ELEMENTARIZATION AND MODULARIZATION - TWO DIDACTICAL AIMS BEING REALIZED BY USING COMPUTERALGEBRA SYSTEMS

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It is obvious to use computeralgebra systems (CAS) as a calculating aid for dedicating tasks as solving equations, computing limits and derivatives, etc. to the computer. However, CAS can also be used for introducing new mathematical concepts, e.g., solving problems for changing rates by computing sequences of differences is much more elementary than using derivatives. So CAS are a valuable help for introducing and elementarizing new mathematical concepts. Using CAS students have the possibility to define functions in a very similar way as they were used in regular maths lessons or textbooks. So CAS are an important instrument for modularizing maths teaching.

1. Introduction

We report about mathematics courses in a class with 12 girls and 3 boys at the age of 16 in the years from 1995 to 1998 at the Stiftsgymnasium Wilhering, which is a private High School near Linz in Austria. The emphasis of the school and the interests of the students are in the study of languages and arts. Because the students are not intrinsically motivated in mathematics, it is our aim, to make new mathematical concepts easy to understand for them by treating and computing them in a very elementary way. All students use a TI-92, which is a pocket calculator for symbolic and numerical manipulations and for graphical visualizations. The students use the TI-92 during maths classes, at home for doing exercises and during maths tests.

2. Elementarization

In the following examples we demonstrate how one can use a computeralgebra system (CAS) for introducing new mathematical contents. The experiences we will describe are made with the TI-92, however the didactical concepts could be transferred to other CAS products.

Exponential functions

In traditional maths courses exponential functions are introduced as functions of the type a^x. Most of the time is spent for proving and training transformation rules or for solving exponential equations. There is not much time for the students to learn about the meaning of a and in which situations exponential functions are necessary for modelling a process and how exponential functions differ from other types of functions.

Using CAS we can concentrate on the modelling process. Dedious and complicated computations can be dedicated to the CAS. Furthermore the modelling process is supported by the possibility of defining sequences recursively. Consider the following example of a population of 1000 individuals with a procedual growing rate of 5 % per year. The growing process of this population can be described recursively as follows: $P_n = P_{n-1} + P_{n-1} \cdot 0.05$ with $P_0 = 1000$. In this recursive description the amount of absolute growth $P_{n-1} \cdot 0.05$ can be easily inspected. It depends on the number of individuals in the past year. This is a characteristic difference of exponential growth to other growing processes, e.g. linear growth where the amount of absolute growth is constant.

In contrast to the closed form $P_n = 1000 \cdot 1.05^n$ of exponential processes the recursive definitions only require addition/subtraction and multiplication/division for modelling. This is very helpful for the students, because they are familiar with these elmentary operations. So it is easier for them to understand the models or to do the modelling process by themselves ([ASPETSBERGER, FUCHS 1996], [SCHNEIDER 1998], [WURNIG 1996]).

Differentialquotient

In traditional maths courses most of the teachers do not use the definition $\lim_{z \to x} \frac{f(x)-f(z)}{x-z}$ for differential quotients for solving problems. It is due to the difficulties occuring by the computation of the limit. Instead of this it is easier to find derivatives. However derivatives are not so illustrative as differential quotients.

Using CAS we can stepwise introduce the definition of differential quotients by generating and analyzing sequences of difference quotients. For solving problems we can use the definition above, because the computation of the limit is dedicated to the CAS. So the process of changing rates is always visual for the students.

In the following we present an example for a stepwise introduction of differential quotients [ASPETSBERGER 1997] treating the problem of velocity [FINNEY, THOMAS, DEMANA, WAITS 1994]).

A rock is thrown straight up with a launch velocity of 64 m/sec. It reaches a height of $s(t) = 64t - 5t^2$ m after t seconds.

a) Compute the average velocity of the rock within the first two seconds.

After entering the definition for the height *s(x)* of the rock, where x denotes the time past since the shooting of the rock, the students can easily plot the graph of the function for a first inspection. For computing the average velocity of the rock for the first two seconds we enter the expression $\frac{s(2)-s(0)}{2}$ (see figure 1).

b) Compute the instantaneous velocity after 2 seconds.

For determing the instantaneous velocity after 2 seconds we compute the average velocity for the time intervals [1;2], [1.5;2], [1.9;2] and so on (see fig. 1). This can be managed easily by substituting in the general formula $\frac{s(2)-s(z)}{2-z}$ for z the starting values of the time intervals. The students can observe, that the average velocities converge to 44. However, they cannot substitute for $z = 2$.

