

Give it a Spin!

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One of the MFAQ (= Most Frequent Asked Questions) after workshops or lectures on the usage of modern technologies like the TI-family, *DERIVE*, MS-EXCEL or others for maths education in Secondary Schools is how to find problems and tasks for exercises and assessment. Many of the more experienced teachers have a collection of problems of their own and it is very understandable that they don't want to throw all their treasures away and start again like at the very begin of their career as a teacher.

I was and still am in the same situation. My standard answer is: you don't have to change everything, take your problem collection and try to achieve another point of view. Then in many cases you will be able to give the problems a spin into the right direction and thus they will adopt a new quality, focus on another aim as usual, open new insights or just let it read a bit uncommon.

Please don't expect results of a deep scientific research on that topic within the next short hour and don't expect any classification of how to give "special spins". I am an ordinary school teacher on a Secondary College for Business Administration in St.Pölten, the capital town of Lower Austria. I also have been busy in in- and pre-service courses for teachers and give a lecture on the use of modern technologies in maths education on the Vienna Technical University. So you can imagine that I am very deep involved in the discussion mentioned above.

On my school we teach mathematics for four years - age 16 - 19, and I decided to follow these four years and selected problems from my recently given written and oral examinations - unchanged for this presentation. I hope to give some insight into the daily work of a school teacher who has been using *DERIVE* and the TI since long. Most of the examples presented here are software independent. I choose the TI, because it proved to be the easiest platform for presentations using the ViewScreen.

When we start using the PC or hand held technologies in maths education we have to make the students accustomed to the machines as quick as possible. I do the same what I did years ago with ordinary pocket calculators: evaluate more complicated numerical expressions and learn them to differ between decimal digits and significant digits. But now the students not only note the results but also learn to compare the given problem with its output on the screen.

$\frac{\sqrt{345,45 + 12,3^2}}{0,234 \cdot 456,4} =$	(4 significant digits)	
$28,1^2 \cdot \left(\sqrt[4]{0,75} + \frac{305,9}{417} \right) =$	(1 decimal digit)	
$\sqrt[4]{\frac{0,9842^3 + 5,32 \cdot 11,924}{\sqrt[3]{0,00015}}} =$	(2 significant digits)	

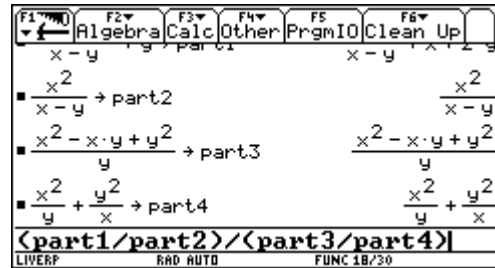
By the way the students do not only practise linearizing the expressions, but also get accustomed with fraction powers for roots and floating point notation.

Then we work with basics in algebra manipulations like working with fractions, binomials and polynomials, solving linear equations etc.

I don't ask my pupils to evaluate a fractional monster like the next one by hands - although we sometimes do it in class room - in an assessment, but they shall be able to edit this expression applying various strategies:

$$\frac{x^2 + y^2}{x - y} + y \quad \frac{x^2 - xy + y^2}{y}$$

$$\frac{x^2}{x - y} : \frac{x^2}{y} + \frac{y^2}{x}$$



One can try to do it in one step or the better choice might be to split the expression and first edit moduls to assemble them at last to obtain the result.

Given are some elements of a binomial's power. Complete and find the task:

$x^4 - 16x^3 + 96x^2 - \dots$

$a^6b^3 + 6a^4b^5 \dots = (\dots)^3$

$\dots - 180x^3z \dots = (\dots)^2$ give more than one solution!!

$729x^{12} - 5832x^{10}y + 19440x^8y^2 \dots =$

(in cooperation with my technology - co-fighter, Tania Koller from my school, the BHAK St.Pölten)

CAS might help to check the conjectures, to factorize larger numbers. We can see that there are problems which don't have a unique solution. We also can add problems which cannot have any solution. Students and teachers as well mostly are fixed to have "the" solution. We should accept that also mathematical problems modelling real life don't have "the" solution in each case.

