THE DERIVE - NEWSLETTER #3

THE BULLETIN OF THE

DERIVE®

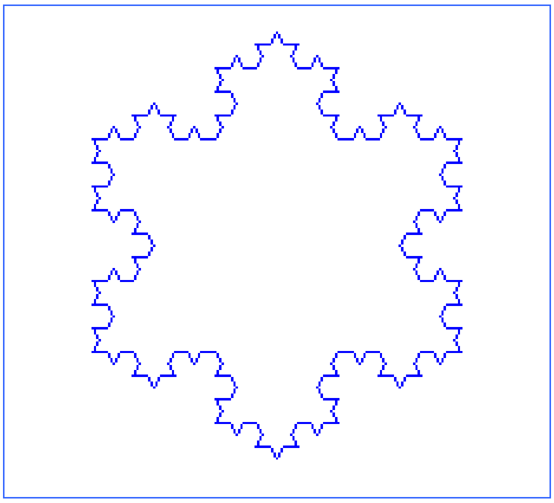
USER GROUP

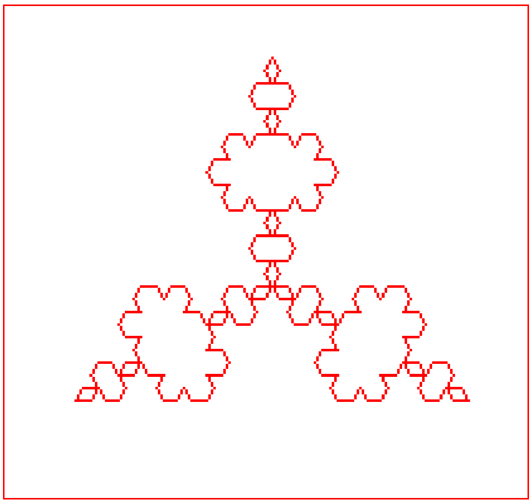
Contents:

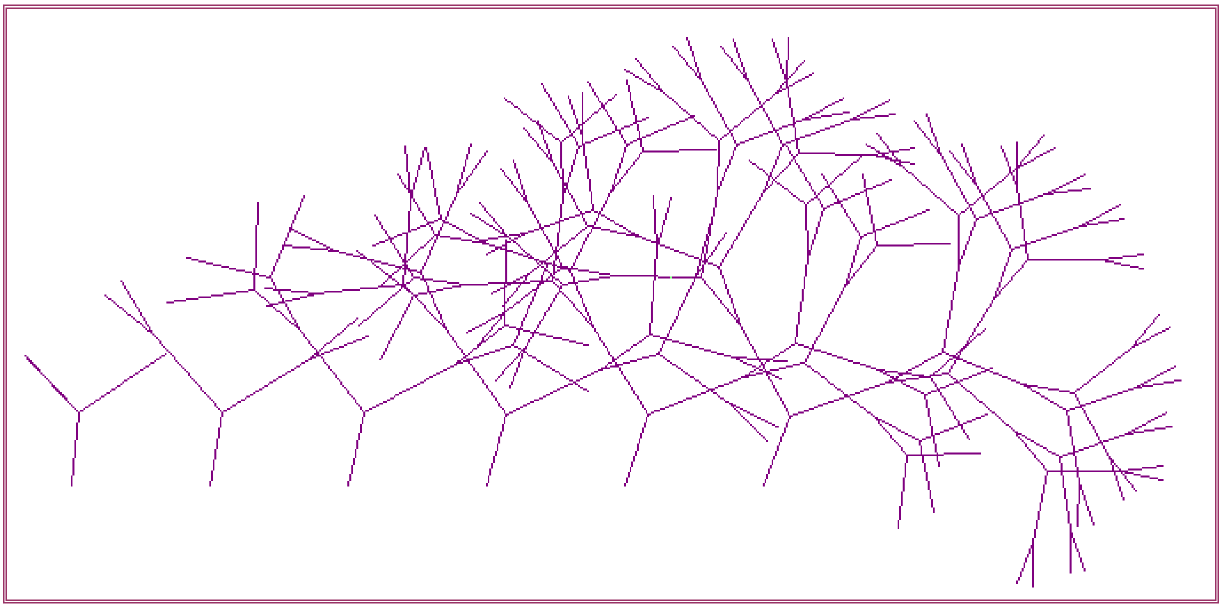
September 1991

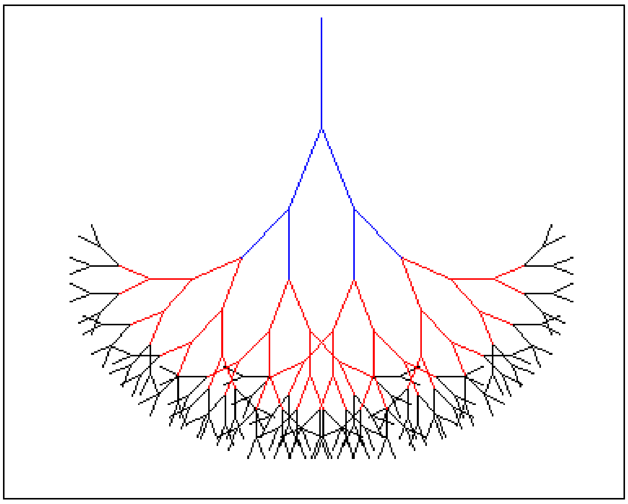
revised reprint 2004

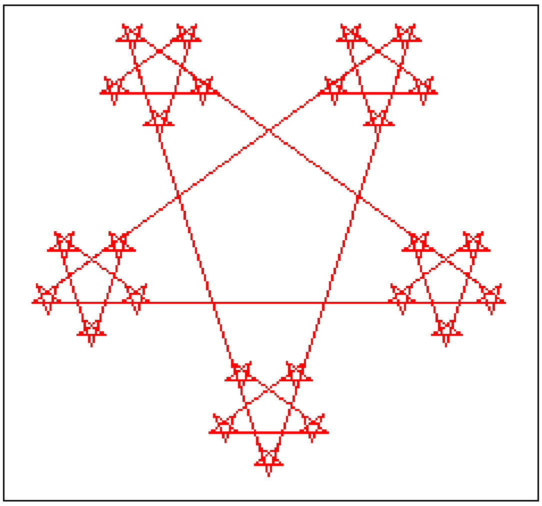
D-N-L #3	From Monsieur Setif´s Treasure Box	D-N-L #3











Lieber Derive Anwender

Die erfreuliche Nachricht zuerst: die DUG wächst weiter. Die Mitgliederzahl ist inzwischen auf über 180 angewachsen. Voraussichtlich wird sich die DUG noch heuer weltweit etablieren. Die Produktentwikkler von DERIVE sind mit einem diesbezüglichen Vorschlag an uns herangetreten. Wir sehen mit Interesse der Entwicklung entgegen.

Außerdem möchte ich Ihr Augenmerk auf die Möglichkeit lenken, DERIVE Consultant von SWHE zu werden. Ein Anmeldeformular liegt dieser Auflage bei.

Viele interessante Zuschriften haben den Herausgeber während des Sommers erreicht. Drei davon möchte ich besonders herausgreifen: Mr. Setif aus Annemasse hat mir insgesamt über 100 Manuskriptseiten überlassen. Sie sind voll mit Ideen, was man mit DERIVE alles behandeln könnte: Kettenbruchentwicklungen, Primzahlprobleme, Folgen und Reihen, Darstellung von Polygonen, Fraktale und noch viel mehr. Der D-N-L hat nun deshalb für einige Zeit eine eigene Kolumne "Mr. Setif's Treasure Box" eingerichtet, um seine Ergebnisse und Anregungen zu publizieren. Zwei Beispiele können Sie mit SYRACUS.MTH und mit einer Auswahl von Fraktalen auf Seite 7, bzw. auf der inneren Umschlagseite finden. Der zweite umfangreiche Beitrag, mit dessen Ausgabe wir im nächsten D-N-L beginnen werden, stammt von Prof. Hodgkinson, Liverpool, und beschäftigt sich mit dem Einsatz von DERIVE beim Erarbeiten des Folgen- und Reihenbegriffs. Dieser Artikel scheint mir deshalb besonders bemerkenswert, weil er viele methodische und didaktische Elemente beeinhaltet. Eine Zuschrift der Herren Appel und Brand zeigt neue Möglichkeiten, allerdings in einem geometrischen Zusammenhang auf.

Die nächste Ausgabe wird wesentlich vielfältiger sein als diese, deren Einheitlichkeit dadurch bedingt ist, daß ich den Beitrag über die Differentialgleichungen abschließen wollte. Ich wünsche Ihnen trotzdem viel Vergnügen mit D-N-L#3 und freue mich über jede Zuschrift, Anregung und/oder Kritik.

Außerdem werden Sie vielleicht im D-N-L#4 eine Überraschung vorfinden!

Dear Derive User,

Let me first of all bring you good news: DUG keeps on increasing and has meanwhile been joined by over 180 members. Most probably DUG will be making its way world-wide before the end of the year. As a matter of fact, the developers of DERIVE have approached us with a proposal relating to such an expansion. Thus we are eagerly awaiting further developments in this matter.

Furthermore I would like to draw your attention to the possibility of becoming a DERIVE Consultant of SWHE. An application form has been enclosed in this issue.