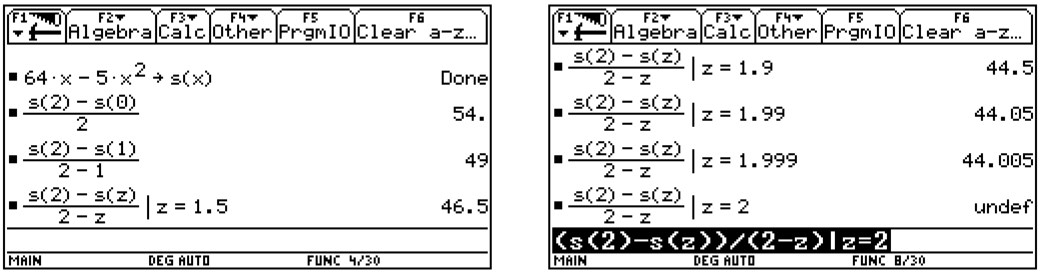


Fig. 1: Generating sequences of difference quotients

For computing the limit of $\frac{s(2)-s(z)}{2-z}$ as z approaches 2 we use the command `limit`. At this stage the students have only an intuitive impression of limits, however we can use the TI-92 for computing the limit. We use the command `limit` as a black box, an exact definition is given afterwards (see for the black box principle [HEUGL, KLINGER, LECHNER 1996]).

c) How high does the rock go and when does it reach its highest point?

On its highest point the velocity of the rock is zero. However we do not know, when this will happen. So we can compute the instantaneous velocity of the rock at various times for instance t = 1, 2, 4, 5 and so on. This leads us to generalize the problem and to compute $\lim\limits_{z \to t} \frac{s(t)-s(z)}{t-z}$ obtaining a general expression $-2 \cdot (5 \cdot t - 32)$ for the velocity of the rock. If we store the expression to the function $v(t)$ we can easily compute the velocity at any time via a function call. This is a very natural introduction of the concept of derivatives, because students understand now why generalization is necessary. By solving the equation $v(t) = 0$ we find out when the rock reaches its highest point. The maximal height can be computed by the function call $s(32/5)$.

Now the students can solve many problems from various application fields using the definition of the differential quotient directly, dedicating the computation of the limit to the CAS.

Integrals

In traditional maths courses many teachers choose the concept of antiderivatives for introducing integrals. They do not use upper and lower rectangular sums because the computation of sums are rather time consuming and the determination of closed forms for sums are quite difficult. However the concept of antiderivatives is not as illustrative as the concept of Riemann sums and the students do not see why antiderivatives are suitable to solve a certain problem. Most of time is spent for computing and training of special integration techniques by hand instead of concentrating on the modelling process for various application areas. So they restrict to problems of computing the area between curves.

Using CAS we can solve problems using upper and lower rectangular sums quite conveniently dedicating the dedious work and determining closed forms to the computer. During the whole process the students have only to concentrate on modelling. An introduction of integrals via Riemann sums on the TI-92 is described in [ASPETSBERGER 1998]. Starting with simple sums of products the students are guided to the general description of Riemann sums and finally to integral functions.

Solving integral problems via Riemann sums with CAS make the process of constructing sums always visible for the students. This is a good training for modelling integrands. Sums are much more elementary than antiderivatives. So the students have a chance to understand what happens. The students do not have to take care of how to compute difficult integrals, so they can solve various problems from different application areas. The computation of the area between graphs is only one special application of the integral concept.

Experiences

By the use of a CAS/TI-92 it is possible to solve problems in a very elementary way and new mathematical concepts can be introduced stepwise. It is easier for students to understand the meaning of a new concept when it is introduced elementary.

However introducing new concepts very elementary may lead to incorrect generalizations. An exact formularion and a theoretical treatment of the new concepts are still not superflous.

One can start with elementary problems from well known application areas for introducing new maths concepts and continue with a phase of exactification.

For solving problems the students use various methods in different representation modes. Experimental attempts are preferred to algebraic methods.

3. Modularization

Using CAS students have the possibility to define functions in a very similar way as they were used in regular maths lessons or textbooks. During problem solving the students can subdivide a problem in a several number of subproblems that can be solved independently. For solving the subproblems the students can use functions which are already implemented in the CAS or define new functions by themselves, generating a set of modules to be used in standard situations. So the use of functions is an important aid for structured problem solving. Again the use of modules by hand is quite laborious and the advantages of functions can be seen hardly by students. So CAS are an important instrument for modularizing maths teaching.

Using functions in maths lessons we have to distinguish between functions which have been defined by the students themselves and functions which are already implemented on the CAS or have been generated by the teacher ([ASPETSBERGER 1996], [HEUGL 1998]). In the second two cases the definitions of the functions are quite complicated and the students have only to know the meaning of the functions and what they can compute using these functions. They use these functions as "black boxes".

Defining functions by the students is typically done in two phases. In the first phase a new problem is solved by hand or by using already well known functions. This phase is called a white box phase, because the students see all the elementary operations and functions which are necessary to solve the problem. Having the problem systematically analysed the students define a new function that solves the problem in a single step. Now the students solve the problem via a function call and do not see the definition anymore. We call this phase a black box phase. The black boxes now are different from the "black boxes" described before, because now the students (hopefully) know the meaning *and* the definition of the functions in principle. The white-box/black-box principle was formulated in [BUCHBERGER 1989] and was described in detail in [HEUGL, KLINGER, LECHNER 1996].

We must be aware of the circumstance that this process is not static in the sense that after a sufficient amount of handcalculations or examples solved (white box phase) we can use a function always as a black box at a later time. Students forget the meaning of functions or the correct syntax and data types of the arguments. For this reason it is necessary to give short repetitions and explanations permanently. So the white box/black box principle must be seen dynamically. A further advantage of the permanent repetitions is that students who did not understand the meaning of a function or how a certain problem should be solved still have a chance of understanding the meaning of the respective function.