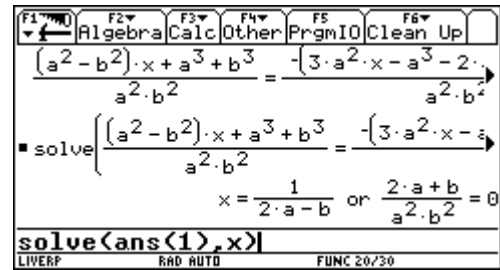
We'll change to linear equations. Combine testing manual calculation skills with responding on new questions.

$$\frac{b-x}{a^2} + \frac{a+x}{b^2} = \frac{b}{a^2} - \frac{3x-a}{b^2} + \frac{b+2a}{a^2b^2}$$

- Enter the equation and explain the TI's "simplification".
- What is the equation's general solution and which conditions are necessary for its validity?
- Find two special solutions!
- What is the TI's output?
- Explain the output's second part using a self chosen example.
- Can you explain where this strange second part is coming from?

Plenty of new - very mathematical, and not too easy to answer - questions are offered only by inspecting the CAS' output. Let's make more use of this new possibilities.

See also the next example how to combine - indispensable? - basics and computer algebra.



$\frac{6}{x^3 + 4x^2 - 9x - 36} - \frac{2}{2x^3 + 19x^2 + 59x + 60} = \frac{9}{2x^3 + 5x^2 - 18x - 45}$
<p>a) You can use the TI for factorizing the denominators, then</p> <p>b) find the domain and</p> <p>c) solve the equation by hands.</p>

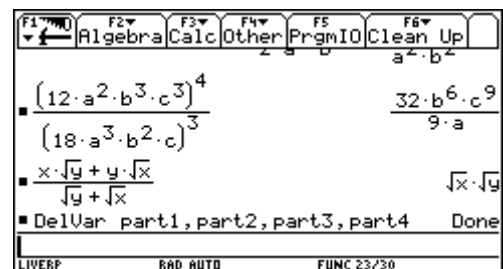
I don't know your experience, but in my teacher's career not learning but applying the power rules is one of the major problems for the students.

It is no problem to use also the support of a CAS to train these basics:

See an example from a "Basics' Test" given in my project class where we differed between basics- and problem solving tests.

A2)	Berechne mit dem TR und begründe das Ergebnis: $\frac{(12a^2 b^3 c^3)^4}{(18a^3 b^2 c)^3}$	(4 Pkte)
A3)	$\frac{x\sqrt{y} + y\sqrt{x}}{\sqrt{y} + \sqrt{x}} =$	Erkläre das Ergebnis des Rechners. (5 Pkte)

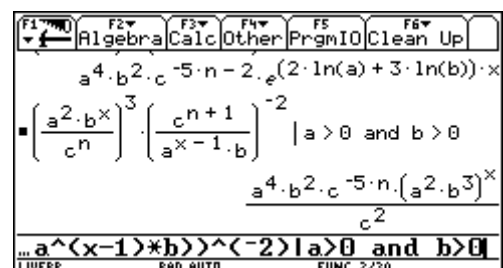
Translation: A2) Evaluate with the pocket calculator and give reasons for the result.
 A3) Give an explanation for the calculator's output.



Sometimes the calculator's results are looking very strange and so it is not necessary to forbid its use:

Collect the result and write it down without use of any denominator:

$$\left(\frac{a^2 b^x}{c^n}\right)^3 \left(\frac{c^{n+1}}{a^{x-1} b}\right)^{-2} =$$



I don't want to present here one more model for a growth or decay function, but working with logarithms only can be interesting enough:

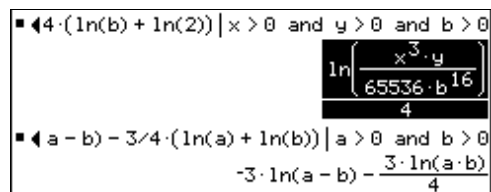
Use the TI: Expand($\ln(9000)$) =
Explain the outcome!

