THE DERIVE - NEWSLETTER #98

ISSN 1990-7079

THE BULLETIN OF THE

DERIVE®

USER GROUP

+ CAS-TI

Contents:

June 2015

Mail from Albert Rich:

Hello Josef,

From your recent email to Michel, I was glad to see that the DNL is alive and well. You must be near celebrating its 25th anniversary.

Since you have expressed an interest in Rubi in the past, I thought you might want to check out her new look at http://www.apmaths.uwo.ca/~arich/

Click on the "Integration Rules" menu item to find and view the appropriate set of rules that apply to a particular integrand type. For example, try finding the trig rules for integrands of the form (a+b tan(c+d x))^n. Eventually I hope to continue this cascading navigation menu tree right down to the specific rule needed to integrate a given expression; thereby mirroring on the website the decision tree Rubi uses to integrate expressions...

Please let me know what you think of the site and any suggestions may have.

Aloha,

Albert

Learn more about "*Rubi*" *i*n the next DNL, Josef.

Mail from Francisco Marcelo Fernandez:

```
Dear Derivians,
     I had a problem calculating a simple integral with Derive. It
is explained in the attached file. May be I am missing something but
I cannot find what.
          Best regards, Marcelo
```

#1: $[a :\in \text{Real } (0, \infty),\ m :\in \text{Integer } (0, \infty)]$

Derive fails to calculate

#2: $$\int_{-\infty}^{\infty} x^{2\cdot m}\cdot \text{EXP}(- a\cdot x^2)\, dx$$

#3: $$\int_{-\infty}^{\infty} e^{- a\cdot x^2}\cdot x^{2\cdot m}\, dx$$

But it can do it with a little help

#4: $$2\cdot\int_{0}^{\infty} x^{2\cdot m}\cdot \text{EXP}(- a\cdot x^2)\, dx$$

#5: $$a^{-(2\cdot m + 1)/2}\cdot\left(m - \frac{1}{2}\right)!$$

Dear DUG Members,

It is end of June now and I am very happy that I can offer DNL#98 in time.

The "Truth Tables …" is one of the many "Contributions waiting to be published" from page 2 which found its way from my stack of more or less ready papers into this newsletter. MacDonald Phillips sent this article long ago (2007) and I am very sorry that I didn't present it earlier. I hope that Don will accept my excuse. 2007 was the time when TI-92, Voyage 200 and TI-89 were full in use while TI-NspireCAS was in its infancy. *DERIVE* does not need a special function or program because TRUTH_TABLE() does a great job. TI-NspireCAS is another case: there is no function implemented and we cannot create own drop down menus. So it was a challenge for me to provide a truth table tool for TI-NspireCAS. Pages 30 and 31 show the result.

It has happened very often that articles come in which are of immediate interest (see Michel's note on the solution of cubics in this issue) or that I become fascinated by a special problem (mail, journal, talk, …) which then gets its self dynamics.

Take as an example the Stern-Brocot Sequence treatment in DNL#98. I read about it in the Scientific American, started *DERIVE*, and then I was caught by programming, browsing my books, searching the web, … and the result is a 10 pages article. I hope that you will enjoy the treatment of the "Misunderstood Sister of the Fibonacci Sequence".

Michel Beaudin sent a toolbox for TI-NspireCAS wich has its history in its *DERIVE* past. Michel had provided *DERIVE*-tools for his students at ETSMTL several years ago. As he and his students changed to TI-Nspire he updated this useful collection for the new technology and added some functions which are implemented in *DERIVE*. All functions, programs and written instructions and provided examples are in French. I asked Michel for permission to produce an English version which was given without hesitating. Many thanks, Michel.

I will meet Michel and a couple of other DUG-members in July in Kalamata, Greece, at ACA 2015. I am sure that we will have an interesting meeting despite the economic problems which Greece and its people must face in these days.

I will give a report in DNL#99.

Always Yours

Josef

Download all *DNL*-DERIVE- and TI-files from

http://www.austromath.at/dug/

In December we will celebrate DNL#100 (= 25 Years DUG). It will be great if some of you – especially members from the early DUG Years – will contribute for this very exceptional issue. All articles, notes, comments, memories, … are very welcome, Josef.

The *DERIVE-NEWSLETTER* is the Bulletin of the *DERIVE* & CAS-*TI User Group*. It is published at least four times a year with a content of 40 pages minimum. The goals of the *DNL* are to enable the exchange of experiences made with *DERIVE*, *TI*-CAS and other CAS as well to create a group to discuss the possibilities of new methodical and didactical manners in teaching mathematics.

Editor: Mag. Josef Böhm
D´Lust 1, A-3042 Würmla
Austria
Phone: ++43-(0)660 3136365
e-mail: nojo.boehm@pgv.at

Contributions:
Please send all contributions to the Editor. Non-English speakers are encouraged to write their contributions in English to reinforce the international touch of the *DNL*. It must be said, though, that non-English articles will be warmly welcomed nonetheless. Your contributions will be edited but not assessed. By submitting articles the author gives his consent for reprinting it in the *DNL*. The more contributions you will send, the more lively and richer in contents the *DERIVE* & CAS-*TI Newsletter* will be.

Next issue: September 2015

Preview: Contributions waiting to be published

Some simulations of Random Experiments, J. Böhm, AUT, Lorenz Kopp, GER
Wonderful World of Pedal Curves, J. Böhm, AUT
Tools for 3D-Problems, P. Lüke-Rosendahl, GER
Hill-Encryption, J. Böhm, AUT
Simulating a Graphing Calculator in *DERIVE*, J. Böhm, AUT
An Interesting Problem with a Triangle, Steiner Point, P. Lüke-Rosendahl, GER
Graphics World, Currency Change, P. Charland, CAN
Cubics, Quartics – Interesting features, T. Koller & J. Böhm, AUT
Logos of Companies as an Inspiration for Math Teaching
Exciting Surfaces in the FAZ / Pierre Charland´s Graphics Gallery
BooleanPlots.mth, P. Schofield, UK
Old traditional examples for a CAS – what´s new? J. Böhm, AUT
Where oh Where is It? (GPS with CAS), C. & P. Leinbach, USA
Mandelbrot and Newton with *DERIVE*, Roman Hašek, CZK
Tutorials for the NSpireCAS, G. Herweyers, BEL
Some Projects with Students, R. Schröder, GER
Dirac Algebra, Clifford Algebra, D. R. Lunsford, USA
A New Approach to Taylor Series, D. Oertel, GER
Henon & Co; Find your very own Strange Attractor, J. Böhm, AUT
Rational Hooks, J. Lechner, AUT
Simulation of Dynamic Systems with various Tools, J. Böhm, AUT
Statistics of Shuffling Cards, H. Ludwig, GER
Charge in a Magnetic Field, H. Ludwig, GER

and others

Impressum:
Medieninhaber: *DERIVE* User Group, A-3042 Würmla, D´Lust 1, AUSTRIA
Richtung: Fachzeitschrift
Herausgeber: Mag. Josef Böhm

Solving Cubics

Michel Beaudin, Montréal, Canada

Hello my friends : if you have time, please take a look at the PDF « Lost_something ». It is about the solutions to the following third degree polynomial equations: $x^3 - 3x + 1 = 0$, $x^3 + 3x + 1 = 0$, $x^3 + 4x^2 + x - 1$ and $27x^3 + 216x^2 + 584x + 528 = 0$. I have used the following CAS : Derive, Maple, Nspire CAS and WolframAlpha Pro.

The first one and third one have three real distinct roots while the second one and fourth one have one real root and two complex conjugates.
In the case of 3 distinct real roots, *Derive* (version 6.1, the last one), is using trig expressions for the answers and this is very clean and clear. And when *Derive* is using Cardano's formula, NO radicals appear in the denominator.

After the *Derive* screen images, you will find the same four examples done with *Maple* 15 (I don't have the latest version). Then screen images from Nspire CAS (OS 4, latest version) will show that Nspire CAS can do a better job with some help : I have defined some functions in my TI-Nspire CAS library « kit_ETS_MB » and the results are similar to those of Derive. I am proud of this.

Solutions by WolframAlpha were printed and scanned and are shown on the pages following the solutions by Nspire CAS. I find easier to use WolframAlpha (Pro) instead of Mathematica… but I have Mathematica 7 on my computer because I need it to run Rubi!

Best regards from Montreal, Canada,

Michel Beaudin

Derive

#1: $\text{SOLVE}(x^3 - 3\cdot x + 1 = 0,\ x)$

#2: $$x = 2\cdot\text{COS}\left(\frac{2\cdot\pi}{9}\right) \vee x = -2\cdot\text{COS}\left(\frac{\pi}{9}\right) \vee x = 2\cdot\text{SIN}\left(\frac{\pi}{18}\right)$$

#3: $\text{SOLVE}(x^3 + 3\cdot x + 1 = 0,\ x)$

#4: $$x = -\frac{(4\cdot\sqrt{5} - 4)^{1/3}}{4} + \frac{(4\cdot\sqrt{5} + 4)^{1/3}}{4} - i\cdot\left(\frac{\sqrt{3}\cdot(4\cdot\sqrt{5} - 4)^{1/3}}{4} + \frac{\sqrt{3}\cdot(4\cdot\sqrt{5} + 4)^{1/3}}{4}\right) \vee x = -$$
$$\frac{(4\cdot\sqrt{5} - 4)^{1/3}}{4} + \frac{(4\cdot\sqrt{5} + 4)^{1/3}}{4} + i\cdot\left(\frac{\sqrt{3}\cdot(4\cdot\sqrt{5} - 4)^{1/3}}{4} + \frac{\sqrt{3}\cdot(4\cdot\sqrt{5} + 4)^{1/3}}{4}\right) \vee x =$$
$$\frac{(4\cdot\sqrt{5} - 4)^{1/3}}{2} - \frac{(4\cdot\sqrt{5} + 4)^{1/3}}{2}$$

#5: $\text{SOLVE}(x^3 + 4\cdot x^2 + x - 1 = 0, x)$

#6: $$x = \frac{2\cdot\sqrt{13}\cdot\text{COS}\left(\frac{\text{ACOT}\left(-\frac{5\cdot\sqrt{3}}{9}\right)}{3}\right)}{3} - \frac{4}{3} \lor x = -\frac{2\cdot\sqrt{13}\cdot\text{SIN}\left(\frac{\text{ATAN}\left(\frac{5\cdot\sqrt{3}}{9}\right)}{3} + \frac{\pi}{3}\right)}{3} - \frac{4}{3} \lor x = \frac{2\cdot\sqrt{13}\cdot\text{SIN}\left(\frac{\text{ATAN}\left(\frac{5\cdot\sqrt{3}}{9}\right)}{3}\right)}{3} - \frac{4}{3}$$

#7: $\text{SOLUTIONS}(27\cdot x^3 + 216\cdot x^2 + 584\cdot x + 528, x)$

#8: $$\left[-\frac{2\cdot(\sqrt{89}-9)^{1/3}}{9} + \frac{2\cdot(\sqrt{89}+9)^{1/3}}{9} - \frac{8}{3}, \frac{(\sqrt{89}-9)^{1/3}}{9} - \frac{(\sqrt{89}+9)^{1/3}}{9} - \frac{8}{3} + i\cdot\left(\frac{\sqrt{3}\cdot(\sqrt{89}-9)^{1/3}}{9} + \frac{\sqrt{3}\cdot(\sqrt{89}+9)^{1/3}}{9}\right), \frac{(\sqrt{89}-9)^{1/3}}{9} - \frac{(\sqrt{89}+9)^{1/3}}{9} - \frac{8}{3} - i\cdot\left(\frac{\sqrt{3}\cdot(\sqrt{89}-9)^{1/3}}{9} + \frac{\sqrt{3}\cdot(\sqrt{89}+9)^{1/3}}{9}\right)\right]$$

Maple

> $solve(x^3 - 3\cdot x + 1 = 0, x);$

$$\frac{1}{2}\left(-4+4\mathrm{I}\sqrt{3}\right)^{1/3} + \frac{2}{\left(-4+4\mathrm{I}\sqrt{3}\right)^{1/3}}, -\frac{1}{4}\left(-4+4\mathrm{I}\sqrt{3}\right)^{1/3} - \frac{1}{\left(-4+4\mathrm{I}\sqrt{3}\right)^{1/3}} + \frac{1}{2}\mathrm{I}\sqrt{3}\left(\frac{1}{2}\left(-4+4\mathrm{I}\sqrt{3}\right)^{1/3} - \frac{2}{\left(-4+4\mathrm{I}\sqrt{3}\right)^{1/3}}\right), -\frac{1}{4}\left(-4+4\mathrm{I}\sqrt{3}\right)^{1/3} - \frac{1}{\left(-4+4\mathrm{I}\sqrt{3}\right)^{1/3}} - \frac{1}{2}\mathrm{I}\sqrt{3}\left(\frac{1}{2}\left(-4+4\mathrm{I}\sqrt{3}\right)^{1/3} - \frac{2}{\left(-4+4\mathrm{I}\sqrt{3}\right)^{1/3}}\right)$$

> $solve(x^3 + 3\cdot x + 1 = 0, x);$

$$-\frac{1}{2}\left(4+4\sqrt{5}\right)^{1/3} + \frac{2}{\left(4+4\sqrt{5}\right)^{1/3}}, \frac{1}{4}\left(4+4\sqrt{5}\right)^{1/3} - \frac{1}{\left(4+4\sqrt{5}\right)^{1/3}} + \frac{1}{2}\mathrm{I}\sqrt{3}\left(-\frac{1}{2}\left(4+4\sqrt{5}\right)^{1/3} - \frac{2}{\left(4+4\sqrt{5}\right)^{1/3}}\right), \frac{1}{4}\left(4+4\sqrt{5}\right)^{1/3} - \frac{1}{\left(4+4\sqrt{5}\right)^{1/3}} - \frac{1}{2}\mathrm{I}\sqrt{3}\left(-\frac{1}{2}\left(4+4\sqrt{5}\right)^{1/3} - \frac{2}{\left(4+4\sqrt{5}\right)^{1/3}}\right)$$

> $solve(x^3 + 4 \cdot x^2 + x - 1 = 0, x);$

$$\frac{1}{6}\left(-260 + 156\,I\sqrt{3}\right)^{1/3} + \frac{26}{3\left(-260 + 156\,I\sqrt{3}\right)^{1/3}} - \frac{4}{3},$$
$$-\frac{1}{12}\left(-260 + 156\,I\sqrt{3}\right)^{1/3} - \frac{13}{3\left(-260 + 156\,I\sqrt{3}\right)^{1/3}} - \frac{4}{3}$$
$$+ \frac{1}{2}\,I\sqrt{3}\left(\frac{1}{6}\left(-260 + 156\,I\sqrt{3}\right)^{1/3}\right.$$
$$\left. - \frac{26}{3\left(-260 + 156\,I\sqrt{3}\right)^{1/3}}\right), -\frac{1}{12}\left(-260 + 156\,I\sqrt{3}\right)^{1/3}$$
$$- \frac{13}{3\left(-260 + 156\,I\sqrt{3}\right)^{1/3}} - \frac{4}{3} - \frac{1}{2}\,I\sqrt{3}\left(\frac{1}{6}\left(-260\right.\right.$$
$$\left.\left. + 156\,I\sqrt{3}\right)^{1/3} - \frac{26}{3\left(-260 + 156\,I\sqrt{3}\right)^{1/3}}\right)$$

> $solve(27*x\char`^3 + 216*x\char`^2 + 584*x + 528 = 0, x);$

$$\frac{2}{9}\left(9 + \sqrt{89}\right)^{1/3} - \frac{4}{9\left(9 + \sqrt{89}\right)^{1/3}} - \frac{8}{3}, -\frac{1}{9}\left(9 + \sqrt{89}\right)^{1/3}$$
$$+ \frac{2}{9\left(9 + \sqrt{89}\right)^{1/3}} - \frac{8}{3} + I\sqrt{3}\left(\frac{1}{9}\left(9 + \sqrt{89}\right)^{1/3}\right.$$
$$\left. + \frac{2}{9\left(9 + \sqrt{89}\right)^{1/3}}\right), -\frac{1}{9}\left(9 + \sqrt{89}\right)^{1/3}$$
$$+ \frac{2}{9\left(9 + \sqrt{89}\right)^{1/3}} - \frac{8}{3} - I\sqrt{3}\left(\frac{1}{9}\left(9 + \sqrt{89}\right)^{1/3}\right.$$
$$\left. + \frac{2}{9\left(9 + \sqrt{89}\right)^{1/3}}\right)$$

TI-Nspire CAS and my contribution: equations $x^3 - 3x + 1 = 0$ and $x^3 + 3x + 1 = 0$.

