	29	0.033333	3.132222	b	-0.517393
4	49	0.025000	3.140625	c	1.815249
5	81	0.020000	3.138000	R²	0.999979
6	113	0.016667	3.133611	Resid	{-0.81524926691,0....
7	149	0.014286	3.136939		

USING DERIVE TO SIMULATE THE BASIC STEPS OF

SHOR'S QUANTUM ALGORITHM

Nelson Urrego

Head of the Department of Mathematics
Gimnasio La Montaña
Bogotá, Colombia

nelsonurrego@glm.edu.co

Abstract

It is known that, at the moment, there is no efficient classic algorithm to factor large integers. However, in 1994 Peter Shor presented a procedure, which exploits quantum parallelism, and is able to efficiently factor large integers. The present work describes the basic steps of factoring Shor's algorithm, and how it can be simulated, in a simple way (for small integers), with a CAS like DERIVE.

Introduction

The problem of factoring large integers is generally considered as intractable from the Classic Theory of Computation. The goal is to find a number k that divides certain number n exactly. If $k = 1$ or $k = n$, we say that k is a trivial factor of n.

As it is known, several cryptosystems (like the famous RSA) base their security on the difficult computing problem for factoring large integers. At the present, the best classic algorithms to factor integers can execute the task over integers up to 150 digits with a quite demanding effort. The running time of these algorithms increases super polynomialy as the integer to be factored grows. One of the best algorithms for factoring (The Number Field Sieve) requires $\exp\left(\Theta(n^{1/3}\log^{2/3} n)\right)$ operations, to factor an n-bit integer. (Nielsen)

In 1994 Peter Shor discovered a quantum algorithm that shows that a quantum computer can solve the factoring problem exponentially faster than the best known algorithms of its counter part classic. With a quantum computer, the task can be accomplished by using $O(n^2 \log n \log\log n)$ operations, which represents an important theoretical result in Computer Science. (Shor)

In the first part of this work, we discuss a particular function, from Modular Arithmetic, that allows an approach to solve the factoring problem.

In the second part we briefly describe the most important stages of Shor's algorithm and how some of them can be simulated with DERIVE. It is clear that some important tools like quantum parallelism can not be simulated in a classic computer, but we can visualize the most important parts of the procedure with an easy example to factor n = 15 and see how the algorithm works.

A trick from Number Theory

Consider the function $f_n(a) = x^a \bmod n,$ where n is the number to factor and x is a random integer co-prime with n. The function $f_n(a)$ is periodic, which means that as you increase the argument a = 0, 1, 2, …, k, their values $f_n(0), f_n(1), f_n(2), ..., f_n(k)$ turn in some pattern for a given x. The number of values between such pattern is called the period or the order of x modulo n, and is denoted by r.

This implies that:

$$x^r \equiv 1 \bmod n \qquad (1)$$

If r is even we have:

$$(x^{r/2})^2 \equiv 1 \bmod n$$

$$(x^{r/2} - 1)(x^{r/2} + 1) \equiv 0 \bmod n \qquad (2)$$

and $(x^{r/2} - 1)(x^{r/2} + 1)$ is a factor of n.

Now unless that $x^{r/2} \equiv \pm 1 \bmod n,$ at least one of the terms on the left hand side in (2) has a non-trivial common factor with n. Hence by computing gcd($x^{r/2}$ – 1,n) and gcd($x^{r/2}$ + 1,n) we have a good chance to find a factor of n. The problem is that, at the present, there is no efficient algorithm to find r on a classic computer.

Steps of Shor's Algorithm

Shor's algorithm is an example of a quantum algorithm, this is, a procedure which can be carried out by using a quantum system.

The main goal of Shor's algorithm is to obtain efficiently the period r of the function $f_n(a) = x^a \bmod n.$ In order to accomplish this, the algorithm exploits quantum parallelism. Let n be the number to be factored, for instance n = 15.

Step 1

Pick a number q such that $2n^2 \le q \le 3n^2$. This guarantees that q is not so large.

This can be programmed with DERIVE:

```
           2                  2
q := 2·n   + RANDOM(3·n )

q := 243
```

Step 2

Pick a number x which is co-prime with n.

```
x := LOOP(x := RANDOM(n), IF(GCD(x, n) = 1, RETURN x, x := x + 1))

x := 13
```

Step 3

Create a single quantum memory register which we divide into two sets called *reg*1 and *reg*2. The combined register is represented by $|\,reg1, reg2\rangle$.