Apply the appropriate logarithmic rule to expand $\ln(b-4)^5$.
a) Which result do you expect?
b) How can you achieve this result with your TI?

Collect the logarithms by hands, try to do your best with the TI and compare the results, explain the different forms of the results.

$$\frac{1}{4}(3\log x + \log y) - 4(\log b + \log 2) =$$

$$-3\log(a - b) - \frac{3}{4}(\log a + \log b) =$$



We should take the chance and force our students to enhance graphical and numerical methods and make these two methods partners of analytical methods with equal rights.

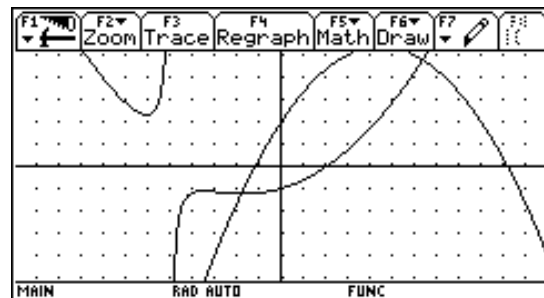
Solve this inequality graphically:

$$g(x) < f(x) \text{ over } \mathbb{N}.$$

For your information:

$$f(x) = -\frac{x^2}{6} + \frac{3x}{2} + 2$$

$$g(x) = \frac{x}{5x+25} + \frac{x^2}{10} + \frac{x}{4} - 1$$



At a test I gave the following simultaneous equations and expected the students to apply any one of the well known methods to solve the system or to calculate the system's determinant after having brought the variables into the right order. I had a nice surprise during the test:

Solve the simultaneous equations for x and y :

$$ax + by = a + 2y - 3x$$

$$2ax - 3by = b - 3y + 2x$$

Find at least two pairs (a,b) , which make the system non soluble!

A female student put the variables into the correct order and immediately noted $(-3,2)$ and $(1,1)$ as answer for the second question. I was surprised and asked her how she had got the answers.

She explained:

$$(a + 3)x + (b - 2)y = a$$

$$(2a - 2)x + (3 - 3b)y = b$$

Take equation 1: for $a = -3$ and $b = 2$ one gets $0 = -3$, which is obviously false and one gets another false statement from equation 2 for $a = 1$ and $b = 1$: $0 = 1$.

Trig equations sometimes need some tricky manipulations. Why not solve them graphically (or numerically using tables):

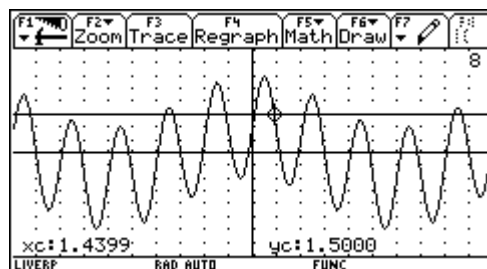
Equations containing trig functions are called "goniometric equations": Solving such equations needs a lot of tricks and can become very tiresome. Using the TI you can try to solve equations of this kind on various manners.

Find at least 4 solutions of

$$2 \sin 2x + \cos \frac{x}{3} = \frac{3}{2}$$

Explain your way of solution and give comments on the solutions. Work in radians but give at least one solution in degrees, too.

How many solutions are there? Can you express a general solution in mathematical terms? (or in words?)



Working with polynomial functions is daily work in school. Let's try again a graphic approach:

Plot the graph of $y(x) = 0.02x^3 - 0.13x^2 - 0.89x + 3.2$

- Translate the graph that it osculates the x -axis. Give the function term.
(one solution is sufficient, extra credits for more)
- Translate the graph that it passes the origin. Give the function term.
(one solution is sufficient, extra credits for more)

or another one:

Find a 4th degree polynomial function with

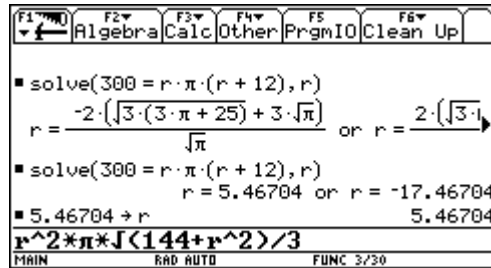
- a double zero at $x = -3$
- a double zero at $x = 4.5$
- one double zero at $x_1 = -2$ and another one at $x_2 = 2$.