Many interesting letters have reached the Editor during the summer. I take the liberty of pointing out three of them which, in my opinion, deserve particular attention: Mr. Setif from Annemasse has let me have a script totalling over 100 pages which are full of ideas of what fields DERIVE may be applied to: continued fractions, prime number problems, sequences and series, polygons, fractals and many more. For this reason D-N-L has installed an individual column, "Mr. Setif's Treasure Box", in which his results and suggestions will be published for some time. Two examples concerning SYRACUS.MTH and a choice of fractals con be found on page 7 and the inside cover, respectively. The second, fairly detailed contribution, which is to be published in the next issue, has been handed in by Prof. Hodgkinson, Liverpool, and deals with the application of DERIVE as being instrumental in establishing the concept of sequences and series. This article seems particularly remarkable to me as it comprises a good many methodical an didactic elements. A letter by Mr. Appel and Mr. Brand points out new possibilities which, however, have to be considered in a geometric context.

Next issue will be considerably more varied than this one, its basic uniformity being mainly due to the fact that I wanted to complete the contribution on differential equations. Despite this, I hope that you will enjoy D-N-L#3 and would like to invite you to send in contributions - ideas or critical comments are equally welcome.

By the way, in D-N-L#4 a surprise may be waiting for you!

The *Derive-News-Letter* is the Bulletin of the *Derive-User-Group*. It is published at least three times a year with a content of 30 pages minimum. The goals of the *D-N-L* is to enable the exchange of experiencies made with *Derive* as well as to create a group to discuss the possibilities of new methodic an didactic manners in teaching Mathematics.
Subscription of the *D-N-L* is restricted to Members of the *Derive-User-Group*. Membership-form is enclosed with the first issue you get.

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Contributions:

Please send all contributions to the above adress. Non-English speakers are encouraged to write their contributions in English to confirm the international touch of the *D-N-L*. At the other hand non-English articles will be warmly welcomed, too. Your contributions will be edited but not refereed. By submitting articles the author gives his consent for reprinting in the *D-N-L*. The more contributions you will send to the Editor the more livelier and richer in content the *Derive-News-Letter* will be.

Preview (Contributions for the next issues):

With ITERATES to Chaos (Part 2)
How to write one´s own DEMO-file
Cyclomania (Hypotrochoids and Epitrochoids)
Joined Venture Derive and Cabri
Laplace Transforms
A Module on Sequences and Series with Derive

(will be published December 1991)

Impressum:
Medieninhaber: DERIVE User Group, A-3042 Würmla, D´Lust 1, AUSTRIA;
Richtung: Fachzeitschrift;
Herausgeber: Mag.Josef Böhm, Adresse wie oben;
Herstellung: eigene Vervielfältigung

DERIVE - User - Forum

Dr. Felix Schumm, Stuttgart

Zum Zeichnen von Strecken zwischen zwei Punkten x1 und x2 habe ich mir in mein USER-File die Funktion aufgenommen, bei der nach SIMPLIFY-PLOT die vorgeschlagenen Grenzen des Parameters -π bis +π nur bestätigt werden müssen.

```
LINE(x1,x2) := x1 + (t/π + 1) (x2 - x1)
```

Aufruf z.B. mit LINE([1,2],[-4,3])

(Felix Schumm presents a function to plot a segment between two points, which needs only pressing the ENTER-key in the 2D-Plot Window to accept the default values for parameter t.)

H. Wunderling, Berlin

... Mit Bewußtsein haben Sie wohl gleich ins erste Heft den Beitrag von Herrn Schmidt (Köln) getan, denn Sie
"... erwarten außerdem den Widerspruch, der mir genehm."

Deshalb widerspreche ich als Lehrer, der seit 2 Jahren DERIVE im Unterricht einsetzt. Ich bin überzeugt, daß Werkzeuge wie DERIVE den Mathematikunterricht erheblich beeinflussen werden und nicht ausgesperrt werden dürfen.

Hier aber erst einmal 3 Dinge, die ich Sie bitte, an die Entwickler von DERIVE **w e i t e r z u g e b e n:**

- Der Zugang zu den einzelnen Menüpunkten sollte vom Lehrer gesperrt werden können. (Ein Konfigurationsprogramm, das dem Schüler unzugänglich ist, müßte dem Lehrer erlauben, genau die Teile aktiv zu halten, die er für seine Schüler auswählt. Z.B. ist der soLve-Befehl didaktisches Gift, wenn Umformungen mit DERIVE geübt werden sollen; oder EIGENVALUES, wenn ....)

- Eine Schullizenz dürfte preislich nicht wesentlich über einer Einzellizenz liegen. Es ist abzusehen, daß Computer-Arbeitsplätze für Schüler vorhanden sein werden und etwas später auch flexiblere Notebook-PCs.