Step 4

Load *reg*1 with integers between 0 and $q-1$ and *reg*2 with zeros

```
reg1 := VECTOR(a + 1, a, -1, q - 1)

reg2 := VECTOR(0·m, m, 0, q - 1)
```

The combined state is represented by $\left|\psi\right\rangle = \frac{1}{\sqrt{q}}\sum_{a=0}^{q-1}\left|a,0\right\rangle$.

Step 5

Apply x^amod n to each number in *reg*1 and write the result in *reg*2. The combined register becomes $\left|\psi\right\rangle = \frac{1}{\sqrt{q}}\sum_{a=0}^{q-1}\left|a, x^a \bmod n\right\rangle$.

```
n := 15

                       a
reg2 := VECTOR(MOD(x  , n), a, 0, q - 1)

reg2 := [1, 13, 4, 7, 1, 13, 4, 7, 1, 13, 4, 7, 1, 13, 4, 7, 1, 13, ·
  7, 1, 13, 4, 7, 1, 13, 4, 7, 1, 13, 4, 7, 1, 13, 4, 7, 1, 13, 4,
  1, 13, 4, 7, 1, 13, 4, 7, 1, 13, 4, 7, 1, 13, 4, 7, 1, 13, 4, 7, 1
  13, 4, 7, 1, 13, 4, 7, 1, 13, 4, 7, 1, 13, 4, 7, 1, 13, 4, 7, 1, 1
  4, 7, 1, 13, 4, 7, 1, 13, 4, 7, 1, 13, 4, 7, 1, 13, 4, 7, 1, 13, ·
  7, 1, 13, 4, 7, 1, 13, 4, 7, 1, 13, 4, 7, 1, 13, 4, 7, 1, 13, 4]
```

In this sample, for $n = 15$, $q = 243$ and $x = 13$, it is easy to conclude that the Period $r = 4$, However we are interested in describing the complete algorithm in a general form and estimate the period by using the discrete Fourier transform.

Step 6

Measure the state in register *reg*2 and obtain certain *k*. In quantum computation this has the consequence of projecting the state of *reg*1 into a superposition of all values a′ such that $x^a \bmod n = k$.

```
                         RANDOM(q - 1)
MeasureReg2 := MOD(x                  , n)
```

The complete state of the quantum register is $|\psi\rangle = \frac{1}{\|A\|}\sum_{a\in A}|a',k\rangle$, where $A = \{a' : x^{a'} \bmod n = k\}$ and $\|A\|$ represents the number of elements in A.

```
                                      b
projectReg1 := VECTOR(IF(MOD(x , n) = MeasureReg2, b, 0), b, 0, q - 1)

ket(t) :=

               1    q - 1
reg1Ket := ——— ·   Σ     ket(t)
              √q    t=0

                           b
z := DIM(SELECT(MOD(x , n) = MeasureReg2, b, reg1))

                                    b
reg1measure := SELECT(MOD(x , n) = MeasureReg2, b, reg1)

  1    z
——— ·  Σ  ket(ELEMENT(reg1measure, d))
 √z   d=1
```

Step 7

Compute the discrete Fourier transform of the projected state in *reg*1:

$$|a'\rangle \rightarrow \frac{1}{\sqrt{q}}\sum_{c=0}^{q-1} e^{2\pi i a' c/q}\,|c\rangle, \text{ then } |\psi\rangle = \frac{1}{\|A\|}\sum_{a\in A}\frac{1}{\sqrt{q}}\sum_{c=0}^{q-1} e^{2\pi i a' c/q}\,|c,k\rangle$$

```
                            1          z    ⎡ 2·π·ί·ELEMENT(reg1measure, d)·c/q⎤
FourierReg1 := VECTOR( ——————————— ·  Σ    ⎣e                                  ⎦, c, 0, q - 1)
                          √(q·z)      d=1

VECTOR([s - 1, |ELEMENT(FourierReg1, s)|], s, 1, q - 1)
```

$$\begin{bmatrix} 0 & 0.4885520725 \\ 1 & 0.02309639933 \\ 2 & 0.0228938444 \\ 3 & 0.02255814073 \end{bmatrix}$$

$$\cdots$$

$$\begin{bmatrix} 239 & 0.02209210406 \\ 240 & 0.02255814073 \\ 241 & 0.0228938444 \\ 1 & 0.02309639933 \end{bmatrix}$$