Plot their graphs. Can you set up any conjecture? If so, then use two other polynomials of other degrees to confirm your conjecture.

The idea to problems like the next one came on one late evening when I prepared a test for my third form. We had repeated and extended the formulae for 2D- and 3D objects and this was part of the test. Next morning I showed this problem to my colleague and she really doubted if the pupils would be able to solve a problem like this having never seen before a similar one:

On the TI-screen of a classmate you see:

- a) What was the problem?
- b) Find the solution for your mate.



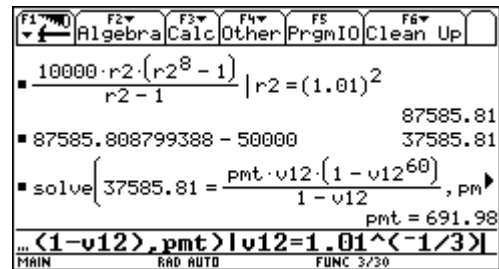
So I was very curious about the "success", and it worked. Surprisingly the weakest pupil was the first one to have the "correct" solution. Why "correct" under quotes? Look carefully! During the test one pupil raised her hand: "Sir, I think that there is something wrong! I am sure that this problem deals with a cone, but in the edit line under the root must be the expression for the cone's height, given by the sideline 12 and the radius. The application of Pythagoras' Theorem seems to be wrong!".

And indeed, as I mentioned before, I set up the test in the late evening and I made a mistake. Three out of 18 pupils found this mistake and earned an extra credit.

MariaLuise Müller, a colleague from another school - BHAK Waidhofen/Ybbs - saw this example at any occasion and tried to transfer the idea to a financial mathematics problem.

The task sounded very clear:

Tell the story of this business!



Back to geometry!

The pupils prepared an electronic collection of formulae on their TI-92 and were allowed to use it. It was very interesting for me to compare the various methods which appeared in the papers of the pupils by solving the next - very easy - problem.

Five spheres (R = 3.17 inch) made of brass are melted down and a cylinder with r = 6,34 inch is casted. What is the cylinder's length?

There were solutions which were completely performed by hands, using the TI-92 only as an old fashioned pocket calculator. Then some used the formulae together with the | - operator for substituting the variables. And fortunately enough there were also some who made use of the function's concept which I hoped them to acquire:

$$\text{solve}(5 * \text{sh}_v(3.17) = \text{cyl}_v(6.34, l), l)$$

In my opinion they not only worked with self provided tools, like the others - that is one aspect - but they also gave this easy problem a new quality in transforming the text into an equation using functions - with more than one variable! I tried to deepen this point of view in some other examples successfully later on.

Working with trig functions in the traditional way (eg problems from surveying technology) achieve a new quality very similar to the standard geometry problems by preparing a toolbox - a Black Box - and applying the appropriate tools. For solving oblique triangles we prepared four functions:

$$\text{sss}(s_1, s_2, s_3), \text{sas}(s_1, a_3, s_2), \text{asa}(a_1, s_3, a_2), \text{ssa}(s_1, s_2, a_1)$$

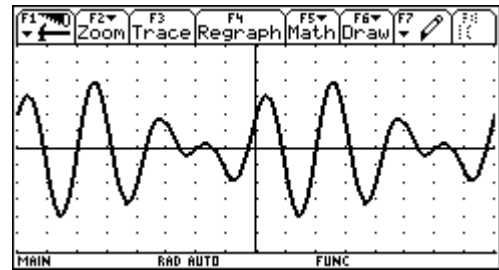
Five points ABCDE form a pentagon:
 AB = 264, BD = 996, AD = 1128, DC = 444,
 $\angle DEA = 122,48^\circ$, $\angle DCB = 73,25^\circ$, $\angle EAD = 33,10^\circ$
 What is the length of segment EC? What is the pentagon's area?