It was not our intention to define very complicated functions. The process of defining functions by the use of other functions was very abstract for some students. Thus, we restricted to few and quite easy user-defined functions. Most of our functions were an abbreviation of a long expression, e.g. how to calculate the angle between two arrays, which is quite complicated and tedious to be entered however easy to be understood

$$angle(a,b) = \cos^{-1}\left(\frac{dotP(a,b)}{norm(a)\cdot norm(b)}\right).$$

In a further example we treat Bernoulli trials [ASPETSBERGER 1998b]:

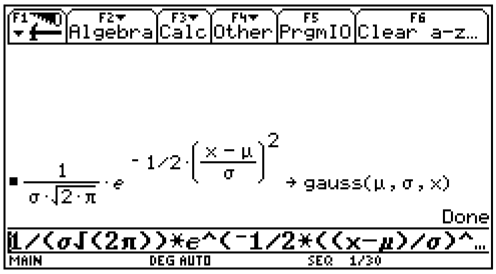
We are investigating experiments repeating n independent trials with exactly two outcomes. The probabilily of each outcome is exactly the same for each trial. The probability that an event will occur on any trial is p. The probability that an event will occur exactly k times on n trials is given by

$$\mathrm{P}(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$$

Working on binomial distributions we use the TI-92 as a calculating aid for computing the binomial coefficients $\binom{n}{k}$. Using the internal function nCr(n,k) of the TI-92 we can compute the binomial coefficient. For determining special values of the binomial distribution function $\mathrm{P}(X=k) = \binom{n}{k} p^k(1-p)^{n-k}$ the students define a simple function bin(n,p,k) = nCr(n,k)*p^k*(1-p)^(n-k). This function can be used for defining a new function binom(n,p,a,b) = Σ(bin(n,p,k),k,a,b) for computing probabilities like $\mathrm{P}(a \le X \le b)$. Solving typical problems within Bernoulli experiments the students can concentrate on modelling the problems and determining important parameters. The computations of the special probabilities like $\mathrm{P}(5 \le X \le 12)$ for a binomial distribution with $n = 20$ and $p = 0.4$ are delegated to the TI-92 via a function call binom(20,0.4,5,12). The tedious tasks of determining certain values of the binomial density function require most of the time in traditional maths courses.

Problems also occured due to the circumstance that CAS or the TI-92 in special are quite intolerant if the students used wrong function names or applied functions to wrong data types for the arguments. The error messages often could not be interpreted correctly. It was really difficult to find errors caused by wrong function definitions, e.g. defining a function gauss describing the density function for normal distributions

$$gauss(\mu,\sigma,x)=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$



problems occured that were quite difficult to be interpreted when using the variable e instead of the special exponential function *e* [ASPETSBERGER 1998b].

In general there is a new chance for functions to be accepted by the students. Using CAS or programmable pocket calculators functions can be seen as an aid for evaluations complicated expressions. In traditional maths courses the students have to evaluate these long expressions by hand by themselves and they do not see the advantages of the use of functions. It is also a step to sturctured problem solving.

4. Experiences

On the TI-92 three different representation forms (algebraic, numeric and graphic) are always available. So the students can solve a problem by computing tables and searching for certain values or by plotting and analyzing graphs. Many problems can be solved numerically with tables and the students do not need to determine a closed form of the algebraic description of the problem. The students have the possibility to choose a representation they like most e.g. for solving problems, for illustration or to get an overview in a certain situation. The permanent change of representation forms can be done by hand too. However the simple realization on the TI-92 is essential for a practical use [PESCHEK 1998]. Experiments with students pointed out, that students rather use tables and graphs than algebraic methods for solving problems. The abstractness of expressions is a major handicap in traditional maths courses when introducing new mathematical concepts.

The CAS is able to handle all the computing problems. It is not neccessary to find tricky ways for solving problems. Introducing new concepts we can start with very elementary and - due to that reason - very illustrative methods. For instance, we solved most problems of calculus using the limit of the quotient of differences. Therefore my students got a better understanding of the concept of a differential quotient and of derivatives. The problem of computing the limits was dedicated to the computer.

The possibility of recovering mathematical contents experimentally is very motivating for many students. The use of a computer gives many opportunities for experiments. However, experiments are quite time consuming and some students prefer traditional methods, because they are more convenient for them.

Using a CAS or the TI-92 we did not save time in the maths courses. There may be two reasons. First we spent much more time for introducing new mathematical concepts. The students tried to solve starting problems with elementary methods to obtain a better understanding for the new type of problems being treated. Secondly, we did not train special transformation rules for e.g. calculus. However it took a lot of time to learn the special techniques being required for an intensive use of the different windows on the TI-92 (home, graph, table, etc.).

The students defined many functions during three years. At the end of this time when the students have to apply all these functions to complex problems it was difficult for the students to remember the right name of the functions or to apply the functions to correct data types for the arguments.

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