$\mathrm{zeros}(x^3 - 3\cdot x + 1, x)$

$$\left\{-\left(\cos\left(\frac{2\cdot\pi}{9}\right) + \sin\left(\frac{2\cdot\pi}{9}\right)\cdot\sqrt{3}\right), \frac{1}{\cos\left(\frac{2\cdot\pi}{9}\right) + \sin\left(\frac{2\cdot\pi}{9}\right)\cdot\sqrt{3} + 1}, \frac{\cos\left(\frac{2\cdot\pi}{9}\right) + \sin\left(\frac{2\cdot\pi}{9}\right)\cdot\sqrt{3} + 1}{\cos\left(\frac{2\cdot\pi}{9}\right) + \sin\left(\frac{2\cdot\pi}{9}\right)\cdot\sqrt{3}}\right\}$$

kit_ets_mb\compact_cubic$(x^3 - 3\cdot x + 1, x)$

$$\left\{2\cdot\sin\left(\frac{\pi}{18}\right), 2\cdot\cos\left(\frac{2\cdot\pi}{9}\right), -2\cdot\cos\left(\frac{\pi}{9}\right)\right\}$$

$\mathrm{exact}(\mathrm{cZeros}(x^3 + 3\cdot x + 1, x))$

$$\left\{\frac{\left(\sqrt{5}+1\right)^{\frac{2}{3}}}{\left(\sqrt{5}+1\right)^{\frac{4}{3}}\cdot 2^{\frac{1}{3}} + 2\cdot\left(\sqrt{5}+1\right)^{\frac{2}{3}} + 2\cdot 2^{\frac{2}{3}}} + \frac{\left(2\cdot\left(\sqrt{5}-1\right)\right)^{\frac{1}{3}}\cdot\left(\left(\sqrt{5}+1\right)^{\frac{2}{3}}\cdot 2^{\frac{1}{3}} + 2\right)\cdot 4^{\frac{2}{3}}\cdot\sqrt{3}}{16}\cdot \boldsymbol{i}, \frac{\left(\sqrt{5}+1\right)^{\frac{2}{3}}}{\left(\sqrt{5}+1\right)^{\frac{4}{3}}\cdot 2^{\frac{1}{3}} + 2\cdot\left(\sqrt{5}+1\right)^{\frac{2}{3}} + 2\cdot} \right.\blacktriangleright$$

kit_ets_mb\compact_cubic$(x^3 + 3\cdot x + 1, x)$

$$\left\{\frac{\left(\sqrt{5}-1\right)^{\frac{1}{3}}\cdot 2^{\frac{2}{3}}}{2} - \frac{\left(2\cdot\left(\sqrt{5}+1\right)\right)^{\frac{1}{3}}\cdot 4^{\frac{2}{3}}}{4}, \frac{\left(2\cdot\left(\sqrt{5}+1\right)\right)^{\frac{1}{3}}\cdot 4^{\frac{2}{3}}}{8} - \frac{\left(\sqrt{5}-1\right)^{\frac{1}{3}}\cdot 2^{\frac{2}{3}}}{4} + \left(\frac{\left(2\cdot\left(\sqrt{5}+1\right)\right)^{\frac{1}{3}}\cdot 4^{\frac{2}{3}}\cdot\sqrt{3}}{8} + \frac{\left(\sqrt{5}-1\right)^{\frac{1}{3}}\cdot\sqrt{3}\cdot 2}{4}\right.\right.\blacktriangleright$$

TI-Nspire CAS and my contribution : equation $x^3+4x^2+x-1=0$.

exact(zeros($x^3+4\cdot x^2+x-1,x$))

$$\left\{\frac{-\left(\cos\left(\frac{\tan^{-1}\left(\frac{5\cdot\sqrt{3}}{9}\right)}{3}+\frac{\pi}{6}\right)\cdot\sqrt{13}+\sin\left(\frac{\tan^{-1}\left(\frac{5\cdot\sqrt{3}}{9}\right)}{3}+\frac{\pi}{6}\right)\cdot\sqrt{39}+4\right)}{3},\frac{-\left(2\cdot\cos\left(\frac{\tan^{-1}\left(\frac{3\cdot\sqrt{3}}{5}\right)}{3}\right)\cdot\sqrt{13}+1\right)}{\cos\left(\frac{\tan^{-1}\left(\frac{5\cdot\sqrt{3}}{9}\right)}{3}+\frac{\pi}{6}\right)\cdot\sqrt{13}+\sin\left(\frac{\tan^{-1}\left(\frac{5\cdot\sqrt{3}}{9}\right)}{3}+\frac{\pi}{6}\right)\cdot\sqrt{39}} \blacktriangleright\right.$$

kit_ets_mb\compact_cubic($x^3+4\cdot x^2+x-1,x$)

$$\left\{\frac{2\cdot\sqrt{13}\cdot\sin\left(\frac{\sin^{-1}\left(\frac{5\cdot\sqrt{13}}{26}\right)}{3}\right)}{3}-\frac{4}{3},\frac{2\cdot\sqrt{13}\cdot\cos\left(\frac{\sin^{-1}\left(\frac{5\cdot\sqrt{13}}{26}\right)}{3}+\frac{\pi}{6}\right)}{3}-\frac{4}{3},\frac{-2\cdot\sqrt{13}\cdot\sin\left(\frac{\sin^{-1}\left(\frac{5\cdot\sqrt{13}}{26}\right)}{3}+\frac{\pi}{3}\right)}{3}-\frac{4}{3}\right\}$$

TI-Nspire CAS and my contribution : equation $27x^3+216x^2+584x+528=0$.

exact(cZeros($27\cdot x^3+216\cdot x^2+584\cdot x+528,x$)) "Error: Resource exhaustion"

kit_ets_mb\compact_cubic($27\cdot x^3+216\cdot x^2+584\cdot x+528,x$)

$$\left\{\frac{2\cdot\left(\sqrt{89}+9\right)^{\frac{1}{3}}}{9}-\frac{2\cdot\left(\sqrt{89}-9\right)^{\frac{1}{3}}}{9}-\frac{8}{3},\frac{-\left(\sqrt{89}+9\right)^{\frac{1}{3}}}{9}+\frac{\left(\sqrt{89}-9\right)^{\frac{1}{3}}}{9}-\frac{8}{3}+\left(\frac{\left(\sqrt{89}+9\right)^{\frac{1}{3}}\cdot\sqrt{3}}{9}+\frac{\left(\sqrt{89}-9\right)^{\frac{1}{3}}\cdot\sqrt{3}}{9}\right)\cdot\boldsymbol{i},\frac{-\left(\sqrt{89}+9\right)^{\frac{1}{3}}}{9}+ \blacktriangleright\right.$$

WolframAlpha

solve(x^3-3x+1=0,x)

Input interpretation:

solve $x^3-3x+1=0$ for x

Results: Approximate forms Step-by-step solution

$$x=-\frac{1}{2}\left(1-i\sqrt{3}\right)\sqrt[3]{\frac{1}{2}\left(-1+i\sqrt{3}\right)}-\frac{1+i\sqrt{3}}{2^{2/3}\sqrt[3]{-1+i\sqrt{3}}}$$

$$x=-\frac{1-i\sqrt{3}}{2^{2/3}\sqrt[3]{-1+i\sqrt{3}}}-\frac{1}{2}\sqrt[3]{\frac{1}{2}\left(-1+i\sqrt{3}\right)}\left(1+i\sqrt{3}\right)$$

$$x=\frac{1}{\sqrt[3]{\frac{1}{2}\left(-1+i\sqrt{3}\right)}}+\sqrt[3]{\frac{1}{2}\left(-1+i\sqrt{3}\right)}$$

Input interpretation:

solve	$x^3+3x+1=0$	for	x

Results:

$$x=\sqrt[3]{\frac{1}{2}\left(\sqrt{5}-1\right)}-\sqrt[3]{\frac{2}{\sqrt{5}-1}}$$

$$x=\frac{1-i\sqrt{3}}{2^{2/3}\sqrt[3]{\sqrt{5}-1}}-\frac{1}{2}\left(1+i\sqrt{3}\right)\sqrt[3]{\frac{1}{2}\left(\sqrt{5}-1\right)}$$

$$x=\frac{1+i\sqrt{3}}{2^{2/3}\sqrt[3]{\sqrt{5}-1}}-\frac{1}{2}\left(1-i\sqrt{3}\right)\sqrt[3]{\frac{1}{2}\left(\sqrt{5}-1\right)}$$

Input interpretation:

solve	$x^3+4x^2+x-1=0$	for	x

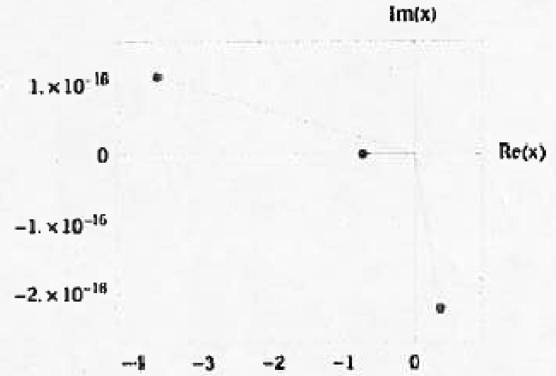
Results:

$x\approx-3.65109$

$x\approx-0.726109$

$$x=-\frac{4}{3}+\frac{1}{3}\left(\frac{13^{2/3}}{\sqrt[3]{\frac{1}{2}\left(-5+3i\sqrt{3}\right)}}+\sqrt[3]{\frac{13}{2}\left(-5+3i\sqrt{3}\right)}\right)$$

Roots in the complex plane:



Input interpretation:

solve	$27x^3+216x^2+584x+528=0$	for	x

Results:

$$x=\frac{2}{9}\left(\sqrt[3]{9+\sqrt{89}}-\frac{2}{\sqrt[3]{9+\sqrt{89}}}\right)-\frac{8}{3}$$

$$x=-\frac{8}{3}+\frac{2\left(1-i\sqrt{3}\right)}{9\sqrt[3]{9+\sqrt{89}}}-\frac{1}{9}\left(1+i\sqrt{3}\right)\sqrt[3]{9+\sqrt{89}}$$

$$x=-\frac{8}{3}+\frac{2\left(1+i\sqrt{3}\right)}{9\sqrt[3]{9+\sqrt{89}}}-\frac{1}{9}\left(1-i\sqrt{3}\right)\sqrt[3]{9+\sqrt{89}}$$

Reactions on Michel’s paper:

Albert Rich:

Hello Michel,

Yes, what is simpler may be in the eyes of the beholder, but Maple and Mathematica returning real solutions involving the imaginary unit certainly isn’t...

Aloha from Hawaii,
Albert

David Jeffrey:

Hello Michel,

The trig formulae are usually called Viete's formulae. If a cubic has 3 real roots, then Viete's formulae give them without needing sqrt(-1).
Congratulations on your simplification of Nspire formulae. I assume that your simplifications could be used on any trig expression, whether it came from a cubic or elsewhere.

Radicals appear in the denominator of Cardano formulae, because in the general case this is necessary to prevent branch cuts jumping to an incorrect solution. In all cases it is clear that Maple simply substitutes in a general Cardano formula, rather than selecting the formula according to the case.

David

David Stoutemyer:

Vietes trig formula CAN involve arcsin or arccos of a quantity having abs greater than 1. See Wikipedia for a hyperbolic alternative for this case.

José Luis Galan:

Michel, congratulations for the simplification in Nspire. And, as in many situations, our old good friend Derive keep being the most appropriate !!!
Best,
José Luis

Josef Böhm:

many thanks to Michel for his impressive cubics' collection which I can accomplish with the WxMaxima and Mathcad 15 solutions.

Thanks also for the additional comments of the CAS-experts.

I'd like to put this in the next DNL's User Forum.

I am trying to program the exact solution of quartics for Nspire but I must admit that I failed until now because of the many nested roots which cannot be handled by the Nspire - at least in my opinion - but I will proceed with my experiments.

Best regards to you all,

Josef

WxMaxima 14.12.1

```
(%i1) solve([x^3-3*x+1=0], [x]);
```

(%o1) $$\left[x=\left(\frac{\sqrt{3}\,\%i}{2}-\frac{1}{2}\right)^{2/3}+\left(-\frac{\sqrt{3}\,\%i}{2}-\frac{1}{2}\right)\left(\frac{\sqrt{3}\,\%i}{2}-\frac{1}{2}\right)^{1/3},\ x=\left(\frac{\sqrt{3}\,\%i}{2}-\frac{1}{2}\right)^{4/3}+\frac{-\frac{\sqrt{3}\,\%i}{2}-\frac{1}{2}}{\left(\frac{\sqrt{3}\,\%i}{2}-\frac{1}{2}\right)^{1/3}},\ x=\left(\frac{\sqrt{3}\,\%i}{2}-\frac{1}{2}\right)^{1/3}+\frac{1}{\left(\frac{\sqrt{3}\,\%i}{2}-\frac{1}{2}\right)^{1/3}}\right]$$

```
(%i2) solve([x^3+3*x+1=0], [x]);
```

(%o2) $$\left[x=\left(\frac{\sqrt{5}}{2}-\frac{1}{2}\right)^{1/3}\left(-\frac{\sqrt{3}\,\%i}{2}-\frac{1}{2}\right)-\frac{\frac{\sqrt{3}\,\%i}{2}-\frac{1}{2}}{\left(\frac{\sqrt{5}}{2}-\frac{1}{2}\right)^{1/3}},\ x=\left(\frac{\sqrt{5}}{2}-\frac{1}{2}\right)^{1/3}\left(\frac{\sqrt{3}\,\%i}{2}-\frac{1}{2}\right)-\frac{-\frac{\sqrt{3}\,\%i}{2}-\frac{1}{2}}{\left(\frac{\sqrt{5}}{2}-\frac{1}{2}\right)^{1/3}},\ x=\left(\frac{\sqrt{5}}{2}-\frac{1}{2}\right)^{1/3}-\frac{1}{\left(\frac{\sqrt{5}}{2}-\frac{1}{2}\right)^{1/3}}\right]$$

```
(%i3) solve([x^3+4*x^2+x-1=0], [x]);
```

(%o3) $\left[x=\frac{13\left(\frac{\sqrt{3}\,\%i}{2}-\frac{1}{2}\right)}{9\left(\frac{13\,\%i}{2\,3^{3/2}}-\frac{65}{54}\right)^{1/3}}+\left(\frac{13\,\%i}{2\,3^{3/2}}-\frac{65}{54}\right)^{1/3}\left(-\frac{\sqrt{3}\,\%i}{2}-\frac{1}{2}\right)-\frac{4}{3},\ x=\left(\frac{13\,\%i}{2\,3^{3/2}}-\frac{65}{54}\right)^{1/3}\left(\frac{\sqrt{3}\,\%i}{2}-\frac{1}{2}\right)+\frac{13\left(-\frac{\sqrt{3}\,\%i}{2}-\frac{1}{2}\right)}{9\left(\frac{13\,\%i}{2\,3^{3/2}}-\frac{65}{54}\right)^{1/3}}-\frac{4}{3},\ x=\left(\frac{13\,\%i}{2\,3^{3/2}}-\frac{65}{54}\right)^{1/3}+\frac{13}{9\left(\frac{13\,\%i}{2\,3^{3/2}}-\frac{65}{54}\right)^{1/3}}-\frac{4}{3}\right]$

```
(%i4) solve([27*x^3+216*x^2+584*x+528=0], [x]);
```

(%o4) $\left[x=-\frac{8\left(\frac{\sqrt{3}\,\%i}{2}-\frac{1}{2}\right)}{81\left(\frac{8\sqrt{89}}{729}+\frac{8}{81}\right)^{1/3}}+\left(\frac{8\sqrt{89}}{729}+\frac{8}{81}\right)^{1/3}\left(-\frac{\sqrt{3}\,\%i}{2}-\frac{1}{2}\right)-\frac{8}{3},\ x=\left(\frac{8\sqrt{89}}{729}+\frac{8}{81}\right)^{1/3}\left(\frac{\sqrt{3}\,\%i}{2}-\frac{1}{2}\right)-\frac{8\left(-\frac{\sqrt{3}\,\%i}{2}-\frac{1}{2}\right)}{81\left(\frac{8\sqrt{89}}{729}+\frac{8}{81}\right)^{1/3}}-\frac{8}{3},\ x=\left(\frac{8\sqrt{89}}{729}+\frac{8}{81}\right)^{1/3}-\frac{8}{81\left(\frac{8\sqrt{89}}{729}+\frac{8}{81}\right)^{1/3}}-\frac{8}{3}\right]$

Mathcad 15

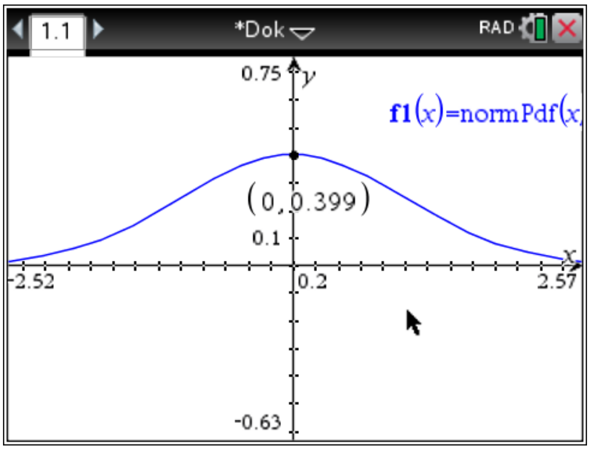
$$x^3-3x+1 \text{ auflösen } \rightarrow \begin{bmatrix} \left(-\frac{1}{2}+\frac{\sqrt{3}\cdot i}{2}\right)^{\frac{-1}{3}}+\left(-\frac{1}{2}+\frac{\sqrt{3}\cdot i}{2}\right)^{\frac{1}{3}} \\ -\frac{\left(-\frac{1}{2}+\frac{\sqrt{3}\cdot i}{2}\right)^{\frac{2}{3}}+1-\sqrt{3}\cdot\left(-\frac{1}{2}+\frac{\sqrt{3}\cdot i}{2}\right)^{\frac{2}{3}}\cdot i+\sqrt{3}\cdot i}{2\cdot\left(-\frac{1}{2}+\frac{\sqrt{3}\cdot i}{2}\right)^{\frac{1}{3}}} \\ -\frac{\left(-\frac{1}{2}+\frac{\sqrt{3}\cdot i}{2}\right)^{\frac{2}{3}}+1-\sqrt{3}\cdot i+\sqrt{3}\cdot\left(-\frac{1}{2}+\frac{\sqrt{3}\cdot i}{2}\right)^{\frac{2}{3}}\cdot i}{2\cdot\left(-\frac{1}{2}+\frac{\sqrt{3}\cdot i}{2}\right)^{\frac{1}{3}}} \end{bmatrix}$$

$$x^3 + 3x + 1 \text{ auflösen } \rightarrow \left[\begin{array}{c} \dfrac{\left(\frac{3}{2} - \frac{\sqrt{5}}{2}\right)^{\frac{1}{3}} - 1}{\left(\frac{\sqrt{5}}{2} - \frac{1}{2}\right)^{\frac{1}{3}}} \\ \dfrac{1 - \left(\frac{3}{2} - \frac{1}{2}\cdot\sqrt{5}\right)^{\frac{1}{3}} + \sqrt{3}\cdot i + \sqrt{3}\cdot\left(\frac{3}{2} - \frac{\sqrt{5}}{2}\right)^{\frac{1}{3}}\cdot i}{2\cdot\left(\frac{\sqrt{5}}{2} - \frac{1}{2}\right)^{\frac{1}{3}}} \\ -\dfrac{\left(\frac{3}{2} - \frac{\sqrt{5}}{2}\right)^{\frac{1}{3}} - 1 + \sqrt{3}\cdot i + \sqrt{3}\cdot\left(\frac{3}{2} - \frac{\sqrt{5}}{2}\right)^{\frac{1}{3}}\cdot i}{2\cdot\left(\frac{\sqrt{5}}{2} - \frac{1}{2}\right)^{\frac{1}{3}}} \end{array}\right]$$

