THE DERIVE - NEWSLETTER #93

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THE BULLETIN OF THE

DERIVE

USER GROUP

+ CAS-TI

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May 2014

Useful Links for Generating Functions

www.mia.uni-saarland.de/Teaching/MFI0708/kap65.pdf

www.math.tugraz.at/~wagner/KombSkr.pdf

www.informatik.uni-bremen.de/~denneberg/Kombinatorik/Skript%20Kombinatorik.pdf

math-www.upb.de/MatheI_02/vorl/woche_16.pdf

www2.s-inf.de/Skripte/Diskrete.2001-SS-Hinrichs.%28JV%29.ErzeugendeFunktionen.pdf

www.mathdb.org/notes_download/elementary/algebra/ae_A11.pdf

www.cut-the-knot.org/blue/GeneratingFunctions.shtml

Links for the Padovan Sequence (very recommendable)

www.mathematik.uni-wuerzburg.de/~dobro/uhalt/u4.pdf

www.had2know.com/academics/perrin-padovan-sequence-plastic-constant-calculator.html

Our friend Wolfgang Pröpper discovered a nice photo in Spiegel-online:

Madeira Impressions

Dear DUG Members,

first of all I'd like to apologize once more for the extra long delay in publishing DNL#93. This is due to several reasons: an extended hiking holiday on the wonderful island of Madeira, a very busy time in preparing the TIME 2014 conference, a very intense exchange of mails with David Halprin with respect to his contribution on the Recursive Series (which has not yet ended ...).

David sent his mathematical CV some time ago. I believe that – especially in connection with his article – it might be of interest for all of us to read what can be done during the life of a mathematician. Many thanks David for your patience and cooperation during our communication. We wish that you can continue your explorations in many fields of mathematics for many years in the future.

David wrote that he had done his investigations in pre CAS times. So his original paper does not contain one single DERIVE – or other CAS – code. You can find some of his mails on this issue in the DNL.

It's funny that I came across an article written by Ian Stewart in a special issue of "Spektrum der Wissenschaft" = "Scientific American" dealing with other Tribonacci numbers, which I then tried to treat with David's findings and my DERIVE-routines.

Then we have another contribution from Roland Schröder involving number theory in generating trochoids.

Finally I wanted to publish a keynote address given by Adrian Oldknow at the occasion of the Gettysburg conference many years ago (1998). Unfortunately it is not included into the proceedings of this conference (which can be downloaded from http://rfdz.ph-noe.ac.at/acdca/acdca-conferences.html. Adrian used the TI-92 in his presentation. But all his examples are still valid using TI-Nspire CAS or any other CAS ...

I have to apologize to Dietmar Oertel for not continuing his papers so far. He sent many updates and additions which keeps me busy bringing all in the right order. Please be patient.

There came also in a couple of requests ready for our User Forum. They will be presented in DNL#94.

I wish the best until summer and hope to meet many of you at TIME 2014 in Krems.

Visit the TIME 2014 website and browse its program and the abstracts:

TIME 2014

1-5 July 2014, Krems, Austria
www.time2014.org

Download all *DNL*-DERIVE- and TI-files from
http://www.austromath.at/dug/

The *DERIVE-NEWSLETTER* is the Bulletin of the *DERIVE* & CAS-*TI User Group*. It is published at least four times a year with a content of 40 pages minimum. The goals of the *DNL* are to enable the exchange of experiences made with *DERIVE*, *TI*-CAS and other CAS as well to create a group to discuss the possibilities of new methodical and didactical manners in teaching mathematics.

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Contributions:
Please send all contributions to the Editor. Non-English speakers are encouraged to write their contributions in English to reinforce the international touch of the *DNL*. It must be said, though, that non-English articles will be warmly welcomed nonetheless. Your contributions will be edited but not assessed. By submitting articles the author gives his consent for reprinting it in the *DNL*. The more contributions you will send, the more lively and richer in contents the *DERIVE* & CAS-*TI Newsletter* will be.

Next issue: June 2014

Preview: Contributions waiting to be published

Some simulations of Random Experiments, J. Böhm, AUT, Lorenz Kopp, GER
Wonderful World of Pedal Curves, J. Böhm, AUT
Tools for 3D-Problems, P. Lüke-Rosendahl, GER
Hill-Encryption, J. Böhm, AUT
Simulating a Graphing Calculator in *DERIVE*, J. Böhm, AUT
Do you know this? Cabri & CAS on PC and Handheld, W. Wegscheider, AUT
An Interesting Problem with a Triangle, Steiner Point, P. Lüke-Rosendahl, GER
Graphics World, Currency Change, P. Charland, CAN
Cubics, Quartics – Interesting features, T. Koller & J. Böhm, AUT
Logos of Companies as an Inspiration for Math Teaching
Exciting Surfaces in the FAZ / Pierre Charland´s Graphics Gallery
BooleanPlots.mth, P. Schofield, UK
Old traditional examples for a CAS – what´s new? J. Böhm, AUT
Truth Tables on the TI, M. R. Phillips, USA
Where oh Where is It? (GPS with CAS), C. & P. Leinbach, USA
Embroidery Patterns, H. Ludwig, GER
Mandelbrot and Newton with *DERIVE*, Roman Hašek, CZK
Tutorials for the NSpireCAS, G. Herweyers, BEL
Some Projects with Students, R. Schröder, GER
Dirac Algebra, Clifford Algebra, D. R. Lunsford, USA
Treating Differential Equations (M. Beaudin, G. Piccard, Ch. Trottier), CAN
A New Approach to Taylor Series, D. Oertel, GER
Cesar Multiplication, G. Schödl, AUT
Henon & Co; Find your very own Strange Attractor, J. Böhm, AUT
Rational Hooks, J. Lechner, AUT
Simulation of Dynamic Systems with various Tools, J. Böhm, AUT
Space Curves with adjustable Curvature and Torsion, P. Trebisz, GER

and others

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Recursive Series of Numbers; An Umbral Look

David Halprin, davrin999@gmail.com

The first three number sequences, that one usually encounters in early school years are the Arithmetic Progression, the Geometric Progression and the Harmonic Progression. Although they are not usually referred to as recursive series, they are, in fact, recursive over the immediately previous term.

One can investigate them in various ways, and, for various reasons: –

e.g. 1) The three most popular goals, are to find the $(n+1)^{\text{th}}$ term, the sum of the first $(n+1)$ terms and the ratio of two neighbouring terms.

e.g. 2) As an illustrative example for this paper, it can be informative to follow the same approach for the geometric sequence, as one does with those sequences, that are recursive over two or more previous terms, as delineated below for Fibonacci, Lucas etc …

In overview, the algorithm is to assume the terms to be the coefficients of a Maclaurin series plus an initial constant term. At any stage, one can equate $x = 1$, provided it does not create a zero denominator. Because it is neater to have the exponent of the $(n+1)^{\text{th}}$ term to be n and the expression for the coefficient to be T_n , then one particular notational convention, that is popular amongst many, is to call the initial constant term, to be the zero$^{\text{th}}$ term, T_0 hence: –

$$S(x) = T_0 + T_1 x + T_2 x^2 + T_3 x^3 + \ldots + T_n x^n .$$

$$T_n = r \cdot T_{n-1}$$

$$S(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots + a_{n-1} x^{n-1} + a_n x^n \qquad n+1 \text{ terms}$$

$$r \cdot S(x) = r \cdot a_0 + r \cdot a_1 x + r \cdot a_2 x^2 + r \cdot a_3 x^3 + \ldots + r \cdot a_{n-1} x^{n-1} + r \cdot a_n x^n$$

$$= a_1 + a_2 x + a_3 x^2 + \ldots + a_n x^{n-1} + a_{n+1} x^n$$

N.B. This method produces its results with the variable x in both the $(n+1)^{\text{th}}$ term and the sum to $(n+1)$ terms. This form is retained for the polynomial series, but for the Fibonacci and Lucas sequences the x is equated to unity.

The two best-known number sequences, recursive over the two previous terms, are Fibonacci and Lucas. There are at least 5 ways of defining them: –

e.g. 1) Each term is the sum of the previous 2 terms etc. .

2) An equation, which determines the $(n+1)^{\text{th}}$ term.

3) The sum to the $(n+1)^{\text{th}}$ term.

4) The limiting value of the ratio of two successive terms.

5) The generating function.

All these types of series can be treated under the one generalisation, called a General Admixture Series, (G.A.S.), defined: –

1.0 $\quad T_n = p \cdot T_{n-2} + q \cdot T_{n-1} \qquad n \geq 2$

where $p = q = 1$ for Fibonacci & Lucas series and $T_0, T_1 = 0, 1$ for Fibonacci or 2, 1 for Lucas series.

So, now, to establish the various equations, that evaluate the $(n+1)^{th}$ term and the generating function.

These methods, being free from Calculus, are a preferable way of expressing the above for use in a computer program, such as Derive for Windows™ , on a simple home-computer.

It will be clear how great an advantage this approach will be for the number sequences.

Assume the existence of a power series, similar to a Maclaurin Series plus a constant first term: –

2.0 $S(x) = T_0 + T_1 \cdot x + T_2 \cdot x^2 + T_3 \cdot x^3 + \ldots + T_{n-1} \cdot x^{n-1} + T_n \cdot x^n$ for n+1 terms.

Now multiply $S(x)$ by $q \cdot x$:

2.1 $q \cdot S(x) \cdot x = q \cdot T_0 \cdot x + q \cdot T_1 \cdot x^2 + q \cdot T_2 \cdot x^3 + \ldots + q \cdot T_{n-1} \cdot x^n$

Now multiply $S(x)$ by $p \cdot x^2$:

2.2 $p \cdot S(x) \cdot x^2 = \quad p \cdot T_0 \cdot x^2 + p \cdot T_1 \cdot x^3 + p \cdot T_2 \cdot x^4 + \ldots + p \cdot T_{n-2} \cdot x^n$

Adding (and applying 1.0):

2.3

$$\begin{aligned} S(x) \cdot (q \cdot x + p \cdot x^2) &= q \cdot T_0 \cdot x + q \cdot T_1 \cdot x^2 + p \cdot T_0 \cdot x^2 + q \cdot T_2 \cdot x^3 + p \cdot T_1 \cdot x^3 + \ldots + p \cdot T_{n-2} \cdot x^n + q \cdot T_{n-1} \cdot x^n \\ &= q \cdot T_0 \cdot x + (p \cdot T_0 + q \cdot T_1) \cdot x^2 + (p \cdot T_1 + q \cdot T_2) \cdot x^3 + \ldots + (p \cdot T_{n-2} + q \cdot T_{n-1}) \cdot x^n \\ &= q \cdot T_0 \cdot x + T_2 \cdot x^2 + T_3 \cdot x^3 + \ldots + T_n \cdot x^n \\ &= q \cdot T_0 \cdot x + S(x) - T_0 - T_1 \cdot x \end{aligned}$$

therefore

2.4 $$S(x) = \frac{T_0 + x\,(T_1 - q \cdot T_0)}{1 - q \cdot x - p \cdot x^2}$$

For Fibonacci, where $T_0 = 0$ and $T_1 = 1$ and $p = q = 1$

2.5 $$S(x) = \frac{x}{1 - x - x^2}$$

For Lucas, where $T_0 = 2$ and $T_1 = 1$

2.6 $$S(x) = \frac{2 - x}{1 - x - x^2}$$

Josef's comment:

I must admit that I was not familiar with "generating functions" and I wondered what the $S(x)$ functions are good for? What have 2.5 and 2.6 to do with Fibonacci and Lucas sequences in common?

Look at the following short *DERIVE* output showing the Taylor expansion of 2.5 and 2.6:

$$\text{TAYLOR}\left(\frac{x}{1 - x - x^2}, x, 10\right)$$

$$55 \cdot x^{10} + 34 \cdot x^9 + 21 \cdot x^8 + 13 \cdot x^7 + 8 \cdot x^6 + 5 \cdot x^5 + 3 \cdot x^4 + 2 \cdot x^3 + x^2 + x$$

$$\text{TAYLOR}\left(\frac{2 - x}{1 - x - x^2}, x, 10\right)$$

$$123 \cdot x^{10} + 76 \cdot x^9 + 47 \cdot x^8 + 29 \cdot x^7 + 18 \cdot x^6 + 11 \cdot x^5 + 7 \cdot x^4 + 4 \cdot x^3 + 3 \cdot x^2 + x + 2$$

Reading the coefficients of the polynomials backwards we can find the elements of the Fibonacci sequence and the Lucas sequence respectively.

David gave an explanation and I will repeat it in other words:

> The *generating function* of a sequence <a_n> is a function with a power series representation at $x = 0$ with the a_n as its coefficients.

There are many applications for generating functions, e.g. enumeration problems. Among others g. F. are used to find explicit expressions for recursively defined sequences. I will show some examples at the end of David's article.

Let's continue with David's contribution.

Suppose we can factorise the denominator of the expression for $S(x)$ into two linear factors

3.0 $$1 - q\cdot x - p\cdot x^2 = (1 - a\cdot x)\cdot(1 - b\cdot x) = 1 - (a+b)\cdot x + a\cdot b\cdot x^2$$

where a and b are the reciprocals of the roots of the quadratic showing

3.1 $$a + b = q,\ a - b = \sqrt{q^2 + 4p},\ a\cdot b = -p$$

Then to take partial fractions: –

3.2 $$\frac{T_0 + x\cdot(T_1 - q\cdot T_0)}{1 - q\cdot x - p\cdot x^2} = \frac{C}{1 - a\cdot x} - \frac{D}{1 - b\cdot x}$$

Whence: –

3.3 $$C = \frac{T_1 - b\cdot T_0}{a - b},\ D = \frac{T_1 - a\cdot T_0}{a - b}$$

Let's do the expansion in partial fractions assisted by *DERIVE*.

$$\frac{x\cdot T_1 + (1 - x\cdot(a + b))\cdot T_0}{a\cdot b\cdot x^2 - x\cdot(a + b) + 1}$$

Simplify > Expand > Radical polynomial

$$\frac{T_1}{(b - a)\cdot(a\cdot x - 1)} + \frac{T_1}{(a - b)\cdot(b\cdot x - 1)} + \frac{b\cdot T_0}{(a - b)\cdot(a\cdot x - 1)} + \frac{a\cdot T_0}{(b - a)\cdot(b\cdot x - 1)}$$

$$\left[\frac{T_1}{(b - a)\cdot(a\cdot x - 1)} + \frac{b\cdot T_0}{(a - b)\cdot(a\cdot x - 1)},\ \frac{T_1}{(a - b)\cdot(b\cdot x - 1)} + \frac{a\cdot T_0}{(b - a)\cdot(b\cdot x - 1)}\right]$$

$$\left[C = \frac{T_1 - b\cdot T_0}{a - b},\ D = -\frac{T_1 - a\cdot T_0}{a - b}\right]$$

David uses the difference of the partial fractions in order to obtain "nicer" expressions for *C* and *D*.

At this moment I'd like to proceed with the special example Fibonacci sequence and demonstrate how to obtain the formula for the n^{th} element of this famous sequence using its generating function.

