Résumé 1 – Part 2

Symbolic and numerical solution of equations and systems of equations, comments and additions to differential equations

1.8 Example of a “symbolic” response

When considering a polynomial equation of degree 5 or more, we can always find, in floating-point arithmetic, all solutions. Let to fix the ideas $x^5+x+3=0$. *Derive* returns the equation, unless you approximate. Since its "`SOLUTIONS`" function returns a matrix, the exact mode can return an empty matrix!

#20: $\mathrm{SOLVE}(x^5 + x + 3 = 0,\ x)$

#21: $x^5 + x = -3$

#22: $x = -0.4753807566 - 1.129701725\cdot i \lor x = -0.4753807566 + 1.129701725\cdot i \lor x = 1.041879539 - 0.8228703381\cdot i \lor x = 1.041879539 + 0.8228703381\cdot i \lor x = -1.132997565$

Figure 1.12

Maple: for a polynomial, the "*RootOf*" structure allows to use each of the roots.

> $solve(x^5 + x + 3 = 0, x);$

$$RootOf(\_Z^5 + \_Z + 3, index = 1), RootOf(\_Z^5 + \_Z + 3, index = 2), RootOf(\_Z^5 + \_Z + 3, index = 3), RootOf(\_Z^5 + \_Z + 3, index = 4), RootOf(\_Z^5 + \_Z + 3, index = 5)$$

Figure 1.13

1.9 Example and common sense

When considering a non-polynomial equation, then symbolic systems do not replace the user's analysis and imagination and common sense are still required. There is no formula to solve the equation $\sin x = 1-\dfrac{x}{6}$. The TI calculator "warns" the user that some solutions may be forgotten:

⚠ $\mathrm{solve}\left(\sin(x)=1-\frac{x}{6},x\right)$	$x=0.988567$ or $x=2.5236$ or $x=6.24272$ or $x=10.2002$ or $x=11.4336$
$\mathrm{exact}\left(\mathrm{solve}\left(\sin(x)=1-\frac{x}{6},x\right)\right)$	$6\cdot\sin(x)+x=6$
$\mathrm{nSolve}\left(\sin(x)=1-\frac{x}{6},x\right)$	0.988567
$\mathrm{nSolve}\left(\sin(x)=1-\frac{x}{6},x=2\right)$	2.5236
$\mathrm{nSolve}\left(\sin(x)=1-\frac{x}{6},x\right)\mid 3<x<7$	6.24272
$\mathrm{nSolve}\left(\sin(x)=1-\frac{x}{6},x,9,11\right)$	10.2002
$\mathrm{nSolve}\left(\sin(x)=1-\frac{x}{6},x,11,\infty\right)$	11.4336

Figure 1.14

A graph of each side of the equation indicates that there are, in total, 5 solutions. We can see from Figure 1.14, that Nspire CX CAS has indeed found (in this case) these 5 solutions when the system switches to approximate mode. In exact mode, it returns only the equation in equivalent form. And guiding the "nsolve"-procedure can be useful (by adding boundaries for possible solutions).

Let's look at the responses of *Maple* and *WolframAlpha* in Figure 1.15.

$> \ solve\left(\sin(x) = 1 - \frac{x}{6}, x\right);$

$$RootOf(6\sin(\_Z) - 6 + \_Z)$$

$> \ fsolve\left(\sin(x) = 1 - \frac{x}{6}, x\right);$

$$0.9885669797$$

$> \ fsolve\left(\sin(x) = 1 - \frac{x}{6}, x = 2\right);$

$$2.523600393$$

WolframAlpha

solve sin(x)=1-x/6 for x

Input interpretation:

solve	$\sin(x) = 1 - \frac{x}{6}$	for	x

Solution over the reals: More digits

$x \approx 0.988567$

$x \approx 2.52360$

$x \approx 6.24272$

$x \approx 10.2002$

$x \approx 11.4336$

Plot:

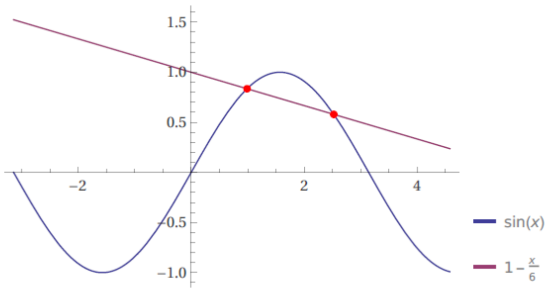

Figure 1.15

And how does *Derive* respond? Initially, a window containing the graphs of each side of the equation shows 5 intersections in total, as indicated earlier:

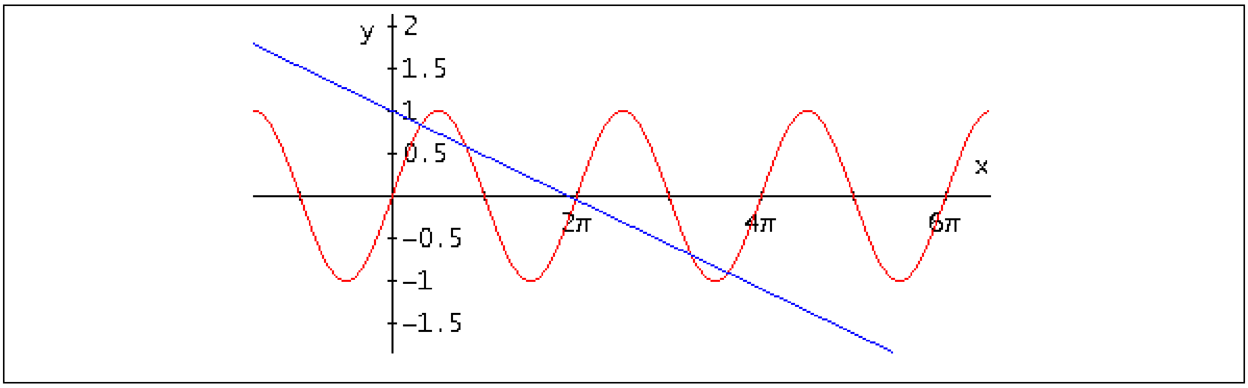

Figure 1.16

In Derive, there has never been something like a "*RootOf*" structure:

#23: $\mathrm{SOLVE}\left(\mathrm{SIN}(x) = 1 - \frac{x}{6},\ x\right)$

#24: $\mathrm{SIN}(x) + \frac{x}{6} = 1$

#25: $\mathrm{NSOLVE}\left(\mathrm{SIN}(x) = 1 - \frac{x}{6},\ x\right)$

#26: $x = 6.242720798$

#27: $\mathrm{NSOLVE}\left(\mathrm{SIN}(x) = 1 - \frac{x}{6},\ x,\ 0,\ 3\right)$

#28: $x = 0.9885669796$

#29: $\mathrm{NSOLVE}\left(\mathrm{SIN}(x) = 1 - \frac{x}{6},\ x,\ 1,\ 3\right)$

#30: $x = 2.523600393$

Figure 1.17

1.10 Polynomial systems

Such systems are solvable using an algorithm (Gröbner basis) fortunately implemented in symbolic systems.

1.11 The function LambertW

This is probably one of the best recent implementations of a mathematical function (2 of the authors were at the TIME-2004 symposium held at ÉTS in July 2004. Many other developers also came to the ACA 2009conference also held at ÉTS in June 2009 as well as to ACA 2019 in July 2019). Let's start with the following example that leads us well on the track of this special function!

1.11.1 Example

We are looking for all real and complex solutions of the equation

$$2^x = x^6.$$

A graph shows two intersections on either side of the origin but a third real (and positive) solution necessarily exists since the exponential function 2^x will eventually dominate the power function x^6. Seen otherwise, since this solution is positive, we can transform the original equation into

$$\frac{\ln x}{x} = \frac{\ln 2}{6}$$

and since function

$$\frac{\ln x}{x}$$

tends to 0 when x tends to infinity, a third solution exists. It can be easily located by making a table of values for example. This is what we would find:

#34:	$\mathrm{NSOLVE}(2^{x} = x^{6},\ x)$	
#35:		$x = -0.9011325528$
#36:	$\mathrm{NSOLVE}(2^{x} = x^{6},\ x,\ 0,\ 2)$	
#37:		$x = 1.140879647$
#38:	$\mathrm{NSOLVE}(2^{x} = x^{6},\ x,\ 2,\ \infty)$	
#39:		$x = 29.21048705$

Figure 1.18

In order to visualize complex solutions, we apply the method introduced in 1.4.2.

Figure 1.19 shows the following. We separate real part and imaginary part. The graphs of the two respective implicit curves were then plotted in the same window. In the neighborhood of the origin we can see four complex solutions in addition to the two real ones close to 0. These solutions can be found numerically soon.