The students have to apply the right functions but also have to select which of the many partial results are important for the calculation. They are freed from boring calculations and solving equations emerging from sine- and cosine rule, but they carefully have to document the process how to reach the results.

The function term of the given graph shows the form

$$y(x) = \sin(ax) + \cos(bx) \text{ with } 0 < \{a, b\} < 4.$$

- a) Read off amplitude and period length.
(x scl = $\pi/2$ and y scl = 0.5)
- b) Try to find the parameters a and b.



We will head towards calculus:

Four sequences are given. Find a conjecture if they are monotonic or not. Give reasons for your conjectures and use **at least two various ways** to illustrate your ideas.

$$a_n = \frac{6n + 2}{27 - 4n}$$

$$a_n = \frac{6 + 5n^2}{1 - 3n^2}$$

$$a_n = \sqrt{5n} (10 - \sqrt{3n})$$

$$a_n = \frac{6}{3\sqrt{n} + 5}$$

Give the proof for one monotonic sequence.

(You can be sure that there is at least one monotonic sequence among the given ones).

Obviously this can be extended for investigating boundaries and limits.

CAS and graphing calculators and programs like DERIVE opened a wide field to "investigate" functions not only in the traditional way (zeros, extremal values and inflection points)

$$y(x) = \frac{|2x + 7|}{4} + \frac{\text{sign}(x - 4) \cdot |x - 1|}{8} - 3$$

- a) Produce a sketch of the graph
- b) Investigate the discontinuities.
- c) Are there other "interesting" places?
- d) Define $y(x)$ as a piecewise defined function and check your definition by superimposing the graphs.

$$\begin{aligned} \sqrt{y10} &= 1/2 \cdot |x + 7/2| + \frac{\text{sign}(x - 4) \cdot |x - 1|}{8} - 3 \\ \sqrt{y11} &= \frac{|2 \cdot x + 7|}{4} + \frac{\text{sign}(x - 4) \cdot |x - 1|}{8} - 3 \\ y12 &= \end{aligned}$$

pnt = 691.98

$$\begin{aligned} \blacksquare y10(x) \mid x < -7/2 & \quad \frac{-3 \cdot x}{8} - 39/8 \\ \blacksquare y11(x) \mid x < -7/2 & \quad \frac{|2 \cdot x + 7|}{4} + \frac{x}{8} - 25/8 \end{aligned}$$

I have best experiences with "descriptions of pictures" with the pictures being graphs of sometimes a bit exotic and strange functions. The students appreciated this work. I have a table full with 60 such functions and each of the students has 5 randomly chosen functions to "describe" including a nice sketch of the graph and a summary of its special points, discontinuities, behaviours etc. In most cases it is not sufficient to only copy the graph from the screen, because eg. very often poles and jumps do not appear as poles and jumps. The screen's resolution produces wrong connections of points and makes necessary an accurate investigation of limits. Scaling has to be adjusted etc.

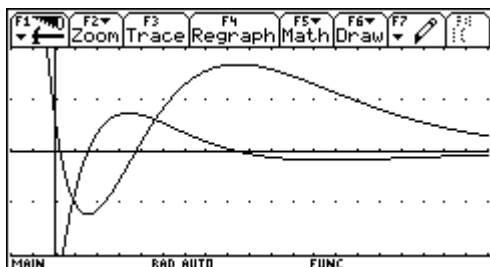
A section of this exotic functions table:

- | | |
|---|---|
| 13. $y = \frac{x^2 - 1}{ x + 1 }$ | 33. $y = x + \left \frac{x^2}{2} - 2 \right $ |
| 14. $y = x - 1 + \frac{x + 1}{ x + 1 }$ | 34. $y = x x - 2(x + x) + 2$ |
| 15. $y = x - 1 + \frac{x^2 - 1}{ x + 1 }$ | 35. $y = x - 3 \cdot x + 1 $ |
| 16. | 36. $y = \frac{x}{2}(2 + x - 2)$ |
| 17. $y = \frac{x - x }{2} + \frac{7}{2}\text{sign}(x + 2)$ | 37. $y = \frac{ 2x - 1}{ x - 1 }$ |
| 32. $y = \text{sign}(4 - x^2)$ | 38. $y = \frac{x}{2} \left(2 + \left \frac{x}{2} - 2 \right \right) - \frac{3}{2}$ |