Remember that the power series of $\frac{1}{1-c\cdot x}=\sum_{n=0}^{\infty}c^n x^n$. So we can extend 3.2 to

$$\frac{T_0+x\cdot(T_1-q\cdot T_0)}{1-q\cdot x-p\cdot x^2}=\frac{C}{1-a\cdot x}+\frac{D}{1-b\cdot x}=C\cdot\sum_{n=0}^{\infty}a^n\cdot x^n+D\cdot\sum_{n=0}^{\infty}b^n\cdot x^n$$
$$=\sum_{n=0}^{\infty}\left(C\cdot a^n+D\cdot b^n\right)\cdot x^n$$

The coefficient of the n^{th} element of the power series for the generating function delivers the n^{th} Fibonacci number – and this coefficient is given by $C\cdot a^n+D\cdot b^n$, too. See now this procedure performed with *DERIVE*.

$$\frac{x}{1-x-x^2}$$

$$\frac{\sqrt{5}-5}{5\cdot(2\cdot x-\sqrt{5}+1)}-\frac{\sqrt{5}+5}{5\cdot(2\cdot x+\sqrt{5}+1)}$$

$$\text{TAYLOR}\left(\frac{\sqrt{5}-5}{5\cdot(2\cdot x-\sqrt{5}+1)}-\frac{\sqrt{5}+5}{5\cdot(2\cdot x+\sqrt{5}+1)},\ x,\ 5\right)$$

$$5\cdot x^5+3\cdot x^4+2\cdot x^3+x^2+x$$

$$\text{SOLUTIONS}(a+b=1\ \wedge\ a\cdot b=-1,\ [a,\ b])=\begin{bmatrix}\frac{\sqrt{5}}{2}+\frac{1}{2} & \frac{1}{2}-\frac{\sqrt{5}}{2}\\ \frac{1}{2}-\frac{\sqrt{5}}{2} & \frac{\sqrt{5}}{2}+\frac{1}{2}\end{bmatrix}$$

$$\left[C=\frac{T_1-b\cdot T_0}{a-b},\ D=-\frac{T_1-a\cdot T_0}{a-b}\right]$$

$$\left[C=\frac{1-b\cdot 0}{a-b},\ D=-\frac{1-a\cdot 0}{a-b}\right]$$

$$\left[C=\frac{1}{a-b},\ D=\frac{1}{b-a}\right]$$

$$\frac{1}{a-b}\cdot a^n-\frac{1}{a-b}\cdot b^n$$

$$\frac{1}{\left(\frac{1}{2}-\frac{\sqrt{5}}{2}\right)-\left(\frac{\sqrt{5}}{2}+\frac{1}{2}\right)}\cdot\left(\frac{1}{2}-\frac{\sqrt{5}}{2}\right)^n-\frac{1}{\left(\frac{1}{2}-\frac{\sqrt{5}}{2}\right)-\left(\frac{\sqrt{5}}{2}+\frac{1}{2}\right)}\cdot\left(\frac{\sqrt{5}}{2}+\frac{1}{2}\right)^n$$

$$\frac{\sqrt{5}\cdot\left(\frac{\sqrt{5}}{2}+\frac{1}{2}\right)^n}{5}-\frac{\sqrt{5}\cdot\left(\frac{\sqrt{5}}{2}-\frac{1}{2}\right)^n\cdot(-1)^n}{5}$$

$$\text{fib}(n):=\frac{\sqrt{5}\cdot\left(\frac{\sqrt{5}}{2}+\frac{1}{2}\right)^n}{5}-\frac{\sqrt{5}\cdot\left(\frac{\sqrt{5}}{2}-\frac{1}{2}\right)^n\cdot(-1)^n}{5}$$

VECTOR(fib(n), n, 0, 10) = [0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55]

David works much more generally and provides much more results:

3.4 $$(a-b)\cdot S(x)=\frac{T_1-b\cdot T_0}{1-a\cdot x}-\frac{T_1-a\cdot T_0}{1-b\cdot x}=$$

3.5 $$=(T_1-b\cdot T_0)(1+a\cdot x+a^2\cdot x^2+...+a^n\cdot x^n)-(T_1-a\cdot T_0)(1+b\cdot x+b^2\cdot x^2+...+b^n\cdot x^n)$$

3.6 $$=(a-b)\cdot T_0+(a-b)\cdot T_1\cdot x+(a^2-b^2)\cdot T_1\cdot x^2-ab(a-b)\cdot T_0\cdot x^2+...$$
$$...+[T_1\cdot(a^n-b^n)-T_0\cdot ab(a^{n-1}-b^{n-1})]\cdot x^n$$

Hence: –

4.0 $$T_n=\frac{[T_1(a^n-b^n)-T_0\cdot a\,b(a^{n-1}-b^{n-1})]}{a-b}$$

Remember $$a=\frac{q+\sqrt{q^2+4p}}{2},\ b=\frac{q-\sqrt{q^2+4p}}{2}$$

If $p<0$, then let $\cos\theta=\frac{q}{2\sqrt{-p}}$ for all q

4.1 $$T_n=\frac{(-p)^{\frac{n-1}{2}}\left\{T_1\sin(n\theta)-T_0\sqrt{-p}\sin[(n-1)\theta]\right\}}{\sin\theta}$$

Hence for Fibonacci: – (from equation 4.0)

4.2 $$F_n=\frac{a^n-b^n}{a-b}$$

and for Lucas: – (again from equation 4.0)

4.3 $$L_n=\frac{(a^n-b^n)+2(a^{n-1}-b^{n-1})}{a-b}=a^n+b^n$$ To be proved.

A verification followed by the proof (considering that T_0, T_1 = 2, 1 and a + b = 1):

$$\left(\frac{\sqrt{5}}{2}+\frac{1}{2}\right)^n+\left(\frac{1}{2}-\frac{\sqrt{5}}{2}\right)^n$$

$$\text{VECTOR}\left(\left(\frac{\sqrt{5}}{2}+\frac{1}{2}\right)^n+\left(\frac{1}{2}-\frac{\sqrt{5}}{2}\right)^n,\ n,\ 0,\ 10\right)$$

[2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123]

$$\frac{(a^n-b^n)+2\cdot(a^{n-1}-b^{n-1})}{a-b}$$

$$\frac{\left(\left(\frac{\sqrt{5}}{2}+\frac{1}{2}\right)^n-\left(\frac{1}{2}-\frac{\sqrt{5}}{2}\right)^n\right)+2\cdot\left(\left(\frac{\sqrt{5}}{2}+\frac{1}{2}\right)^{n-1}-\left(\frac{1}{2}-\frac{\sqrt{5}}{2}\right)^{n-1}\right)}{\left(\frac{\sqrt{5}}{2}+\frac{1}{2}\right)-\left(\frac{1}{2}-\frac{\sqrt{5}}{2}\right)}$$

$$\left(\frac{\sqrt{5}}{2}-\frac{1}{2}\right)^n\cdot(-1)^n+\left(\frac{\sqrt{5}}{2}+\frac{1}{2}\right)^n$$

$$\text{VECTOR}\left(\left(\frac{\sqrt{5}}{2}-\frac{1}{2}\right)^n\cdot(-1)^n+\left(\frac{\sqrt{5}}{2}+\frac{1}{2}\right)^n,\ n,\ 0,\ 10\right)$$

[2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123]

$$\frac{a^n - b^n - 2\cdot a\cdot b\cdot(a^{n-1} - b^{n-1})}{a - b} - (a^n + b^n)$$

$$\frac{a^n - (1-a)^n - 2\cdot a\cdot(1-a)\cdot(a^{n-1} - (1-a)^{n-1})}{a - (1-a)} - (a^n + (1-a)^n)$$

0

Taking 4.0 for the proof (considering that $a\cdot b = -1$) gives:

$$\frac{a^n - b^n + 2\cdot(a^{n-1} - b^{n-1})}{a - b} - (a^n + b^n)$$

$$\frac{a^n - (1-a)^n + 2\cdot(a^{n-1} - (1-a)^{n-1})}{a - (1-a)} - (a^n + (1-a)^n)$$

$$\frac{2\cdot(a^2 - a - 1)\cdot(a\cdot(1-a)^n + a^n\cdot(a-1))}{a\cdot(1-a)\cdot(2\cdot a - 1)}$$

$a^2 - a - 1$ is zero because a is a root of the denominator of equation 2.6. Q.E.D.

Performing the proof without CAS:

Let

$$\frac{T_1(a^n - b^n) - T_0 \cdot a\,b(a^{n-1} - b^{n-1})}{a-b} = (a^n + b^n) \quad \Big| \quad T_0 = 2, T_1 = 1$$

$$(a^n - b^n) - 2a\,(a^{n-1} - b^{n-1}) - (a-b)(a^n + b^n) = 0$$

$$a^n(1 - 2b - a + b) + b^n(-1 + 2a - a + b) = 0$$

$$a^n(1 - (a+b)) + b^n((a+b) - 1) = 0 \quad \Big| \quad a + b = q = 1$$

$0 = 0$ Q.E.D.

But let us have a more general look: –

If

5.0 $k(a-b) + T_{0.} \cdot b = T_0 \cdot a - k(a-b)$

$\therefore$ if $a \neq b$, then $T_0 = 2k$ and therefore

5.1 $k(a+b)(a-b) = T_1 \cdot (a-b)$ therefore

5.2 $k(a+b) = T_1,\ T_0 = 2k$

Let us see some consequent series of Lucas type: –

(1) $p = 1, q = 1, k = 2, T_0 = 4, T_1 = 2$ a multiple of Lucas

(2) $p = 1, q = 2, k = 2, T_0 = 4, T_1 = 4$ see below

(3) $p = 2, q = 1, k = 2, T_0 = 4, T_1 = 2$ see below

(4) $p = 1, q = 2, k = 0.5, T_0 = 1, T_1 = 1$ see below

Just for fun, we will try with *DERIVE* again.

For (1) we can apply *a* and *b* from above because *p* = *q* = 1, for (2) we have to recalculate *a* and *b* using the algorithm from above:

(1) The multiple of Lucas

$$\text{VECTOR}\left(2\cdot\left(\left(\frac{\sqrt{5}}{2}+\frac{1}{2}\right)^n+\left(\frac{1}{2}-\frac{\sqrt{5}}{2}\right)^n\right), n, 0, 5\right) = [4, 2, 6, 8, 14, 22]$$

(2) I start again with the partial fraction decomposition:

$$\frac{T_0 + x\cdot(T_1 - q\cdot T_0)}{1 - q\cdot x - p\cdot x^2}$$

$$\frac{4\cdot(x-1)}{x^2+2\cdot x-1}$$

$$\frac{2\cdot(1-\sqrt{2})}{x-\sqrt{2}+1}+\frac{2\cdot(\sqrt{2}+1)}{x+\sqrt{2}+1}$$

$$\text{TAYLOR}\left(\frac{2\cdot(1-\sqrt{2})}{x-\sqrt{2}+1}+\frac{2\cdot(\sqrt{2}+1)}{x+\sqrt{2}+1}, x, 0, 5\right)$$

$$164\cdot x^5+68\cdot x^4+28\cdot x^3+12\cdot x^2+4\cdot x+4$$

$$\text{SOLUTIONS}(a+b=2 \wedge a\cdot b=-1, [a, b]) = \begin{bmatrix}\sqrt{2}+1 & 1-\sqrt{2}\\ 1-\sqrt{2} & \sqrt{2}+1\end{bmatrix}$$

$$\left[C = \frac{T_1 - b\cdot T_0}{a-b}, D = -\frac{T_1 - a\cdot T_0}{a-b}\right]$$

$$\left[C = \frac{4\cdot(1-b)}{a-b}, D = \frac{4\cdot(a-1)}{a-b}\right]$$

$$\frac{4\cdot(1-b)}{a-b}\cdot a^n + \frac{4\cdot(a-1)}{a-b}\cdot b^n$$

$$\frac{4\cdot(1-(1-\sqrt{2}))}{(\sqrt{2}+1)-(1-\sqrt{2})}\cdot(\sqrt{2}+1)^n + \frac{4\cdot((\sqrt{2}+1)-1)}{(\sqrt{2}+1)-(1-\sqrt{2})}\cdot(1-\sqrt{2})^n$$

$$2\cdot(\sqrt{2}-1)^n\cdot(-1)^n + 2\cdot(\sqrt{2}+1)^n$$

$$\text{VECTOR}(2\cdot(\sqrt{2}-1)^n\cdot(-1)^n + 2\cdot(\sqrt{2}+1)^n, n, 0, 5) = [4, 4, 12, 28, 68, 164]$$

Series (3)

$$\text{VECTOR}(2\cdot(-1)^n + 2\cdot 2^n, n, 0, 5) = [4, 2, 10, 14, 34, 62]$$

Series (4)

$$\text{VECTOR}\left(\frac{1}{2}\cdot(\sqrt{2}+1)^n+\frac{1}{2}\cdot(1-\sqrt{2})^n, n, 0, 5\right) = [1, 1, 3, 7, 17, 41]$$

These are all of Lucas type of series because: –

$$T_n = k\,(a^n+b^n),\ T_0 \neq 0,\ T_1 = k\,(a+b) = k\cdot q,\ T_0 = 2k$$

If we wish to have three consecutive terms forming a Pythagorean triple, we take: –

6.0 $T_{n-2}^2 + T_{n-1}^2 = T_n^2$ where

6.1 $p \cdot T_{n-2} + q \cdot T_{n-1} = T_n$

6.2 If $p = \frac{T_{n-2}}{T_n},\ q = \frac{T_{n-1}}{T_n}$ then

6.3 $\frac{T_{n-2}^2}{T_n} + \frac{T_{n-1}^2}{T_n} = T_n$ Q.E.D.

e.g. 3, 4, 5, 6.4, …; 5, 12, 13, 256/13, …; 8, 15, 17, 375/17, ...

Now to consider the special case where the roots of the quadratic denominator of the generating function are equal, i.e. $a = b$. Rather than rework the whole exercise, use the rule of L'Hospital.

7.0 $T_n = \frac{[T_1(a^n - b^n) - T_0 \cdot a \cdot b(a^{n-1} - b^{n-1})]}{a-b}$

7.1 $\lim_{b \to a} T_n = T_1 \cdot n \cdot a^{n-1} - T_0 \cdot b(n \cdot a^{n-1} - b^{n-1})$

which in case of a Fibonacci series, reduces to $T_n = n$.

with $T_0 = 0$, $T_1 = 1$, $p = -1$, $q = 2$, $a = b = 1$ we obtain 0, 1, 2, 3, 4, 5 etc.

$$\text{TAYLOR}\left(\frac{1}{(1-x)^2}, x, 0, 5\right) = 6 \cdot x^5 + 5 \cdot x^4 + 4 \cdot x^3 + 3 \cdot x^2 + 2 \cdot x + 1$$

Possibly, we may have already thought of the natural numbers as constituting a recursive series, but not realised how much they have in common with the Fibonacci series. This will also be shown later with Tribonacci series etc..

Now for some examples of series with rational values for a, b. In order that the discriminant be a perfect square,

Let $\Delta = (2m + 1)^2$, where $p = m^2 + m$, $q = 1$, $k = 1$ and m is any integer greater than zero, for a Lucas-type of series.

NAME	m	p	q	a	b	SERIES
BiLucas	1	2	1	2	–1	2, 1, 5, 7, 17, 31, 65, 127, 257, 511, 1025, ...
HexaLucas	2	6	1	3	–2	2, 1, 13, 19, 97, 211, 793, ...
DodecaLucas	3	12	1	4	–3	2, 1, 25, 37, 337, 781, ...
IcosaLucas	4	20	1	5	–4	2, 1, 41, 61, 881, ...
find your own						...

The generating functions for DodecaLucas and IcosaLucas together with the closed formulae for their n^{th} element:

$$\text{TAYLOR}\left(\frac{2-x}{1-x-12\cdot x^2}, x, 0, 5\right) = 781\cdot x^5 + 337\cdot x^4 + 37\cdot x^3 + 25\cdot x^2 + x + 2$$

$$\text{VECTOR}(4^n + (-3)^n, n, 0, 10) = [2, 1, 25, 37, 337, 781, 4825, 14197, 72097, 242461, 1107625]$$

$$\text{TAYLOR}\left(\frac{2-x}{1-x-20\cdot x^2}, x, 0, 5\right) = 2101\cdot x^5 + 881\cdot x^4 + 61\cdot x^3 + 41\cdot x^2 + x + 2$$

$$\text{VECTOR}(5^n + (-4)^n, n, 0, 10) = [2, 1, 41, 61, 881, 2101, 19721, 61741, 456161, 1690981, 10814201]$$

Now for some Fibonacci-type with *a* and *b* being complex, (see FibComp-n in table below).