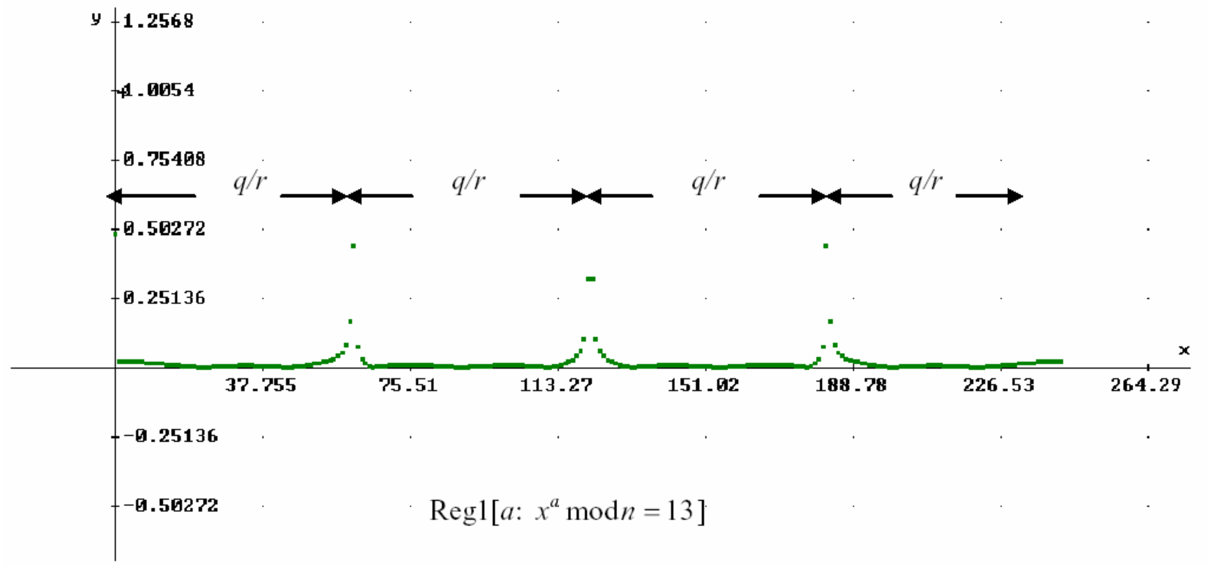


Figure 1 is the graph of register 1 which contains the set $\{a: x^a \; mod \; \text{n} = 13\}$ in the horizontal axis and ln(pr(measure a)) in the vertical axis.

Here the discrete Fourier transform projects the state of *reg*1 into a superposition

$$|\psi\rangle = \frac{1}{\|A\|}\sum_{a\in A}\frac{1}{\sqrt{q}}\sum_{c=0}^{q-1} e^{2\pi i a' c/q}\,|\,c,k\rangle.$$

Step 8

By measuring the state of *reg*1 we obtain samples of the discrete Fourier transform. The result is certain number t which is multiple of q/r (Figure 1). We repeat $O(\ln q)$ times the steps 1 - 7 to estimate period r.

In this case $r = 4$, therefore by computing $\gcd(13^{4/2} - 1, 15)$ and $\gcd(13^{4/2} + 1, 15)$ we obtain the desired factors 3 and 5 respectively.

Conclusion

Recently Shor's algorithm has been implemeted by using new technologies like the employ of spins in liquid solution. There is still a long way to cover before we can see a functional quantum computer, but several theoretical results have risen in the last decade, involving the discovery of quantum algorithms, which allow us approaching problems, which are not feasible for classic computers. Such results, along with other important outcomes in quantum cryptography (Urrego) make the quantum information processing a fascinating field to explore and to study.

Bibliography

Nielsen, Michael & Chuang, Isaac (2000). *Quantum Computation and Quantum Information.* Cambridge

Shor, Peter (1994). *Algorithms for Quantum Computation: Discrete Logarithms and Factoring.* Proceedings 35th Annual Symposium on Foundations of Computer Science, pp 124-134

Urrego, N. *Seguridad Criptográfica y Criptográfica Cuántica.* Ingeniería y Universidad. Vol 14. Enero 2003

Williams Collin & Clearwater Scott (1997). *Exploration in Quantum Computing.* Springer

I found some interesting websites, Josef.