- Bei der Behandlung des zentralen Grenzwertsatzes der Stochastik habe ich bemerkt, daß die neue Funktion ITERATES auch syntaktisch falsch gesetzte Klammern zu verarbeiten beginnt; der Editor fängt die Fehler nicht ab!

z.B. statt:

```
ITERATES (VECTOR(IF(j=1 OR i=23,0,1/3(ELEMENT(v,i-1)+ELEMENT(v,i)+
ELEMENT(v,i+1))),i,1,23),v,v1,20)
```

wird auch akzeptiert:

```
ITERATES (VECTOR(IF(j=1 OR i=23,0,1/3(ELEMENT(v,i-1)+ELEMENT(v,i)+
ELEMENT(v,i+1)),i,1,23)),v,v1,20)
```

oder auch:

```
ITERATES (VECTOR(IF(j=1 OR i=23,0,1/3(ELEMENT(v,i-1)+ELEMENT(v,i)+
ELEMENT(v,i+1)),i,1,23),v,v1,20))
```

(Mr Wunderling demonstrates that Derive accepts parentheses which are set syntactically incorrect. In my opinion Derive internally counts the number of opening and closing parentheses and does not check, if their position make sense or not.)

Dipl.-Ing. K. Schmidt, Köln

Die Reaktionen auf meine Zitate sind genau so erfolgt, wie ich es erwartet habe. Nach dem Auto ist der Computer die zweite "Heilige Kuh" der Menschen. Es ist schade, daß Sie nicht gebracht haben, was auf der Tagung über "Opportunities and Risks of Artificial Intelligence" in Hamburg 1989 festgestellt wurde. Durch die Künstliche Intelligenz werden vor allem zwei Entwicklungen verschärft: das Verschwinden der Verantwortung und die schleichende Erosion von Qualifikation und Wissen. Einmal soll also der Computer alle Entscheidungen treffen und damit die Verantwortung übernehmen, weil die meisten Menschen nicht denken wollen, zum Teil auch gar nicht können. Zum anderen sagen sie sich: warum soll ich mir noch Mühe geben, der Computer macht doch alles. Sie spielen mit dem Rechner, lernen aber nicht, wie man es macht. Es gibt sogar Informatiker, die diese Richtung propagieren. Schon wird in Fachzeitschriften vor bestimmten Expertensystemen gewarnt, weil sie gedankenlos genutzt werden. Ich war 19 Jahre "Knowledge Engineer". Man glaubt gar nicht, was für verrückte Ideen ich manchmal programmieren oder verbessern mußte, wobei es Ärger mit Akademikern aller Grade gab.

Andererseits darf man nicht vergessen, daß die Derive-User-Group ein exklusiver Klub von hochgradigen Experten ist, von denen vielleicht mancher nicht weiß, wie es in der Praxis zugeht. Gute mathematische Kenntnisse haben leider nur sehr, sehr wenige Menschen, die dazu die entsprechende Veranlagung haben müssen. Für den hohen Stand von Technik und Naturwissenschaften sind es viel zu wenige. Z.B. brechen bei den Studenten der Informatik (gehört ja zur Praktischen Mathematik) ca 70% (1989 waren es in Frnnkfurt sogar 88%) das Studium ab.

Robert Setif, Annemasse, Frankreich

1. Voici quelques fichiers (files) *.MTH. Beaucoup de mes fonctions sont lentes et lourdes inutilement, mais elles sont plutôt destinées á des apprentis en programmation. En effet les exemples des fichiers *.MTH de DERIVE ne sont pas toujours faciles á comprendre avec un niveau mathématique insuffisant.

2. Je commence un gros fichier sur les fractions continues pour des approximation de √2, √3, ... Je sais bien que DERIVE peut les calculer directement, mais c'est pour le plaisir de la programmation. Ces fractions continues vous intéressent, elles?

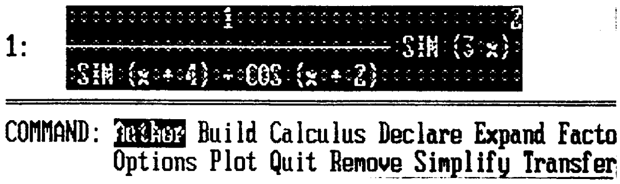
3. DERIVE 2.xx est vraiment WONDERFUL avec sa nouvelle capacité de programmer. Cependant il me semble qu'il manque un équivalent de WHILE condition ou UNTIL condition, ce qui permettrait de stopper une suite d' iterations dont on ne connaît pas le nombre á l'avance (voir par exemple dans SYRACUS.MTH). Il manque aussi une fonction qui calculerait des nombres alèatoires (RND) ∈ N ou ∈ Q ou ∈ R. Il faudrait de même une fonction MOD dans N écrite en langage machine plus rapidement que celle de MISC.MTH. Ce dont bien être possible puisque NEXT_PRIME existe. Pourriez vous transmettre ces 3 revendications "syndicales" a Hawaii? J' imagine que vous en avez déjà d'autres à envoyer?

4. Peut-être pourriez-vous aussi publier dans votre bulletin européen de DERIVE des articles concernant MuMATH?
   A propos de MuMATH, je n' ai pas reçu de Newsletter MuMath depuis novembre 88.

D-N-L: To the first part of this letter see the Letter of the Editor! I've sent Mr Setif's wishes to Hawaii and here is Mr. A. Rich's answer: ".... good suggestions. They will be added to the Version 3 wish list." Maybe there are some DERIVE-Users who would like to correspond with Mr Setif about MuMath. I would be glad to knot the wires.