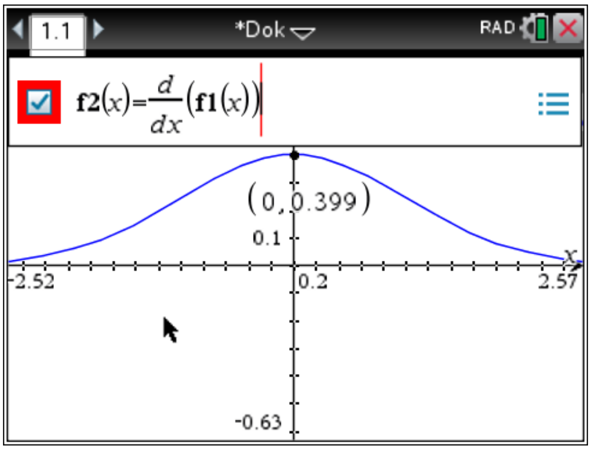
$$x^3 + 4x^2 + x - 1 \text{ auflösen } \rightarrow \left[\begin{array}{c} \dfrac{13}{9\cdot\left(-\frac{65}{54} + \frac{13i\cdot\sqrt{3}}{18}\right)^{\frac{1}{3}}} + \left(-\frac{65}{54} + \frac{13i\cdot\sqrt{3}}{18}\right)^{\frac{1}{3}} - \frac{4}{3} \\ -\dfrac{13}{18\cdot\left(-\frac{65}{54} + \frac{13i\cdot\sqrt{3}}{18}\right)^{\frac{1}{3}}} - \dfrac{\left(-\frac{65}{54} + \frac{13i\cdot\sqrt{3}}{18}\right)^{\frac{1}{3}}}{2} - \dfrac{4}{3} - \dfrac{\sqrt{3}\cdot\left(-\frac{65}{54} + \frac{13i\cdot\sqrt{3}}{18}\right)^{\frac{1}{3}}\cdot i}{2} + \dfrac{13i\cdot\sqrt{3}}{18\cdot\left(-\frac{65}{54} + \frac{13i\cdot\sqrt{3}}{18}\right)^{\frac{1}{3}}} \\ -\dfrac{13}{18\cdot\left(-\frac{65}{54} + \frac{13i\cdot\sqrt{3}}{18}\right)^{\frac{1}{3}}} - \dfrac{\left(-\frac{65}{54} + \frac{13i\cdot\sqrt{3}}{18}\right)^{\frac{1}{3}}}{2} - \dfrac{4}{3} + \dfrac{\sqrt{3}\cdot\left(-\frac{65}{54} + \frac{13i\cdot\sqrt{3}}{18}\right)^{\frac{1}{3}}\cdot i}{2} - \dfrac{13i\cdot\sqrt{3}}{18\cdot\left(-\frac{65}{54} + \frac{13i\cdot\sqrt{3}}{18}\right)^{\frac{1}{3}}} \end{array}\right]$$

$$27x^3 + 216x^2 + 584x + 528 \text{ auflösen} \rightarrow \left[\begin{array}{c} \dfrac{\left(\dfrac{128\cdot\sqrt{89}}{59049}+\dfrac{10880}{531441}\right)^{\frac{1}{3}}}{\left(\dfrac{8\cdot\sqrt{89}}{729}+\dfrac{8}{81}\right)^{\frac{1}{3}}} - \dfrac{8}{81\cdot\left(\dfrac{8\cdot\sqrt{89}}{729}+\dfrac{8}{81}\right)^{\frac{1}{3}}} - \dfrac{8}{3} \\ \dfrac{8 - 81\cdot\left(\dfrac{128\cdot\sqrt{89}}{59049}+\dfrac{10880}{531441}\right)^{\frac{1}{3}} - 432\cdot\left(\dfrac{8\cdot\sqrt{89}}{729}+\dfrac{8}{81}\right)^{\frac{1}{3}} + 8i\cdot\sqrt{3} + 81i\cdot\sqrt{3}\cdot\left(\dfrac{128\cdot\sqrt{89}}{59049}+\dfrac{10880}{531441}\right)^{\frac{1}{3}}}{162\cdot\left(\dfrac{8\cdot\sqrt{89}}{729}+\dfrac{8}{81}\right)^{\frac{1}{3}}} \\ -\dfrac{432\cdot\left(\dfrac{8\cdot\sqrt{89}}{729}+\dfrac{8}{81}\right)^{\frac{1}{3}} + 81\cdot\left(\dfrac{128\cdot\sqrt{89}}{59049}+\dfrac{10880}{531441}\right)^{\frac{1}{3}} - 8 + 8i\cdot\sqrt{3} + 81i\cdot\sqrt{3}\cdot\left(\dfrac{128\cdot\sqrt{89}}{59049}+\dfrac{10880}{531441}\right)^{\frac{1}{3}}}{162\cdot\left(\dfrac{8\cdot\sqrt{89}}{729}+\dfrac{8}{81}\right)^{\frac{1}{3}}} \end{array}\right]$$

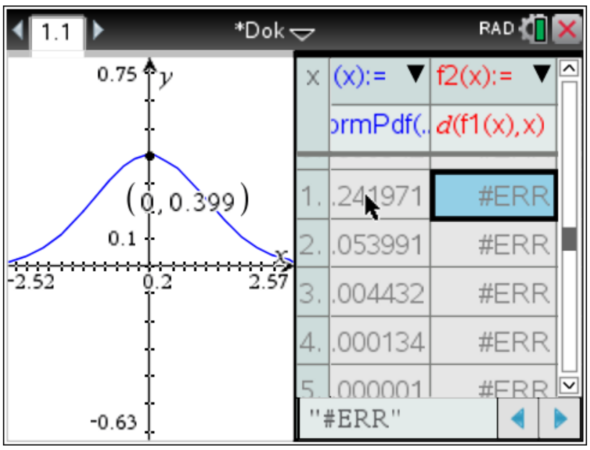
Wilfried Zappe, Germany: Problems with normpdf on TI-Nspire



Analysing the graph we can detect the maximum but not the inflection point.

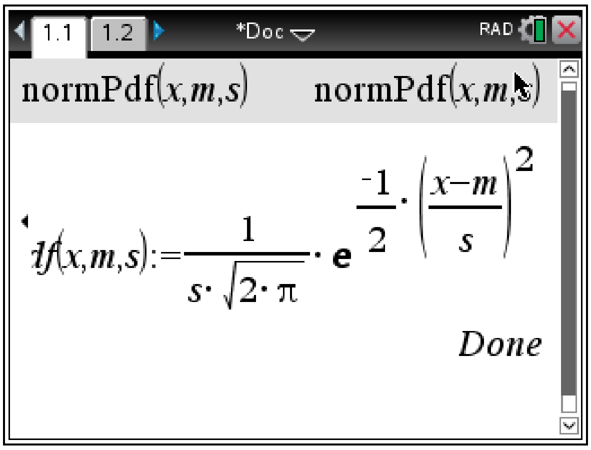


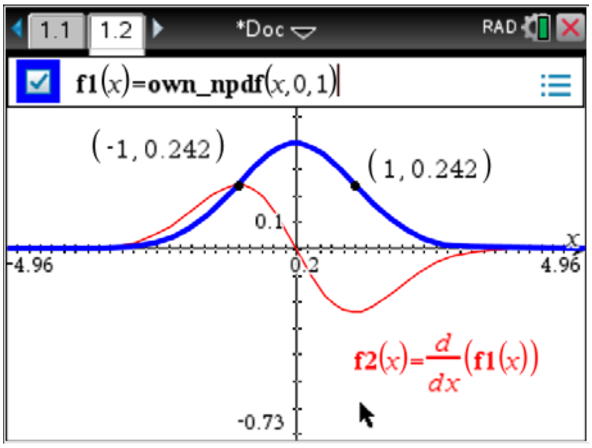
It is not possible to plot the first derivative of the distribution function.



It seems to be that normpdf internally is not defined as a function.

One has to define his/her own distribution function. Then it works as you can see on the next screen shots.

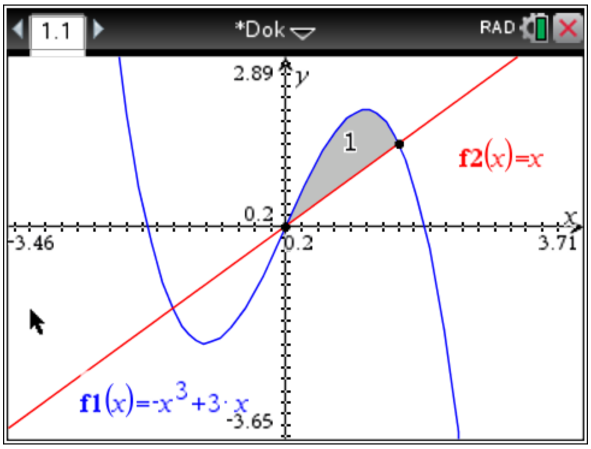


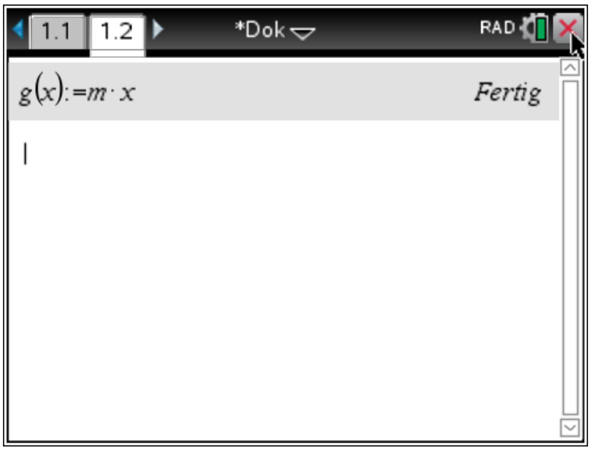


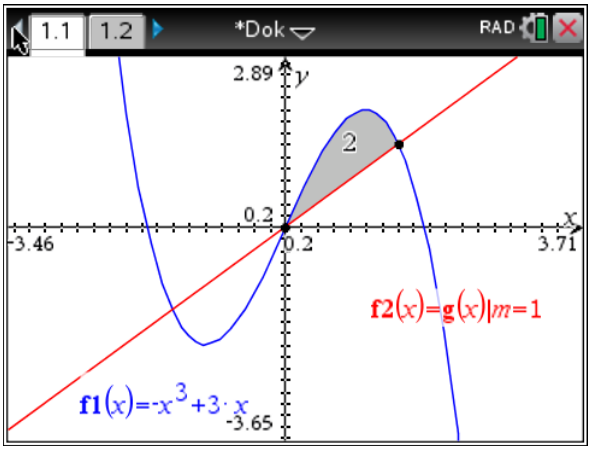
Hubert Langlotz, Germany: Just another bug?

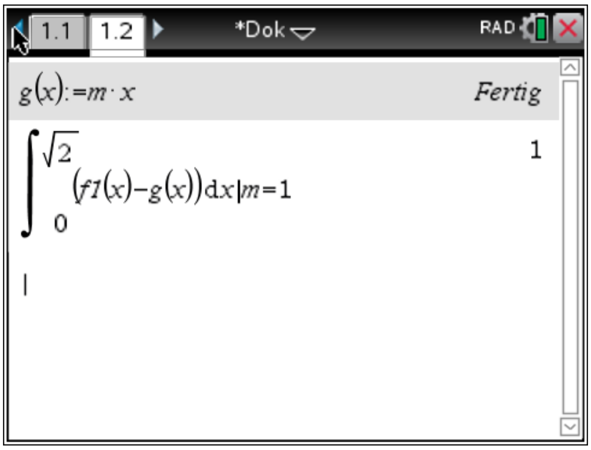
While correcting the papers of our final examination I found another bug in OS 4.0.

Its real bad, because the use of the tool “bounded area” was a possible way to solve this problem in the exam!









DNL:

Same happens in version 3.9. Interestingly enough it works properly when introducing a slider for m. Then everything is ok.

The Mathematician and the Clockmaker

Josef Böhm, Würmla, Austria

In the German issue of *Scientific American, Spektrum der Wissenschaft*, from May 2015 I found an interesting article about a sequence: "The Misunderstood Sister of the Fibonacci Sequence" written by Jean-Paul Delahaye.

This sequence is defined by $s_0 = 0,\ s_1 = 1,\ s_{2n} = s_n,\ s_{2n+1} = s_n + s_{n+1}$.

It is called Stern[1]-Brocot[2] Sequence. This sequence offers many very remarkable properties which are worth to be investigated.

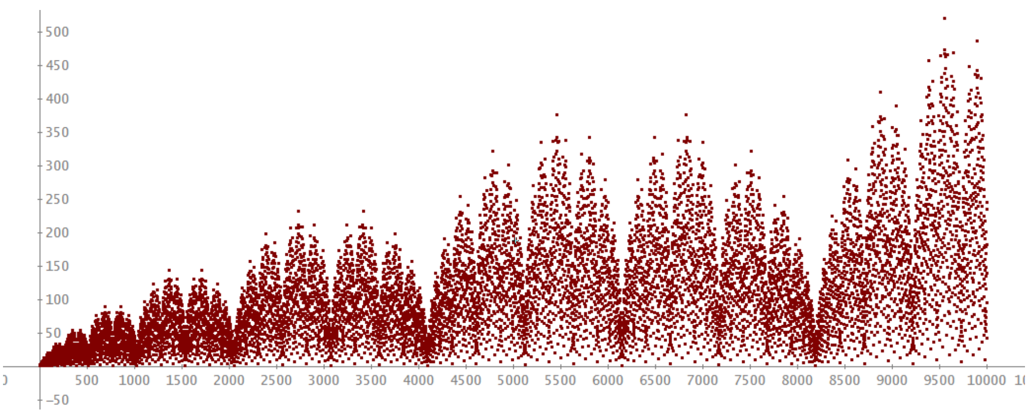
The following short program generates *n* elements of the sequence:

```
st_bro(n, seq, el, k) :=
  Prog
    seq := [1]
    k := 2
    Loop
      If k > n exit
      el := IF(MOD(k, 2) = 0, seq↓(k/2), seq↓((k - 1)/2) + seq↓((k - 1)/2 + 1))
      seq := APPEND(seq, [el])
      k :+ 1
    seq

st_bro(20) = [1, 1, 2, 1, 3, 2, 3, 1, 4, 3, 5, 2, 5, 3, 4, 1, 5, 4, 7, 3]
```

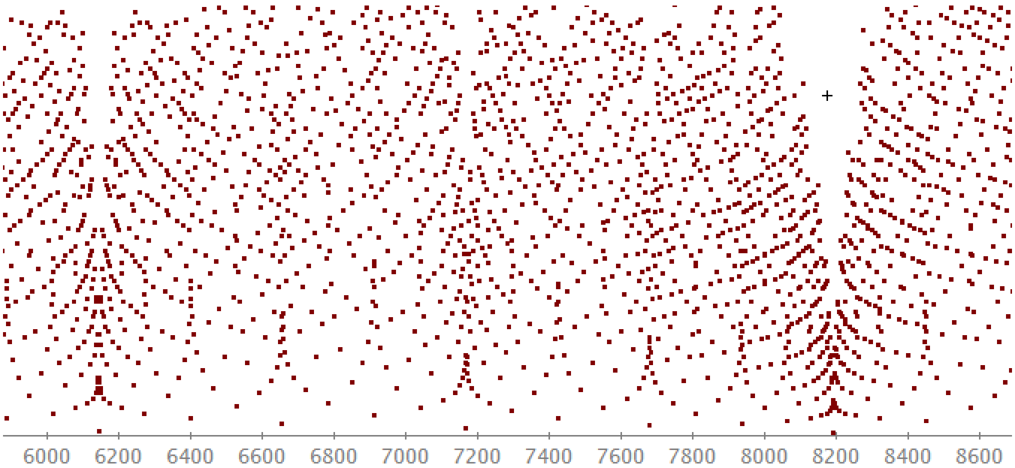
There is no special structure visible. Things will change when we produce a graph presenting the elements versus their position numbers in the sequence:

```
st_bro_pts(n, k, seq, el, pts) :=
  Prog
    seq := [1]
    pts := [[1, 1]]
    k := 2
    Loop
      If k > n exit
      el := IF(MOD(k, 2) = 0, seq↓(k/2), seq↓((k - 1)/2) + seq↓((k - 1)/2 + 1))
      seq := APPEND(seq, [el])
      pts := APPEND(pts, [[k, el]])
      k :+ 1
    pts

st_bro_pts(10000)
```

DERIVE makes it possible to plot the first 10 000 points:



Zooming in we can find a fractal similar structure:



I will follow the article in the journal and form groups of the elements of size 2^k – omitting the 0.
I store one sequence (5000 elements) as `st_seq`.

```
st_seq := st_bro(5000)

VECTOR(st_seq[2^k, ..., 2^(k + 1) - 1], k, 0, 5)
```

```
[1]
[1, 2]
[1, 3, 2, 3]
[1, 4, 3, 5, 2, 5, 3, 4]
[1, 5, 4, 7, 3, 8, 5, 7, 2, 7, 5, 8, 3, 7, 4, 5]
[1, 6, 5, 9, 4, 11, 7, 10, 3, 11, 8, 13, 5, 12, 7, 9, 2, 9, 7, 12, 5, 13, 8, 11, 3, 10, 7, 11, 4, 9, 5, 6]
```

I don't need the centred order of the groups of elements. So I produce the first 12 groups and select them one after the other with maximum 32 elements:

```
grps := VECTOR(st_seq[2^k, ..., 2^(k + 1) - 1], k, 0, 12)

[1]
[1, 2]
[1, 3, 2, 3]
[1, 4, 3, 5, 2, 5, 3, 4]
[1, 5, 4, 7, 3, 8, 5, 7, 2, 7, 5, 8, 3, 7, 4, 5]
[1, 6, 5, 9, 4, 11, 7, 10, 3, 11, 8, 13, 5, 12, 7, 9, 2, 9, 7, 12, 5, 13, 8, 11, 3, 10, 7, 11, 4, 9, 5, 6]
[1, 7, 6, 11, 5, 14, 9, 13, 4, 15, 11, 18, 7, 17, 10, 13, 3, 14, 11, 19, 8, 21, 13, 18, 5, 17, 12, 19, 7, 16, 9
[1, 8, 7, 13, 6, 17, 11, 16, 5, 19, 14, 23, 9, 22, 13, 17, 4, 19, 15, 26, 11, 29, 18, 25, 7, 24, 17, 27, 10, 23
[1, 9, 8, 15, 7, 20, 13, 19, 6, 23, 17, 28, 11, 27, 16, 21, 5, 24, 19, 33, 14, 37, 23, 32, 9, 31, 22, 35, 13, 3
[1, 10, 9, 17, 8, 23, 15, 22, 7, 27, 20, 33, 13, 32, 19, 25, 6, 29, 23, 40, 17, 45, 28, 39, 11, 38, 27, 43, 16,
[1, 11, 10, 19, 9, 26, 17, 25, 8, 31, 23, 38, 15, 37, 22, 29, 7, 34, 27, 47, 20, 53, 33, 46, 13, 45, 32, 51, 19
[1, 12, 11, 21, 10, 29, 19, 28, 9, 35, 26, 43, 17, 42, 25, 33, 8, 39, 31, 54, 23, 61, 38, 53, 15, 52, 37, 59, 2
```

The last list is created by:

$$\text{grps}_{12,1,[1,\ \ldots,\ 32]}$$

By the way, appending a 1 to all groups of numbers we create palindromes.