Related websites:

http://arxiv.org/pdf/quant-ph/9508027v2.pdf (Peter Shor's original paper)

http://de.wikipedia.org/wiki/Shor-Algorithmus (in German)

http://www.worldcat.org/title/quantum-computation-and-quantum-information/oclc/43641333 (Nielsen & Chang's book)

http://en.wikipedia.org/wiki/Shor%27s_algorithm

http://katzgraber.org/teaching/FS08/files/herrigel.pdf

http://www.cs.washington.edu/education/courses/cse599d/06wi/lecturenotes11.pdf

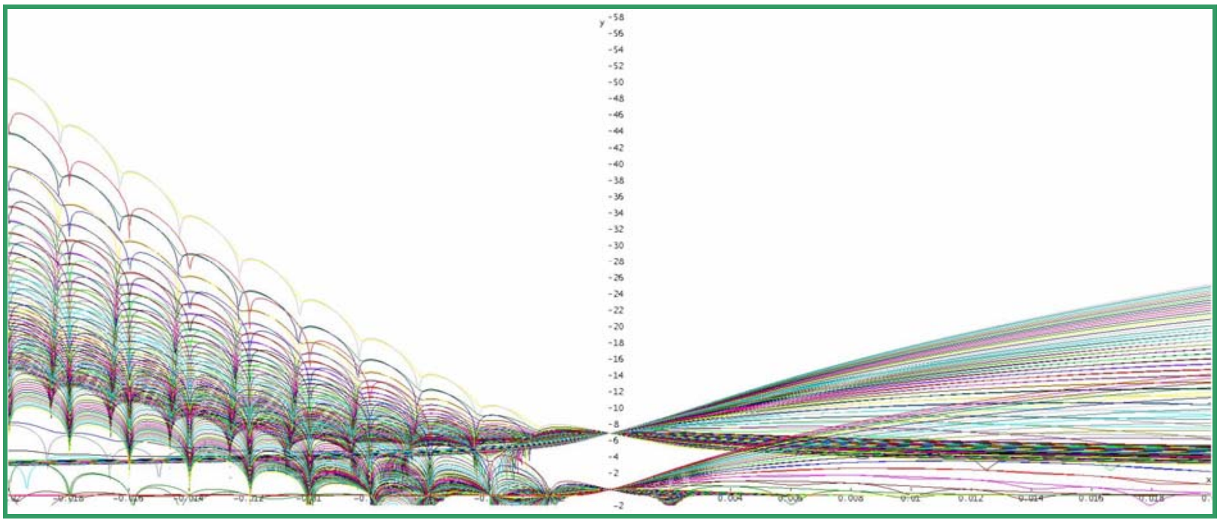
http://www.newscientist.com/blog/technology/2007/09/how-quantum-computer-factorises-numbers.html (A nice short explanation)