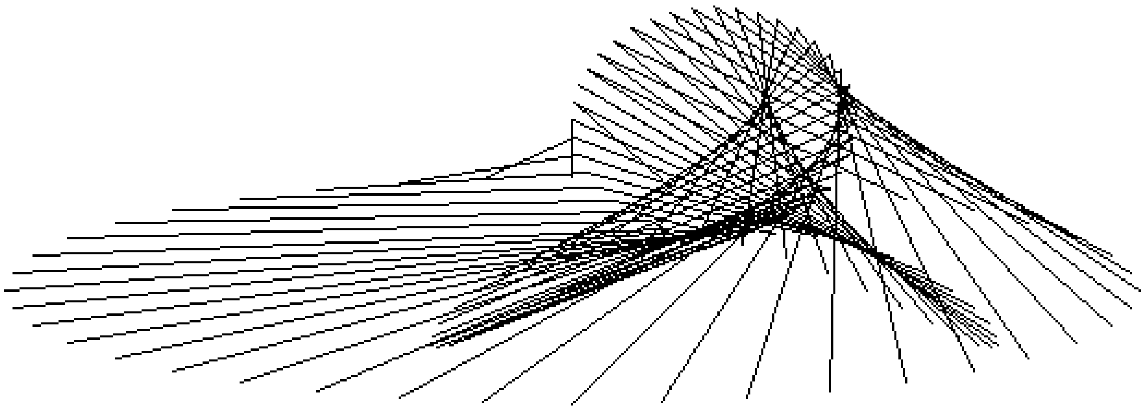
Std. W. Kramp, Berlin

Im VGA-Grafikmodus wird bei mir die aktuelle Funktion falsch abgebildet (siehe Bild). Wie läßt sich der inverse Ausdruck der aktuellen Funktion abstellen oder der Fehler beheben?



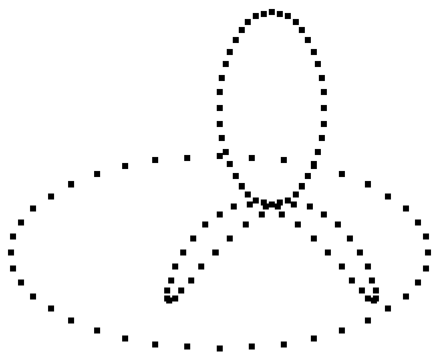
D-N-L: Another answer from A. Rich, concerning one question of Mr Lauer in D-N-L#2, page 6: "memory full" in GCD(123456,9876):

"due to stack overflow caused by an excessive number of recursive function calls".



Here is an example of a "needle graphic", produced by connecting the points of three curves.

(Plot Options for the points: Discrete Large)



$$\text{VECTOR}\left(\begin{bmatrix} 2\cdot\text{COS}(t) - 0.5 & -\text{SIN}(t) \\ 0.5\cdot\text{SIN}(t) & \text{COS}(t) + 1.5 \\ \text{COS}(t + 1) & 0.5\cdot\text{SIN}(2\cdot t) \end{bmatrix}, t, 0, 2\cdot\pi, \frac{\pi}{20}\right)$$

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WOULD YOU BE INTERESTED IN BECOMING A DERIVE CONSULTANT?

Soft Warehouse Europe (SWHE), the European representative of the developer of DERIVE, is looking for knowledgeable DERIVE users who have experience in giving lectures, seminars, or workshops on DERIVE and/or in writing articles about the system. If you meet these requirements and if you would like to offer your expertise to SWHE, authorized DERIVE dealers, conference organizers, publishers, and others, we would welcome your application to become a DERIVE consultant.

We offer to a DERIVE Consultant:

- listing in the "European DERIVE Consultant Directory" - this document will be regularly distributed to authorized DERIVE dealers and will be sent to people or institutions requesting it (conference organizers, publishers, etc.),
- a free upgrade service for the DERIVE software and manual - including all non-announced, minor improvements on the system,
- one year free membership to the DERIVE User Group - including the receipt of the DERIVE News Letter,
- regular information about DERIVE-related publications and presentations.

We accept from a DERIVE Consultant:

- active promotion of DERIVE through lectures, seminars, workshops, articles or books,
- the readiness to cooperate with authorized DERIVE dealers, conference organizers, publishers, etc. (honorarium to be agreed individually between the partners),
- the readiness to speak to potential new users who are seeking an independent opinion about DERIVE (authorized DERIVE dealers may give prospective customers/purchasers names and addresses of their nearest DERIVE consultants).

We further expect the applicant to be a registered DERIVE user.

How to become a DERIVE Consultant:

Return the enclosed DERIVE Consultant Application Form together with all requested documents.