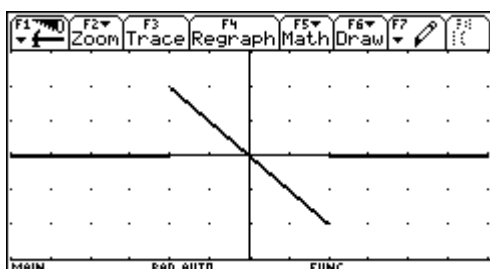
It is my opinion that we can disclaim sophisticated limits' rules and make use of the CAS' implemented functions to demonstrate the various kinds of discontinuities:

$$y(x) = \frac{3x^2}{|x^2 - 3|} \text{sign}(3x - 8)$$

Use CAS to investigate the discontinuities.
 Find the limits for x tending to $\pm\infty$. Explain the meaning of the results for the graph.
 Sketch the graph.



Decide: which of the graphs is the function and which one is the derivative? Give reasons for your decision.



You see the graph of function $y(x)$. Add the graph of its continuous antiderivative, passing $P(-3, -2)$.

Then we have the wide field of "curve discussions". In summer term 2000 I had not time enough to cover this calculus application during the lessons. We very carefully had introduced and studied the concept of the derivative and I tried to change a math lesson into a test:

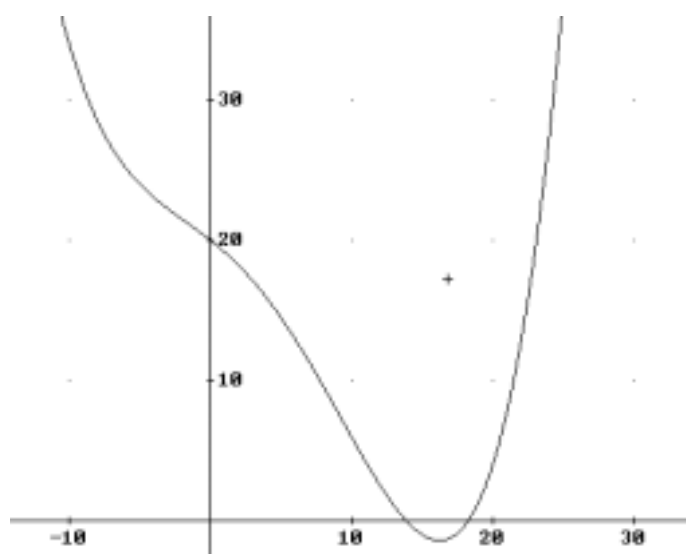
An - in pre TI-times - important calculus application in school mathematics were the so called "Curve Discussions". The students had to calculate (by solving a lot of equations)

- the zeros
- the local extremal values (maximum and minimum points, i.e. points with horizontal tangents).
- the "inflection points" (which are points with curvature = 0; curvature = instantaneous rate of change of the slope)

a) Use this "recipe" to obtain the special points explained above for

$$y(x) = \frac{x^4 - 8x^2 - 16}{25}$$

- b) Do you recognize another special feature of the graph?
 c) Give an explanation for this form.
 d) Produce a sketch of the graph and mark and label the special points from above.



Or the other way round: find the function term of a 4th degree polynomial, knowing an inflection point at You know all these problems, leading to a system of linear equations.

I prefer presenting a graph and giving the task to find an approximating function. The students have to figure out suitable data, read off the numbers and compare their results with the given graph. Calculating is no problem any longer. (I'd like to remind many of you on Paul Drijvers' ghost!)