This can be calculated from either $T_n = \dfrac{a^n - b^n}{a-b}$ or by using a substitution $\cos(\theta) = \dfrac{q}{2\sqrt{-p}}$ which leads to two representations in trig-form which better illustrates the periodicity either using (*a*, *b*) or (*p*, *q*):

$$T_n = \frac{(ab)^{\frac{n-1}{2}} \cdot \sin(n \cdot \text{phase}(a))}{\sin(\text{phase}(a))} \text{ or } T_n = \frac{(-p)^{\frac{n-1}{2}} \cdot \sin\left(n \cdot \cos^{-1}\dfrac{q}{2\sqrt{-p}}\right)}{\sin\left(\cos^{-1}\dfrac{q}{2\sqrt{-p}}\right)}.$$

NAME	p	q	a	b	θ	SERIES
FibComp-1	–2	2	$1+i$	$1-i$	$\frac{\pi}{4}$	0, 1, 2, 2, 0, –4, –8, ...
FibComp-2	–13	4	$2+3i$	$2-3i$	$\cos^{-1}\left(\frac{2}{\sqrt{13}}\right)$	0, 1, 4, 3, –40,–199, ...
FibComp-3	–1	1	$\frac{1}{2}\left(1+i\sqrt{3}\right)$	$\frac{1}{2}\left(1-i\sqrt{3}\right)$	$\frac{\pi}{3}$	0, 1, 1, 0, –1, –1, 0, 1, ...
FibComp-4	–1	$\sqrt{3}$	$\frac{1}{2}\left(\sqrt{3}+i\right)$	$\frac{1}{2}\left(\sqrt{3}-i\right)$	$\frac{\pi}{6}$	$0, 1, \sqrt{3}, 2, \sqrt{3}, 1, 0, 1, \ldots$
FibComp-5	$-\frac{11}{2}$	$-\frac{7}{2}$	$\frac{1}{4}\left(-7+i\sqrt{39}\right)$	$\frac{1}{4}\left(-7-i\sqrt{39}\right)$	$\cos^{-1}\left(-\frac{7\sqrt{22}}{44}\right)$	$0, 1, -\frac{7}{2}, \frac{27}{4}, -\frac{35}{8}, \ldots$

I will use *DERIVE* again to generate the presented Fibo-type series using these three representations of the explicit formulae:

$$\text{T}(a, b, n) := \frac{a^n - b^n}{a - b}$$

$$\text{TT}(a, b, n) := \frac{(a\cdot b)^{(n-1)/2}\cdot \text{SIN}(n\cdot \text{PHASE}(a))}{\text{SIN}(\text{PHASE}(a))}$$

$$\text{TTT}(p, q, n) := \frac{(-p)^{(n-1)/2}\cdot \text{SIN}\left(n\cdot \text{ACOS}\left(\dfrac{q}{2\cdot\sqrt{(-p)}}\right)\right)}{\text{SIN}\left(\text{ACOS}\left(\dfrac{q}{2\cdot\sqrt{(-p)}}\right)\right)}$$

```
VECTOR(T(1 + i, 1 - i, k), k, 0, 10) = [0, 1, 2, 2, 0, -4, -8, -8, 0, 16, 32]
VECTOR(TT(1 + i, 1 - i, k), k, 0, 10) = [0, 1, 2, 2, 0, -4, -8, -8, 0, 16, 32]
VECTOR(TTT(-2, 2, k), k, 0, 10) = [0, 1, 2, 2, 0, -4, -8, -8, 0, 16, 32]
VECTOR(T(2 + 3·i, 2 - 3·i, k), k, 0, 10) = [0, 1, 4, 3, -40, -199, -276, 1483, 9520, 18801, -48556]
VECTOR(TT(2 + 3·i, 2 - 3·i, k), k, 0, 10) = [0, 1, 4, 3, -40, -199, -276, 1483, 9520, 18801, -48556]
VECTOR(TTT(-13, 4, k), k, 0, 10) = [0, 1, 4, 3, -40, -199, -276, 1483, 9520, 18801, -48556]
```

$$\text{VECTOR}\left(T\left(\frac{1}{2}\cdot(1+i\cdot\sqrt{3}),\ \frac{1}{2}\cdot(1-i\cdot\sqrt{3}),\ k\right),\ k,\ 0,\ 10\right) = [0, 1, 1, 0, -1, -1, 0, 1, 1, 0, -1]$$

$$\text{VECTOR}\left(TT\left(\frac{1}{2}\cdot(1+i\cdot\sqrt{3}),\ \frac{1}{2}\cdot(1-i\cdot\sqrt{3}),\ k\right),\ k,\ 0,\ 10\right) = [0, 1, 1, 0, -1, -1, 0, 1, 1, 0, -1]$$

```
VECTOR(TTT(-1, 1, k), k, 0, 10) = [0, 1, 1, 0, -1, -1, 0, 1, 1, 0, -1]
```

You are invited to check the remaining two “Fib-Comps” and to create your own series.

It has been shown, thus far, that if a series is to be described a ‘Fibonacci-type’ then its n^{th} term must have the form

$$T_n = \frac{a^n - b^n}{a-b}.$$

If a series is to be described as a ‘Lucas-type’ then its n^{th} term must be of the form

$$T_n = k\cdot(a^n + b^n).$$

So that the numerator of the n^{th} term in a Fibonacci series does not contain the term with factor T_0, then T_0 must be 0, however in some problems this may appear to be not so, but to simplify calculations, it may be assumed; viz: –

If we are given 2, 3, 5, 8, and asked to find a further terms, we could call $T_0 = 2$ and $T_3 = 8$ then

$$T_4 = \frac{3(a^4-b^4) - 2a\cdot b\,(a^3-b^3)}{a-b} = \frac{(9\sqrt{5}+4\sqrt{5})}{\sqrt{5}} = 13 \qquad \left(\text{with } a = \frac{1+\sqrt{5}}{2}, b = \frac{1-\sqrt{5}}{2}\right)$$

or we could call $T_6 = 8$ whence $T_7 = \dfrac{832\sqrt{5}}{64\sqrt{5}} = 13.$

The Sum of a Recursive Series

If the recursion is $T_n = p\cdot T_{n-2} + q\cdot T_{n-1}$ then

8.0 $\quad p\cdot T_{Sum} = p\cdot T_0 + p\cdot T_1 + p\cdot T_2 + \ldots + p\cdot T_{n-1} + p\cdot T_n$

8.1 $\quad q\cdot T_{Sum} = q\cdot T_0 + q\cdot T_1 + q\cdot T_2 + q\cdot T_3 + \ldots + q\cdot T_n$

8.2 $\quad (p+q)\cdot T_{Sum} = q\cdot T_0 + T_2 + T_3 + T_4 + \ldots + T_n + T_{n+1} + p\cdot T_n$

8.3 $\quad = q\cdot T_0 + T_{Sum} - T_0 - T_1 + T_{n+1} + p\cdot T_n$

8.4 $\quad = (q-1)\cdot T_0 + T_{Sum} - T_1 + p\cdot T_n + T_{n+1}$

therefore

8.5 $\quad T_{Sum} = \dfrac{(q-1)\cdot T_0 + p\cdot T_n + T_{n+1} - T_1}{p+q-1} \qquad \text{iff } p+q \neq 1$

For Fibonacci Series and Lucas Series:

$$T_{Sum} = T_n + T_{n+1} - 1\,.$$

It's just for fun, demonstrating the generalized sum formulae with TI-NspireCAS (applying equations 3.1 for 8.5):

$f(a,b,n):=\frac{a^n-b^n}{a-b}$	*Fertig*
$fibsum(a,b,n):=\frac{-a\cdot b\cdot f(a,b,n)+f(a,b,n+1)-1}{a+b-1-a\cdot b}$	*Fertig*
$\mathrm{seq}\left(f\left(\frac{1+\sqrt{5}}{2},\frac{1-\sqrt{5}}{2},k\right),k,0,10\right)$	$\{0,1,1,2,3,5,8,13,21,34,55\}$
$\mathrm{sum}(\{0,1,1,2,3,5,8,13,21,34,55\})$	143
$f\left(\frac{1+\sqrt{5}}{2},\frac{1-\sqrt{5}}{2},10\right)+f\left(\frac{1+\sqrt{5}}{2},\frac{1-\sqrt{5}}{2},11\right)-1$	143
$fibsum\left(\frac{1+\sqrt{5}}{2},\frac{1-\sqrt{5}}{2},10\right)$	143
© Let's try the complex Fibonacci:	
$\mathrm{seq}(f(2+3\cdot i,2-3\cdot i,k),k,0,10)$	$\{0,1,4,3,-40,-199,-276,1483,9520,18801,-48556\}$
$\mathrm{sum}(\{0,1,4,3,-40,-199,-276,1483,9520,18801,-48556\})$	-19259
$fibsum(2+3\cdot i,2-3\cdot i,10)$	-19259

$\mathrm{sum}(\{0,1,4,3,-40,-199,-276,1483,9520,18801,-48556\})$	-19259
$fibsum(2+3\cdot i,2-3\cdot i,10)$	-19259
$l(a,b,n):=a^n+b^n$	*Fertig*
$lucsum(a,b,n):=\frac{-a\cdot b\cdot l(a,b,n)+l(a,b,n+1)+2\cdot a+2\cdot b-3}{a+b-1-a\cdot b}$	*Fertig*
$\mathrm{seq}\left(l\left(\frac{1+\sqrt{5}}{2},\frac{1-\sqrt{5}}{2},k\right),k,0,10\right)$	$\{2,1,3,4,7,11,18,29,47,76,123\}$
$\mathrm{sum}(\{2,1,3,4,7,11,18,29,47,76,123\})$	321
$l\left(\frac{1+\sqrt{5}}{2},\frac{1-\sqrt{5}}{2},10\right)+l\left(\frac{1+\sqrt{5}}{2},\frac{1-\sqrt{5}}{2},11\right)-1$	321
$lucsum\left(\frac{1+\sqrt{5}}{2},\frac{1-\sqrt{5}}{2},10\right)$	321
© Let's try DodecaLucas:	
$\mathrm{seq}(l(4,-3,k),k,0,10)$	$\{2,1,25,37,337,781,4825,14197,72097,242461,1107625\}$
$\mathrm{sum}(\{2,1,25,37,337,781,4825,14197,72097,242461,1107625\})$	1442388
$lucsum(4,-3,10)$	1442388

Recursion over Previous Three Terms

Let us now consider a general admixture series (GAS) where three terms are combined in a nominated proportion to produce the fourth term. We shall call it a Tribonacci-type series: –

$T_n = p \cdot T_{n-3} + q \cdot T_{n-2} + r \cdot T_{n-1}$ where terms T_0, T_1 and T_2 are chosen arbitrarily.

Here are two Tribonacci series generated by an ITERATES-construct:

```
S(T0, T1, T2, p, q, r, n) := (ITERATES([w₂, w₃, p·w₁ + q·w₂ + r·w₃], w, [T0, T1, T2], n))↓↓1

T(T0, T1, T2, p, q, r, n) := (S(T0, T1, T2, p, q, r, n))
                                                        n + 1

S(1, 1, 2, 1, 1, 1, 10) = [1, 1, 2, 4, 7, 13, 24, 44, 81, 149, 274]

T(1, 1, 2, 1, 1, 1, 5) = 13

S(1, 1, 2, 24, -26, 9, 10) = [1, 1, 2, 16, 116, 676, 3452, 16276, 72956, 316276, 1340252]

T(1, 1, 2, 24, -26, 9, 13) = 94314676
```

Let there be a series $S(x)$

9.0 $\quad S(x) = T_0 + T_1 \cdot x + T_2 \cdot x^2 + T_3 \cdot x^3 + \ldots + T_{n-1} \cdot x^{n-1} + T_n \cdot x^n$ for n+1 terms.

Now multiply $S(x)$ by $r \cdot x$, $q \cdot x^2$ and $p \cdot x^3$ respectively:

9.1 $\quad r \cdot S(x) \cdot x = r \cdot T_0 \cdot x + r \cdot T_1 \cdot x^2 + r \cdot T_2 \cdot x^3 + r \cdot T_3 \cdot x^4 + \ldots + r \cdot T_{n-2} \cdot x^{n-1} + r \cdot T_{n-1} \cdot x^n$

9.2 $\quad q \cdot S(x) \cdot x^2 = q \cdot T_0 \cdot x^2 + q \cdot T_1 \cdot x^3 + q \cdot T_2 \cdot x^4 + \ldots + q \cdot T_{n-3} \cdot x^{n-1} + q \cdot T_{n-2} \cdot x^n$

9.3 $\quad p \cdot S(x) \cdot x^3 = p \cdot T_0 \cdot x^3 + p \cdot T_1 \cdot x^4 + \ldots + p \cdot T_{n-4} \cdot x^{n-1} + p \cdot T_{n-3} \cdot x^n$

Adding equations 9.1, 9.2 and 9.3 and then solving for $S(x)$:

9.4 $$S(x) \cdot (r \cdot x + q \cdot x^2 + p \cdot x^3) = r \cdot T_0 \cdot x + r \cdot T_1 \cdot x^2 + q \cdot T_0 \cdot x^2 + \underbrace{(r \cdot T_2 + q \cdot T_1 + p \cdot T_0)}_{T_3} \cdot x^3 + \ldots + T_n \cdot x^n$$

$$= r \cdot T_0 \cdot x + r \cdot T_1 \cdot x^2 + q \cdot T_0 \cdot x^2 + S(x) - T_0 - T_1 \cdot x - T_2 \cdot x^2$$

Therefore

9.5 $$S(x) = \frac{T_0 + (T_1 - r \cdot T_0) \cdot x + (T_2 - r \cdot T_1 - q \cdot T_0) \cdot x^2}{1 - r \cdot x - q \cdot x^2 - p \cdot x^3}$$

We can show that 9.5 is the generating function for the Tribonacci series:

```
                                                                    2
                           T0 + (T1 - r·T0)·x + (T2 - r·T1 - q·T0)·x
TS(T0, T1, T2, p, q, r, x) := ─────────────────────────────────────────────
                                                   2       3
                                    1 - r·x - q·x  - p·x

                                     1
TS(1, 1, 2, 1, 1, 1, x) = - ──────────────────
                              3    2
                             x  + x  + x - 1

TAYLOR⎛        1                ⎞          7        6        5       4       3       2
      ⎜- ──────────────── , x, 7⎟ = 44·x  + 24·x  + 13·x  + 7·x  + 4·x  + 2·x  + x + 1
      ⎜    3    2               ⎟
      ⎝   x  + x  + x - 1       ⎠

                                                   7          6         5         4        3       2
TAYLOR(TS(1, 1, 2, 24, -26, 9, x), x, 7) = 16276·x  + 3452·x  + 676·x  + 116·x  + 16·x  + 2·x  + x + 1
```

Suppose that we can factorise the denominator into three linear factors, where *a*, *b* and *c* are reciprocals of the roots of the cubic: –

9.6 $1 - r \cdot x - q \cdot x^2 - p \cdot x^3 = (1 - a \cdot x)(1 - b \cdot x)(1 - c \cdot x)$

9.7 $r = a + b + c,\ q = -a\,b - b\,c - c\,a,\ p = a\,b\,c$

There are four cases to consider for the n^{th} term T_n

Case 1: Three different roots $a \neq b$, $b \neq c$, $a \neq c$:

9.8 $$T_n = \frac{a^n(-T_2 + (b+c)T_1 - bcT_0)}{(a-b)(c-a)} + \frac{b^n(-T_2 + (a+c)T_1 - acT_0)}{(a-b)(b-c)} + \frac{c^n(-T_2 + (a+b)T_1 - abT_0)}{(b-c)(c-a)}$$

Take the example from above with $a = 2$, $b = 3$, $c = 4$, $T_0 = T_1 = 1$, $T_2 = 2 \rightarrow p = 24$, $q = -26$, $r = 9$

$$\text{TSN1(T0, T1, T2, a, b, c, n)} := \frac{a^n\cdot(\text{T0}\cdot b\cdot c - \text{T1}\cdot(b + c) + \text{T2})}{(a - b)\cdot(a - c)} + \frac{b^n\cdot(\text{T0}\cdot a\cdot c - \text{T1}\cdot(a + c) + \text{T2})}{(a - b)\cdot(c - b)} + \frac{c^n\cdot(\text{T0}\cdot a\cdot b - \text{T1}\cdot(a + b) + \text{T2})}{(a - c)\cdot(b - c)}$$

```
VECTOR(TSN1(1, 1, 2, 2, 3, 4, k), k, 0, 10) = [1, 1, 2, 16, 116, 676, 3452, 16276, 72956, 316276, 1340252]
```

Case 2: Two roots are equal: $a = b \neq c$:

9.9 See expression for TSN2 below.

$$\text{TSN2(T0, T1, T2, a, c, n)} := \frac{c^n\cdot(\text{T0}\cdot a^2 - 2\cdot\text{T1}\cdot a + \text{T2})}{(a - c)^2} - \frac{a^{n-1}\cdot(n + 1)\cdot(\text{T0}\cdot a\cdot c - \text{T1}\cdot(a + c) + \text{T2})}{c - a} - \frac{a^{n-1}\cdot(\text{T0}\cdot a\cdot c\cdot(3\cdot a - 2\cdot c) + \text{T1}\cdot(c^2 - 3\cdot a^2) + \text{T2}\cdot(2\cdot a - c))}{(a - c)^2}$$

```
VECTOR(TSN2(1, 1, 2, 2, 3, k), k, 0, 10) = [1, 1, 2, 10, 50, 214, 818, 2902, 9794, 31942, 101714]
VECTOR(TSN2(0, 1, 2, 1, 5, k), k, 0, 10) = [0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10]
S(0, 1, 2, 5, -11, 7, 10) = [0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10]
```

The second example shows that we can generate the sequence of the natural numbers as a Tribonacci series $T_n = 5T_{n-3} - 11T_{n-2} + 7T_{n-1}; T_0 = 1, T_1 = 1, T_2 = 2$.

Case 3: All three roots are equal: $a = b = c$:

9.10 $$T_n = a^{n-2}\left(\frac{(n+1)(n+2)(a^2T_0 - 2aT_1 + T_2)}{2} + (n+1)(-3a^2T_0 + 5aT_1 - 2T_2) - 3a^2T_0 - 3aT_1 + T_2\right)$$

Examples: (1) $a = 2$; ($p = 8$, $q = -12$, $r = 6$) with 0, 1, 1 as the first three terms

(2) $a = 1$; ($p = 1$, $q = -3$, $r = -3$)

Let's take three examples for (2), dependent on the choice of the first three terms.

```
VECTOR(TSN3(0, 1, 1, 2, k), k, 0, 10) = [0, 1, 1, -6, -40, -160, -528, -1568, -4352, -11520, -29440]
VECTOR(TSN3(0, 0, 1, 1, k), k, 0, 10) = [0, 0, 1, 3, 6, 10, 15, 21, 28, 36, 45]
VECTOR(TSN3(0, 1, 2, 1, k), k, 0, 10) = [0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10]
VECTOR(TSN3(0, 1, 4, 1, k), k, 0, 10) = [0, 1, 4, 9, 16, 25, 36, 49, 64, 81, 100]
```

0, 0, 1, ... is not particularly interesting – the differences are increasing by 1, which leads to a quadratic,

0, 1, 2, ... gives the natural numbers, again,

0, 1, 4, ... gives the squares of the natural numbers.