We need one more step to find the property which I want to show. Look at the numbers of the first column, the second column, the third column, and so on:

```
[1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]

[2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12]

[2, 3, 4, 5, 6, 7, 8, 9, 10, 11]

[3, 5, 7, 9, 11, 13, 15, 17, 19, 21]

[2, 3, 4, 5, 6, 7, 8, 9, 10]

[5, 8, 11, 14, 17, 20, 23, 26, 29]

[3, 5, 7, 9, 11, 13, 15, 17, 19]

[4, 7, 10, 13, 16, 19, 22, 25, 28]

[2, 3, 4, 5, 6, 7, 8, 9]

[7, 11, 15, 19, 23, 27, 31, 35]

[5, 8, 11, 14, 17, 20, 23, 26]

[8, 13, 18, 23, 28, 33, 38, 43]
```

This seems not to be too spectacular: all columns form arithmetic sequences. Now notice the differences of these sequences:

[0, 1, 1, 2, 1, 3, 2, 3, 1, 4, 3, 5, …] – Ooops:

```
st_bro(12) = [1, 1, 2, 1, 3, 2, 3, 1, 4, 3, 5, 2]
```

this is again the Stern-Brocot Sequence.

What's about the sums of the groups? (You may remember that the sums in the rows of the Triangle of Pascal are the powers of 2.)

$\text{VECTOR}(\Sigma(\text{grps}_{n,1}), n, 12)$

```
[1, 3, 9, 27, 81, 243, 729, 2187, 6561, 19683, 59049, 177147]

FACTOR([1, 3, 9, 27, 81, 243, 729, 2187, 6561, 19683, 59049, 177147], Number)
```

$$\left[1, 3, 3^2, 3^3, 3^4, 3^5, 3^6, 3^7, 3^8, 3^9, 3^{10}, 3^{11}\right]$$

You may try to proof this result! (According to the author of the article it should not be too difficult.)

$\text{VECTOR}(\text{MAX}(\text{grps}_{n,1}), n, 12)$

```
[1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233]
```

I am sure that you can recognize these numbers! Yes, you are right, it is the Fibonacci Sequence.

The above mentioned Triangle of Pascal is a very well known structure of numbers. We can discover two nice properties which are possibly not so well known.

```
pas(p) := VECTOR([VECTOR(COMB(n, k), k, 0, n)], n, 0, p)
```

It's no problem to present the Triangle:

We observe the numbers appearing in the increasing diagonals – the numbers of the same colours.

```
          ⎡         [1]          ⎤
          ⎢       [1, 1]         ⎥
          ⎢     [1, 2, 1]        ⎥
pas(7) =  ⎢    [1, 3, 3, 1]      ⎥
          ⎢   [1, 4, 6, 4, 1]    ⎥
          ⎢ [1, 5, 10, 10, 5, 1] ⎥
          ⎢[1, 6, 15, 20, 15, 6, 1]⎥
          ⎣[1, 7, 21, 35, 35, 21, 7, 1]⎦
```

```
⎡             [1]              ⎤
⎢            [1, 1]            ⎥
⎢          [1, 2, 1]           ⎥
⎢         [1, 3, 3, 1]         ⎥
⎢       [1, 4, 6, 4, 1]        ⎥
⎢     [1, 5, 10, 10, 5, 1]     ⎥
⎢   [1, 6, 15, 20, 15, 6, 1]   ⎥
⎣ [1, 7, 21, 35, 35, 21, 7, 1] ⎦
```

Let *DERIVE* do the job, separating theses numbers.

$$\text{diags}(p,\ m) := \text{VECTOR}\left(\left[\text{VECTOR}\left((\text{pas}(p))_{n-k,1,1+k},\ k,\ 0,\ \text{IF}\left(\text{MOD}(n,\ 2) = 1,\ \frac{n}{2},\ \frac{n}{2} - 1\right)\right)\right],\ n,\ m\right)$$

```
ds := diags(20, 20)
```

`diags(p,m)` produces the first *m* increasing diagonal sequences of a Pascal triangle with *p* rows.

These are the last 5 diagonals:

```
⎡        [1, 14, 78, 220, 330, 252, 84, 8]         ⎤
⎢      [1, 15, 91, 286, 495, 462, 210, 36, 1]      ⎥
⎢     [1, 16, 105, 364, 715, 792, 462, 120, 9]     ⎥
⎢  [1, 17, 120, 455, 1001, 1287, 924, 330, 45, 1]  ⎥
⎣[1, 18, 136, 560, 1365, 2002, 1716, 792, 165, 10] ⎦
```

Now let's calculate the sum of the elements in the 20 rows of the above matrix (which are the diagonals):

$$\text{VECTOR}(\Sigma(ds_{k,1}),\ k,\ \text{DIM}(ds))$$

```
[1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584, 4181, 6765]
```

We meet again the … Fibonacci Numbers. Let's count the odd numbers in the diagonals. You may start with the first coloured diagonals above. I ask again the CAS for doing the job:

$$\text{VECTOR}(\Sigma(\text{MOD}(ds_{k,1}, 2)), k, \text{DIM}(ds))$$

$$[1, 1, 2, 1, 3, 2, 3, 1, 4, 3, 5, 2, 5, 3, 4, 1, 5, 4, 7, 3]$$

Do you recognize this sequence? Isn't this funny?

Can you imagine that there is a relationship between the binary representation of a number, continued fractions and the elements of the Stern-Brocot Sequence? I'll show you.

Let's take the binary representation of 100[3]:

$$\text{DUAL}(100) = [1, 1, 0, 0, 1, 0, 0]$$

I count the ones and the zeros how they appear in the list: 2, 2, 1, 2, and close with a 0 (for no one is following). Then I use the list [**2, 2, 1, 2, 0**] to form a continued fraction:

$$0 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{2}}}} = \frac{7}{19}$$

You might ask: "Ok, but where is the connection with the Stern-Brocot Sequence?" Wait just one moment. We look for elements s_{100} and s_{101} of the sequence.

$$(\text{st\_bro}(101))_{[100,\ 101]} = [7, 19]$$

s_{100} is the numerator and its successor s_{101} the denominator of the fraction. To go for sure we'll take a second example:

$$\text{DUAL}(503) = [1, 1, 1, 1, 1, 0, 1, 1, 1]$$

$$(\text{st\_bro}(504))_{[503,\ 504]} = [23, 6]$$

I wrote a small function for calculating the continued fraction and apply it immediately:

```
fr(x, f, i) :=
  Prog
    f := 1/x↓1
    i := 2
    Loop
      If i = DIM(x) exit
      f := 1/(f + x↓i)
      i :+ 1
    x↓i + f
```

$$\text{fr}([5, 1, 3]) = \frac{23}{6}$$

I am quite sure that you didn't expect this relationship.

We didn't mention the most remarkable property of this sequence until now. It was the reason for Stern to introduce his sequence in 1858.

We all know the usual way to count the rational numbers using the (1, 2, 3, …) × (1, 2, 3, …) table and counting the fractions in the diagonals. There are rational numbers which appear several times by cancellation (1/3 = 2/6 = 3/9 = …). Omitting these fractions disturbs the system and makes it difficult to find the fraction on say position n.

The Stern-Brocot Sequence helps solving this problem. We calculate a new sequence

$$\frac{s_0}{s_1}, \frac{s_1}{s_2}, \frac{s_2}{s_3}, \ldots, \frac{s_{n-1}}{s_n}, \ldots$$

$$\text{rat\_numbs}(n) := \text{VECTOR}\left(\frac{(\text{st\_bro}(n+1))_k}{(\text{st\_bro}(n+1))_{k+1}}, k, n\right)$$

$$\text{rat\_numbs}(20) = \left[1, \frac{1}{2}, 2, \frac{1}{3}, \frac{3}{2}, \frac{2}{3}, 3, \frac{1}{4}, \frac{4}{3}, \frac{3}{5}, \frac{5}{2}, \frac{2}{5}, \frac{5}{3}, \frac{3}{4}, 4, \frac{1}{5}, \frac{5}{4}, \frac{4}{7}, \frac{7}{3}, \frac{3}{8}\right]$$

This gives all rational numbers. And the n^{th} rational number in this sequence is the fraction $\frac{s_{n-1}}{s_n}$ which can easily be calculated using the binary forms described above. Which is the 1000^{th} rational number in this list?

$$\text{DUAL}(1000) = [1, 1, 1, 1, 1, 0, 1, 0, 0, 0]$$

$$\text{fr}([5, 1, 1, 3, 0]) = \frac{11}{39}$$

It should be 11! Let's check!

$$(\text{st\_bro}(1000))_{1000} = 11$$

Remembering the palindromes formed by the 2^k – element groups of the original sequence it is not so astonishing that the sequence of fractions – all rational numbers in a nice order – will also appear in pretty groups with 2^k elements each:

$$\text{r\_n} := \text{rat\_numbs}(200)$$

$$\text{VECTOR}\left(\left[\text{r\_n}_{[2\text{^}k, \ldots, 2\text{^}(k+1)-1]}\right], k, 0, 4\right)$$

$$\begin{bmatrix} [1] \\ \left[\frac{1}{2}, 2\right] \\ \left[\frac{1}{3}, \frac{3}{2}, \frac{2}{3}, 3\right] \\ \left[\frac{1}{4}, \frac{4}{3}, \frac{3}{5}, \frac{5}{2}, \frac{2}{5}, \frac{5}{3}, \frac{3}{4}, 4\right] \\ \left[\frac{1}{5}, \frac{5}{4}, \frac{4}{7}, \frac{7}{3}, \frac{3}{8}, \frac{8}{5}, \frac{5}{7}, \frac{7}{2}, \frac{2}{7}, \frac{7}{5}, \frac{5}{8}, \frac{8}{3}, \frac{3}{7}, \frac{7}{4}, \frac{4}{5}, 5\right] \end{bmatrix}$$

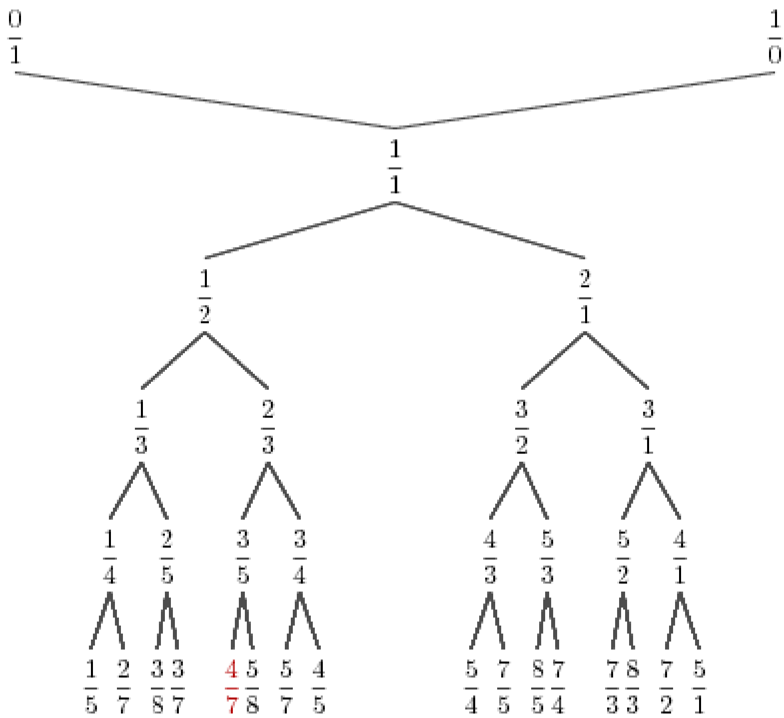
The fractions form a binary tree which is called *Calting-Wilf-Tree*.

We can sort the elements in increasing order:

```
VECTOR([SORT(r_n                          )], k, 0, 4)
                [2^k, ..., 2^(k + 1) - 1]
```

$$\begin{bmatrix} [1] \\ \left[\frac{1}{2}, 2\right] \\ \left[\frac{1}{3}, \frac{2}{3}, \frac{3}{2}, 3\right] \\ \left[\frac{1}{4}, \frac{2}{5}, \frac{3}{5}, \frac{3}{4}, \frac{4}{3}, \frac{5}{3}, \frac{5}{2}, 4\right] \\ \left[\frac{1}{5}, \frac{2}{7}, \frac{3}{8}, \frac{3}{7}, \frac{4}{7}, \frac{5}{8}, \frac{5}{7}, \frac{4}{5}, \frac{5}{4}, \frac{7}{5}, \frac{8}{5}, \frac{7}{4}, \frac{7}{3}, \frac{8}{3}, \frac{7}{2}, 5\right] \end{bmatrix}$$

Compare with the graph of this binary tree (Reference [4])



Performing a binary search say for $\frac{8}{5}$ and starting at $\frac{1}{1}$, we have to move right, left, right and left again on the branches (= RLRL). Each finite string consisting of Rs and Ls defines exactly one rational number. Irrational numbers are defined by infinite sequences of Rs and Ls.

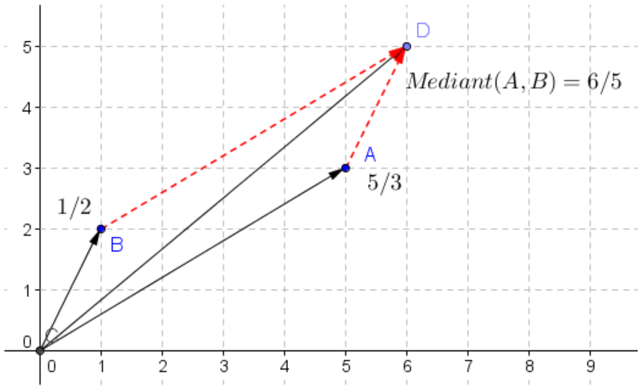
Task for the reader:

Find out, which number is defined by the sequence RLRLRLRLRL … .

And here is another one: $RL^0RLR^2LRL^4RLR^6LRL^8$ … (R^2 = RR and L^4 = LLLL etc.)

This binary tree can be constructed by calculating the *mediant* of two consecutive elements of the sequence. The mediant of two fractions $\frac{p}{q}$ and $\frac{p'}{q'}$ is $\frac{p+p'}{q+q'}$. We start with fractions $\frac{0}{1}$ and $\frac{1}{0}$. We know very well that the latter is no fraction but in this case it is helpful to take it as one.

The mediant has a nice representation as the sum of two vectors:



Let's follow the "mediant – idea" with *DERIVE*:

```
med(a, b) := (NUMERATOR(a) + NUMERATOR(b)) / (DENOMINATOR(a) + DENOMINATOR(b))
```

$$\text{med}\left(\frac{1}{4}, \frac{1}{3}\right) = \frac{2}{7}$$

```
meds(frs, nfrs) :=
  Prog
    frs := REVERSE(REST(REVERSE(frs)))
    nfrs := [FIRST(frs)]
    Loop
      If DIM(frs) = 1 exit
      nfrs := APPEND(nfrs, [med(frs↓1, frs↓2), frs↓2])
      frs := REST(frs)
    nfrs := APPEND(nfrs, [1/IF(nfrs = [0], 1, nfrs↓2), "1/0"])
```

meds([0, 1/0]) = [0, 1, 1/0]

$$\text{meds}([0,\ 1,\ 1/0]) = \left[0,\ \frac{1}{2},\ 1,\ 2,\ 1/0\right]$$

$$\text{meds}\left[0,\ \frac{1}{2},\ 1,\ 2,\ 1/0\right] = \left[0,\ \frac{1}{3},\ \frac{1}{2},\ \frac{2}{3},\ 1,\ \frac{3}{2},\ 2,\ 3,\ 1/0\right]$$

This is a good occasion to apply the ITERATES-command:

ITERATES(meds(ٱ), ٱ, [0, 1, 1/0], 2)

$$\left[[0,\ 1,\ 1/0],\ \left[0,\ \frac{1}{2},\ 1,\ 2,\ 1/0\right],\ \left[0,\ \frac{1}{3},\ \frac{1}{2},\ \frac{2}{3},\ 1,\ \frac{3}{2},\ 2,\ 3,\ 1/0\right]\right]$$

But we would like to see the binary tree growing. So let's build a matrix of the vectors created above including the very first row:

```
meds_rows(n) := APPEND([[[0, 1/0]]], VECTOR([v], v, ITERATES(meds(ٱ), ٱ, [0, 1, 1/0], n)))
meds_rows(3)
```

$$\begin{bmatrix} [0,\ 1/0] \\ [0,\ 1,\ 1/0] \\ \left[0,\ \frac{1}{2},\ 1,\ 2,\ 1/0\right] \\ \left[0,\ \frac{1}{3},\ \frac{1}{2},\ \frac{2}{3},\ 1,\ \frac{3}{2},\ 2,\ 3,\ 1/0\right] \\ \left[0,\ \frac{1}{4},\ \frac{1}{3},\ \frac{2}{5},\ \frac{1}{2},\ \frac{3}{5},\ \frac{2}{3},\ \frac{3}{4},\ 1,\ \frac{4}{3},\ \frac{3}{2},\ \frac{5}{3},\ 2,\ \frac{5}{2},\ 3,\ 4,\ 1/0\right] \end{bmatrix}$$

It looks good, but it is not really the tree we would like to see, because the new rows are containing the elements of the previous rows, too. They need to be eliminated. I remember a *DERIVE* – command `REMOVE_ELEMENTS` – and this should do the job:

$$\text{bin\_tree(n) := APPEND([[[0, 1/0]]], VECTOR}\left(\left[\text{REMOVE\_ELEMENTS((meds\_rows(n))}_{k+1,1}\text{, (meds\_rows(n))}_{k,1}\text{)}\right]\text{, k, n + 1)}\right)$$

bin_tree(4)

$$\begin{bmatrix} [0,\ 1/0] \\ [1] \\ \left[\frac{1}{2},\ 2\right] \\ \left[\frac{1}{3},\ \frac{2}{3},\ \frac{3}{2},\ 3\right] \\ \left[\frac{1}{4},\ \frac{2}{5},\ \frac{3}{5},\ \frac{3}{4},\ \frac{4}{3},\ \frac{5}{3},\ \frac{5}{2},\ 4\right] \\ \left[\frac{1}{5},\ \frac{2}{7},\ \frac{3}{8},\ \frac{3}{7},\ \frac{4}{7},\ \frac{5}{8},\ \frac{5}{7},\ \frac{4}{5},\ \frac{5}{4},\ \frac{7}{5},\ \frac{8}{5},\ \frac{7}{4},\ \frac{7}{3},\ \frac{8}{3},\ \frac{7}{2},\ 5\right] \end{bmatrix}$$

And this is really the binary tree from above which can be used for the binary search.