A Famous Sequence (Working with ITERATES)

In original DNL#3 the MOD-function had to be preloaded from MISC.MTH (together with FLOOR and many others which became later part of the Derive core. This is the MOD-function from those times:

$$\text{MOD}(a,\ b) := \lim_{x\to a+}\left(\frac{b}{2} - \frac{b\cdot \text{ATAN}\left(\text{COT}\left(\frac{\pi\cdot x}{b}\right)\right)}{\pi}\right)$$

```
      sy(n) :=
        If MOD(n, 2)
#1:        n/2
           3·n + 1
```

The element following number *n*:

```
#2:   sy(5) = 16

#3:   sy(8) = 4

      nsy(n) :=
        If n = 1
           0
#4:        If sy(n) = 1
              1
              1 + nsy(sy(n))

#5:   nsy(15) = 17
```

The number of iterations which are necessary to come to an end of the sequence initialized by *n.*

$$\#6:\quad \text{syr(n)} := \text{ITERATES}([1 + v_1,\ \text{sy}(v_2)],\ v,\ [0,\ n],\ \text{nsy(n)})$$

$$\#7:\quad \text{syr(3)} = \begin{bmatrix} 0 & 3 \\ 1 & 10 \\ 2 & 5 \\ 3 & 16 \\ 4 & 8 \\ 5 & 4 \\ 6 & 2 \\ 7 & 1 \end{bmatrix}$$

The 3n+1 Sequence starts with a number $n \in$ N. If *n* is even you may halve it, otherwise multiply by three and add 1. Perform the same with this newly created element and go on. The sequence leads to a power of 2.

The complete sequence:

```
#9:   syra(n) := ITERATES(sy(a), a, n, nsy(n))
#10:  syra(5) = [5, 16, 8, 4, 2, 1]
#11:  syra(15) = [15, 46, 23, 70, 35, 106, 53, 160, 80, 40, 20, 10, 5, 16, 8, 4, 2, 1]
#12:  DIM(syra(27)) = 112
```

$$\#13:\quad \text{maxsy(n)} := \text{ITERATES}([1 + v_1,\ \text{sy}(v_2),\ \text{MAX}(v_2,\ v_3)],\ v,\ [0,\ n,\ 1],\ \text{nsy(n)})$$

$$\#14:\quad \text{maxsy(34)} = \begin{bmatrix} 0 & 34 & 1 \\ 1 & 17 & 34 \\ 2 & 52 & 34 \\ 3 & 26 & 52 \\ 4 & 13 & 52 \\ 5 & 40 & 52 \\ 6 & 20 & 52 \\ 7 & 10 & 52 \end{bmatrix}$$

The matrix shows the maximum element after *k* steps (1st col) in the last column.

In the next issue I will show some improvements Mr. Setif has asked for!

With ITERATES to CHAOS

Dr. Felix Schumm, Stuttgart

Inhalt:

1. Einleitung

Im folgenden Beitrag soll gezeigt werden, wie man die Befehle `ITERATES`, `VECTOR` und `ELEMENT` günstig zur Lösung von Iterationsproblemen einsetzen kann, und wie die erhaltenen Ergebnisse mit DERIVE graphisch dargestellt werden können. Es wird versucht, von einfachen Beispielen ausgehend zu komplizierteren Befehlskombinationen langsam fortzuschreiten.

The following contribution shall demonstrate how to apply ITERATES, VECTOR and ELEMENT for solving iteration problems. We will try to proceed from easy examples to more complex combinations of commands. ELEMENT is still available in recent Derive versions but can be replaced by the sub-script command sub (↓).

2. Iteration mit einer Variablen

2.1 Allgemeine Betrachtungen

Einer Zahl x_{alt} werde mittels eines Funktionsterms $f(x_{alt})$ ein neuer Wert x_{neu} zugeordnet.

$$x_{neu} = f(x_{alt}), \quad \text{bzw.} \quad x_{neu} \overset{f:}{\leftarrow} x_{alt}$$

Anschließend wird die Zahl x_{neu} mit dem gleichen Funktionsterm erneut abgebildet, bzw. als ein neues x_{alt} interpretiert. Führt man diesen Prozess n-mal durch, so spricht man von n-facher Iteration.

Für $n = 3$ gilt:

$$x_{neu} = f(f(f(x_{alt}))), \quad \text{bzw.} \quad x_{neu} \overset{f:}{\leftarrow} x \overset{f:}{\leftarrow} x \overset{f:}{\leftarrow} x_{alt}$$

Programmiert wird eine Iteration durch eine n-fache Schleife an deren Ende der Inhalt der Variablen x_{neu} jeweils der Variablen x_{alt} zugewiesen wird.

x_{neu} := $f(x_{alt})$

LET $x_{alt} := x_{neu}$

Schleife n-mal durchlaufen.
n-fold Loop

In BASIC lautet ein entsprechendes Programm etwa so:

```
xalt = startwert
FOR i = 0 TO n
  PRINT " Iterationsstufe: "i"; x-Wert:"xalt
  xneu = f(xalt)
  xalt = xneu
NEXT i
```

Man benötigt also einen Funktionsterm $f(x)$, einen Startwert x_0 für die Variable x und die Iterationstiefe n.