#40:	$\mathrm{RE}(2^{x + i\cdot y} = (x + i\cdot y)^{6})$	
#41:		$2^{x}\cdot\mathrm{COS}(y\cdot\mathrm{LN}(2)) = (x^{2} - y^{2})\cdot(x^{4} - 14\cdot x^{2}\cdot y^{2} + y^{4})$
#42:	$\mathrm{IM}(2^{x + i\cdot y} = (x + i\cdot y)^{6})$	
#43:		$2^{x}\cdot\mathrm{SIN}(y\cdot\mathrm{LN}(2)) = 2\cdot x\cdot y\cdot(3\cdot x^{4} - 10\cdot x^{2}\cdot y^{2} + 3\cdot y^{4})$

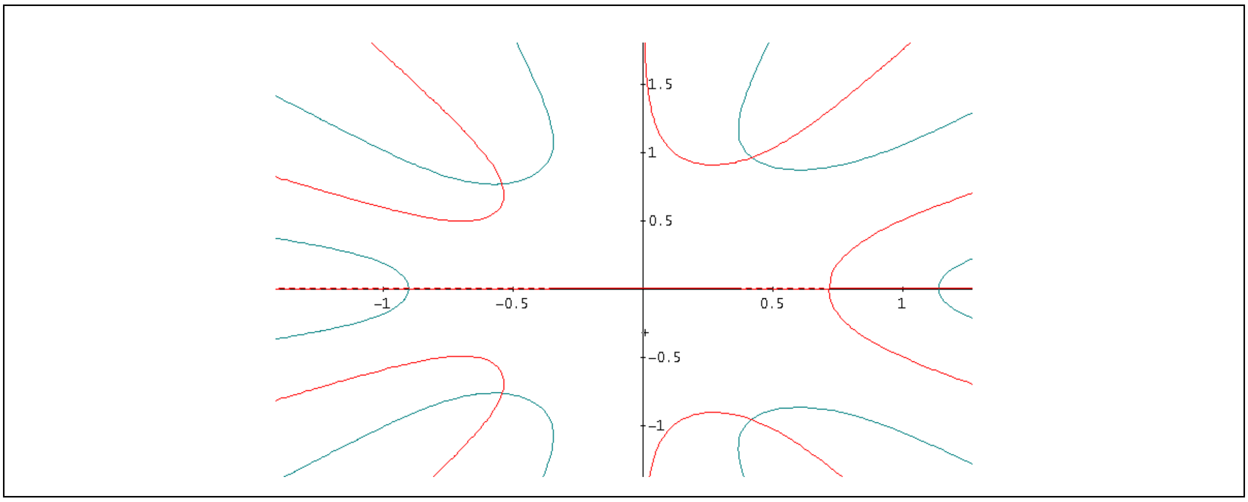

Figure 1.19

How to explain what *Maple* does with the same equation? Its *Solver* uses a famous function that seems to have been invented to solve equations like $z\,e^z = w$. In order to understand what is going on, let's assume that α designates one of the six sixth roots of 1, so we have to solve the equation $\alpha\,2^{x/6} = x$ which is equivalent to the following:

$$\alpha e^{\frac{x\ln(2)}{6}} = x \Leftrightarrow xe^{\frac{-x\ln(2)}{6}} = \alpha \Leftrightarrow -\frac{x\ln(2)}{6}e^{\frac{-x\ln(2)}{6}} = -\alpha\frac{\ln(2)}{6}\,.$$

So we have to solve an equation of type $X\,e^X = Y$ lösen. In our example: $X = -\dfrac{x\ln(2)}{6}$ and $Y = -\dfrac{\alpha\ln(2)}{6}$ with $\alpha \in \left\{\pm 1, \pm\frac{1}{2} \pm i\frac{\sqrt{3}}{2}\right\}$.

Note that the function $x \mapsto x e^x$ is not bijective but will become so by considering the two parts on either side of its absolute minimum: by inverting, we obtain two branches denoted by $W_0(x)$ ("main branch") and $W_{-1}(x)$ as shown in Figure 1.20. Only these two branches provide real values (although they also give complex values when $x < -e^{-1}$). The other branches W_k for integer k always give complex values.