All of these can be anticipated by substituting for T_0, T_1 and T_2 in equation 9.10.

$$\text{TSN3}(0, 0, 1, 1, n) = \frac{n\cdot(n-1)}{2}$$

$$\left(\text{TABLE}\left(\frac{n^2}{2} - \frac{n}{2}, n, 0, 10\right)\right)\downarrow\downarrow 2 = [0, 0, 1, 3, 6, 10, 15, 21, 28, 36, 45]$$

$$\text{TSN3}(0, 1, 2, 1, n) = n$$

$$\text{TSN3}(0, 1, 4, 1, n) = n^2$$

Case 4: a is the real root, b, c are complex roots with $b, c = k\cdot(\cos(\theta)\pm i\cdot\sin(\theta))$; $k = |b|$

9.11
$$T_n = \frac{a^n(2k\cdot T_1\cdot\cos\theta - T_2 - k^2\cdot T_0)\cdot\sin\theta + k^{n-1}\cdot T_1(k^2\cdot\sin((n-2)\theta) - a^2\cdot\sin(n\theta))}{\sin\theta\cdot(2a\cdot k\cdot\cos\theta - k^2 - a^2)} - \\ - \frac{k^{n-1}\cdot T_2(k\cdot\sin((n-1)\theta) - a\cdot\sin(n\theta)) + a\cdot k^n\cdot T_0(k\cdot\sin((n-2)\theta) + a\cdot\sin((n-1)\theta))}{\sin\theta\cdot(2a\cdot k\cdot\cos\theta - k^2 - a^2)}$$

e.g. $a = 1$, $b = 1 + i$, $c = 1 - i$ hence $p, q, r, = 2, -4, 3$ and $T_0, T_1, T_2 = 0, 0, 1$ gives:

$$T_n = 1 + (\sqrt{2})^n\left(\sqrt{2}\sin\left(\frac{(n-1)\pi}{4}\right) - \sin\left(\frac{n\pi}{4}\right)\right) = 0, 0, 1, 3, 5, 5, 1, -7, ...$$

If we choose a different set of three initial terms 1, 1, 2 then we have

$$T_n = 2 + (\sqrt{2})^n\left(\sqrt{2}\sin\left(\frac{(n-1)\pi}{4}\right) - \sin\left(\frac{n\pi}{4}\right)\right) = 1, 1, 2, 4, 6, 6, 2, -6, ...$$

See which results are provided by *DERIVE*:

(TSN4 is defined according equation 9.11 with ABS(b) = k and PHASE(b) = θ.)

$$\text{TSN4}(0, 0, 1, 1, 1 + i, n) = 1 - 2^{n/2}\cdot\text{COS}\left(\frac{\pi\cdot n}{4}\right)$$

$$\text{VECTOR}\left(1 - 2^{n/2}\cdot\text{COS}\left(\frac{\pi\cdot n}{4}\right), n, 0, 10\right) = [0, 0, 1, 3, 5, 5, 1, -7, -15, -15, 1]$$

$$\text{TSN4}(1, 1, 2, 1, 1 + i, n) = 2^{n/2}\cdot\left(\sqrt{2}\cdot\text{COS}\left(\frac{\pi\cdot n}{4} + \frac{\pi}{4}\right) - 2\cdot\text{COS}\left(\frac{\pi\cdot n}{4}\right) + \text{SIN}\left(\frac{\pi\cdot n}{4}\right)\right) + 2$$

$$\text{VECTOR}(\text{TSN4}(1, 1, 2, 1, 1 + i, n), n, 0, 10) = [1, 1, 2, 4, 6, 6, 2, -6, -14, -14, 2]$$

If we now examine the traditional Tribonacci series, where $p = q = r = 1$ and the series appears as 1, 1, 2, 4, 7, 13, 24, 44, ..., then we have a lot more to calculate.

According to 9.5 a, b, c are the reciprocals of the roots of the cubic:

$$\mathrm{VECTOR}\left(\frac{1}{k},\ k,\ \mathrm{SOLUTIONS}(1 - x - x^2 - x^3 = 0,\ x)\right)$$

These bulky expressions are roots *a*1 (real) and *a*2, *a*3 (complex):

$$a1 := -\frac{3}{(3\cdot\sqrt{33} - 17)^{1/3} - (3\cdot\sqrt{33} + 17)^{1/3} + 1}$$

$$a2 := \frac{3\cdot((3\cdot\sqrt{33} - 17)^{1/3} - (3\cdot\sqrt{33} + 17)^{1/3} - 2)}{2\cdot((586 - 102\cdot\sqrt{33})^{1/3} + (102\cdot\sqrt{33} + 586)^{1/3} - (3\cdot\sqrt{33} - 17)^{1/3} + (3\cdot\sqrt{33} + 17)^{1/3} + 3)} - \frac{3\cdot\sqrt{3}\cdot i\cdot((3\cdot\sqrt{33} - 17)^{1/3} + (3\cdot\sqrt{33} + 17)^{1/3})}{2\cdot((586 - 102\cdot\sqrt{33})^{1/3} + (102\cdot\sqrt{33} + 586)^{1/3} - (3\cdot\sqrt{33} - 17)^{1/3} + (3\cdot\sqrt{33} + 17)^{1/3} + 3)}$$

$$a3 := \frac{3\cdot((3\cdot\sqrt{33} - 17)^{1/3} - (3\cdot\sqrt{33} + 17)^{1/3} - 2)}{2\cdot((586 - 102\cdot\sqrt{33})^{1/3} + (102\cdot\sqrt{33} + 586)^{1/3} - (3\cdot\sqrt{33} - 17)^{1/3} + (3\cdot\sqrt{33} + 17)^{1/3} + 3)} + \frac{3\cdot\sqrt{3}\cdot i\cdot((3\cdot\sqrt{33} - 17)^{1/3} + (3\cdot\sqrt{33} + 17)^{1/3})}{2\cdot((586 - 102\cdot\sqrt{33})^{1/3} + (102\cdot\sqrt{33} + 586)^{1/3} - (3\cdot\sqrt{33} - 17)^{1/3} + (3\cdot\sqrt{33} + 17)^{1/3} + 3)}$$

Does formula 9.11 hold (in its CAS-realisation?)

```
VECTOR(TSN4(1, 1, 2, a1, a2, n), n, 0, 15)
[1, 1, 2, 4, 7, 13, 24, 44, 81, 149, 274, 504, 927, 1705, 3136, 5768]
```

Yes, it works! But by approximation only, simplifying these nested roots is too much, even for *DERIVE*.

What I (Josef) found out:

I tried TSN1 (three different real roots treating Case 4):

```
VECTOR(TSN1(0, 0, 1, 1, 1 + i, 1 - i, n), n, 0, 10) = [0, 0, 1, 3, 5, 5, 1, -7, -15, -15, 1]
VECTOR(TSN1(1, 1, 2, 1, 1 + i, 1 - i, n), n, 0, 10) = [1, 1, 2, 4, 6, 6, 2, -6, -14, -14, 2]
VECTOR(TSN1(1, 1, 2, a1, a2, a3, n), n, 0, 10)
[1, 1, 2, 4, 7, 13, 24, 44, 81, 149, 274]
```

and was surprised to obtain the correct series.

One more example: $a = 2$, $b, c = -3 \pm 4i \rightarrow p, q, r = -4, 50, -13$, $T_0, T_1, T_2 = 1, 2, 3$

Here is this series generated in various ways:

```
S(1, 2, 3, 50, -13, -4, 10) = [1, 2, 3, 12, 13, -58, 663, -1248, -6527, 75482, -279477]
VECTOR(TSN1(1, 2, 3, 2, -3 + 4·i, -3 - 4·i, n), n, 0, 10)
[1, 2, 3, 12, 13, -58, 663, -1248, -6527, 75482, -279477]
VECTOR(TSN4(1, 2, 3, 2, -3 + 4·i, n), n, 0, 10)
[1, 2, 3, 12, 13, -58, 663, -1248, -6527, 75482, -279477]
```

The Sum of a Number Sequence

The sum to n terms for such a series may be deduced: –

Let S_n represent the sum to n terms for Tribonacci-type series: –

10.0 $S_n = T_0 + T_1 + T_2 + T_3 + \ldots + T_{n-2} + T_{n-1} + T_n$

10.1 $r \cdot S_n = r \cdot T_0 + r \cdot T_1 + r \cdot T_2 + r \cdot T_3 + \ldots + r \cdot T_n$

10.2 $q \cdot S_n = \quad q \cdot T_0 + q \cdot T_1 + q \cdot T_2 + \ldots + q \cdot T_{n-1} + q \cdot T_n$

10.3 $p \cdot S_n = \quad p \cdot T_0 + p \cdot T_1 + \ldots + p \cdot T_{n-2} + p \cdot T_{n-1} + p \cdot T_n$

10.4 $(p+q+r) \cdot S_n = (q+r) \cdot T_0 + r \cdot T_1 + (S_n - T_0 - T_1 - T_2) + T_{n+1} + (p+q) \cdot T_n + p \cdot T_{n-1}$

10.5 $$S_n = \frac{(q+r-1) \cdot T_0 + (r-1) \cdot T_1 - T_2 + (p+q) \cdot T_n + p \cdot T_{n-1} + T_{n+1}}{p+q+r-1} \quad \text{iff} \quad p+q+r \neq 1$$

This equation simplifies for particular examples e. g. Standard Tribonacci

10.6 $$S_n = \frac{T_0 - T_2 + T_n + T_{n+2}}{2} = \frac{T_n + T_{n+2} - 1}{2}$$

```
Σ([1, 1, 2, 4, 7, 13, 24, 44, 81, 149, 274]) = 600

              1
sttrib(n) := ─·(TSN1(1, 1, 2, a1, a2, a3, n) + TSN1(1, 1, 2, a1, a2, a3, n + 2) − 1)
              2

sttrib(10)

600
```

Limiting Ratio of Successive Terms in these Number Series

To find limiting ratio, R, as n increases indefinitely

11.0 $$R = \lim_{n\to\infty} \frac{T_n}{T_{n-1}}$$

11.1 $$R_{\text{Fib}} = \lim_{n\to\infty} \frac{a^n - b^n}{a^{n-1} - b^{n-1}}$$

However $0 < |b| < 1$ therefore

11.2 $$R_{\text{Fib}} = \frac{a^n}{a^{n-1}} = a$$

However, for a more general look at the G.A.S. we can see that for certain values of $p > q + 1$ we find that b has a modulus greater than unity, so the previous argument does not obtain. Instead we use identical methods for all series, whether recursive over the last two terms or over the last three terms.

Fibonacci-type Series:

11.3 $b < a$; i.e. $b = k \cdot a$ where $0 < |k| < 1$

11.4 $$R_{\text{FibType}} = \lim_{n\to\infty} \frac{a^n - b^n}{a^{n-1} - b^{n-1}} = \lim_{n\to\infty} \frac{a^n(1-k^n)}{a^{n-1}(1-k^{n-1})} = a$$

Similarly, with the more complicated ratios in Tribonacci-type series:

We choose whichever is the largest of the roots and hence find that the limiting value is the value of the largest root. Take the case where $a > b > c$, our ratio is of the form

11.5 $$R_{\text{TribType}} = \lim_{n\to\infty} \frac{k\cdot a^n + l\cdot b^n + m\cdot c^n}{k\cdot a^{n-1} + l\cdot b^{n-1} + m\cdot c^{n-1}} = \lim_{n\to\infty} \frac{a\left(1 + \frac{l\cdot d^n}{k} + \frac{m\cdot e^n}{k}\right)}{1 + \frac{l\cdot d^{n-1}}{k} + \frac{m\cdot e^{n-1}}{k}} = a$$

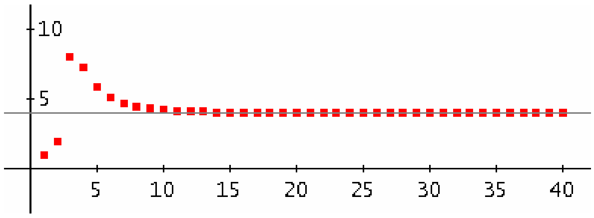
Similarly, when two roots are equal $a = b$ *and* $a > c$ then the limit is a, but if $a < c$ then the limit is c.

11.6 When all roots are equal then $a = b = c$ is the limiting ratio.

11.7 When the real root is a and b, c are complex then the limiting ratio is the larger out of a and ABS(b).

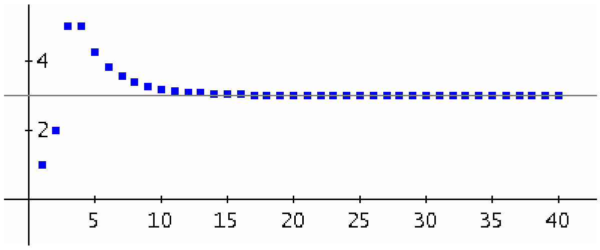
$a, b, c = 2, 3, 4$

$$\mathrm{TABLE}\left(\frac{\mathrm{TSN1}(1,\ 1,\ 2,\ 2,\ 3,\ 4,\ k)}{\mathrm{TSN1}(1,\ 1,\ 2,\ 2,\ 3,\ 4,\ k-1)},\ k,\ 1,\ 40\right)$$



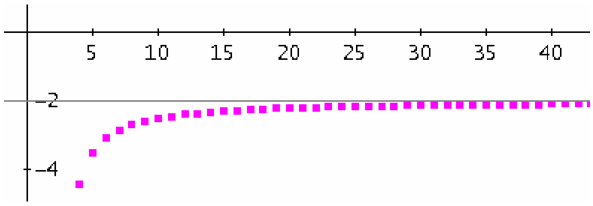
$a, = b = 2, c = 3$

$$\mathrm{TABLE}\left(\frac{\mathrm{TSN2}(1,\ 1,\ 2,\ 2,\ 3,\ k)}{\mathrm{TSN2}(1,\ 1,\ 2,\ 2,\ 3,\ k-1)},\ k,\ 1,\ 40\right)$$



$a = b = c = -2$

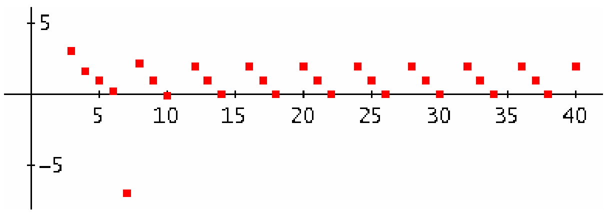
$$\mathrm{TABLE}\left(\frac{\mathrm{TSN2}(1,\ 1,\ 2,\ 2,\ 3,\ k)}{\mathrm{TSN2}(1,\ 1,\ 2,\ 2,\ 3,\ k-1)},\ k,\ 1,\ 40\right)$$



$a = 1, b = 1 + i, c = 1 - i$

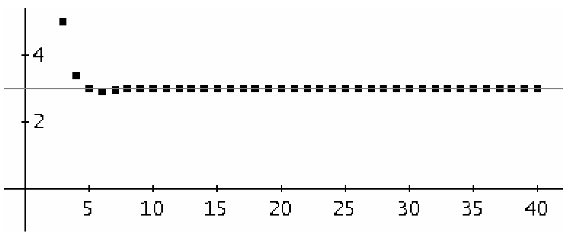
$$\mathrm{TABLE}\left(\frac{\mathrm{TSN4}(0,\ 0,\ 1,\ 1,\ 1+i,\ k)}{\mathrm{TSN4}(0,\ 0,\ 1,\ 1,\ 1+i,\ k-1)},\ k,\ 1,\ 50\right)$$

divergent!



$a = 3, b = 1 + i, c = 1 - i$

$$\mathrm{TABLE}\left(\frac{\mathrm{TSN4}(0,\ 0,\ 1,\ 3,\ 1+i,\ k)}{\mathrm{TSN4}(0,\ 0,\ 1,\ 3,\ 1+i,\ k-1)},\ k,\ 1,\ 40\right)$$



Inspecting the last two examples from above you can observe that 11.7 only holds if *a* > *k* (*k* = abs(*b*)). Otherwise the limit of the ratio does not exist. There are much more points outside of the detail of the graph.

Recursion over *m* Previous Terms

Now to a further observation and concomitant conundrum/challenge. Let us consider further number series, that are progressively recursive over an increasing number of terms, so that the denominator is a polynomial of degree *m*, whose 'factors' may be real or complex and accordingly the expressions for T_n will depend on the precise nature of these roots of the polynomial denominator of $S(x)$, and can be summarised in a chart: –

The denominator of a real polynomial of the m^{th} degree has *m* factors, which may be all real, all imaginary, or mixed real and imaginary. Now consider them one at a time.