I think it would be even better to present a hand sketch. Many variations are possible: have different scales on x- and y-axis, ask to use only two or three points to force the students to consider the slope or any other attribute, too. Make it not too easy reading off the coordinates of sufficient points. Add poles and/or horizontal asymptotes. I can assure you that the students appreciate this type of problems. But I also must admit that grading the tests has become more difficult and time consuming as ever before.

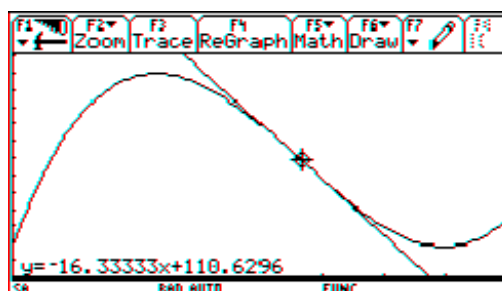
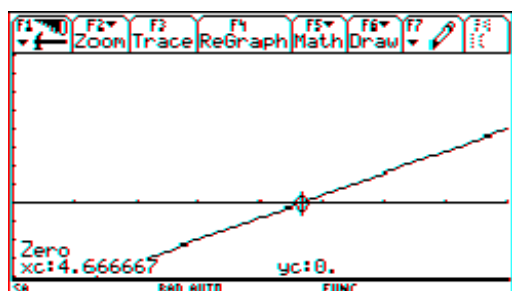
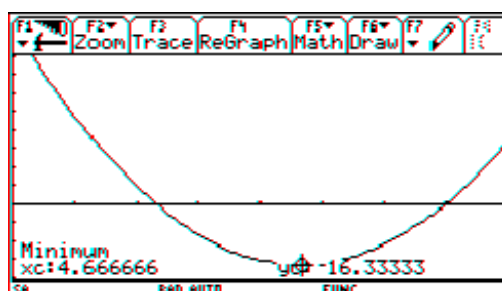
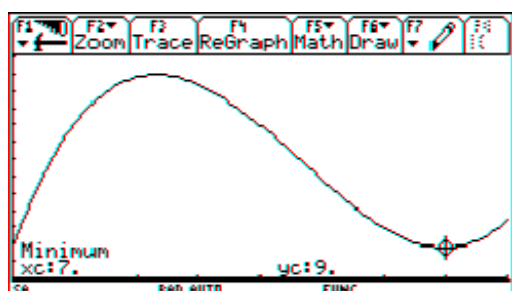
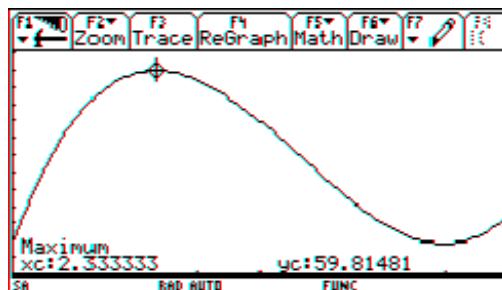
This is the function of the graph presented above:

$$\frac{x^4 - 15 \cdot x^3 - 100 \cdot x^2 - 2000 \cdot x + 50000}{2500}$$

Eleonore Eisler, a colleague from Tulln, Lower Austria and member of our project group in investigating possible changes in assessment using technologies in maths teaching gave the following task in the frame of a "problem solving test" (following ideas from Hildegard Urban-Woldron, Lower Austria):

A cubic is given by the following screen shots. Try to find the cubic's equation on three different ways and use at least three of the given figures. (Explain the figures).

X	Y		
1.	45.		
2.	59.		
3.	57.		
4.	45.		
5.	29.		
6.	15.		
7.	9.		
8.	17.		



I'd like to finish with a very recent assessment example from my last end examination.

The concentration M [grams/liter] of a 6-hour allergy medicine in a body is modeled by

$$M(t) = 12 - 4 \ln(t^2 - 4t + 6), \quad 0 \leq t \leq 6,$$

where t is the time in hours since the allergy medication was taken. Find the average level of concentration in the body over the 6-hour period.