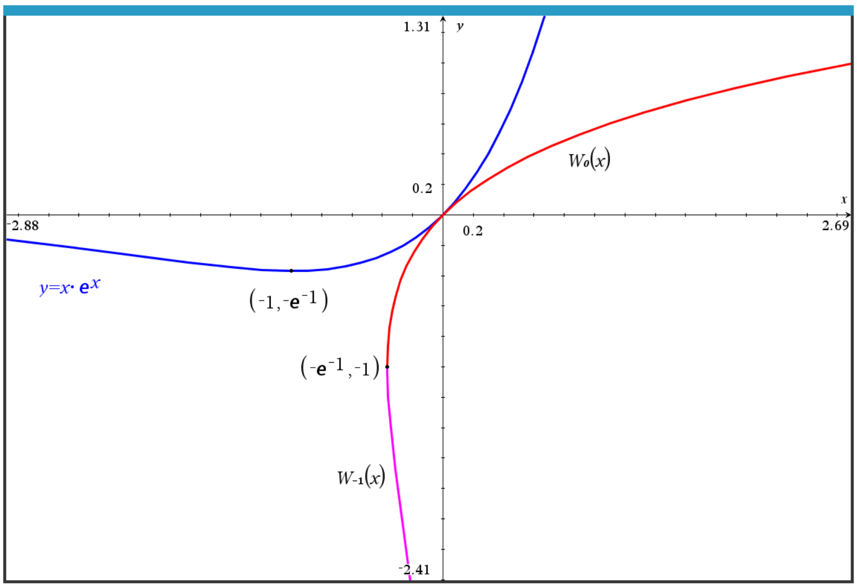

Figure 1.20

The different branches of the LambertW function are separated as follows. The curve (given here in parametric form)

$$\begin{cases} x = -t\cot t \\ y = t \end{cases}, \quad -\pi < t < \pi$$

to which we add the point (1, 0) separates the main branch $W_0(x)$ from the two branches $W_1(x)$ and $W_{-1}(x)$.). The interval $]-\infty, -1]$ separates the branches $W_1(x)$ and $W_{-1}(x)$. Finally, the other branches are separated by the curves

$$\begin{cases} x = -t\cot t \\ y = t \end{cases}, \quad 2k\pi < \pm t < (2k+1)\pi, \quad k = 1, 2, 3, \cdots$$

The *LambertW* function is therefore "multiform" and is therefore the reciprocal function of $y = x e^x$. We can now continue our example. We can now continue our example.

So, the solutions of the equation $2^x = x^6$ are $x = -\dfrac{6}{\ln(2)}\mathrm{Lambert}W\left(k, -\dfrac{\alpha \ln(2)}{6}\right)$. *Maple* gives:

> $solve(2^x = x^6, x);$

$$-\frac{6\,\mathrm{LambertW}\left(\frac{1}{6}\ln(2)\right)}{\ln(2)}, -\frac{6\,\mathrm{LambertW}\left(-\frac{1}{6}\ln(2)\right)}{\ln(2)},$$

$$-\frac{6\,\mathrm{LambertW}\left(-1, -\frac{1}{6}\ln(2)\right)}{\ln(2)},$$

$$-\frac{6\,\mathrm{LambertW}\left(-\frac{1}{6}\ln(2)\left(\frac{1}{2} - \frac{1}{2}\,\mathrm{I}\sqrt{3}\right)\right)}{\ln(2)},$$

$$-\frac{6\,\mathrm{LambertW}\left(-\frac{1}{6}\ln(2)\left(\frac{1}{2} + \frac{1}{2}\,\mathrm{I}\sqrt{3}\right)\right)}{\ln(2)},$$

$$-\frac{6\,\mathrm{LambertW}\left(-\frac{1}{6}\ln(2)\left(-\frac{1}{2} - \frac{1}{2}\,\mathrm{I}\sqrt{3}\right)\right)}{\ln(2)},$$

$$-\frac{6\,\mathrm{LambertW}\left(-\frac{1}{6}\ln(2)\left(-\frac{1}{2} + \frac{1}{2}\,\mathrm{I}\sqrt{3}\right)\right)}{\ln(2)}$$

$$W(z)e^{W(z)} = z \qquad y e^y = z \Leftrightarrow y = W_k(z)$$

http://www.orcca.on.ca/LambertW/

Figure 1.21

When we program this function with its different branches, we can then evaluate it. We did this it in the kit_ets_mb library. One finds it via its function "*ProductLog*":in *Maple* and in *Mathematica* as well:

$$\mathbf{ra\_0}(\alpha):=\frac{-6}{\ln(2)}\cdot \mathbf{kit\_ets\_mb\backslash lambertw}\left(0,\frac{-\alpha\cdot\ln(2)}{6}\right) \blacktriangleright \text{Terminé}$$