The nub of the problem centres around the splitting of a fraction into partial fractions. The denominator of the fraction is a real polynomial of m^{th} degree with $(m+1)$ terms. The partial fractions must have a variety of monomial denominators mostly, which, for some cases, may be squared or cubed. So, in essence, one deems the polynomials to be factorised, so one needs to consider the hypothetical number of cases of the 'roots' of a given polynomial.

Let the general univariate polynomial be $P(x)$

$$P(x) = 1 + a_1x + a_2x^2 + a_3x^3 + ... + a_nx^n = 1 + \sum_{r=1}^{n} a_r x^r$$

The various cases for the 'roots' subsume: –

1) all are real, equal, different etc ...

2 for *m* even, all can be conjugate complex pairs, equal, different etc ...

3) a mixture of real and conjugate complex pairs

Let *R* = the number of cases for real 'roots, *U* = the number of cases for conjugate complex 'roots' and *M* = the number of cases for an admixture of real and imaginary roots.

Let *T* = *R* + *U* + *M* = the total number of cases.

DEGREE	NAME	REAL	UNREAL	MIXED	TOTAL
1	Monomial	1	0	0	1
2	Quadratic (Fibonacci, Lucas, etc.)	2	1	0	3
3	Cubic (Tribonacci)	3	0	1	4
4	Quartic (Quattraci)	5	2	2	9
5	Quintic (Pentonacci)	7	0	5	12
6	Sextic (Hexonacci)	11	3	9	23
7	Septic (Septonacci)	15	0	16	31
8	Octic (Octonacci)	22	5	27	54
9	Nonic	30	0	43	73
10	Decic	42	7	69	118
11	Undecic	56	0	103	159
12	Diodecic	77	11	123	211

Let's check this for degree 4:

There are 5 possibilities for only real solutions:
(r1, r2, r3, r4), (r1 = r2, r3 = r4), (r1 = r2 = r3, r4), (r1 = r2, r3, r4) and (r1 = r2 = r3 = r4)

There are 2 possibilities for only complex solutions:
(c1,2, c3,4) and (c1,2 = c3,4)

And finally there are 2 possibilities for mixed solutions:
(r1, r2, c1,2) and (r1 = r2, c1,2)

which makes 9 different cases totally.

The question remains to find a formula to express the number of solutions for each value of m.

Also, is it possible to have a more general solution for T_n, which covers for all, (or some more than at present), of the several solutions for a particular value of m, whether, or not, some roots are equal?

David's contribution kept me very busy for a while – one reason for the delay of publishing DNL#93 – and we exchanged many mails between Austria and Australia. I sent question, David replied answers, updates and *DERIVE* files. You can find one of his mails below.
I am very grateful for his careful proof reading. There are so many expressions which had to be copied from David's pdf-file (generated from of Word Perfect document). Moreover I wanted to illustrate several steps providing examples calculated with *DERIVE* or TI-Nspire CAS.

It would be great if somebody of the *DERIVE-TI-Nspire* community would feel inspired to continue investigating the G.A.S.

I felt inspired to do some Internet-research for Generating Functions. I found a lot and I found interesting facts in some of my textbooks (about Discrete and Combinatorial Mathematics). As I announced on page 5 I will present other applications of generating functions.

David's mail from March 9, 2014:

Hi Josef
Thanks. I had several working copies, and also I wrote up an extra couple. I find that Derive is indispensable for much of the work, nevertheless it is always necessary to check out some `pen and paper', whereby some simplification usually comes to light. It's mainly to do with factoring, that Derive cannot do in some instances.

Incidentally, I have almost completed a rejoinder, with a very special couple of series, recursive over four terms, based on a geometrical finding of Charles Dupin in Compte Rendus 1848. He used them for estimating any missing figures in a statistical table, very successfully. I have discovered that they can represent many known series e.g Any arithmetic progression (A.P.), a series of the squares of any A.P. Similarly with the cubes, fourth power and fifth power e.g. $\sum_{k=1}^{n}(a+k\cdot d)^k$.

I will present three examples for applying generating functions without giving any explanation. You can find them in enumeration problems:

(1) In how many ways can we pay 50 Cents using 1, 2 and 5 Cent coins?[1]

The generating function is $g(x)=\dfrac{1}{(1-x)(1-x^2)(1-x^5)}$ and the solution is the coefficient of x^{50} of the Taylor expansion.

```
g(x) :=            1
        ─────────────────────────
                    2         5
        (1 - x)·(1 - x )·(1 - x )

                          10      9      8      7      6      5      4      3      2
TAYLOR(g(x), x, 10) = 10·x   + 8·x  + 7·x  + 6·x  + 5·x  + 4·x  + 3·x  + 2·x  + 2·x  + x + 1

POLY_COEFF(TAYLOR(g(x), x, 100), x, 50) = 146

24 sec
```

(2) While shopping one Saturday, Mildred buys 12 candies for her children, Carl, Wade, Pat and Jane. In how many ways can she distribute the candies, so that Carl and Wade get no more than 4, Wade gates at least one, Carl gets at least two, Pat and Jane get at least three candies? [2]

The solution is the number of all integer solutions of the system:

$$c+w+j+p=12$$
$$2\le c\le 4,\ 1\le w\le 4, p\ge 3, j\ge 3$$

The generating function is $c(x)=(x^2+x^3+x^4)(x+x^2+x^3+x^4)(x^3+x^4+x^5+x^6)$ and the number of distributions is the coefficient of x^{12}.

```
          2    3    4        2    3    4     3    4    5    6 2
c(x) := (x  + x  + x )·(x + x  + x  + x )·(x  + x  + x  + x )

POLY_COEFF(c(x), x, 12) = 19
```

Is this the correct answer? 19 distributions are not so many. It should be possible to write them all down:

Carl	Wade	Pat	Jane	Total	Carl	Wade	Pat	Jane	Total
2	1	3	6	12	3	1	3	5	12
2	1	4	5	12	3	1	4	4	12
2	1	5	4	12	3	1	5	3	12
2	1	6	3	12	3	2	3	4	12
2	2	3	5	12	3	2	4	3	12
2	2	4	4	12	3	3	3	3	12
2	2	5	3	12	4	1	3	4	12
2	3	3	4	12	4	1	4	3	12
2	3	4	3	12	4	2	3	3	12
2	4	3	3	12					

Do you find more distributions? Did I forget one?

(3) Find the generating function for the number of ways an advertising agent can purchase 5 minutes of air time if time slots for commercials come in blocks of 30, 60, 120 or 180 seconds. [2]

Generalise for *n* minutes.

Let 30 seconds represent one time unit. Then the solution is the number of integer solutions to the equation

$$a + 2b + 4c + 6d = 10 \quad (a, b, c, d \geq 0)$$

The generating function is

$$t(x) = (1 + x + x^2 + \ldots)(1 + x^2 + x^4 + \ldots)(1 + x^4 + x^8 + \ldots)(1 + x^6 + x^{12} + \ldots)$$
$$= \frac{1}{(1-x)(1-x^2)(1-x^4)(1-x^6)}$$

The coefficient of x^{10} is the number of partitions of 10 into 1's, 2's, 4's, and 6's, the answer to the problem.

```
                               1
t(x) := ─────────────────────────────────────────
                         2         4         6
        (1 - x)·(1 - x )·(1 - x )·(1 - x )

                          10       9        8       7       6       5       4       3       2
TAYLOR(t(x), x, 10) = 16·x   + 11·x  + 11·x  + 7·x  + 7·x  + 4·x  + 4·x  + 2·x  + 2·x  + x + 1
```

Correct? Check for 5 minutes (= 6 time units):

180	**120**	**60**	**30**
1			
	1	1	
	1		2
		3	
		2	2
		1	4
			6

There are seven possible ways to form 5 minutes (6 units) and 7 is the coefficient of x^6.

[1] math-www.upb.de/MatheI_02/vorl/woche_16.pdf

[2] Ralph P. Grimaldi, *Discrete and Combinatorial Mathematics*, Addison Wesley 1999
Jiri Matousek, Jaroslav Nesetril, *Diskrete Mathematik,* Springer 2002

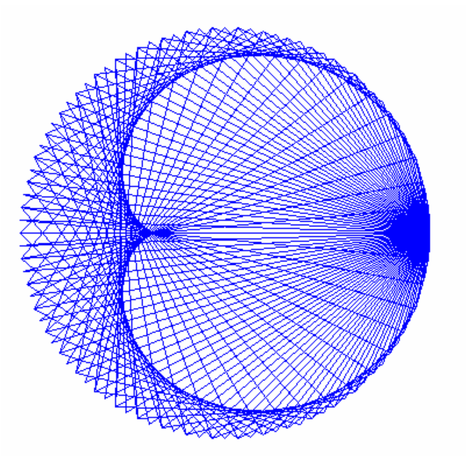
Light in the Coffee Cup

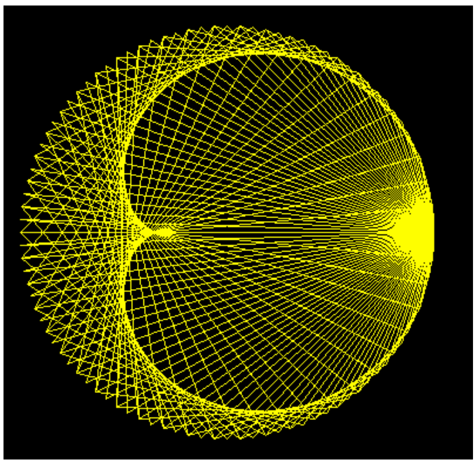
Roland Schröder

When in the morning the lamp is sending its light into the coffee cup and if there is a lucky constellation of light source and the positions of cup and observer a nice picture on the surface of the coffee similar to the picture given below can be detected by the careful observer. It is not possible to see the many delicate lines but their heart-shaped envelope, the cardiod. The bright spot on the right border of the cup symbolizes the light source (which is over the cup). The rays sent by the light are reflected by the border of the cup. And where many reflected rays are superimposed a white heart-shaped curve can be perceived very well.

The graph can be produced using dynamic geometry programs, too. Here we will use *DERIVE*. The command for producing the graph is a very short one:

$$\mathrm{VECTOR}\left(\begin{bmatrix} 1 & 0 \\ 1 & \dfrac{k\cdot\pi}{50} \\ 1 & \dfrac{k\cdot\pi}{25} \end{bmatrix}, k, 1, 100\right)$$



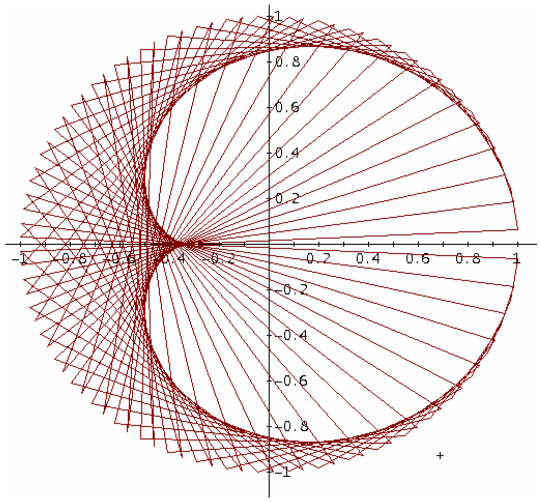


The points must be "connected" and what is very important: we are working with polar coordinates. The "negative" graph was produced applying the „Paint"-tool on the graph depicted on the left hand side. (Apply the 5th option in the Edit-menu on a the graph copied into the Algebra window. Later I will show how to achieve this in another way, too.) It is interesting deleting the *DERIVE*-graph step by step. You can follow the run of each single ray separately.

The cardiod is generated by a punctual light source and parallel light rays generate a "double cardiod", a so called nephroid ("kidney formed"). It is said that it cannot be produced by a dynamic geometry software. We will not check this here but we will show that *DERIVE* has not reached the end of its abilities. For demonstrating this we will plot the nephroid using another approach i.e. We start with the *DERIVE* command

$$\mathrm{VECTOR}([1, \mathrm{MOD}(2^n,101)\cdot 2\pi/101], n, 1, 101)$$

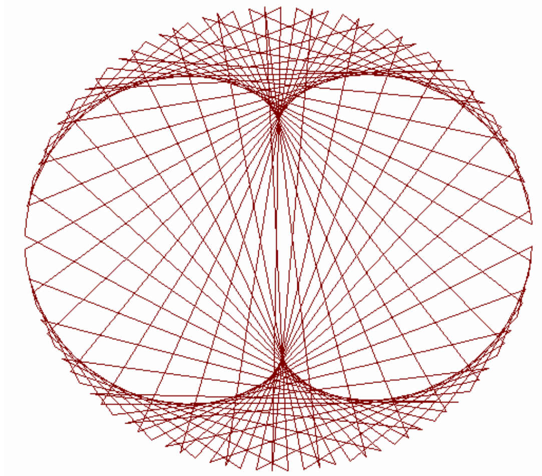
(in polar coordinates with points connected)

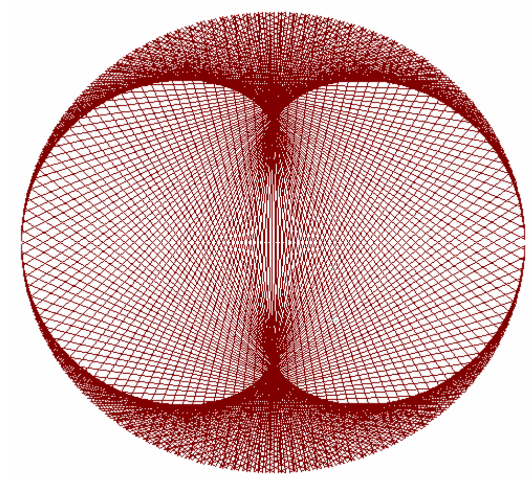


The light rays are coming from all directions now, but we obtain a cardioid again. Number 101 has a special role in this representation. Vary this number in order to find out its importance. Another fact will become clear when deleting the graph:

It is one closed polygon. We change the last *DERIVE* command at one position and we will obtain the nephroid:

VECTOR([1, MOD(3^n,101)·2π/101], n, 1, 101)



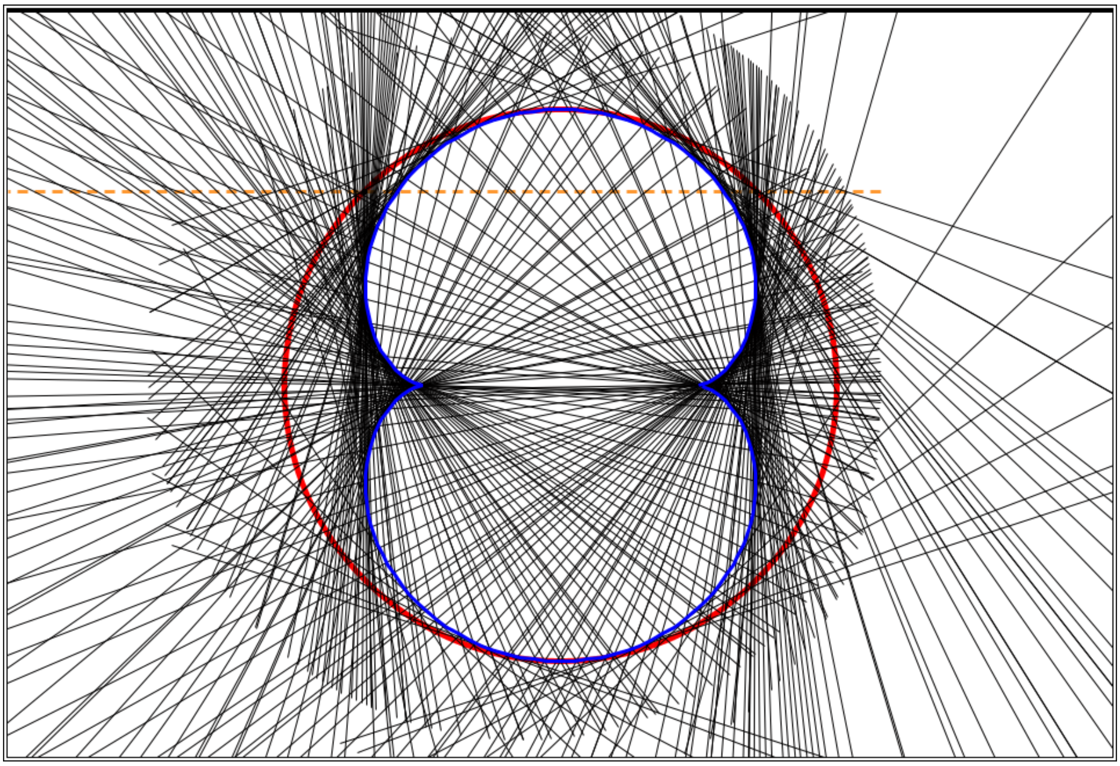


Now the *DERIVE* user will proceed full of enthusiasm producing four- and five leafed clovers (three leafed clovers cause problems).

All these graphs are closed polygons which connect *p* points on the circumference of a circle. The properties of *p* must be investigated by experts in number theory. The second coordinates, which are the arc lengths of the rotations form so called modular sequences of the powers of two, three, four, ...)