I will close with another property of this really remarkable sequence:

The *simplicity* of a fraction $\frac{p}{q}$ is defined as $\frac{1}{p \cdot q}$. I omit the first row of the `bin_tree` generated matrix and calculate the *simplicities* of all elements:

$$\text{simpls(n) := VECTOR}\left(\left[\text{VECTOR}\left(\frac{1}{\text{DENOMINATOR(w}_k\text{)·NUMERATOR(w}_k\text{)}}\text{, k, DIM(w)}\right)\right]\text{, w, VECTOR(v}_1\text{, v, REST(bin\_tree(n)))}\right)$$

simpls(4)

$$\begin{bmatrix} [1] \\ \left[\frac{1}{2},\ \frac{1}{2}\right] \\ \left[\frac{1}{3},\ \frac{1}{6},\ \frac{1}{6},\ \frac{1}{3}\right] \\ \left[\frac{1}{4},\ \frac{1}{10},\ \frac{1}{15},\ \frac{1}{12},\ \frac{1}{12},\ \frac{1}{15},\ \frac{1}{10},\ \frac{1}{4}\right] \\ \left[\frac{1}{5},\ \frac{1}{14},\ \frac{1}{24},\ \frac{1}{21},\ \frac{1}{28},\ \frac{1}{40},\ \frac{1}{35},\ \frac{1}{20},\ \frac{1}{20},\ \frac{1}{35},\ \frac{1}{40},\ \frac{1}{28},\ \frac{1}{21},\ \frac{1}{24},\ \frac{1}{14},\ \frac{1}{5}\right] \end{bmatrix}$$

Do you recognize a special feature of the matrix? Add the elements in the rows!

$$\text{simpls\_sums(n)} := \text{VECTOR}\left(\sum\left(\text{VECTOR}\left(\frac{1}{\text{DENOMINATOR}(w_k)\cdot\text{NUMERATOR}(w_k)}, k, \text{DIM}(w)\right)\right), w, \text{VECTOR}(v_1, v, \text{REST(bin\_tree(n))})\right)$$

simpls_sums(10) = [1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]

Finally Jean-Paul Delahay presents another very simple formula to generate an element r_{n+1} of the sequence of all rational numbers from r_n:

$$r_{n+1} = \frac{1}{1 + 2\cdot\lfloor r_n \rfloor - r_n},$$

which gives us another opportunity to generate this sequence applying *DERIVE*'s ITERATES:

$$\text{st}(x) := \frac{1}{1 + 2\cdot\text{FLOOR}(x) - x}$$

$$\text{st}\left(\frac{8}{5}\right) = \frac{5}{7}$$

$$\text{st\_its}(n) := \text{ITERATES}\left(\frac{1}{1 + 2\cdot\text{FLOOR}(x) - x}, x, 1, n\right)$$

st_its(20)

$$\left[1, \frac{1}{2}, 2, \frac{1}{3}, \frac{3}{2}, \frac{2}{3}, 3, \frac{1}{4}, \frac{4}{3}, \frac{3}{5}, \frac{5}{2}, \frac{2}{5}, \frac{5}{3}, \frac{3}{4}, 4, \frac{1}{5}, \frac{5}{4}, \frac{4}{7}, \frac{7}{3}, \frac{3}{8}, \frac{8}{5}\right]$$

[1] Moritz Abraham Stern(1807 – 1894) was a German mathematician. He introduced this sequence 1858 by inventing this way to count the rational numbers.

[2] Achille Brocot (1817 – 1878) discovered this sequence – completely independent of Stern – by investigating transmission ratios of gearwheels. He was a clockmaker (see reference [4]).

[3] This is a tool from an earlier DERIVE Newsletter

I will close now presenting the Stern-Brocot Sequence and some of its properties. It was interesting – and sometimes a little bit challenging programming tools for this paper. Please note that there are a lot of resources in the web dealing with this "Misunderstood Sister of the Fibonacci Sequence".

References

[1] Jean-Paul Delahaye, *Spektrum der Wissenschaft*, May 2015

[2] http://www.cut-the-knot.org/blue/Stern.shtml

[3] http://rosettacode.org/wiki/Stern-Brocot_sequence

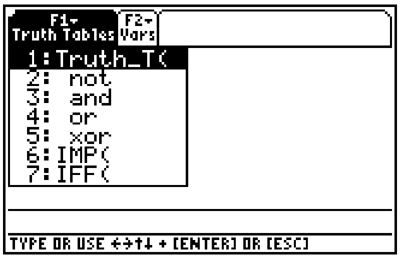
[4] http://www.ams.org/samplings/feature-column/fcarc-stern-brocot

[5] http://www.emis.de/journals/INTEGERS/papers/o39/o39.pdf

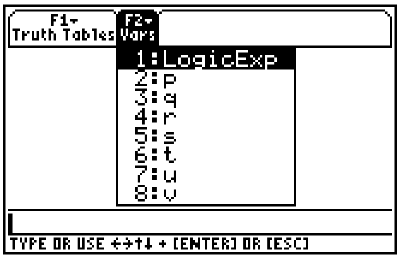
Truth Tables on the TI-89

by MacDonald Phillips
don.phillips@gmail.com

Truth Tables on the TI-89 is a simple program to use. Simply put in the logical expression and press ENTER. First, however, run the menu() program that is located in the Truth folder. Under F1: Truth Tables, is the main program and logical connectors.

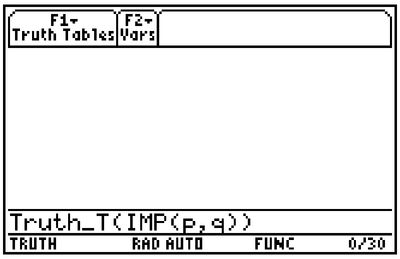


Besides the logical connectors included on the TI-89 (not, and, or, xor), I created two functions, one for implication and the other for if-and-only-if. IMP() takes two arguments, IMP(p,q), meaning if p then q. IFF() also takes two arguments, IFF(p,q), meaning if and only if p then q.

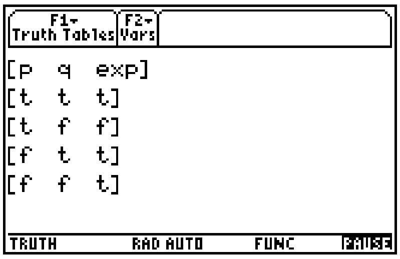


Under F2: Vars is a listing of the common letters used as variables in symbolic logic and the last logical expression used.

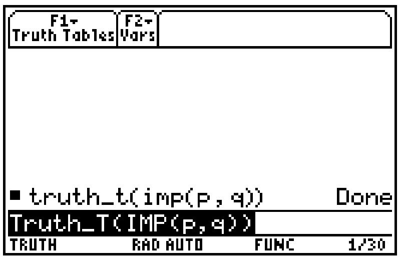
The truth table for implication is created thus:



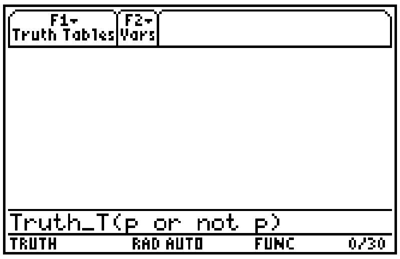
And the output is:

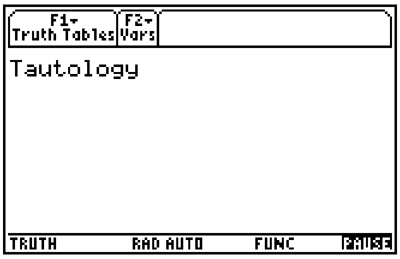


You are given a list of the logical variables and the expression entered (exp) and the possible truth values of the variables and resulting truth value of the entered expression. Press ENTER to display the rest of the truth table and/or to exit the program and return to the HOME screen.



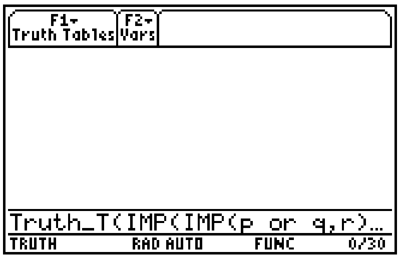
Some simple logical expressions are (unfortunately) automatically reduced to true or false by the TI-89. If that is the case, then either Tautology or Self-Contradiction will be displayed instead of the truth table.



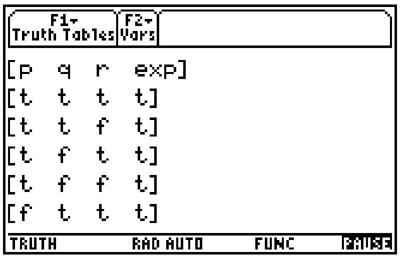


Complex arguments may also be evaluated by the program. For instance, evaluate the argument if(if(p or q then r) and not r then not p and not q).

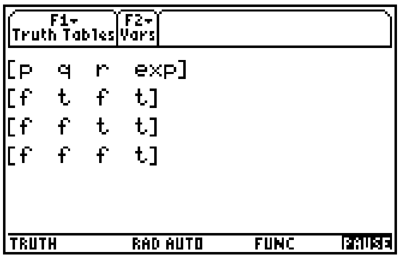
This can be entered as Truth_T(IMP(IMP(p or q, r) and not r, not p and not q).



The truth table is:



and



As can be seen, the truth value of the argument is true no matter what the true values of the variables; the argument is therefore a tautology.

(Note: The program will support any number of logical variables, but only about 7 or 8 can be displayed in a row in the output screen. To display the truth table on the HOME screen just type in tt and press ENTER.)

Reference: *Logic: An Introduction*, Lionel Ruby, J. B. Lippincott Company, 1960.

...it is unworthy of excellent men to lose hours
like slaves in the labor of computation.
Gottfried Wilhelm Leibniz

It is much easier for *DERIVE* users because the function TRUTH_TABLE is ready made implemented. The logic operators AND, OR, XOR, NOT, IMP, and IFF (logic equivalence) are available.

Let's create the truth table for the last example in Don's paper:

The first line

```
TRUTH_TABLE(p,q,r,(((p OR q) imp r) and not r) imp (not p and not q))
```

displays as:

```
TRUTH_TABLE(p,q,r,(((p ∨ q) → r) ∧ ¬ r) → (¬ p ∧ ¬ q))
```

p	q	r	(p ∨ q → r) ∧ ¬ r → ¬ p ∧ ¬ q
true	true	true	true
true	true	false	true
true	false	true	true
true	false	false	true
false	true	true	true
false	true	false	true
false	false	true	true
false	false	false	true

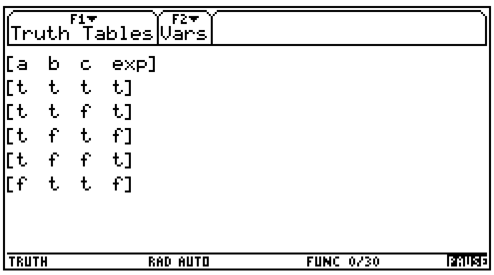
Not all operators can be taken from the mathematical toolbar, some must be written explicitly:

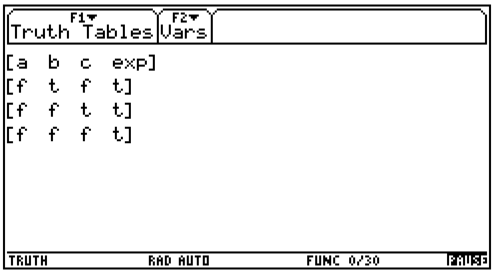
```
[p xor q,p iff q]=
```

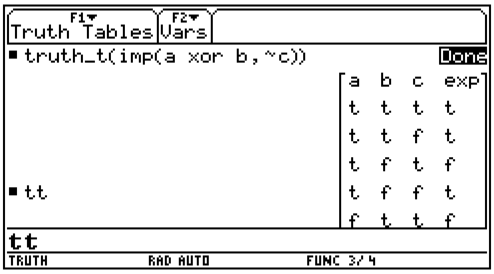
displays and simplifies as follows:

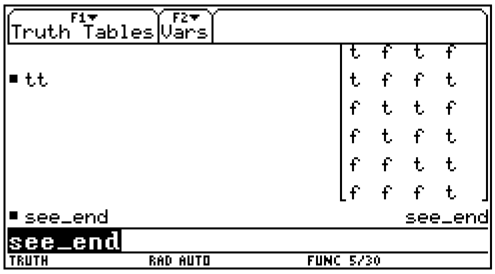
```
[p ⊻ q, p ↔ q] = [(¬ p ∧ q) ∨ (p ∧ ¬ q), (¬ p ∨ q) ∧ (p ∨ ¬ q)]
```

Although Don's contribution reads about a tool for the TI-89 it is no problem at all to run this program on the other TI-devices, too(TI-92, Voyage 200). See V 200 screenshots:









Usually I don’t reprint TI-programs but I assume that many of you don’t have Graph Link for the TI-92/V 200 in the computers any more. Don’s program is very sophisticated from some points of view. He needs some auxiliary functions collaborating with two programs. The functions are (in alphabetical order):

`delelem(list,i)` returns `list` with the ith element removed.

```
delelem(lst,i)
Func
augment(left(lst,i-1),mid(lst,i+1,dim(lst)-i))
EndFunc
```

`iff(ä,ë)` and `iff(ä,ë)` define logic equivalence and implication

```
iff(ä,ë)
Func
(not ä or ë) and (ä or not ë)
EndFunc


imp(ä,ë)
Func
not ä or ë
EndFunc
```

`memberg(list/mat,e)` returns true if e is an element of list/mat, else false

```
memberq(list,e)
Func
Local  i:If  getType(list)="MAT":mat▸list(list)→list:For
i,1,dim(list):If  when(e=list[i],true,false,false):Return
true:EndFor:Return  false:EndFunc
```

RmDup(list) returns list with duplicate elements removed - Bhuvanesh Bhatt [1]

```
rmdup(list)
Func
Local  i:For  i,dim(list),1,⁻1:If  memberq(left(list,i-1),list[i])
:delelem(list,i)→list:EndFor:list:EndFunc
```

subst(u,o,n) substitutes elements of n for elements in o

```
subst(u,o,n)
Func
Local  a,d,i
dim(o)→d
For  i,1,d
o[i]=n[i]→a
u|a→u
EndFor
Return  u
EndFunc
```

VarList(xpr) returns all variables in expression xpr - Bhuvanesh Bhatt

```
varlist(xpr)
Func
Local  tmp
If  getType(xpr)="MAT":mat▸list(xpr)→xpr:If getType(xpr)="LIST"
Then:If  dim(xpr)=1:xpr[1]→xpr:EndIf
If  when(part(xpr)=0,true,false,false):Return  {xpr}:If
part(xpr)=1:Return  rmdup(varlist(part(xpr,1))):
If when(part(xpr)>= 2,true,false,false)
Then:augment(varlist(part(xpr,1)),varlist(seq(part(xpr,ïï),ïï,2,
part(xpr))))→tmp:Return  rmdup(tmp):EndIf
EndFunc
```

menu() generates a specific menu bar on the top of the screen with drop down menus

```
menu()
Prgm
Custom
Title  "Truth Tables"
Item  "Truth_T("
Item  " not ":Item  " and "
Item  " or ":Item  " xor "
Item  "IMP(":Item  "IFF("
Title  "Vars"
Item  "LogicExp"
Item  "p":Item  "q"
Item  "r":Item  "s"
Item  "t":Item  "u"
Item  "v"
EndCustm
CustmOn
EndPrgm
```

truth_t(xpr) is the main program

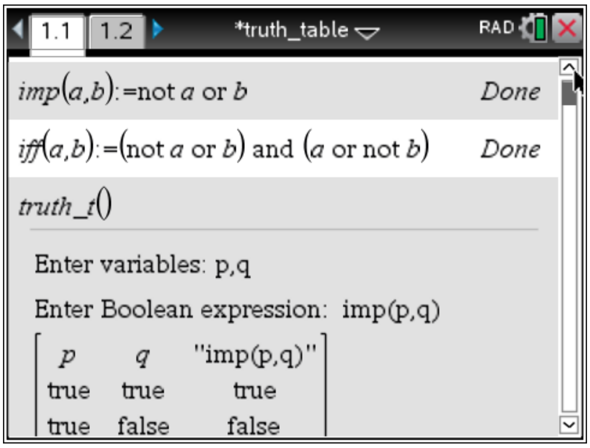
```
truth_t(xpr)
Prgm
Local  i_,j_,p_,l_,m_,k_,v_,n_,o_,q_,c_,md_,xp_
xpr→logicexp
If when(xpr=true,true,false,false) Then
Disp  "Tautology"
Pause
ClrIO:DispHome
ElseIf when(xpr=false,true,false,false) Then
Disp  "Self-Contradiction"
Pause
ClrIO:DispHome
Else
varlist(xpr)→v_
SortA  v_
augment(v_,{exp})→v1 :dim(v_)→n_
2^n_→i_:{}→m_
For  j_,1,n_
i_/2^j_→p_
seq(x_,y_,1,p_)→l_:seq(l_,x_,1,Ø,⁻1)→l_
mat▸list(l_)→l_
If when(j_>1,true,false,false) Then
For  k_,1,j_-1
augment(l_,l_)→l_
EndFor
EndIf
augment(m_,l_)→m_
EndFor
m_|1=t and Ø=f→m_:(list▸mat(m_,i_))ᵀ→m_
m_|t=true and f=false→m_
{}→o_
For  j_,1,i_
subst(xpr,v_,mat▸list(m_[j_]))→q_:augment(o_,{q_})→o_
EndFor
list▸mat(o_,1)→o_:list▸mat(v1,n_+1)→v1
augment(v1;augment(m_,o_))→tt
subst(tt,{true,false},{t,f})→tt
ceiling(i_/5)→c_:mod(i_,5)→md_
For  j_,1,c_
ClrIO
Disp  tt[1]
If j_=c_ and md_>Ø Then
For  k_,1,md_
Disp  tt[(j_-1)*5+k_+1]
EndFor
Else
For  k_,1,5
Disp  tt[(j_-1)*5+k_+1]
EndFor
EndIf
Pause
EndFor
ClrIO:DispHome
EndIf
EndPrgm
```

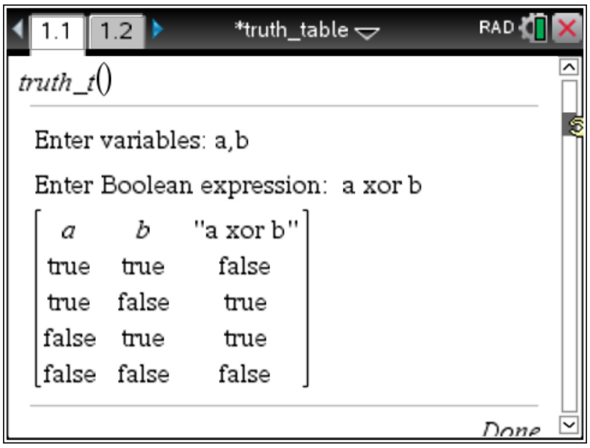
Function varlist is of special interest because it extracts all variables in use of the given logic expressions.