$$\left\{\mathbf{ra\_0}(1),\mathbf{ra\_0}(-1),\mathbf{ra\_0}\left(\frac{1}{2}+\frac{i\cdot\sqrt{3}}{2}\right),\mathbf{ra\_0}\left(\frac{1}{2}-\frac{i\cdot\sqrt{3}}{2}\right),\mathbf{ra\_0}\left(\frac{-1}{2}+\frac{i\cdot\sqrt{3}}{2}\right),\mathbf{ra\_0}\left(\frac{-1}{2}-\frac{i\cdot\sqrt{3}}{2}\right)\right\}$$

$$\blacktriangleright \{1.14088,-0.901133,0.420866+0.961768\cdot i,0.420866-0.961768\cdot i,-0.540072+0.768772\cdot i,-0.540072-0.768772\cdot i\}$$

$$\mathbf{ra\_1}(\alpha):=\frac{-6}{\ln(2)}\cdot \mathbf{kit\_ets\_mb\backslash lambertw}\left(-1,\frac{-\alpha\cdot\ln(2)}{6}\right) \blacktriangleright \textit{Terminé}$$

$$\mathbf{ra\_1}(1) \blacktriangleright 29.2105$$

Figure 1.22

We can even add several other complex solutions (which can be seen by enlarging the window given by *Derive* in Figure 1.19).

$$\left\{\mathbf{ra\_1}(-1),\mathbf{ra\_1}(1),\mathbf{ra\_1}\left(\frac{1}{2}+\frac{i\cdot\sqrt{3}}{2}\right),\mathbf{ra\_1}\left(\frac{1}{2}-\frac{i\cdot\sqrt{3}}{2}\right),\mathbf{ra\_1}\left(\frac{-1}{2}+\frac{i\cdot\sqrt{3}}{2}\right),\mathbf{ra\_1}\left(\frac{-1}{2}-\frac{i\cdot\sqrt{3}}{2}\right)\right\}$$

$$\blacktriangleright \{33.4659+34.0702\cdot i,29.2105,30.1715-12.4531\cdot i,30.1715+12.4531\cdot i,34.8737+44.0603\cdot i,31.8633+23.6579\cdot i\}$$

$$\mathbf{ra\_2}(\alpha):=\frac{-6}{\ln(2)}\cdot \mathbf{kit\_ets\_mb\backslash lambertw}\left(2,\frac{-\alpha\cdot\ln(2)}{6}\right) \blacktriangleright \textit{Terminé}$$

$$\left\{\mathbf{ra\_2}(-1),\mathbf{ra\_1}(1),\mathbf{ra\_2}\left(\frac{1}{2}+\frac{i\cdot\sqrt{3}}{2}\right),\mathbf{ra\_2}\left(\frac{1}{2}-\frac{i\cdot\sqrt{3}}{2}\right),\mathbf{ra\_2}\left(\frac{-1}{2}+\frac{i\cdot\sqrt{3}}{2}\right),\mathbf{ra\_2}\left(\frac{-1}{2}-\frac{i\cdot\sqrt{3}}{2}\right)\right\}$$

$$\blacktriangleright \{39.8563-91.628\cdot i,29.2105,42.4937-128.677\cdot i,41.2725-110.207\cdot i,39.0541-82.2787\cdot i,40.5925-100.934\cdot i\}$$

Figure 1.23

1.11.2 Simplifications using LambertW

If we stick to solving equations, then any equation reducible to the form $ye^y = z$, where z is a given (possibly complex) number and where y is the unknown to be found, can be solved by taking the LambertW function on each side of this equation. In other words, using "W" to denote any branch of index "k" of the Lambert function $W(k, z)$, then we have

$$W(ye^y) = y.$$

In particular one has (what was noticed besides from the figure 1.21)

$$e^{W(z)} = \frac{z}{W(z)}.$$

1.11.3 Example

It is easy to find by eye the solutions 2 and 4 of the equation $x^2 = 2^x$. The third real solution (as well as 2 and 4), as well as all complex solutions are easy to find via LambertW. Indeed, the equation is equivalent to $x = \pm e^{x\ln(2)/2}$ which in turn is equal to

$$-x\frac{\ln 2}{2}e^{-x\ln(2)/2} = \pm\frac{\ln 2}{2}.$$

But then again $-x = -\frac{2}{\ln 2}W\left(\pm\frac{\ln 2}{2}\right)$ where "W" denotes a LambertW branch. Let us note that

$-\frac{\ln 2}{2} > -\frac{1}{e}$ and therefore the branches k = 0 and k = -1 will be used to evaluate the value of