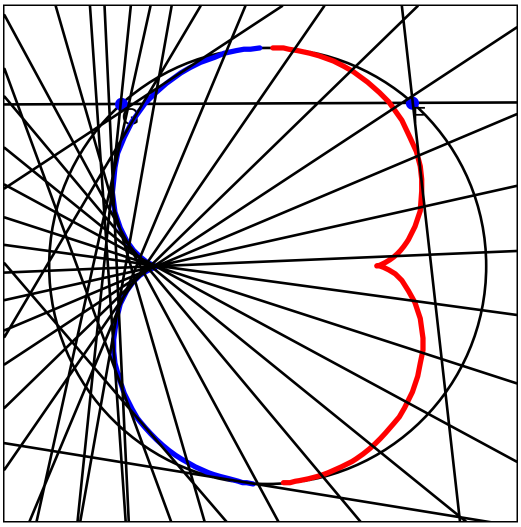
Roland wrote about problems producing the nephroid by a dynamic geometry program. I tried TI-NspireCAS and found the nephroid as envelope of the reflected rays – but could not produce the envelope as a curve. It could have been possible to add the curve using its parameter representation which is:

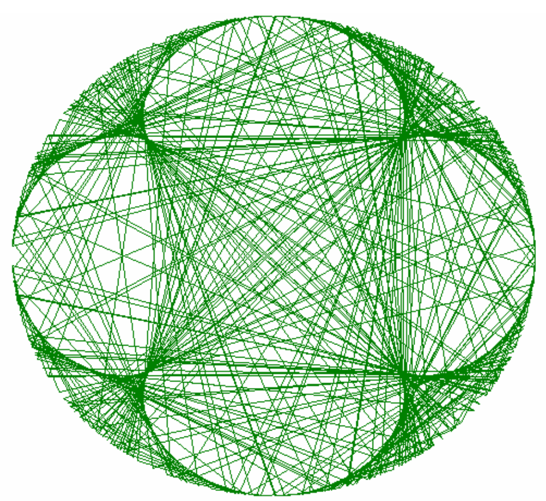
$$(6a\cos\varphi - 4a\cos^3\varphi, 4a\sin^3\varphi)$$



See the TI-NspireCAS plot above.

The right side picture is a screen shot from Geometry Expressions. This program finds the envelope in two parts as we have two families of reflected rays.
Try to find the four-leafed clover as it is shown below.
Try to find the three leafed clover, too.





Josef

Thanks for your attached extract from DNL90, which I read with great interest. I have been interested in Fresnel Integrals and Cornu's Spiral, (also known as Klothoid, Euler's Spiral and Railway Transition Curve), for many years. When my first son was a pre-teen, I constructed a table-top railway track for a train set. I designed the curves, as closely as I could, to be Klothoids, due to the need to provide even 'wear and tear' on the rails as the train cornered.

This brought to mind many other seemingly overlooked occurrences. I strongly feel that there are many neglected facts of mathematics and/or science. One of the main reasons for this neglect is due to the seemingly exponential explosion of information in the last 100+ years. Curricula of schools and universities have had to make space for so much more course contents, that much material perhaps, in some cases, was wrongfully eliminated.

THIS IS A SHORT LIST OF WHAT I HAVE INVESTIGATED OVER THE YEARS

I have researched a vast amount of the mathematics of the 19th Century by working my way through countless journals from UK, USA and Europe, in my search for topics of special interest to me, until I noticed in the early 1900s, that I didn't understand the titles, let alone the contents, of most of those 20th Century papers. Thuswise, I was able to notice the absence of development of many topics, that, in my humble opinion, **"cried out"** for continuation of development.

e.g.1) Intrinsic Geometry, (using Intrinsic Coordinates),

e.g.2) Other coordinate systems in the plane and their interrelationships,

e.g.3) Osculants, penosculants and orders-of-contact,

e.g.4) Asymptotes,

e.g.5) Deviation (aberrancy) and higher order qualities.

etc.

There have been several serendipitous events/occurrences during, and since, university days, without which I would have never been where I am today, mathematically speaking.

One of my professors was the highly respected Felix Adelbert Behrend. To me, the most memorable parts of his class were the chapters on **"Logic"** and **"The Analysis of the Quadratic Form in Two Cartesian Coordinates"**, (The Conic Sections, real, imaginary and/or degenerate). I added to it with my findings on the **"Absolute Invariants"** and their geometric meanings, such as area, latus rectum, diameter, etc..

Another fortuitous event occurred while in the Melbourne Public Library. I came across an article in an earlier edition of Encyclopaedia Britannica on Curves (Special), in which a *`strange'* new coordinate system, named for Cesàro, was used to define some curves, without any explanation of its meaning. It took me a few years to find anything worthwhile on Ernesto Cesàro, but it was only a reference to an Italian language publication of his series of lectures, given at the Royal Neapolitan University in the last decade of the 19th Century.

Then one day in 1969, while in a city book store, I had the **"effrontery"** to ask them if they had anything on Cesàro. Lo and behold, they found reference to a 1968 publication of Gerhard Kowalewski's 1901 German translation of Cesàro's book, which the Johnson Reprint Corp. had the wisdom to choose as worthwhile for reprinting. I ordered it and consequently started on a translation of the planar section, thus introducing me to the immense power of Cesàro's Intrinsic Coordinates.

Later on, this was helped enormously by the discovery of an original German language publication by Kowalewski, some 30 years later on Lie Transformations, in the first third of which, he expounded the foundations of Intrinsic Geometry as a necessity for his findings Lie. However, serendipitously for me, he used a more modern notation and terminology than Cesàro used, especially with the use of infinitesimals, and some of his proofs were more rigorous and easily understandable.

I have a copy of `Opere Scelte', a three-volume compendium of Cesàro's papers. Disappointingly, it has very few papers on Intrinsic Geometry, so I had to order in those papers from various universities around the world.

There have been many rewards for me from Cesàro's work, including the **"Angle of Contingency"**, **"Circular Asymptotes"**, many higher ordersof-contact for osculants, than the traditional tangent, curvature and deviation (aberrancy).

I have written a review of Norman Wildberger's strange new book on his idiosyncratic version of Geometry, **"Divine Proportions, (2005)"**.

I made a surprisingly unsuspected discovery of the Froude Number, when formulating a simple equation for Sand Dunes.

I was able to help a senior lecturer in the Chemistry dept. of Melb. Uni. in 1980, who presented me with a problem, that needed a computer-assisted solution. After telling him that I could not help him, he gave me a couple of sheets with trigonometric equations and diagrams to just look at it, in case. I showed it to my 15-year-old son, with whom I shared ownership of a Euro Apple II and he said that if I did the math., he would do the programming in Apple Basic, and we solved the equations much to the joy of the lecturer.

In 1979, my son Geoffrey and I, together with a few friends, founded the Apple Users Society of Melbourne (AUSOM). Eventually it grew considerably to have 2000 members and many Special Interest Groups (SIGs), one of which was the Math. Sig, which I ran.

In 1987, I bought an XT PC and joined the Melbourne PC User Group, for whom I wrote a software review on an interesting product. It is called **"Expert Thinker"** and is a Logic Problem Solver, Theorem Prover and Predicate Calculus to Clause Form Converter. It is an expert system, which uses a special algorithm, which is complete for first order predicate calculus.

In 1989, an industrial chemist, (a member of AUSOM's Math. SIG), who worked for a paint manufacturer, presented me with a problem, that he needed solving, with or without a computer. It dealt with colloidal particles of dyes and their brightness and shade. It was an algorithm for an analytical solution of the conversion of absolute colour coordinates of standard and sample to coordinates of strength difference, brightness difference and shade difference. I was able to convert the problem to the scalar product of vectors, which were either collinear or orthogonal; voila!

A water-sprinkler problem, published by a math lecturer in USA. I provided a simpler solution in Intrinsic Coordinates and also identified the curve, which had not been done by the author.

I foresee a new analysis of Cricket results, (and maybe Tennis and other sports), over the decades, based on the analysis of Baseball results by the American, Steven Jay Gould, using the Bell Curve, the number of standard deviations above the mean etc..

Some of my math. findings are:-

A symbolic representation of John Horton Conway's Fractran set of 14 fractions as a **"prime producing machine"**. (This topic was presented by J.H. Conway at the Felix Behrend Memorial lecture in 1999.)

Fractional Derivatives (inspired by Oliver Heaviside),

Fractional Iterates (with the assistance of Professor George Szekeres in 1989, recommended by his friend Paul Erdös),

The River Meander & other sinuous curves, and their relationship to the Elastica. This was helped by the 1951 work of Hermann von Schelling and his analysis of Random Walks and Most Probable Paths.

I shall attach a copy, since it demonstrates some variations of shape, that your natural equations in DNL 90 have demonstrated.

"Glove Osculants" and a homologous recursive series for the **"Super Osculants"**.

Subsequently I discovered the work of:-

Benjamin Peirce, who taught math. using his **"Circular Coordinates"**.

I spent 3 days in the Houghton Library at Harvard in 1984, going through most of 21 boxes of personal letters, poetry and other writings of Benjamin Peirce, and I now have many photocopies of much of his engrossing work.

Thanks to Charles Dupin, we have the Telegraph Curve, a previously unknown derived curve. Based on his work, I formulated the Paragraph Curve, which ranges from being a Parallel Curve to that of a Telegraph Curve, being a hybrid of both, as well as their inverses. There is the possibility of Dupin's methodology to be used in today's statistics, having escaped modern day techniques.

William Watson, who investigated many coordinate systems.

Thomas Hill, who investigated many coordinate systems.

Moritz Cantor, whose doctoral dissertation was a little-known coordinate system.

Karol Taubner and Vilmos Fest, who wrote their prize-winning essays in 1844 on the quadratic form in intrinsic coordinates, apparently way back before Matrix Algebra was available, since neither of them used matrices.

William Whewell, who used a different pair of intrinsic coordinates (s,ö), and who presented them and many plane curves in the Cambridge Philosophical Transactions. I have a set of 2 books on Whewell by Isaac Todhunter, and a great many photocopies of much of his writings.

I have written a Seance paper, which is a `Mathematics Fiction' to compare with good Science Fiction. The maths is flawless; only the setting is fiction.

I have written "Vermischte Geschichte" a selection of humorous epitaphs of famous mathematicians, referring to their achievements.

Yet another fortuitous event was, when attending Open Day at Melbourne Uni. in 1967, I was looking around the book room and I saw the Science Review annual, in which was a paper **"Time is the Essence"** by Dewey B. Larson. This entranced me so much, that I contacted the research dept. of Encyclopaedia Britannica for information on the author and/or any of his publications. In 1968 I received the address of the publisher of 2-3 of Larson's books and within a few weeks I was corresponding with the author. Within a few months, I was introduced, by mail, to a group of academics, who were studying Larson's Reciprocal System of Theory (RST). I especially befriended Frank Meyer, a Professor of Physics at Wisconsin University, and we corresponded frequently.

By 1979, I was able to take a vacation in USA and attend an RST conference at Wisconsin Uni., hosted by Professor Meyer, where I met a large group of academics from many states of USA and also Canada, who all shared an interest in this new paradigm. RST was able to predict Quasars and Pulsars two years before they were discovered, as well as explain, by alternate means to Relativity, the Perihelion of Mercury problem, and the bending of light rays during an eclipse, both of which Newton's equations could not do accurately or at all resp. and which only Relativity had done until RST made its appearance in the 1950s.

I attended several further conferences and have been a member of the board of trustees for over 25 years, even serving as vice president for one term. I have presented papers when attending, and many other papers for the quarterly journal over the years. My friendship with Dewey Larson extended over 22 years until his death in 1990, aged 90.

My focus of interest in RST was primarily on the mathematics, and my latest paper exemplifies this, being an original finding on the actual spin, (the gyroscopic motion), of atoms and subatomic particles.

I re-introduced the almost unknown concept of **"Intractance"** and its higher order counterparts. This paper represents an extension of Newtonian mathematics, however, perhaps not as far-reaching as Professor Moti Milgrom's MOND. (Yet to be investigated).

I am half-way through a more difficult extension to Newton's equations, for neither linear motion, nor gyroscopic motion; in other words, planetary orbits, with the hope that it will explain the Perihelion of Mercury more accurately as well as Einstein's relativity Equations, without recourse to tensors. The D.E.s are challenging, however!

Also, I opine that because String Theory has evolved over the years with several successive definitions for the actual string, (the latest being a heterotic string, yet is not free from cognitive dissonances), then it is worth trying out a new replacement for the string, coming from RST, known by the aficionados as the **"Oscillating Space Unit"**.

An interesting side issue, is a homologous series of never-beforeseen plane curves, which I have named for Larson, and which can be plotted as planar curve triplets by projecting the spinning atoms and subatomic particles onto the three orthogonal planes, or as a 3-D shape.

There is an interesting paradigm by Robert Oros Di Bartini, which, in some respects, resembles RST, however it is difficult for me to come to grips with it; it uses Combinatorial Topology. It was first published in Doklady Akademii Nauk SSSR 1965, later translated in part by the Americans, and the balance of it translated about 1970 at my behest and expense by someone chosen by the `chief' of the language laboratory at Melbourne University.

As a result of Felix Behrend's erudite expositions,

1) I have written a logic paper, which continued on from where he left off, by using £ukasiewicz notation, resulting in the solution of many logic problems, with greater facility. (a la Wff'n Proof by Professor Layman Allen of Michigan University)

2) I have written papers, wholly, or partly, on:-

a) **"The Analysis of the Quadratic Form in Three Cartesian Coordinates"**, (The Quadric Surfaces), including many absolute invariants and their meanings, such as surface area, volume etc..

b) **"The Analysis of the Quadratic Form in Two Intrinsic Coordinates, (ñ,s)"** (Cesàro).

c) **"The Analysis of the Quadratic Form in Two Intrinsic Coordinates, (s,ö)"** (Whewell).

d) **"The Analysis of the Quadratic Form in Two Intrinsic Coordinates, (ñ,ö)"** (Euler).

e) **"The Analysis of the Quadratic Form in Two Cartesian Coordinates under deformation"**

3) A *`general solution'* of Euler's Equation in **"Calculus of Variations in the Plane"**, thus enabling a direct solution of planar problems to be written straightaway in intrinsic coordinates. (No solving a D.E. is necessary.)

4) Many derived curves, especially an infinite set of families of **"Base Curve and Pursuit Curve"** in closed form. Similarly with the Pseudo-Pursuit Curve.

5) Anallagmatic Curves and the discovery of the innermost relationship between the equation to the curve and the coordinate system, to which it is referred, resulting in their theoretical breakdown to Prime Curves and Composite Curves.

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Josef,

Many thanks for the 9 page part 1, just received 13:45 Sunday afternoon, when I was about to complete my correction to FibComp Table, so I shall send that together with a few extra bits and pieces, and then I shall take much pleasure in going thru your proof with all the extra insertions from Derive etc.

I note how you show great preference for Derive's ability to make things easier, which is the case most of the time.

I, however, use its simplifications with a degree of caution, since on many occasions it cannot do what a human can do with pen and paper. I prepared this paper on Recursive Series with pen and paper completely many years ago. I look for patterns so as to regroup expressions, sometimes replacing them temporarily with a single variable, so as to manipulate them with great facility.

Coincidentally, in the last month I have been working on three papers, simultaneously, dealing with recursion over 4, 6 & 8 terms resp, with incredible unsuspected findings. It will be a while before I complete them.

I had never intended to do any more on recursion, but for the serendipitous picking up a photocopy of two papers by Dupin from Compte Rendus 1847, which I have had in my filing cabinet for many years.

A French friend, Alain, who I see about once or twice a year, used to be a maths. professor in Paris at the Sorbonne. Last month I was due to visit him and I thought to take Charles Dupin's papers to him for his evaluation.

Anyhow, we had other matters to discuss, so it was not looked at. When I got home 'the penny dropped' and I realised that Dupin's geometrical finding/linking of data from a statistical table, was, in fact, a finite recursive series of 4 and 6 terms resp..

So I decided to treat them as definitions of an infinite series and therefore investigate their properties" :- nth term, sum to n terms etc.. I was `gob-smacked' at what I discovered.

Ausgezeichnet Dupin!!!

Third International DERIVE/TI-92 Conference
Keynote Address

Why CAS must mean more than symbolic manipulation by computer (or calculator) - or - the case of the People versus Symbolic Manipulators.

Adrian Oldknow, a_oldknow@compuserve.com
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Abstract: Taking examples from pure mathematics (such as Geometry) and applied mathematics (such as Modelling) the talk illustrates the power of using computer symbolic manipulation (CSM) in problem-solving, but also highlights some of the dangers of using it without the other tools (such as numerical and graphical), associated with Computer Algebra System (CAS) software packages, to validate results.