[1] http://www.technicalc.org/bbhatt/
This is a very rich resource for TI-89/92 & V 200 including many useful links to other websites

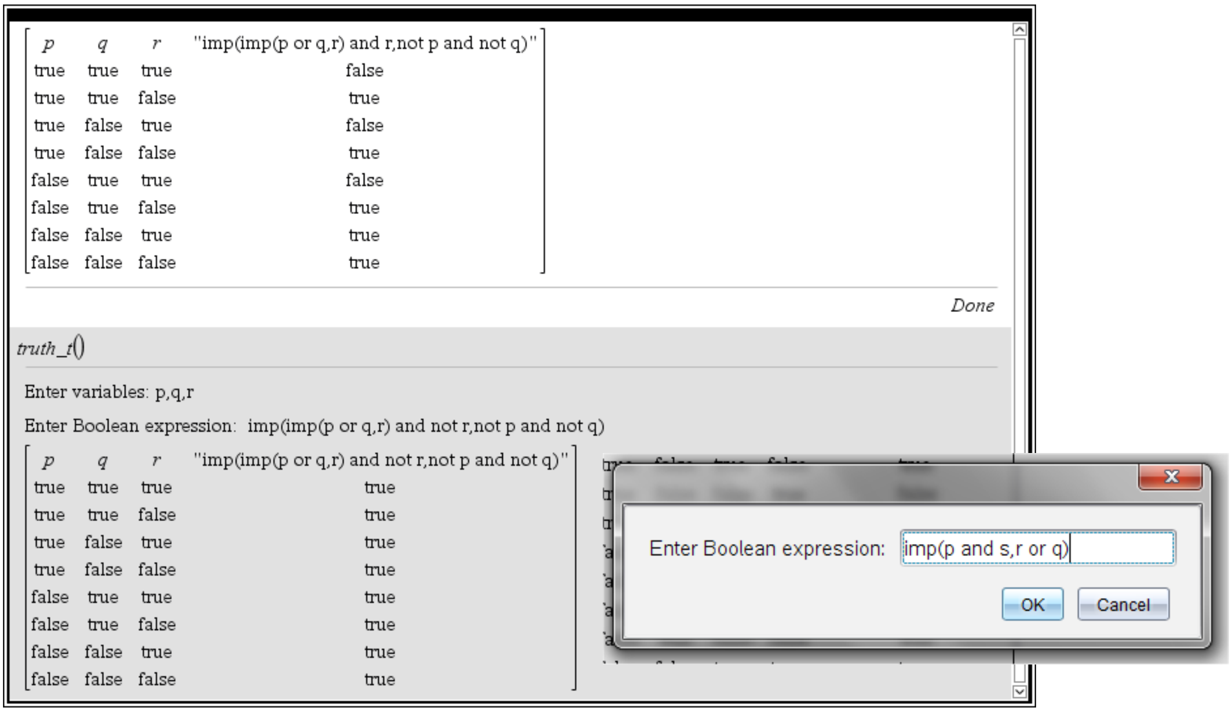
But how is it with TI-Nspire? TI-NspireCAS has neither built in the truth_table function nor can it run the TI-89 (92, V 200) program. It is necessary to write an own program. See how it works:

There are two screen shots in handheld view:





followed by screen shots from the PC:



We will solve an example from [2]:

Let *p, q, r* denote the primitive statements given as:

p: Roger studies.

q: Roger plays tennis.

r Roger passes the maths exam.

Now let p_1, p_2, p_3 denote the premises:

p_1: If Roger studies then, he will pass the exam.

p_2: If Roger doesn't play tennis then he'll study.

p_3: Roger failed the exam.

We want to determine whether the argument $(p_1 \wedge p_2 \wedge p_3) \to q$ is valid.

We rewrite the premises as:

p_1: $p \to r$

p_2: $\neg q \to p$

p_3: $\neg r$

Finally we generate the truth table for the implication:

$$((p \to r) \wedge (\neg q \to p) \wedge \neg r) \to q$$

Done

truth_t()

Enter variables: p,q,r

Enter Boolean expression: imp(imp(p or q,r) and not r,not p and not q)

p	*q*	*r*	"imp(imp(p or q,r) and not r,not p and not q)"
true	true	true	true
true	true	false	true
true	false	true	true
true	false	false	true
false	true	true	true
false	true	false	true
false	false	true	true
false	false	false	true

Done

The truth table presents a tautology. So we can say that $(p_1 \wedge p_2 \wedge p_3) \to q$ is a valid argument.

KIT_ETS_MB for NspireCAS (and for DERIVE, of course)

Michel Beaudin (Montréal, CAN) and Josef Böhm (Würmla, AUT)

You might remember the *Integration of Piecewise Continuous Functions* contributed by Michel Beaudin, Frédérick Henri and Geneviève Savard in DNL#91. The Canadian colleagues provided a collection of utility functions in several KIT_ETS_... files for the TI-NspireCAS. We must not forget Chantal Trottier who is author of ETS_specfunc.tns. All these tools can be downloaded from the ETSMTL website.

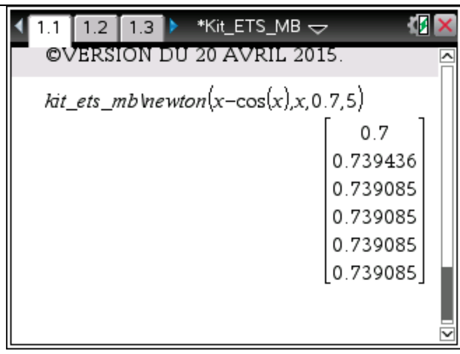
Michel set an update of his papers on the web (in French, of course). One paper is a list of the functions together with their syntax provided for some of his courses at the ETSMTL. The other shows some examples how to apply the functions.

It was my idea not only to combine both papers in an English version but also add examples for all functions appearing in the list. What you can find in the following is the first part of this list together with appropriate examples. Hope that you enjoy it and that you will find it useful, Josef.

(Needless to say, that a *DERIVE*-appendix will not be missed.)

newton(f,x,a,n)

n times application of Newton's method for solving an equation of form *f*(*x*) = 0 with *a* as initial value (first estimation).

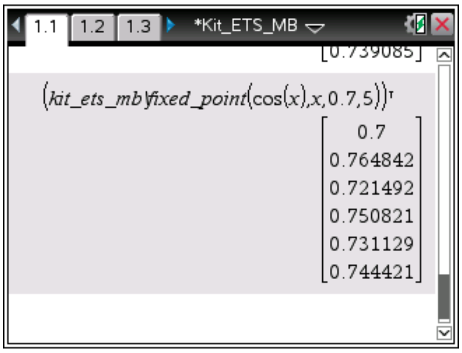


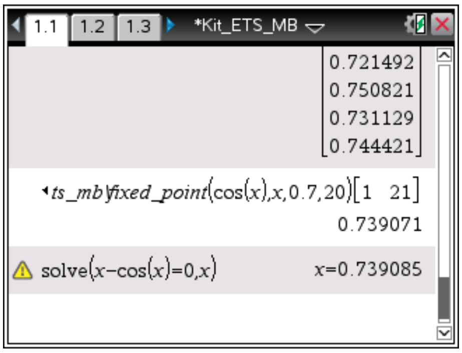
fixed_point(f,x,a,n)

n times application of the fix point method for solving an equation of form *x* = f(*x*) with *a* as initial value (first estimation).

I transposed the result in order to receive a column which is easier to be read.

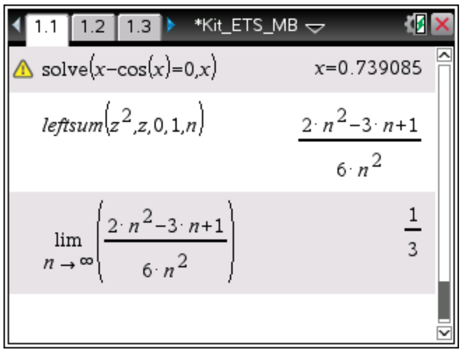
We observe that this method converges much slower than Newton's method. I present the result after 20 iterations.





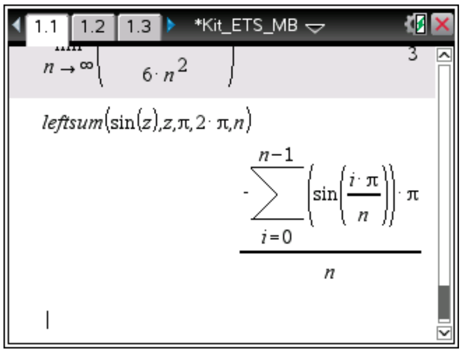
leftsum(f,x,a,b,n)
rightsum(f,x,a,b,n)
midsum(f,x,a,b,n)

give the respective sums using n subdivisions for estimating the integral $\int_a^b f(x)\,dx$.



Unfortunately *NspireCAS CX* is unable to find a closed form for some sums (eg. for the trig functions).

We can calculate the sums for large values of n numerically.



Nspire produces the sum and returns it in exact form by presenting all summands.

The closed form for the left sum is given by $-\frac{\pi}{n}\cot\left(\frac{\pi}{2n}\right)$.

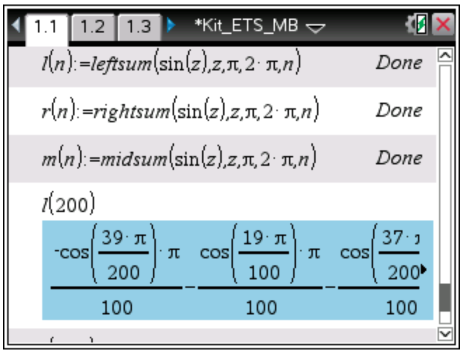
Unlike *NspireCAS* is *DERIVE* able to find the closed forms for the approximating sums of trig functions as you can see below.

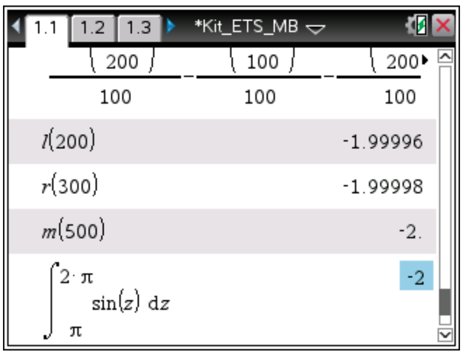
```
[F(x) := SIN(x), a := π, b := 2·π]
[LSUM_VAL(n), RSUM_VAL(n), MSUM_VAL(n)]
```

$$\left[-\frac{\pi\cdot\mathrm{COT}\left(\frac{\pi}{2\cdot n}\right)}{n},\ -\frac{\pi\cdot\mathrm{COT}\left(\frac{\pi}{2\cdot n}\right)}{n},\ -\frac{\pi}{n\cdot\mathrm{SIN}\left(\frac{\pi}{2\cdot n}\right)}\right]$$

$$\lim_{n\to\infty}[\mathrm{LSUM\_VAL}(n),\ \mathrm{RSUM\_VAL}(n),\ \mathrm{MSUM\_VAL}(n)]$$

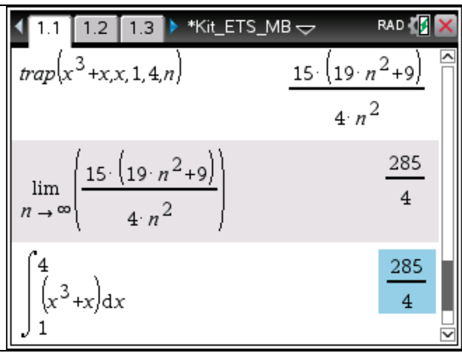
$$[-2,\ -2,\ -2]$$





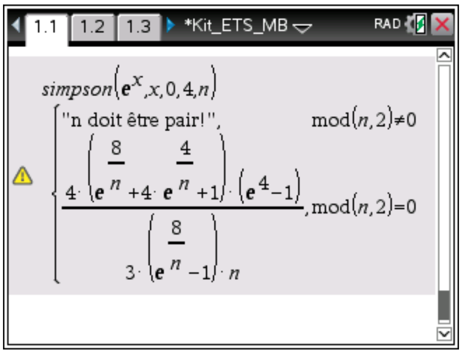
trap(f,x,a,b,n)

Trapezium method for approximating a defined integral.

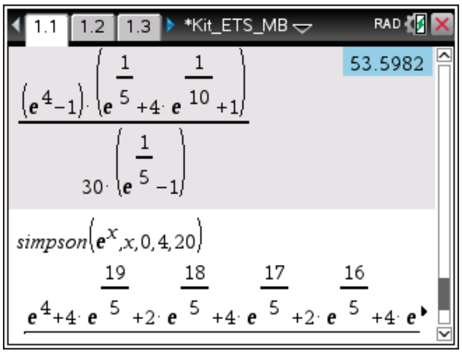


simpson(f,x,a,b,n)

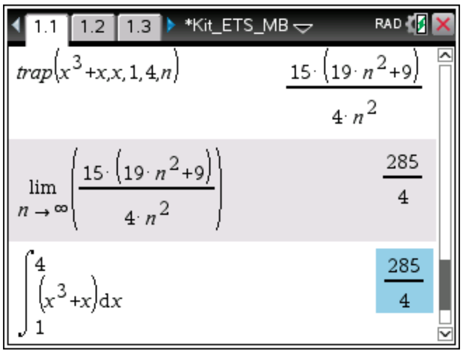
Simpson method for numerical integration (*n* must be an even number).

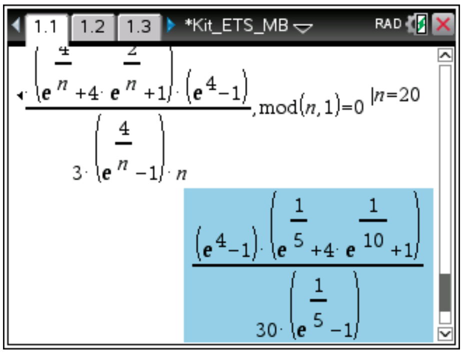


I don't receive an answer in closed form even when I enter 2*n* subdivisions (even).

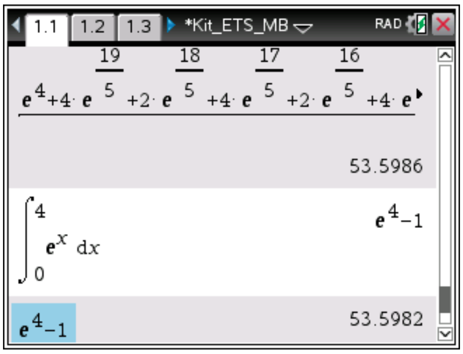


Entering $n = 20$ directly in the function gives the sum explicitly with all its summands.





I substitute for $n = 20$ and receive a pretty expression which an be approximated



Approximation results in the same value.

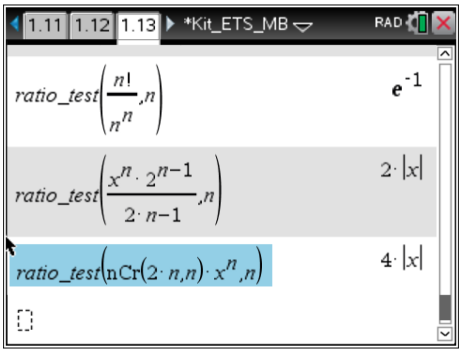
Let's compare with *DERIVE*:

$$\text{SIMPSON}(e^x,\ x,\ 0,\ 4,\ 2\cdot n) = \frac{(e^4 - 1)\cdot(e^{2/n} + 4\cdot e^{1/n} + 1)}{3\cdot n\cdot(e^{2/n} - 1)}$$

$$\lim_{n\to\infty} \text{SIMPSON}(e^x,\ x,\ 0,\ 4,\ 2\cdot n) = e^4 - 1$$

ratio_test(t,n)

Gives the $\lim\limits_{n\to\infty}\dfrac{t(n)}{t(n+1)}$ for testing the convergence of the sum of the series $\sum\limits_{n=1}^{\infty} t(n)$. The series is convergent if the limit is less 1, it is divergent if greater 1. If the ratio test result is equal 1 then we cannot decide (and possibly try other tests).



$\sum\limits_{n=1}^{\infty}\dfrac{n!}{n^n}$ is convergent.

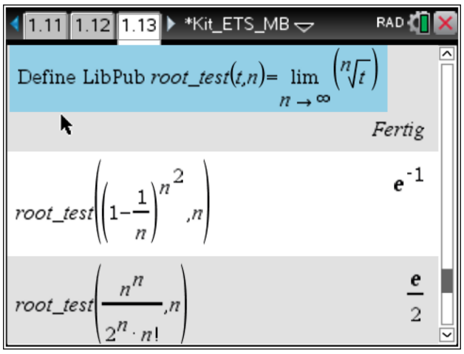
The second series is convergent for $|x| < ½$ and the third one is convergent for $|x| < ¼$.

When I read about the ratio test I remembered that I had learned about a root test, too – some decades ago.

The sum of a series converges if $\lim\limits_{n\to\infty}\sqrt[n]{t} < 1$.

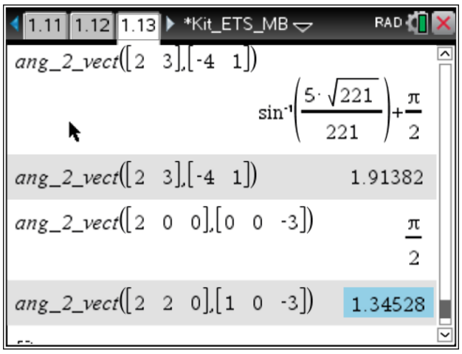
Defining the function *root_test*(t,n) was an easy job.

First series is convergent, the second one is not.



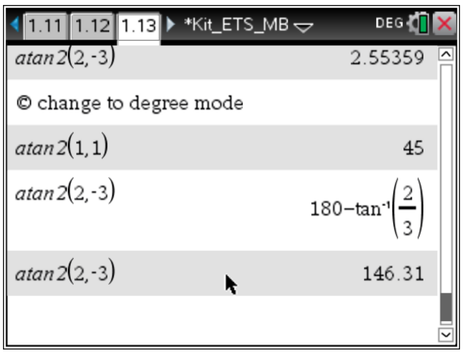
ang_vect(**u**,**v**)

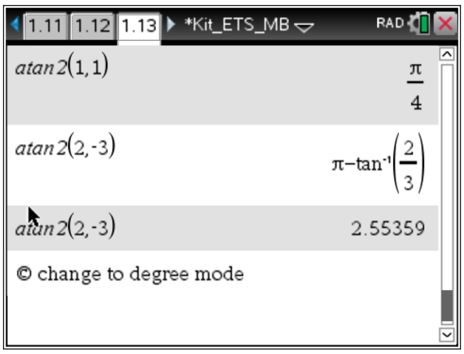
returns the angle (between 0 and π) between the two vectors **u** and **v** $\left(\mathbf{u},\ \mathbf{v}\in\mathbb{R}^2 \text{ or } \mathbb{R}^3\right)$.



atan(y,x)

returns the angle θ ($-\pi \le \theta \le \pi$) of the vector [x,y].
Notice the order of entering the coordinates.