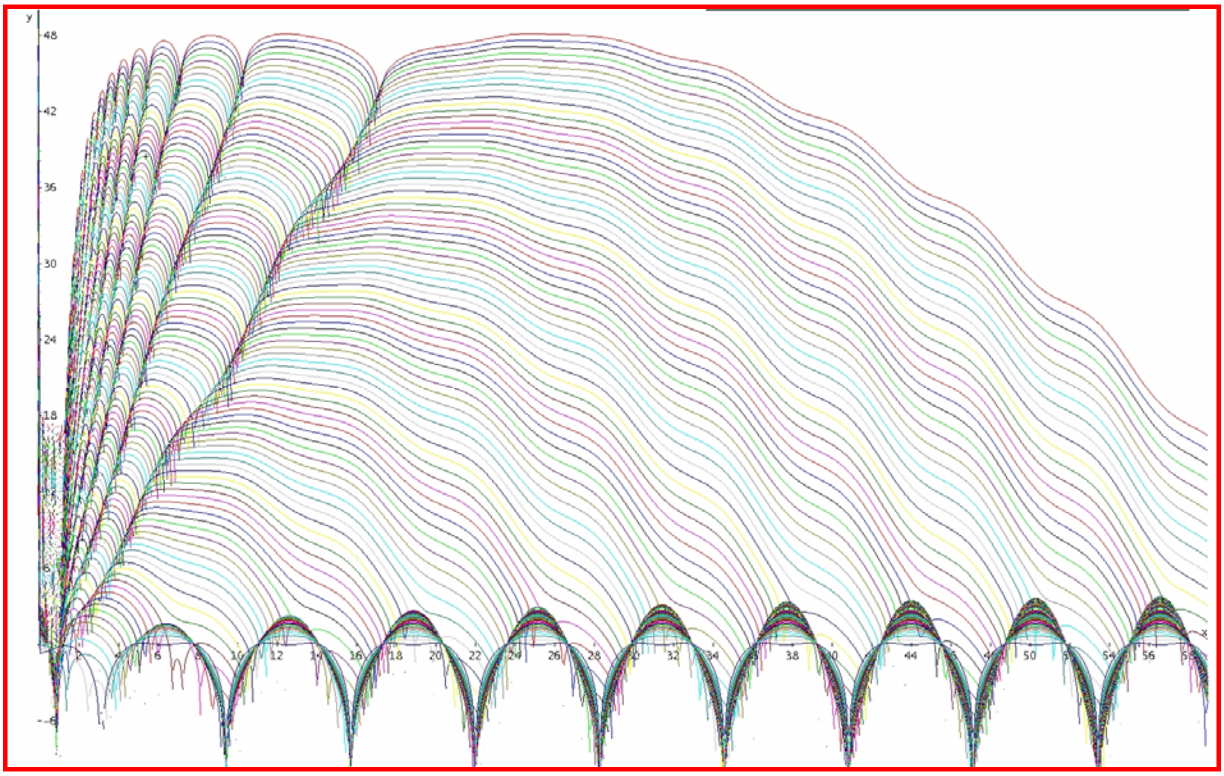
About Quantum Computers

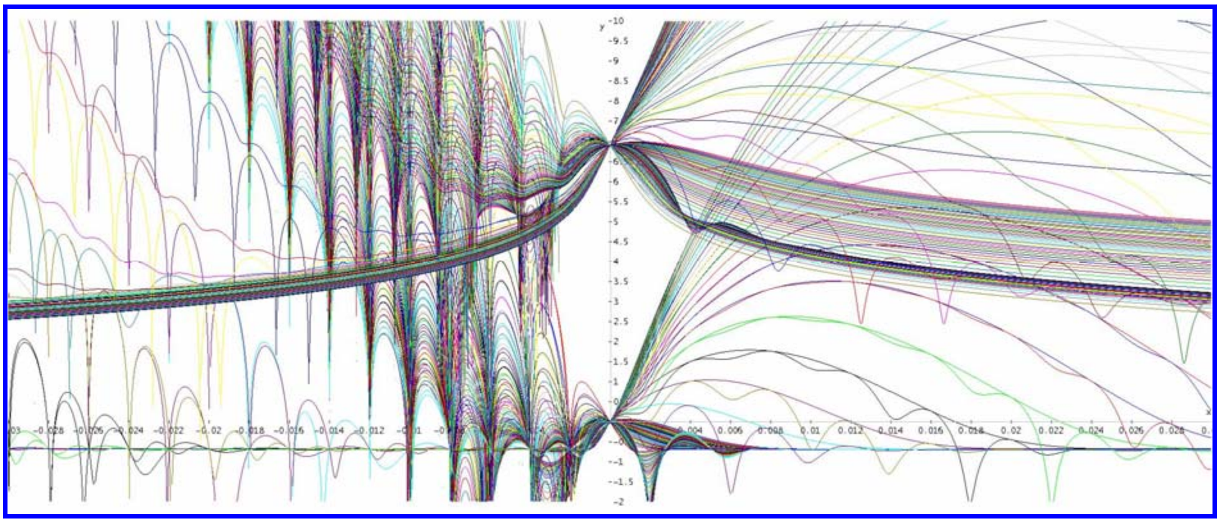
http://en.wikipedia.org/wiki/Quantum_computer

http://computer.howstuffworks.com/quantum-computer1.htm (How QuantumCompters Work)

Samples from Dietmar Oertel's exciting Taylor Series Gallery







In the last DNL we mentioned the TRIBONACCI-Sequence (Cubus Simus contribution).
I invited writing a function/program to produce this sequence.