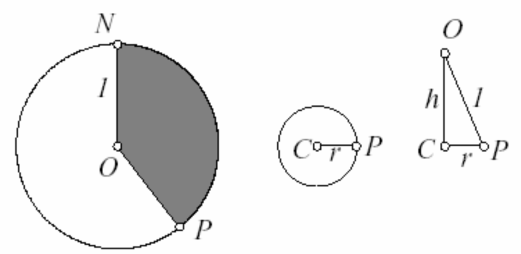
Preliminary remarks to the jury

First I must thank the conference chairs, Carl and Bert, for inviting me to give this opening keynote address - it is both a pleasure and an honour. Some of you here were, I think, at another conference in Koblenz last September - the third International Conference on Teaching Mathematics with Technology (ICTMT-3), chaired by Prof. Wolfgang Fraunholz. There Bert was scheduled to give a keynote entitled: **What is the appropriate role, if any, of hand-held computer symbolic algebra in the teaching and learning of mathematics?** (Oldknow & Waits, 1997) Unfortunately at the last minute Bert had to return to the USA for personal reasons, and so my invitation to give that keynote was with considerably less notice than I received on this occasion - just at breakfast the day before the talk! That story also allows me to get in a "plug" for the next ICTMT conference which is to be held in Plymouth, UK from August 9 - 12, 1999 with Prof. John Berry as chair.

Exhibit 1: a single cone

To illustrate that talk I took an example which I first used in a graphing calculator workshop with my colleague, Warwick Evans, at the first ICTMT conference in 1993 in Birmingham, UK with Prof. Leone Burton as chair. The basic problem is a simple optimisation one.

A segment is cut from a circular piece of paper of unit radius, and folded to form a cone with a circular base of radius *r*. What is the value of *r* if the cone is to have maximum volume?



Well that is the kind of problem - an application of differential calculus - which is meat and drink (with apologies to vegetarians and tea-totallers) to computer symbolic manipulation (CSM). The slant-height of the cone is the radius of the original circle i.e. 1, so the vertical height h is given by Pythagoras: $h = \sqrt{(1 - r^2)}$ and so the volume is $1/3\ \pi\ r^2\ h =$
$= 1/3\ \pi\ r^2 \sqrt{(1 - r^2)}$, and it is sufficient to find the extrema of the function $v(r) = r^2 \sqrt{(1 - r^2)}$. Using the "calculus differentiate" function of the TI-92 we have the derivative as:

$$2\ r \sqrt{(1 - r^2)} - r^3/\sqrt{(1 - r^2)}$$

and using the "zeros" function we find the extrema as: $\{0\ , -\sqrt{(6)}/3\ , \sqrt{(6)}/3\}$. From our knowledge of the problem we recognise the first as a minimum and the second as unrelated to the physical problem, hence we know that the third value is the required value of *r* for the cone with the maximum volume.

This use of the CSM aspect of a CAS serves to illustrate the issues which divide academics on its use. On the one hand it liberates the problem-solver from the tedium of algebraic manipulation and the application of the rules of calculus - thus allowing the intelligent human to be the supervisor, defining the strategy, and the dumb computer to be the slave performing the humdrum algorithms (without making the kind of slips which characterise human manipulations). On the other hand it is that very human manipulative performance, with accuracy, which has been long regarded as the hallmark of a good mathematician - a skill which many academics still hold very dear. Later I shall return to a variation of this same cone problem.

Connections with the Accused

Now I just want to wander down memory lane a bit to remind you and I how I come to be in the position of addressing you today. My background is that of a smart kid who was very slick at doing just those kinds of "advanced level" manipulations accurately and fast - and so getting very high scores on the UK public examinations at 15 and 17, culminating in the high point of my academic career: an open scholarship in mathematics at Oxford University. Evelyn Waugh, a former student at the same college, Hertford, wrote a book called "Decline and Fall" (Waugh, 1928), which neatly sums up my own undergraduate career! Computers were, of course, more or less unheard of among Oxford mathematicians in the 1960s.

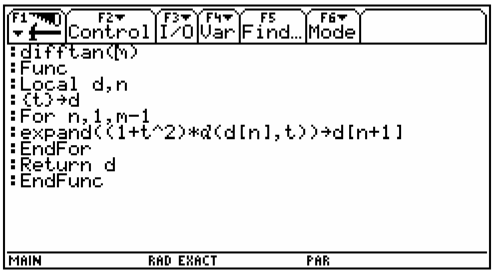
After teaching mathematics in schools, I took up a post in 1970 as a lecturer in mathematics and computer science. The speed with which people were able to trade-up jobs at that boom time in the computer industry accounted for the need to draft in ignorant, but willing, staff into the lower paid jobs, such as teaching! So I found myself learning *Basic* programming over the summer holiday from (Kemeny & Kurtz, 1964) - still never having seen or used a computer! Like many before me, I cut my programming teeth on problems such as writing routines for performing integer arithmetic with numbers of arbitrary length. After a year I got sent on a day-release Master's course in Computer Science where I first came across (Knuth, 1968) and the ideas of programming languages other than *Basic*, such as *LISP*; of data structures other than matrices, such as threaded lists; and of operations with data other than numbers, such as symbolic manipulation.

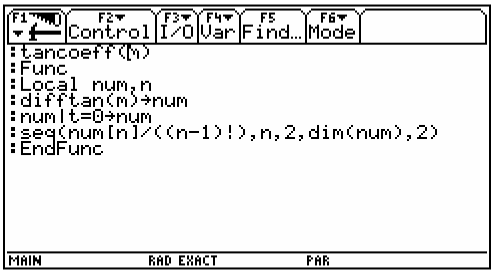
Later, as primitive classroom computers first became available, I did, in a small way, some pioneering work in their application to mathematics teaching in the UK. Through my contacts with colleagues abroad I became aware of the development of mu-Math for MS-DOS. Advertised as "a college education for $200" I readily parted with my own money to get a copy on 5.25" floppies, written in mu-Lisp, for my Z-80 based DOS machine (and later for the graphic additions in Basic!). So OK, you have some software which comfortably copes with the majority of questions on A-level math examination papers (the UK equivalent of AP) – but how can you use it constructively to do something you haven't done before? As far as I do anything recognisable as "research", it is this kind of question I try to address in my own work.

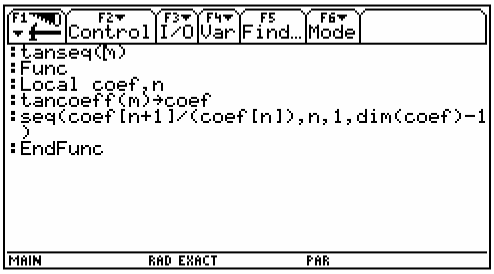
Exhibit 2: expansions of tan

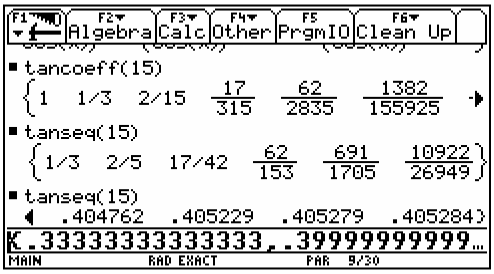
The aspect that took my fancy was the ability to compute many terms of the Taylor expansion of a function. I guessed I knew a fair amount about the theory of sine, cosine, logarithmic and exponential functions - but was curious about why I didn't remember anything about the series expansion of the tangent function. Using "taylor(tan(x),x,7,0)" on the TI-92 gives:
$17x^7/315 + 2x^5/15 + x^3/3 + x$ where the coefficients are expressed as rational numbers in lowest terms. I wanted to be able to study the ratios of the successive terms of these polynomials.

But the degree 7 version doesn't give much information. So it would be nice to have at least terms up to degree 15. However, even using *Derive for Windows* on my nice Pentium based laptop we run into one of the principal problems using CAS systems - that of knowing just how long to wait when an answer does not appear! I leave you to test this for yourself, but certainly my old Z80 DOS machine showed no signs of ever getting there! Now the derivative of tan(x) is $\sec^2(x)$, and very soon the higher derivatives become enormously involved. But we know from Pythagoras that we can rewrite: $\sec^2(x) = 1 + \tan^2(x)$. If we write t for tan(x) then $dt/dx = 1 + t^2$, and denote the n-th derivative of tan(x) w.r.t. x in terms of t by d(n,t) then we have the recurrence: $d(n+1,t) = d/dt(d(n,t))\cdot(1+ t^2)$ for n = 0,1,2,.... where d(0,t) = t . With the old version of *mu-Math* it was possible to write procedures in the programming language *mu-Lisp* to carry out such an algorithm. Here I guess we can use the vector functions of *Derive*, but TI-92 programs, or better still functions, might be easier to follow:









Here the list "d" holds the successive derivatives of tan(x) as polynomials in t.
The list "num" substitutes t = 0 for the Taylor expansion around x = 0 to get the numerators of the non-zero coefficients (the odd powers of x), which are divided by the appropriate factorials to arrive at the list "tancoeff(m)" of the actual coefficients from which the successive ratios are computed in the function "tanseq(m)".

A call to "tanseq(15)" gives: {1/3 , 2/5 , 17/42, 62/153 , 691/1705 , 10922/26949 }

but it is hard in this representation to see if these ratios are converging. So clearly the facility for the user to write algorithms is an important aspect of CAS if we are to be able to use human intellect to help CAS when it gets stuck in a blind alley! But we also need the numerical facility to convert rational numbers to decimal approximations. Doing this on the TI-92, and rounding to 5 d.p. yields:

{.33333 , .4 , .40476 , .40523 , .40528 , .40528 }

and we have convergence to 5 decimals - but to what number? Let's call it k. Then we have found that the long term behaviour of the Taylor series for tan(x) is that the ratio of successive terms tends to kx^2.

This means that there is an underlying geometric progression with common ratio kx^2 and first term x - and this is easy to find by summation to infinity (assuming convergence) as $x/(1 - kx^2)$ which can be put into partial fractions as:

$$1/(2p)\ [1/(1-px) - 1/(1+px)]\ ,\ \text{where } k = p^2\ .$$

But now we just have to remember what the actual problem was, and that tan(x) has a bunch of singularities, the first of which occur at $x = \pi/2$ and $x = -\pi/2$! So it's pretty obvious that p must be equal to $2/\pi$, giving $k = p^2 = 4/\pi^2 \approx 0.4052847$. So we have found a remarkably good approximation to tan(x) as $P_1(x) = (\pi^2/8)\ [(\pi/2 - x)^{-1} - (\pi/2 + x)^{-1}]$ as the graphing functions of e.g. *Derive* or the TI-92 soon reveal. But we need not stop there. We can study the Taylor expansion of the difference between tan(x) and $P_1(x)$ - and find that this, too, has an underlying GP whose sum is given by the function:

$$P2(x) = (\pi^2/8)\ [(3\pi/2 - x)^{-1} - (3\pi/2 + x)^{-1}]$$

and by induction that we can write $\tan(x) = \Sigma\ Pr(x)$ where r runs from 1 to infinity.
where $P_r(x) = (\pi^2/8)\ [((2r-1)\pi/2 - x)^{-1} - ((2r-1)\pi/2 + x)+]\ ,\ r = 1,2,3....$

I have to admit I felt pretty pleased with myself when I made this discovery - so much that I used it in a talk I gave to the Sussex branch of the Mathematical Association. So you can image my dismay when one of the academics in the audience said that he remembered that form of series expansion from a course of complex analysis he used to give to postgraduates - and that I should look up the work of the Swedish mathematician Gösta Mittag-Leffler (1846-1927) who had beaten me to it by about 90 years! So at least I was saved from the ignominy of claiming an original result. But I was certainly guilty of using CSM like George Bernard Shaw's drunkard used a lamp-post - more for support than illumination. And that comes back to what it takes to be a mathematician - accurate symbol manipulation (by man or machine) is necessary, but not sufficient, to make advances. CSM is just another tool, like the astronomer's telescope and the biologist's microscope, which produces data from which information may, or may not, be extracted. I should have been smart enough to have thought about the singularities of tan(x) before I set off on my naive voyage on the good ship "Symbol Manipulation".

Exhibit 3: Sprinkling water on the grass

The first two exhibits were clearly from mainstream "pure mathematics", in the sense in which we use that term in the UK in pre-university education. That is to say practising the "applicable" techniques of the calculus either without a context or within a very contrived one. The next exhibits move first to the field of "applied mathematics" and then to that field which most people would agree to be "pure" - geometry.

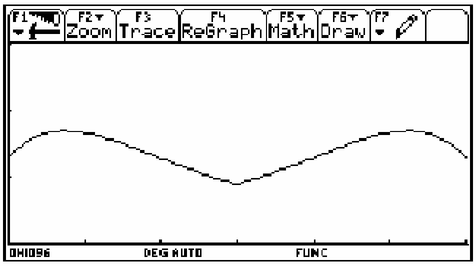
In the UK our weather pattern is quite changeable now, and there are times in the summer when we do have long periods without rain. As you know we like our gardens, and, particularly our carefully tended patches of flat green grass known as "lawns". So in dry weather we need to water these to keep them green. A common type of water sprinkler has a bar studded with holes which can oscillate from side-to-side. A simple turn of a knob alters the sprinkling pattern between large sweeps left, large sweeps right, large sweeps both sides and small sweeps both sides. Closer inspection reveals that the mechanism is based on a common design in mechanical engineering, called the four-bar linkage (Oldknow, 1997a).

In the quadrilateral *ABCD*, the frame *AB* is fixed. The rotor arm *AC*, called the driver crank, is driven round a circle at constant velocity by a small water turbine. The sprinkler bar turns freely about *B* and is aligned with the follower crank *DB*. The coupler *CD* is loosely jointed at *C* and *D*. The knob alters the length of *CD*. So we need to design a mechanism in which the lengths $AB = d$, $AC = a$ and $CD = b$ are constant parameters constrained by physical dimensions of the artefact and $BD = c$ is a variable parameter for which there are 4 values which correspond to the required water sprinkling pattern.

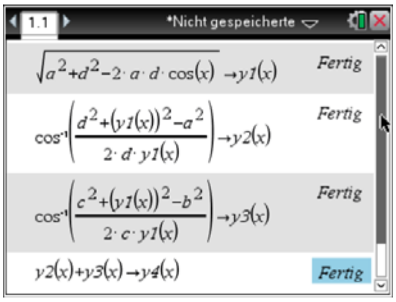
If we take the angle $\alpha = \angle BAC$ as the independent variable, then we seek to find the angle $\beta = \angle ABD$ as a function of α and the parameters a,b,c,d. To do this we have to solve the quadrilateral, which means dividing it into triangles, such as ΔABC and ΔBCD. In ΔABC we know the sides $AB = d$ and $AC = a$, as well as the angle $\angle BAC = \alpha$ - and so we can find the length $BC = e$ and angle $\angle ABC = \gamma$, e.g. from the cosine rule applied twice. In ΔBCD we now know the three sides $BC = e$, $CD = b$, $BD = c$ and so we can find the angle $\angle CBD = \delta$, again from the cosine rule. Hence we can write a function to define the angle β as a function of angle α. Alternatively on the TI-92 we can just enter the function definitions for e, γ, δ, β in the function graphing screen as y1(x), y2(x), y3(x) and y4(x):

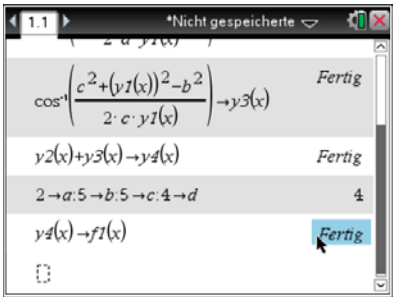
```
y1(x)=√(a^2+d^2-2*a*d*cos(x))
y2(x)=cos⁻¹((d^2+y1(x)^2-a^2)/(2*d*y1(x)))
y3(x)=cos⁻¹((c^2+y1(x)^2-b^2)/(2*c*y1(x)))
y4(x)=y2(x)+y3(x)
```

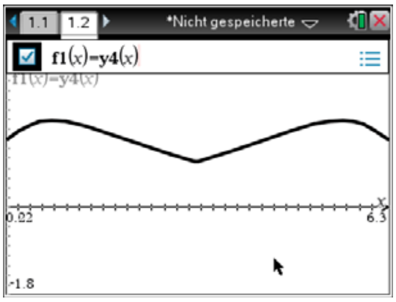
and with suitable values stored in a,b,c,d we can graph the response curve as the angle x describes a full rotation from 0 to 2π. The following graph is obtained from values: $a = 2$, $b = 5$, $c = 5$, $d = 4$.