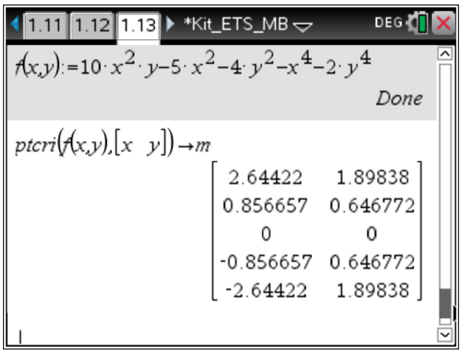
In radiants (above) and in degree mode (left).

ptcri(f,**v**)

returns the matrix containing the critical points of function f of two variables **v** = [x,y]. f should be a polynomial.

Given is $f(x,y) = 10x^2y - 5x^2 - 4y^2 - x^4 - 2y^4$.

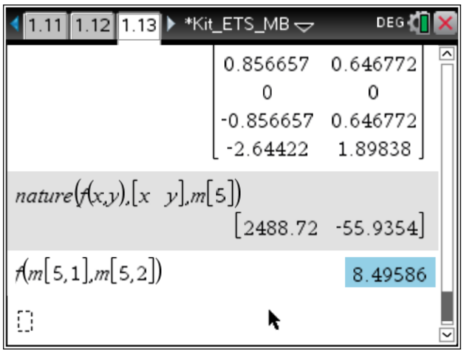
The critical points can be found by solving the system of equations $\nabla f(x,y) = [f_x, f_y] = [0,0]$.



nature(f,**v**,**vo**)

The nature of a critical point **vo** = [a,b] can be decided by evaluating the determinant of the Hessian matrix D together with the value of the 2nd partial derivative f_{xx} in the respective point **vo**.

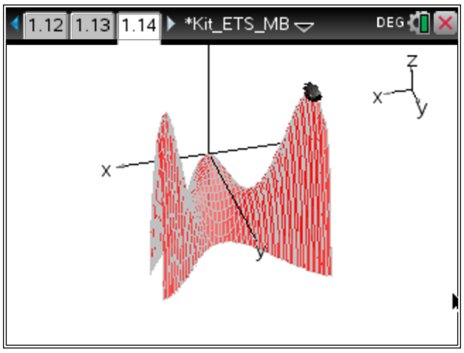
This function returns [$D(a,b)$, $f_{xx}(a,b)$].



In the example we would like to find the nature of the fifth point in the list of critical points. As $D > 0$ and $f_{xx}<0$ f has a relative maximum at (-2.644,1.898) with function value 8.496.

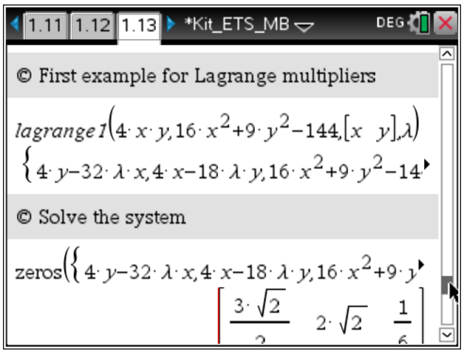
The 3D-plot of the function confirms the calculations.

It is in fact a global maximum. The maximum point is realized in the graph as a sphere with its centre at (-2.64, 1.90, 8,50) and radius 0.3.



lagrange1(f,g,**v**,λ)

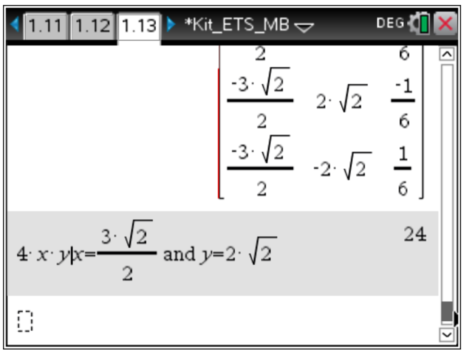
simplifies to a list of equations which can be used to apply the method of Lagrange multipliers with one constraint. f is the objective function, g the constraint (in form $g = 0$), **v** is the vector of the variables and λ the multiplier.



Example:
What is the rectangle of maximum area that can be inscribed in the ellipse given by $16x^2 + 9y^2 = 144$?

The vertex of the rectangle where x, y > 0 is at $\left(\frac{3\sqrt{2}}{2}, 2\sqrt{2}\right)$ and the maximum area is 24.

We can see the complete calculation on the PC-screen.



-

© First example for Lagrange multipliers

$lagrange1(4\cdot x\cdot y, 16\cdot x^2+9\cdot y^2-144, [x\ \ y], \lambda)$ $\quad \{4\cdot y-32\cdot\lambda\cdot x, 4\cdot x-18\cdot\lambda\cdot y, 16\cdot x^2+9\cdot y^2-144\}$

© Solve the system

$\text{zeros}(\{4\cdot y-32\cdot\lambda\cdot x, 4\cdot x-18\cdot\lambda\cdot y, 16\cdot x^2+9\cdot y^2-144\}, \{x,y,\lambda\})$

$$\begin{bmatrix} \frac{3\cdot\sqrt{2}}{2} & 2\cdot\sqrt{2} & \frac{1}{6} \\ \frac{3\cdot\sqrt{2}}{2} & -2\cdot\sqrt{2} & \frac{-1}{6} \\ \frac{-3\cdot\sqrt{2}}{2} & 2\cdot\sqrt{2} & \frac{-1}{6} \\ \frac{-3\cdot\sqrt{2}}{2} & -2\cdot\sqrt{2} & \frac{1}{6} \end{bmatrix}$$

$4\cdot x\cdot y|x=\frac{3\cdot\sqrt{2}}{2} \text{ and } y=2\cdot\sqrt{2}$ $\quad 24$

Example:

The Cobb-Douglas production function for a special product is given by $f(x,y)=100x^{3/4}y^{1/4}$ where x represents the units of labor (at € 150 per unit) and y represents the units of capital (at € 250 per unit). The total expenses for labor and capital is limited to € 50.000. What is the maximum production level for this good?

© Second example for Lagrange multipliers

$cd(x,y):=100\cdot x^{\frac{3}{4}}\cdot y^{\frac{1}{4}}$ $\quad$ *Done*

$lagrange1(cd(x,y), 150\cdot x+250\cdot y-50000, [x\ \ y], \lambda)$

$$\left\{\frac{75\cdot y^{\frac{1}{4}}}{x^{\frac{1}{4}}}-150\cdot\lambda, \frac{25\cdot x^{\frac{3}{4}}}{y^{\frac{3}{4}}}-250\cdot\lambda, 150\cdot x+250\cdot y-50000\right\}$$

⚠ $\text{zeros}(lagrange1(cd(x,y), 150\cdot x+250\cdot y-50000, [x\ \ y], \lambda), \{x,y,\lambda\})$ $\quad \begin{bmatrix} 250 & 50 & \frac{5^{\frac{3}{4}}}{10} \end{bmatrix}$

$cd(250,50)$ $\quad 5000\cdot 5^{\frac{3}{4}}$

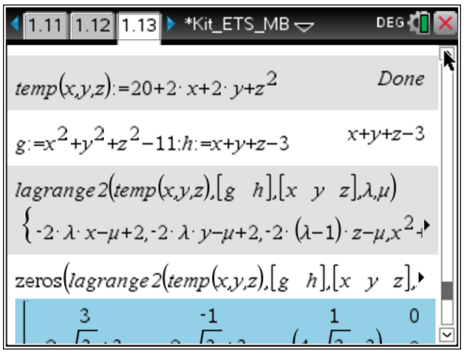
$cd(250,50)$ $\quad 16718.5$

$\frac{3}{4}$ $\quad 0.33437$

50 units of capital and 250 units of labor guarantee the maximum production of 16.718 product units. Economists call 0.334 the *marginal productivity of money* (i.e. each additional spent € results in 0.334 additional product units.

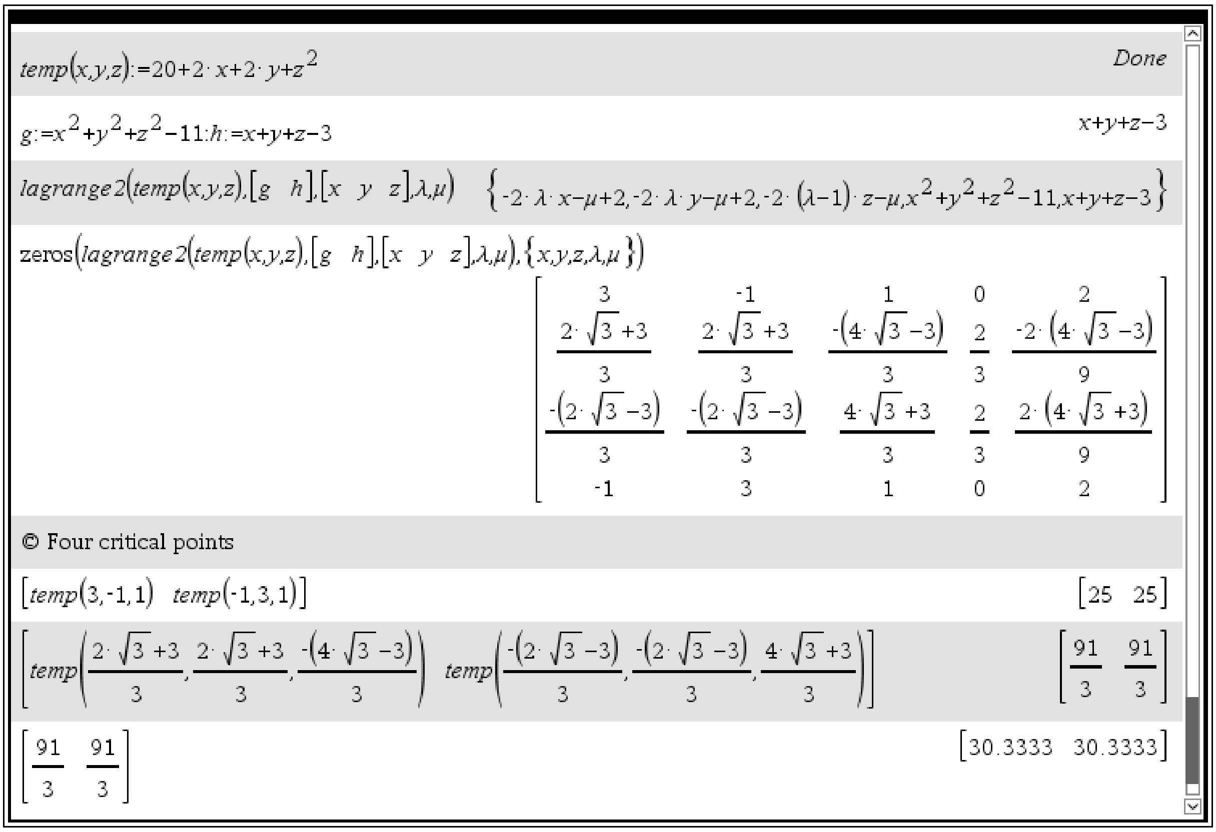
lagrange2(f,**g**,**v**,λ,μ)

simplifies to a list of equations which can be used to apply the method of Lagrange multipliers with two constraints and three variables. f is the objective function, **g** the vector of the constraints (in form $g = 0$), **v** is the vector of the variables and λ and μ are the multipliers.



Example:

$temp(x,y,z) = 20 + 2x + 2y + z^2$ describes the temperature at each point on the sphere with its centre in the origin and radius √11. What are the extreme temperatures on the intersection curve of the sphere and the plane $x + y + z = 3$?



The maximum temperature is 30.33 and the minimum temperature is 25.

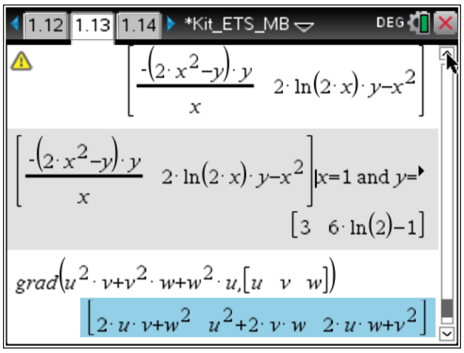
(Note: The examples with the Lagrange multipliers are from *Larson, Hostetler, Edwards*: *Multivariable Calculus*.)

grad(f,**v**)

returns the gradient of function f. **v** is the vector of variables.

Find first the gradient of $f(x,y) = y^2 \ln(2x) - x^2 y$ at point (1,3) and then the gradient of $f(u,v,w) = u^2 v + v^2 w + w^2 u$.

Enter : grad(y^(2)*ln(2*x) – x^(2)*y,[x,y]).

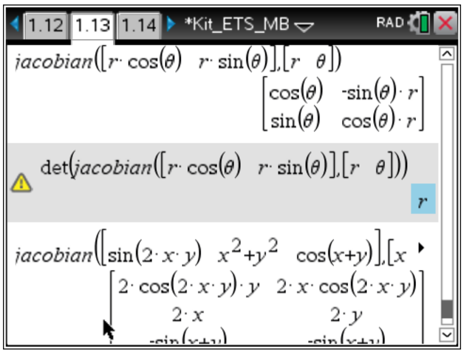


jacobian(**F**,**v**)

simplifies to the Jacobi matrix of the vector field **F**. **v** is the vector of variables.

Find the first Jacobian for x = r cos(θ) and y = r cos(θ) and then the Jacobian determinant.

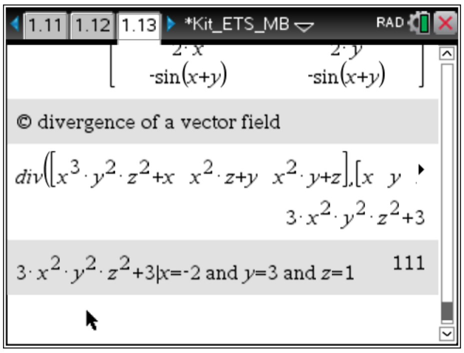
Find the Jacobian for **F** = $\begin{pmatrix} \sin(2xy) \\ x^2 + y^2 \\ \cos(x+y) \end{pmatrix}$.



div(**F**,**v**)

simplifies to divergence of the vector field **F**. **v** is the vector of variables.

Find the divergence of **F** = $[x^3 y^2 z^2 + x, x^2 z + y, x^2 y + z]$ at point (-2,3,1).

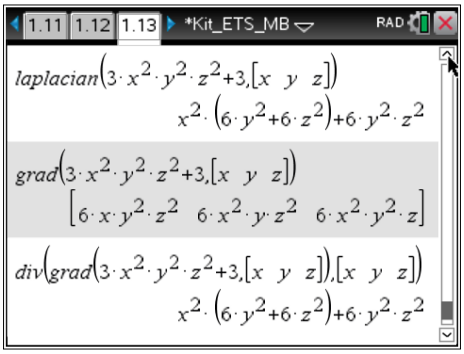


laplacian(**F**,**v**)

applies the Laplace-operator ∇^2 on the scalar field f with **v** representing the vector of variables.

Find the Laplacian of $f = 3x^2 y^2 z^2 + 3$.

Show that $\nabla^2 f = div(grad(f))$.



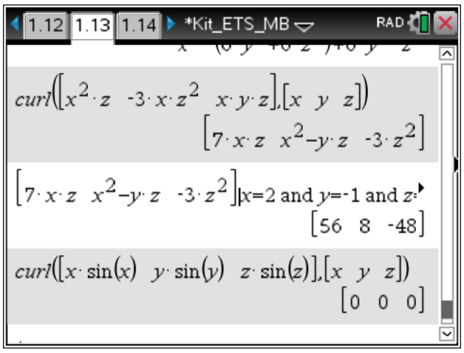
curl(**F**,**v**)

returns the rotation (in German *rot*(**F**,**v**)) of a vector field of three components.

Find the curl of **F** = $[x^2z, -3xz^2, xyz]$ at (2, -1, 4)

Find the rotation of $[x \sin x, y \sin y, z \sin z]$.

(Vector fields with *curl* = zero vector are called *irrotational*. **F** is conservative iff its *curl* is **0**.)

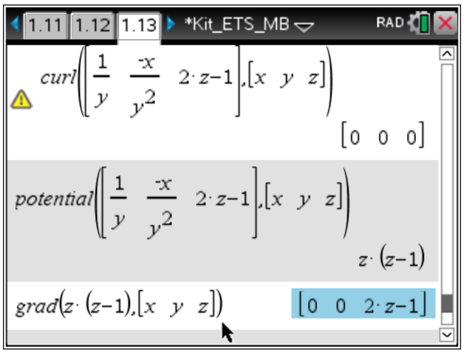


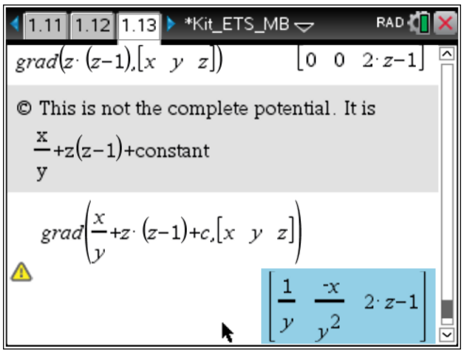
potential(**F**,**v**)

returns the potential of a vector field.

Check always the answer by calculating the gradient of the result.

First check if the curl of F = $\left[\frac{1}{y}, \frac{-x}{y^2}, 2z-1\right]$ = 0 and if so, find its potential.





Find the potential of $\mathbf{G}(x, y, z)=[y^2z^3, 2xyz^3, 3xy^2z^2]$.

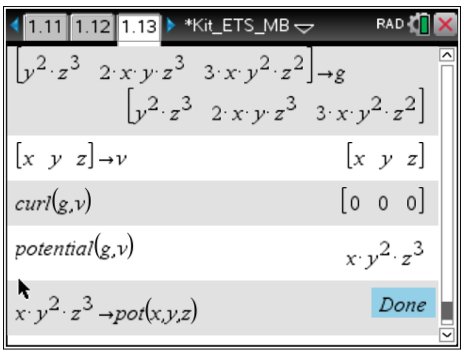
It is easy to see – without CAS – that it is given by xyz^3.

Verify that the gradient of the result gives **G** from above.

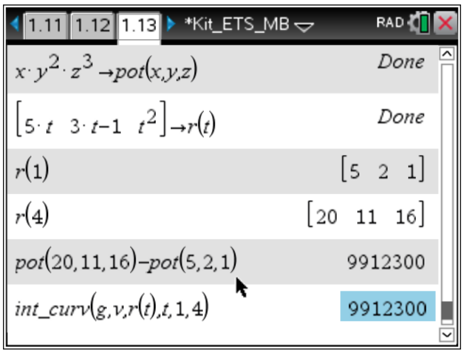
It is important that certain line integrals can be calculated by means of the potential.

Calculate the line integral in this vector field along the curve given by $\mathbf{r}(t) = [5t, 3t-1, t^2]$, $1 \le t \le 4$.