With DERIVE we can do this performing a *RECURSION* or performing an *ITERATION*:

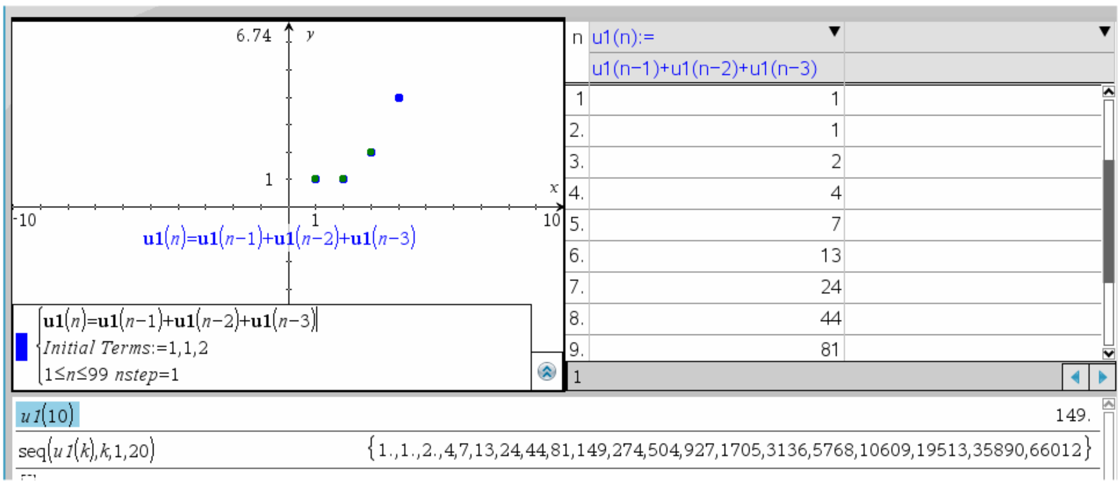
$$a_n = a_{n-1} + a_{n-2} + a_{n-3} \quad for \;\; n \geq 4$$
$$a_1 = a_2 = 1,\; a_3 = 2$$

```
tribo(n) :=
  If n = 0
     0
     If n = 1
        1
        If n = 2
           1
           If n = 3
              2
              tribo(n - 1) + tribo(n - 2) + tribo(n - 3)

VECTOR(tribo(k), k, 1, 10)

[1, 1, 2, 4, 7, 13, 24, 44, 81, 149]
```

Recursion (above) and iteration (below)

```
#1:  trib_seq(n) := (ITERATES([w_2, w_3, w_1 + w_2 + w_3], w, [1, 1, 2], n - 1))↓↓1

#2:  trib_seq(15)

#3:  [1, 1, 2, 4, 7, 13, 24, 44, 81, 149, 274, 504, 927, 1705, 3136]

#4:  trib_el(n) := (trib_seq(n))_n

#5:  trib_el(15) = 3136
```

With TI-Nspire we could write a program / function or we use the appropriate mode in the Graphs Application and produce the function table.



The sequence can be called in the Calculator App using the `seq`-command.

Dear Josef,

I cannot find an example of a function with an arbitrary number of arguments.

Thank you very much.

Best regards.

Robert Setif

I don't know if you are looking for an easy example or a more sophisticated one.
I add an example using DERIVE's FOO-function.

#1: foo v := [AVERAGE(v), VARIANCE(v)]

#2: $\text{foo}(1, 4, 8) = \left[\frac{13}{3}, \frac{37}{3}\right]$

#3: $\text{foo}(1, 4, 8, 10, 15) = \left[\frac{38}{5}, \frac{293}{10}\right]$

Dear Josef,
Thank you very much for your fast answer.
If you have time, please would you send me several or at least one example of a more sophisticated one.
Best regards

It would be great if anybody could send another example!

Dear Josef,

I need to construct a matrix of all zeros (32 × 32) and then go through the point plot that I make and depending on the location of the point, update the matrix by adding 1 to the corresponding position in the matrix. Nothing seems to happen. The matrix stays all zeros. I tried the small example to try and debug my larger program. I attached the small file. I tried several strategies with the same non-result. What am I missing? Is there a utility file that I need to load?

$A := \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 9 \end{bmatrix}$

$A_{2,3} := A_{2,3} + 1$

$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 9 \end{bmatrix}$

$\text{REPLACE}(A_{2,3} + 1, A_2, 3)$

$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 9 \end{bmatrix}$

See my answer:

#1: $A := \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 9 \end{bmatrix}$

#2: $A_{2,3} := A_{2,3} + 1$

Now simplify

#3: $A_{2,3} := 8$

The element has changed as expected:

#4: $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 8 & 9 \end{bmatrix}$

Or more directly:

Enter expression #2 again and press the =-button in the entry line
for immediately simplifying:

#5: $A_{2,3} := 9$

#6: $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 9 & 9 \end{bmatrix}$