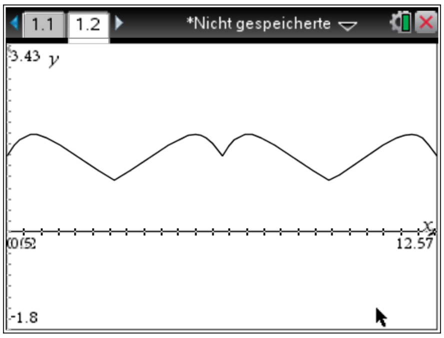


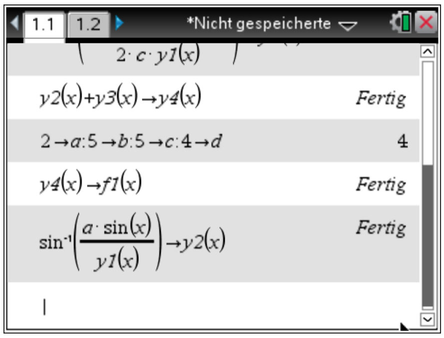
As you can see I did it with TI-NspireCAS. It is pretty the same procedure (Josef):





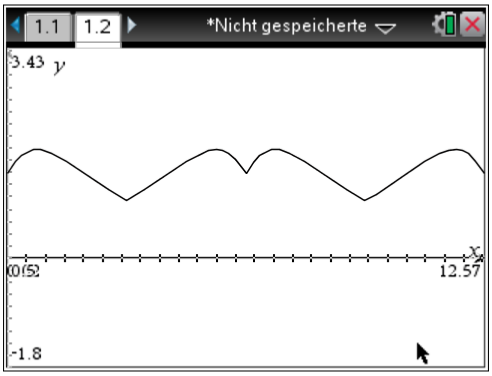


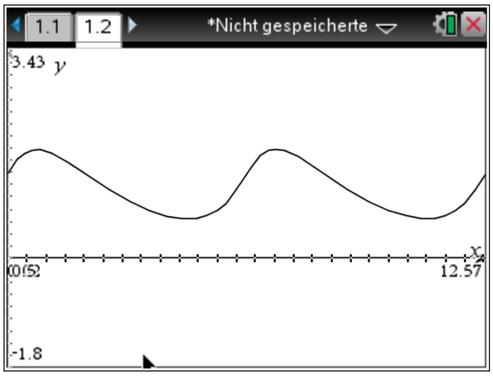




At this point we could feel quite happy with ourselves - although we have used symbolic definitions of the functions, when numerical values are substituted we get a respectable looking graph. But what happens if we go between 0 and 4π? Can you interpret this graph physically? The motion should be smooth, so clearly we are encountering problems between π and 2π (left picture below).

Here we should give thought to the range of the inverse cosine function! We have used the numerical and graphical tools to validate our symbolic model - and found it wanting! We can also attempt to debug the problem and by using the sine rule in ΔABC instead, redefine y2(x) giving the smooth periodic (but non-sinusoidal) graph shown below.





So clearly the validation of a model cannot done just by reworking the symbolic manipulations! It is essential to compare results calculated from them with values measured or independently computed. Here the numeric, programming, graphing, geometry and statistics functions of the TI-92 can all be used to this end.

Exhibit 4: Analytic plane geometry

There are many ways of converting objects in 2D or higher dimensions into algebraic objects, such as Cartesian coordinates, homogeneous coordinates, vectors and complex numbers. As soon as any degree of geometric complexity is encountered the corresponding algebraic expressions can easily become quite unwieldy. Take for example an algebraic attempt to solve Fermat's problem - given points *A,B,C* to find the position of a point *P* for which the sum of distances *AP*+*BP*+*CP* is minimal. We know *A,B,C* are coplanar so we can assume that *P* lies in that plane and so try using 2D Cartesian coordinates. We can choose *A*, say as origin (0,0) and *AB* as *x*-axis with *B* at (2,0) say. Then *C*(*p,q*) can be anywhere in the plane, though we can restrict it without loss of generality to lie in the upper half-plane $q>0$.

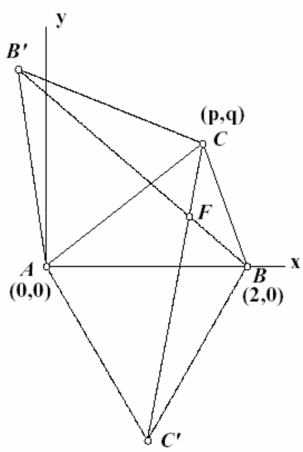
The problem then becomes to find values of *x,y* so that *P*(*x,y*) minimises the function:

$$f(x,y) = \sqrt{(x^2 + y^2)} + \sqrt{((x-2)^2 + y^2)} + \sqrt{((x-p)^2 + (y-q)^2)}$$

and we just have to solve the pair of simultaneous (non-linear) equations:

$$df/dx = 0 = df/dy \text{ for } x \text{ and } y.$$

Well, I haven't managed to do that on *Derive* or the TI-92! But, of course, we can substitute numeric values for the parameters *p, q* e.g. $p = 1/2$, $q = 2$ and plot the surface $z = f(x,y)$ to see if it has a minimum. Oddly enough even trying to solve the differentials now doesn't seem to be any more tractable! But this problem was solved by Fermat using "pure" geometry.



We just have to find the vertices *C'* and *B'* such that *ABC'* and *ACB'* are equilateral triangles described outwards from *ABC*. Then find the intersection *F* of *CC'* and *BB'*. Provided no angle of the triangle *ABC* exceeds 120° then *F* is the solution, other-wise it is the vertex with the largest angle (Coxeter, 1969). It is easy to find that the coordinates of *C'* are $(1, -\sqrt{3})$, and rotating *C* through 60° about *A* we get *B'* as $((p - q\sqrt{3})/2\ ,\ (p\sqrt{3} + q)/2)$.

But even finding the coordinates of *F*(*xf, yf*) seems too much for the TI-92, though not *Derive*, which gives:

$$xf = \sqrt{3}(\sqrt{3}p^2 + 20(2q + \sqrt{3}) + q(\sqrt{3}q + 2))/(3(p^2 - 2p + q^2 + 2\sqrt{3}q + 4))$$

and something equally horrid for *yf*. Substituting theses values of *xf* and *yf* in the derivatives still does not produce the expected zero values - but many terms involving the SIGN function. Even defining the inequalities for p and q so that F is internal to ABC fails to do the trick: $q > 0$, $q > -\sqrt{3}p$, $q > \sqrt{3}(p-2)$ and $(p-1)^2 + (y + 1/\sqrt{3})^2 > 4/3$. But putting in numerical values for p and q, such as 1/2, 2, again produces the expected zeros for the derivative.[*]

What this tells us is that is not difficult to find even simple problems where the symbolic manipulations soon become either more than the CAS systems can handle, or produce unwieldy and unenlightening results. So the researcher has to work with the CAS in a symbiotic mode, often employing a variety of tools such as a CAS package, a programming language and dynamic geometry software (*Cabri* or *Sketchpad*) together with pencil and paper, and a lot of thought! As part of such a team, CAS certainly has a large part to play.

I have been working both alone, and with a collaborator, Prof. Brian Griffiths of Southampton University, in the field of triangle geometry. Many of our discoveries have relied upon finding some remarkable and surprising factorisations using CSM that we could not possibly have performed manually - but to get there we had to help the CSM very considerably! These results will not be described here in detail, but references to such work include (Griffiths & Oldknow, 1998), (Oldknow, 1995a), (Oldknow, 1995c), (Oldknow, 1996), (Oldknow, 1997b). One of my proudest moments was doing the inevitable ego-kick search on "oldknow" with a web-crawler. To my amazement some nice guy has included some of my discoveries on his website.

One thing worth mentioning here is the lack of references to the use of CSM in academic papers in pure mathematics. In his address at the International Mathematica Symposium in Southampton, 1995, Prof. J.H. Davenport of Bath University said that nearly all the results of recent research in his own field of number theory and encryption would have been impossible to find without CSM, but that it was not in the nature of academic publication, nor the psyche of pure mathematicians, to give away details of how results were discovered - just to prove their validity! This secretiveness makes it much harder for mathematics departments to justify their claims for improved computer resources!

Exhibit 5: A pair of cones

So my final piece of evidence for the prosecution against the unvalidated use of CSM comes from the same cone problem as exhibit 1. Here we found that taking a sector of a circle, which accounts for $\sqrt{(6)}/3 \approx 82\%$ of the circle, produces the cone of maximum volume. But in these green and conservationist times we need to realise that we could make another cone, albeit a small one, with the little piece that's left. So that raises the question: how should we divide the unit circle into two sectors so that the sum of the volumes of the resulting cones is maximum?

Well the answer seems very obvious, doesn't it? But we might as well see if CSM can produce it for us. We can easily verify that if the radius of the base circle of one cone is r, then that of the other cone is $1 - r$. So the total volume is a multiple of the function:

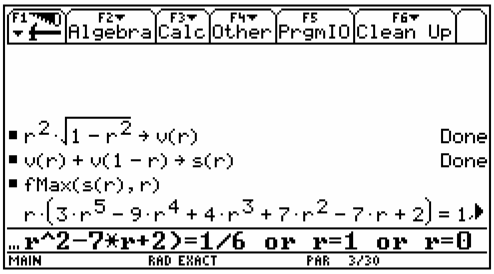
$$s(r) = v(r) + v(1-r) \text{ where } v(r) = r^2\sqrt{(1-r^2)}, \text{ and its derivative is:}$$

$$2r\sqrt{(1-r^2)} - r^3/\sqrt{(1-r^2)} + 2(r-1)\sqrt{(-r(r-2))} - (r-1)^3/\sqrt{(-r(r-2))}$$

Asking for the zeros of this on the TI-92 in exact mode gives: {1/2} as expected.

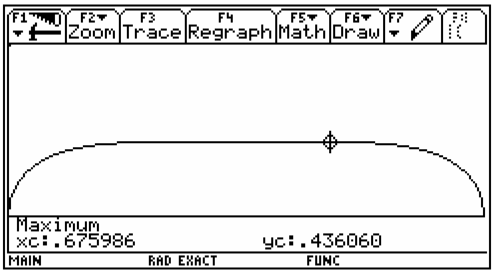
[*] Heinz Rainer Geyer provided a contribution treating the Fermat Point in DNL#16 from 1994.

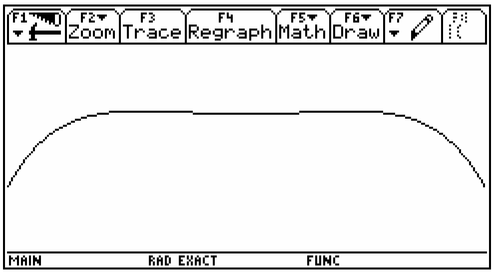
But remarkably FMAX(s(r),r) returns:



where the 6th degree polynomial: $18\,r^6 - 54\,r^5 + 24\,r^4 + 42\,r^3 - 42\,r^2 + 12\,r - 1$ cannot be factored - and the value $r = 1/2$ does not appear! Just to confirm the expected result we can graph the function s(x) on the domain [0,1] and check that its maximum is on the line of symmetry at $x = 1/2$ as in the next figure. But hang on! This function is incredibly flat – are we sure it always increasing on [0,1/2]?

If we use F5 Math Maximum we don't get $x = 1/2$. Try zooming in to get more detail. (Oldknow, 1995b).





Well, that's the clincher - unthinking and unvalidated use of CSM must be guilty as charged. Not only has it goofed up on finding a maximum - it's actually dished in a minimum! What, then, are the values of r which produce the maxima? Well suppose we switch to AUTOMATIC or APPROXIMATE mode and find the zeros of the derivative again, we get: {.324013851832 , 1/2 , .675986148169} as the set of values of r for the extrema of s(r) , whereas if we find the approximate zeros of the 6th degree polynomial above we find there are an additional 4 spurious values induced by the manipulations: { -1.00392 , .14798 , .85202 , 2.00393} of which 2 are outside the domain of r. But our 6 roots appear in 3 pairs which each sum to one. I leave it the reader to show that the sixth degree polynomial can be expressed as:

$$(x - u)(x - (1 - u))\,(x - v)(x - (1 - v))\,(x - w)(x - (1 - w)) = (x^2 - x + a)\,(x^2 - x + b)\,(x^2 - x + c)$$

where: $a + b + c = -5/3$, $ab + bc + ca = -2/3$ and $abc = -1/18$

so that a, b, c are the roots of: $x^3 + 5/3\,x^2 - 2/3\,x + 1/18 = 0$

The value of a we need is close to 0.219 , and so u is given by: $u = 1/2\,(1 - \sqrt{(1 - 4a)})$.

Applying Cardan's algorithm (which was built into *mu-Math*!) we find:

$$x = 1/2 - 1/6\sqrt{\{29 - 36\,[\,\sqrt[3]{(-1121/2916 + \sqrt{191/324}\,i)} + \sqrt[3]{(-1121/2916 - \sqrt{191/324}\,i)}\,]\}}$$

But so what - I'm beginning to manipulate the symbols for the sake of it! It must be addictive.

Summing Up for the prosection

Using examples from my own experience I have tried to show (a) just how powerful even the small CAS crammed into the TI-92 can be in attacking a wide range of problems, but (b) that CSM is just one part of the CAS and that it in realistic situations it is very unwise to rely upon it alone either to produce the results in the fashion sought, or a result at all! But in this audience I am teaching my peers to suck eggs - none of you would do anything like this!

The Verdict: Guilty as charged

Well - a foregone conclusion, really. I have set up an "Aunt Sally" that any jury in the land would convict.

The Sentence

Our problem is to communicate to those unfamiliar with the full range of facilities in a CAS that these can be used to aid mathematical enquiry - but that the human is always playing the supervisory role, defining the strategy, choosing the tools, monitoring the results, validating the solutions, refining the problem etc. Academic mathematicians seem to be able to be as frightened, suspicious and challenged by CSM (which they call CAS) as others have been about the use of calculators for arithmetic. We must be sensitive to their concerns - and try to take the whole mathematics community with us into the millennium when computer assistance is seen as just as commonplace in working at, and researching in, mathematics as telescopes, microscopes and other instruments are by colleagues in the sciences. This view undoubtedly has implications for what should be the knowledge, content and skills of mathematical education for the future. There have been some tentative steps in this direction - but I guess each country will just have to proceed at its own pace (Oldknow & Flower, 1996), (Sutherland, 1997), (Oldknow & Waits, 1997). One thing is sure - to be well briefed to plead sound cases in our own countries it is essential that we share information about what is going on in other countries - and that is one major role for international conferences, such as this.

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Two more Tribonacci Sequences – the *plastic number*

Josef Böhm

It is really funny that I found in one of my old "Spektrum Dossiers" from 2003 a contribution written by Ian Stewart "*The plain sister of the Golden Number*". Ian Stewart presents the so called *Padovan* Sequence defined by

$T_n = T_{n-2} + T_{n-3}$, $T_0 = T_1 = T_2 = 1$:   1, 1, 1, 2, 2, 3, 4, 5, 7, 9, 12, 16, 21, ...

The limit of the ratio of neighbouring elements is approximately 1,324718 (the so called *plastic number*). *Perrin* found a sequence following the same rule, starting with 3, 0, 2 giving the same limit for the ratio.

Let's try with David's findings and my *DERIVE* tools:

The generating functions according 9.5 are: $S(x) = \dfrac{1+x}{1-x^2-x^3}$ and $\dfrac{3-x^2}{1-x^2-x^3}$ respectively.

$$\mathrm{TAYLOR}\left(\frac{1+x}{1-x^2-x^3},\ x,\ 10\right)$$

$$12\cdot x^{10} + 9\cdot x^9 + 7\cdot x^8 + 5\cdot x^7 + 4\cdot x^6 + 3\cdot x^5 + 2\cdot x^4 + 2\cdot x^3 + x^2 + x + 1$$

```
VECTOR(TSN1(1, 1, 1, b1, b2, b3, n), n, 0, 20)
[1, 1, 1, 2, 2, 3, 4, 5, 7, 9, 12, 16, 21, 28, 37, 49, 65, 86, 114, 151, 200]
```

$$\left(\mathrm{VECTOR}\left(\frac{\mathrm{TSN1}(1, 1, 1, b1, b2, b3, n+1)}{\mathrm{TSN1}(1, 1, 1, b1, b2, b3, n)},\ n,\ 0,\ 50\right)\right)_{[49,\ 50,\ 51]}$$

```
[1.324717959, 1.324717956, 1.324717956]
```

$$\mathrm{TAYLOR}\left(\frac{3-x^2}{1-x^2-x^3},\ x,\ 10\right) = 17\cdot x^{10} + 12\cdot x^9 + 10\cdot x^8 + 7\cdot x^7 + 5\cdot x^6 + 5\cdot x^5 + 2\cdot x^4 + 3\cdot x^3 + 2\cdot x^2 + 3$$

```
VECTOR(TSN1(3, 0, 2, b1, b2, b3, n), n, 0, 20)
```

$$\left[3,\ -1.146144438\cdot 10^{-12},\ 2,\ 3,\ 2,\ 5,\ 5,\ 7,\ 10,\ 12,\ 17,\ 22,\ 29,\ 39,\ 51,\ 68,\ 90,\ 119,\ 158,\ 209,\ 277\right]$$

Proof: The *Padovan* Sequence can be defined as $T_n = T_{n-1} + T_{n-5}$, too.
Do you find out where the *plastic number* is hidden in the *DERIVE* file above?