Then apply function *int_curv* contained the library.



int_curve(**F**,**v**,**r**,*t*,*a*,*b*) calculates the line integral in the vector field **F** (variables **v**) along the parameter curve **r**(t) with $a \le t \le b$.

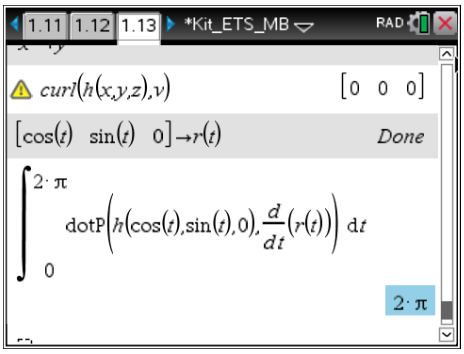


The vector field $\mathbf{H}(x, y, z) = \dfrac{1}{x^2+y^2}[-y,x,0]$ is not conservative because its line integral over a certain curve does not pass the origin. But its rotation (*curl*) is the zero vector **0**.

The domain of **H** are the points in space except those on the z-axis. We choose as closed path the circle $x^2 + y^2 = 1$, $z = 0$ (which is a circle in the xy-plane).

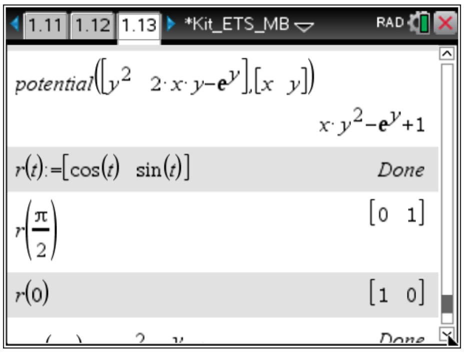
Show that the line integral is 2π.

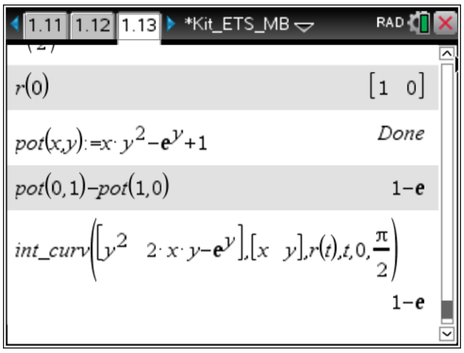
We find the integral without applying *int_curv*.



I add three more examples:

(1) Integrate the vector field $[y^2, 2xy - e^y]$ along $[\cos(t), \sin(t)]$, $0 \le t \le \pi/2$.





This was an example in the xy-plane. Let's do a similar problem in space:

(2) Find the work done by the force field $\mathbf{F}(x,y,z) = \left[-\dfrac{x}{4}, \dfrac{y}{2}, \dfrac{1}{4}\right]$ on a particle as it moves along a helix like space curve given by $\mathbf{r}(t) = \left[\cos(t),\, 2\sin(t),\, \dfrac{t}{2}\right]$, $0 \le t \le 4\pi$.

$potential\left(\begin{bmatrix}\frac{-x}{4} & \frac{y}{2} & \frac{1}{4}\end{bmatrix},[x\ \ y\ \ z]\right)$	$\frac{-x^2}{8}+\frac{y^2}{4}+\frac{z}{4}$
$pot(x,y,z):=\frac{-x^2}{8}+\frac{y^2}{4}+\frac{z}{4}$	*Done*
$r(t):=\begin{bmatrix}\cos(t) & 2\cdot\sin(t) & \frac{t}{2}\end{bmatrix}$	*Done*
$r(0)$	$[1\ \ 0\ \ 0]$
$r(4\cdot\pi)$	$[1\ \ 0\ \ 2\cdot\pi]$
$pot(1,0,2\cdot\pi)-pot(1,0,0)$	$\frac{\pi}{2}$
$int\_curv\left(\begin{bmatrix}\frac{-x}{4} & \frac{y}{2} & \frac{1}{4}\end{bmatrix},[x\ \ y\ \ z],r(t),t,0,4\cdot\pi\right)$	$\frac{\pi}{2}$

(3) The components of a force **G** are given by $\left[\frac{x+y-3z}{(x+y+z)^3},\frac{a\cdot x+b\cdot y+c\cdot z}{(x+y+z)^3},\frac{d\cdot x+e\cdot y+f\cdot z}{(x+y+z)^3}\right]$.

Find the coefficients *a, b, c, d, e* and *f* such that the force has potential *u* and find **G** and *u.*

$fx:=\frac{x+y-3\cdot z}{(x+y+z)^3}:fy:=\frac{a\cdot x+b\cdot y+c\cdot z}{(x+y+z)^3}:fz:=\frac{d\cdot x+e\cdot y+f\cdot z}{(x+y+z)^3}$	$\frac{d\cdot x+e\cdot y+f\cdot z}{(x+y+z)^3}$
$potential([fx\ \ fy\ \ fz],[x\ \ y\ \ z])$	undef
$\left[\int fx\,dx\ \ \int fy\,dy\ \ \int fz\,dz\right]$	$\left[\frac{-(x+y-z)}{(x+y+z)^2}\ \ \frac{-((a+b)\cdot x+2\cdot b\cdot y+(b+c)\cdot z)}{2\cdot(x+y+z)^2}\ \ \frac{-((d+f)\cdot x+(e+f)\cdot y+2\cdot f\cdot z)}{2\cdot(x+y+z)^2}\right]$
$\text{solve}\left(1=\frac{a+b}{2}\text{ and }1=b\text{ and }1=\frac{-(b+c)}{2},\{a,b,c\}\right)$	$a=1$ and $b=1$ and $c=-3$
$\text{solve}\left(1=\frac{d+f}{2}\text{ and }1=\frac{e+f}{2}\text{ and }1=-f,\{d,e,f\}\right)$	$d=3$ and $e=3$ and $f=-1$
© the force ist given by:	
$[fx\ \ fy\ \ fz]\|a=1$ and $b=1$ and $c=-3$ and $d=3$ and $e=3$ and $f=-1$	$\left[\frac{x+y-3\cdot z}{(x+y+z)^3}\ \ \frac{x+y-3\cdot z}{(x+y+z)^3}\ \ \frac{3\cdot x+3\cdot y-z}{(x+y+z)^3}\right]$
© and its potential is $-\frac{x+y-z}{(x+y+z)^2}$	
⚠ $grad\left(\frac{-(x+y-z)}{(x+y+z)^2},[x\ \ y\ \ z]\right)$	$\left[\frac{x+y-3\cdot z}{(x+y+z)^3}\ \ \frac{x+y-3\cdot z}{(x+y+z)^3}\ \ \frac{3\cdot x+3\cdot y-z}{(x+y+z)^3}\right]$

The last example cannot be solved by means of Michel's tools. We integrate *fx* wrt *x* (because it is the partial derivative of the possible potential. We integrate *fy* and *fz* wrt to *y* and *z* and perform a comparison of the coefficients.

newtons(**u**,**v**,**v0**,*n*) applies Newton's method to find a solution of the system **u** of more variables (given in vector **v**) with initial solution vector **v0** performing *n* iterations.

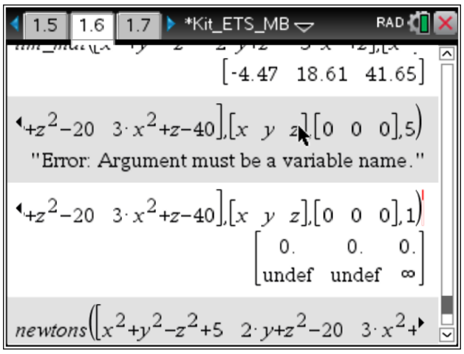
The calculation process can last a longer time.

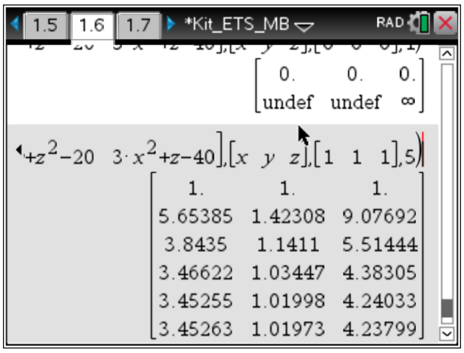
It can also happen that one is not successful with the chosen initial vector. Then try another one.

Example:

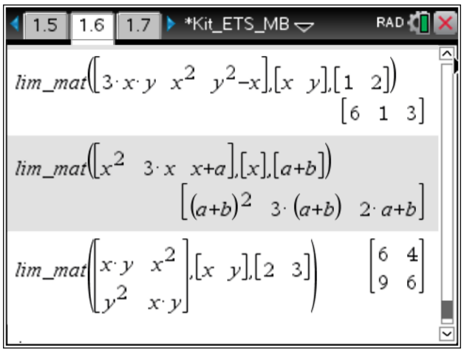
Solve the system

$$x^2 + y^2 - z^2 = -5$$
$$2y + z^2 = 20$$
$$3x^2 + z = 40$$





lim_mat(**mat**,**v**,**v0**) returns the limit of **mat** when **v** tends to **v0**. **mat** can be a scalar vector field, a vector or a matrix.



newtons needs the ITERATES-command which is not a Nspire built-in function. It was not so easy to transfer this great *DERIVE*-function to the Nspire-technology. In early TI-92 days this was done by Terence Etchells – as far as I do remember. You may find the discussion between Michel and me interesting on this issue which is published in *DERIVE*-Newsletter #90.

The next screen shot shows the PC-screen containing some more examples of applying *newtons* and *lim_mat* as well.

$$newtons\left(\left[(\sin(x))^2-y \quad 2\cdot x+y^2-4\right],[x \quad y],[1 \quad 1],5\right) \quad \begin{bmatrix} 1. & 1. \\ 1.41477 & 1.08523 \\ 1.50108 & 1.00235 \\ 1.5044 & 0.995609 \\ 1.50439 & 0.995597 \\ 1.50439 & 0.995597 \end{bmatrix}$$

$$lim\_mat\left(\left[(\sin(x))^2-y \quad 2\cdot x+y^2-4\right],[x \quad y],[1.50439 \quad 0.995597]\right) \quad [\,^{-}3.21936\text{E-}7 \quad ^{-}0.000007]$$

$$newtons\left(\left[x^3-2\cdot x^2+y-10 \quad 1-x^3+2\cdot z \quad 100+x^4-y^2+z^2-100\right],[x \quad y \quad z],[1 \quad 1 \quad 1],1\right) \quad \begin{bmatrix} 1. & 1. & 1. \\ 5.2 & 15.2 & 6.3 \end{bmatrix}$$

$$newtons\left(\left[x^3-2\cdot x^2+y-10 \quad 1-x^3+2\cdot z \quad 100+x^4-y^2+z^2-100\right],[x \quad y \quad z],[1 \quad 1 \quad 1],3\right) \quad \begin{bmatrix} 1. & 1. & 1. \\ 5.2 & 15.2 & 6.3 \\ 3.77992 & 9.13151 & 12.2054 \\ 2.92322 & 8.33689 & 8.14282 \end{bmatrix}$$

$$lim\_mat\left(\left[x^3-2\cdot x^2+y-10 \quad 1-x^3+2\cdot z \quad 100+x^4-y^2+z^2-100\right],[x \quad y \quad z],[2.923 \quad 8.337 \quad 8.143]\right)$$
$$[6.22305 \quad ^{-}7.6879 \quad 69.8016]$$

$$newtons\left(\left[x^3-2\cdot x^2+y-10 \quad 1-x^3+2\cdot z \quad 100+x^4-y^2+z^2-100\right],[x \quad y \quad z],[3 \quad 8 \quad 8],5\right) \quad \begin{bmatrix} 3. & 8. & 8. \\ 2.51596 & 8.26064 & 6.46543 \\ 2.36098 & 8.11734 & 5.99154 \\ 2.34531 & 8.10186 & 5.94932 \\ 2.34516 & 8.10169 & 5.94893 \\ 2.34516 & 8.10169 & 5.94893 \end{bmatrix}$$

Michel sent a rich collection of more utility functions and examples how to apply them. We will publish his materials in the next newsletters.

He also sent a DERIVE file containing some of his functions. I accomplished this collection of tools and tried to use the same function names and parameters as well. You can find the respective file on the next two pages.

I added three examples from above performed with DERIVE.

References:

[1] *Gunter & Kusmin*, Aufgabensammlung zur Höheren Mathematik, VEB Deutscher Verlag der Wissenschaften, Berlin 1964

[2] *T. Arens a.o.*, Mathematik, Spektrum 2009

[3] *Larson, Hostetler, Edwards,* Multivariable Calculus, Houghton Mifflin Company 2002

Michel's Tools prepared for DERIVE

kitETS_DERIVE.mth

$$\text{NEWTON}(f,\ x,\ x0,\ n) = \text{ITERATES}\left(x - \frac{d}{dx}\ f,\ x,\ x0,\ n\right)$$

$$\text{fixed\_point}(f,\ x,\ x0,\ n) := \text{ITERATES}(f,\ x,\ x0,\ n)$$

$$\text{RIGHTSUM}(g,\ x,\ a,\ b,\ n) := \sum_{i=1}^{n} \left(\lim_{x \to a + i\cdot(b - a)/n} g\right)\cdot\frac{b - a}{n}$$

$$\text{LEFTSUM}(g,\ x,\ a,\ b,\ n) := \sum_{i=0}^{n-1} \left(\lim_{x \to a + i\cdot(b - a)/n} g\right)\cdot\frac{b - a}{n}$$

$$\text{MIDSUM}(g,\ x,\ a,\ b,\ n) := \sum_{i=1}^{n} \left(\lim_{x \to a + (i - 1/2)\cdot(b - a)/n} g\right)\cdot\frac{b - a}{n}$$

$$\text{TRAP}(g,\ x,\ a,\ b,\ n) := \frac{(b - a)\cdot\left(\left(\sum_{i=1}^{n} \lim_{x \to i\cdot(b - a)/n + a} g\right) + \sum_{i=0}^{n-1} \lim_{x \to i\cdot(b - a)/n + a} g\right)}{2\cdot n}$$

```
SIMPSON(g, x, a, b, n) :=
  If MOD(n, 2) ≠ 0
     "n pair S.V.P.!!!"
     (2·MIDSUM(g, x, a, b, n/2) + TRAP(g, x, a, b, n/2))/3
```

ratio_test is implemented in DERIVE

$$\text{root\_test}(u,\ n) := \lim_{n \to \infty} u^{1/n}$$

$$\text{ang\_2\_vect}(u,\ v) := \text{ACOS}\left(\frac{u\cdot v}{|u|\cdot|v|}\right)$$

atan2(y,x) is implemented in DERIVE

$$\text{ptcri}(f,\ v) := \text{SOLUTIONS}(\partial(f,\ v_1) \land \partial(f,\ v_2),\ [v_1,\ v_2])$$

$$\text{hessian}(f,\ v) := \partial(f,\ v_1,\ 2)\cdot\partial(f,\ v_2,\ 2) - \partial(\partial(f,\ v_1),\ v_2)^2$$

$$\text{nature}(f,\ v,\ v0) := \left[\lim_{v \to v0} \text{hessian}(f,\ v),\ \lim_{v \to v0} \partial(f,\ v_1,\ 2)\right]$$

NEWTONS(u,v,v0,n) is implemented

$$\text{lagrange1}(f,\ g,\ v := [x,\ y],\ \lambda) := \text{APPEND}(\text{GRAD}(f - \lambda\cdot g,\ v),\ [g])$$

$$\text{lagrange2}(f,\ g,\ v := [x,\ y,\ z],\ \lambda,\ \mu) := \text{APPEND}(\text{GRAD}(f - \lambda\cdot g_1 - \mu\cdot g_2,\ v),\ g)$$

GRAD(f,v), DIV(F,v), LAPLACIAN(F,v) and CURL(F,v) are implemented, see Online Help.

$$\text{jacobian}(f,\ v) := \text{VECTOR}(\text{VECTOR}(\partial(f_j,\ v_i),\ i,\ \text{DIM}(v)),\ j,\ \text{DIM}(f))$$

lim_mat is not necessary, LIM-function is sufficient for all cases!

$$\text{potential}(f,\ v) := \int_0^1 (\lim_{v \to t\_ \cdot v} f) \cdot v \, dt\_$$

$$\text{int\_curv}(ch,\ ve,\ co,\ pa,\ a,\ b) := \int_a^b (\lim_{ve \to co} ch) \cdot \frac{d}{d\ pa} co \, dpa$$

Three examples from above worked with *DERIVE*:

$$\text{ptcri}(10 \cdot x^2 \cdot y - 5 \cdot x^2 - 4 \cdot y^2 - x^4 - 2 \cdot y^4,\ [x,\ y]) = \begin{bmatrix} 0 & 0 \\ 0 & i \\ 0 & -i \\ 0.8566568714 & 0.6467721990 \\ -0.8566568714 & 0.6467721990 \\ 2.644224301 & 1.898384431 \\ -2.644224301 & 1.898384431 \\ 3.902022956 \cdot i & -2.545156630 \\ -3.902022956 \cdot i & -2.545156630 \end{bmatrix}$$

$$[\text{temp}(x,\ y,\ z) := 20 + 2 \cdot x + 2 \cdot y + z^2,\ g := x^2 + y^2 + z^2 - 11,\ h := x + y + +z - 3]$$

SOLUTIONS(lagrange2(temp(x, y, z), [g, h], [x, y, z], λ, μ), [x, y, z, λ, μ])

$$\begin{bmatrix} -1 & 3 & 1 & 0 & 2 \\ 3 & -1 & 1 & 0 & 2 \\ \frac{2 \cdot \sqrt{3}}{3} + 1 & \frac{2 \cdot \sqrt{3}}{3} + 1 & 1 - \frac{4 \cdot \sqrt{3}}{3} & \frac{2}{3} & \frac{2}{3} - \frac{8 \cdot \sqrt{3}}{9} \\ 1 - \frac{2 \cdot \sqrt{3}}{3} & 1 - \frac{2 \cdot \sqrt{3}}{3} & \frac{4 \cdot \sqrt{3}}{3} + 1 & \frac{2}{3} & \frac{8 \cdot \sqrt{3}}{9} + \frac{2}{3} \end{bmatrix}$$

$$r(t) := \left[\text{COS}(t),\ \text{SIN}(2 \cdot t),\ \frac{t}{2}\right]$$

$$\text{int\_curv}\left(\left[-\frac{x}{4},\ \frac{y}{2},\ \frac{1}{4}\right],\ [x,\ y,\ z],\ r(t),\ t,\ 0,\ 4 \cdot \pi\right)$$

$$\text{int\_curv}\left(\left[-\frac{x}{4},\ \frac{y}{2},\ \frac{1}{4}\right],\ [x,\ y,\ z],\ r(t),\ t,\ 0,\ 4 \cdot \pi\right) = \frac{\pi}{2}$$