$W\left(-\frac{\ln 2}{2}\right)$ and the branch $k = 0$ will be used to evaluate $W\left(\frac{\ln 2}{2}\right)$. The real solutions are thus found among the 3 following values:

$$-\frac{2}{\ln 2}W\left(0,\frac{\ln 2}{2}\right), -\frac{2}{\ln 2}W\left(0,\frac{-\ln 2}{2}\right) \text{ and } -\frac{2}{\ln 2}W\left(-1,\frac{-\ln 2}{2}\right).$$

The first answer gives 0.766665... . Now let's note: in $W(y\, e^y)$ we replace y by – ln(x) and using the fact that for non-zero x we always have exp(ln(x)) = x - and thus exp(-ln(x)) = 1/x -, we find that $W\left(-\frac{\ln x}{x}\right) = -\ln x.$ But then the image of the k = 0 branch consisting of the real numbers ≥ –1 and since –ln(2) > –1, we can write

$$-\frac{2}{\ln 2}W\left(0,\frac{-\ln 2}{2}\right) = -\frac{2}{\ln 2}\cdot(-\ln 2) = 2.$$

An because the image of the branch k = –1 consists of the real numbers ≤ –1 and –ln 4 < –1, we can write

$$-\frac{2}{\ln 2}W\left(-1,\frac{-\ln 2}{2}\right) = -\frac{2}{\ln 2}W\left(-1,\frac{-2\ln 2}{4}\right) = -\frac{2}{\ln 2}W\left(-1,\frac{-\ln 4}{4}\right) = -\frac{2}{\ln 2}\cdot(-\ln 4) = 4.$$

1.11.4 Example

In if we now replace y by ln(x) in $W\left(y e^y\right) = y$, and still use the fact that for non-zero x we always have exp(ln(x)) = x, we find that $W(x\ln x) = \ln x.$ This shows that the (positive) solution to the equation $x^{x^3} = 3$ is $\sqrt[3]{3}.$ Indeed, a calculation shows that we end up with $x^3 \ln x = \ln 3,$ hence $x^3 \ln\left(x^3\right) = 3\ln 3$, hence $W(x^3\ln(x^3)) = W(3\ln 3) = \ln 3$ and thus $\ln\left(x^3\right) = \ln 3$ and finally $x = \sqrt[3]{3}.$

1.12 Remark and definition

In the case of numerically given solutions, which "magic" methods do symbolic systems use? Probably, it is a "mix" of different methods.

Two of them we will present now. The first one is the fixed-point method and the second one is Newton's method. Each of these methods will be extended to systems of equations and the use of a calculator is very useful, even necessary here.

A function g is said to have a fixed point if there exists a number r such that g(r) = r. If, moreover, function g is differentiable, we will say that this fixed point is an attractor if $|g'(r)| < 1.$

Thus, it is easy to find that the fixed points of the function

$$g(x) = \frac{x(4+x^2)}{1+4x^2}$$

are 0, 1 and –1, and that 0 is not an attractor. Then we can then ask ourselves if, by iterating $g(x)$, it will converge or not? Thus, starting "close" to 0, say 0.2, we would find that it converges but not to 0 but rather to 1: the "ITERATES" function of *Derive* has an optional last argument: Precision is limited to 14 digits in *TI-Nspire* (we see in figure 1.25 the display at "float 6")::

"ITERATES" is implemented in *Derive*:

```
#1:   g(x) := x·(4 + x^2) / (1 + 4·x^2)

#2:   ITERATES(g(x), x, 0.2, 10)

#3:   [0.2, 0.6965517241, 1.062374033, 0.9880277022, 1.002411363, 0.9995184273, 1.000096342, 0.9999807326, 1.000003853,
        0.9999992292, 1.000000154]

#4:   ITERATES(g(x), x, 0.2)

#5:   [0.2, 0.6965517241, 1.062374033, 0.9880277022, 1.002411363, 0.9995184273, 1.000096342, 0.9999807326, 1.000003853,
        0.9999992292, 1.000000154, 1, 1]
```

Figure 1.24

"iterates" is part of the kit_ets_mb library for *TI-Nspire*:

$$g(x):=\frac{x\cdot\left(4+x^2\right)}{1+4\cdot x^2}$$ *Terminé*

kit_ets_mb\iterates$(g(x),x,0.2,10)$

$[0.2 \quad 0.696552 \quad 1.06237 \quad 0.988028 \quad 1.00241 \quad 0.999518 \quad 1.0001 \quad 0.999981 \quad 1. \quad 0.999999 \quad 1.]$

Figure 1.25

To be continued