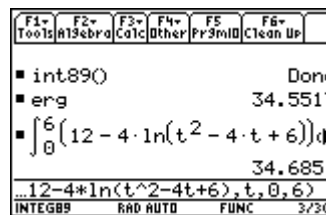
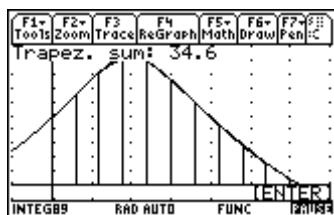
- Use the Trapezoidal Rule (6 strips calculating by hands)
- Use the Trapezoidal Rule (12 strips using `integ()`)
- Use CAS to find the exact result.
- What is the error percentage of a) and b). How much improves the result by doubling the number of strips?
- In various textbooks one can find a formula for estimating the error E occurring by applying the trapezoidal rule

$$E \leq \frac{(b - a)^3}{12n^2} \cdot \max |f''(x)|.$$

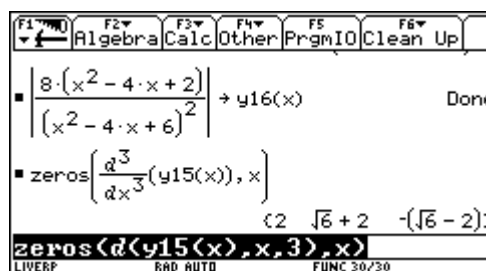
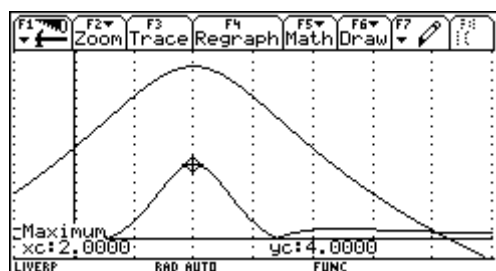
Give the error estimation for a).

- Find the extremal value for $M(t)$. Explain the importance of this point in connection with the model function.

Idea from (Calculus, Larson & Edwards, Houghton Mifflin 1999, ISBN 0-395-91683-6)^[3]



Two screen shots from a TI-89 to demonstrate, how the students used $\text{integ}()$ ^{[1],[2]}.



In this case I gave a spin by using technology to extend the questions in a direction which would not be possible without. I disclaimed the difficult integration and was satisfied enough if the students were able to recognize and to apply the integration concept. They worked with a program after having shown, that they knew what the black box $\text{integ}()$ could do for them. Then they were forced to use the graphing capabilities to find the answer for task e). By the way they also should show how to work with percentages.

Let me close giving some hints and tips how to "spin" your traditional problems:

- Include the graphing capabilities for additional tasks - forcing visualization.
- Encourage and ask for numerical and graphical solutions, ask for more than one way of solution, accept heuristic methods, encourage and ask for "Try and Error Methods".
- Let the pupils produce self made tools and give opportunities to apply them.
- Provoke functional thinking.
- Find the problem to presented solutions.
- Let the pupils set up conjectures - but also ask sometimes for the proofs.
- Include the various ways of the CAS' output into the task, interpret the output and compare with calculation by hands, make the CAS built-in simplification rules to a mathematical subject.
- Use the CAS-capabilities for additional questions (and keep staying within the curriculum!!)
- But over all take care that you don't give problems which set the calculator in the center of the problem, we should remain guided by mathematics and not by the manual.

This is the end of my presentation. I hope that it was not too unorganized. My idea and hope were and are to encourage you to explore the many traditional problems and tasks for giving them a major or minor spin, turning them round to receive new and sometimes challenging problems for students and teachers.

I repeat that the teachers' job will become more difficult, but it also might become more satisfying observing how the use of technology - independent of the used soft- and hardware - could shift the aims, the competencies, the skills, the understanding and last but not least the students' attitude towards mathematics.

- [1] Einführung des Integralbegriffs mit dem TI-92, J Böhm & W Pröpper, bk-teachware SR-13
- [2] Exploring Integration with the TI-89/92, J Böhm & W Pröpper, bk-teachware SL-12, to appear in 2000
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