2.2 Umsetzung in DERIVE

Die Iterationsschleife ist in DERIVE kompakt programmierbar mit dem Befehl `ITERATES`.

Es sei:

x	die Variable, die iteriert wird,
f	irgend ein Term in x,
$x0$	der Startwert von x und
n	die Iterationstiefe.

Nach Eingabe von

```
F(X):= ........
ITERATES(F(X),x,x0,n)
```

liefert `SIMPLIFY` (oder `APPROXIMATE`) die Ergebnisse der einzelnen Iterationen in Form eines Vektors mit folgender Bedeutung der Komponenten:

```
┌                  ┐
│x0                │
│f(x0)             │
│f(f(x0)           │
│ ...              │
│ ...              │
│f(...(f(x0)...)   │
└                  ┘
```

`x0`	Startwert (*Initial value*) – keine Iteration (*no iteration*)
`f(x0)`	x_{neu} (für 1-mal iteriert)
`f(f(x0)`	x_{neu} (für 2-mal iteriert)
`...`	
`...`	
`f(...(f(x0)...)`	x_{neu} (für n-mal iteriert)

2.3 Beispiel

Für die Wachstumsrate einer Population gilt unter bestimmten Voraussetzungen folgende Differentialgleichung: (*Differential equation for modeling the growth rate of a population*)

$$\frac{dx}{dt} = a\,x - b\,x^2$$

t:	Zeit - *time*
x:	momentane Anzahl von Individuen (bzw. momentane Dichte von Individuen je Flächen- oder Volumseinheit) *instantaneous number of idividuals (density of individuals per unit of area or volume)*
$a\,x$	Vermehrungsrate proportional zur Anzahl der schon vorhandenen Individuen *growth rate is proportional to number of existing individuals*
$b\,x^2$:	Abnahme durch gegenseitige Behinderung bei großer Individuenanzahl. *decrease because of a negative mutual influence for a large number of individuals*

Die Differentialgleichung läßt sich leicht integrieren und ergibt eine Lösung, die für $t \to +\infty$ immer die Asymptote $x = a/b$ hat. (*DNL* #2, S. 25). Läßt man die Zeit (und damit auch *x*!) nicht kontinuierlich anwachsen (biologisch bedeutet dies, daß nur ganzzahlige Nachkommen möglich sind), so ändern sich die Lösungsmöglichkeiten drastisch und man erhält ein Iterationsproblem. Aus der Differentialgleichung wird zunächst die zugehörige Differenzengleichung:

The DE can easily be integrated and gives a solution, which for t → +∞ has always y = a/b as asymptote. (DNL#2, p 25). Taking discrete steps for t and x – instead of contineous ones – which has the biologic interpretation that only an integer number of succeeders is possible the variety of solutions changes dramatically and we are facing an iteration problem. First of all we make a "difference equation" from the given differential equation:

$$\frac{Dx}{Dt} = a\,x - b\,x^2$$ wobei *Dt* ein endliches Zeitintervall bedeutet.

$$Dx = \left(a\,x - b\,x^2\right)\cdot Dt$$

$$x_{neu} = x_{alt} + Dx$$

$$x_{neu} = x_{alt} + (a\cdot x_{alt} + b\cdot x_{alt}{}^2)\cdot Dt$$

$$x_{neu} = (1 + a\cdot Dt)\cdot x_{alt} - b\cdot Dt\cdot x_{alt}{}^2$$

Der Funktionsterm, den wir untersuchen wollen, lautet also:

```
f(x):=(1+a*Dt)*x-b*Dt*x^2
```

Zur Vereinfachung wählen wir das Zeitintervall `Dt = 1/(b-a)` und erhalten:

$$f(x) := (1 + a\cdot Dt)\cdot x - b\cdot Dt\cdot x^2$$

$$f(x) := \left(1 + a\cdot\frac{1}{b-a}\right)\cdot x - b\cdot\frac{1}{b-a}\cdot x^2 \qquad \to f(x):=r(x-x^2) \text{ mit } r = \frac{b}{b-a}$$

$$\frac{b\cdot x^2}{a-b} + \frac{b\cdot x}{b-a}$$

Zeichnet man sich die Zuordnung $x_{neu} = f(x_{alt})$ auf, so ergibt dies eine abwärts gerichtete Parabel mit den Nullstellen (0|0) und (1|0) und dem Scheitel (0.5|*r*/4). Man erkennt, daß nur Werte $0 < x_{alt} < 1$ für den Start sinnvoll sind, da sonst schon im ersten Iterationsschritt die x_{neu}-Werte negativ oder 0 würden, was für unser Problem keinen Sinn macht.
Man kann die Frage stellen, ob es Startwerte gibt, bei denen die Individuenanzahl völlig unverändert bleibt. Für diese nicht notwendig stabilen Gleichgewichtswerte oder Fixpunkte der Abbildung *f*(*x*) gilt:

The sketch of $x_{neu} = f(x_{alt})$ shows a downward directed parabola with zeros (0|0) and (1|0) and vertex (0.5|r/4). It is obvious that only values $0 < x_{alt} < 1$ make sense, because otherwise the first iteration would result in 0 or in negative values. One can ask now it there are initial values which leave the number of individuals unchanged. So we are looking for the fixponts:

$x_{fix} = f(x_{fix})$ und damit $x_{fix} = r\,x_{fix}\,(1 - x_{fix})$

und man erhält sofort nach Auflösen dieser Gleichung: $x_{fix} = \dfrac{r-1}{r}$ (und den Trivialfall $x_{fix} = 0$).

Naheliegenderweise interessieren uns nur positive x_{fix}-Werte, d.h. wir werden nur r-Werte mit $|r| > 1$ untersuchen.

Nun zur Programmierung in DERIVE. Als Voreinstellung wählen wir:

```
Option Notation (decimal, digits 3)
Option Precision (approximate)
```

und geben ein:

```
#1:   Precision := Approximate
#2:   NotationDigits := 3
                     2
#3:   f := r·(x - x )
                r - 1
#4:   x_fix := ─────
                  r
#5:   ITERATES(f, x, x0, n)
#6:   [n := 13, r := 2]
#7:   x_fix = 0.5
```

Wir berechnen den Gleichgewichtswert für $r = 2$ und erhalten 0.5.
Als Startwert definieren wir willkürlich 0.1, markieren die Zeile #5 und führen **Simplify** durch. Dann erhält man die einzelnen Stufen der Iteration.

```
#8:   x0 := 0.1
#9:   [0.1, 0.18, 0.295, 0.416, 0.485, 0.499, 0.499, 0.5, 0.5, 0.5, 0.5, 0.5, 0.5, 0.5]
#10:  x0 := 0.9
#11:  [0.9, 0.18, 0.295, 0.416, 0.485, 0.499, 0.499, 0.5, 0.5, 0.5, 0.5, 0.5, 0.5, 0.5]
```

Wir können aber auch einen anderen Startwert wählen (z.B. $x0 = 0.9$) und führen anschließend wieder Zeile #5 aus: Gleichgültig, welchen Wert man für x0 ($0 < x0 < 1$) ansetzt, immer läuft die Iteration auf den Gleichgewichtswert $x_{fix} = 0.5$ hin.

We can choose several different initial values x0, approximating or simplifying expression #5 always converges to $x_{fix} = 0.5$.

Dies ist jedoch nicht für jedes $|r| > 1$ der Fall! Wählen wir z.B. $r = 3.2$, so ergibt sich $x_{fix} = 0.687$. Dieser Fixwert wird jedoch nicht angenommen, vielmehr schwanken die Ergebnisse, wenn man lange genug iteriert, unabhängig vom Anfangswert $x0$ zwischen den beiden Zahlen (= Attraktoren) 0.799 und 0.513. Der Zuwachs, bzw. die Abnahme von x schießen jeweils gewissermaßen über den Gleichgewichtszustand x_{fix} hinaus. Die entsprechenden Eingaben in DERIVE sind (für die erste Iteration gilt noch immer $x0 = 0.9$):

```
#12:  r := 3.2
#13:  [0.9, 0.288, 0.656, 0.721, 0.642, 0.735, 0.623, 0.751, 0.597, 0.769, 0.567, 0.785,
        0.539, 0.795]
#16:  ITERATES(f, x, 0.1, n) = [0.1, 0.288, 0.656, 0.721, 0.642, 0.735, 0.623, 0.751,
        0.597, 0.769, 0.567, 0.785, 0.539, 0.795]

#17:  ITERATES(f, x, 0.5, n) = [0.5, 0.8, 0.512, 0.799, 0.512, 0.799, 0.513, 0.799,
        0.513, 0.799, 0.513, 0.799, 0.513, 0.799]

#18:  ITERATES(f, x, 0.8, n) = [0.8, 0.512, 0.799, 0.512, 0.799, 0.513, 0.799, 0.513,
        0.799, 0.513, 0.799, 0.513, 0.799, 0.513]
```

Was geschieht für $x0 = x_{fix}$?

```
#14:  0.687

#15:  ITERATES(f, x, (3.2 - 1)/3.2, n) = [0.687, 0.687, 0.687, 0.687, 0.687, 0.687, 0.687,

        0.687, 0.687, 0.687, 0.687, 0.687, 0.687, 0.687]
```

Die Begründung, warum für $r > 3$ der Gleichgewichtszustand $x_{fix} = (r–1)/r$ unstabil wird, soll hier nicht weiter verfolgt werden, da man diese in vielen Artikeln zum Thema CHAOS unter dem Schlagwort "Feigenbaum" (oder "Bifurkation") nachlesen kann. Erwähnt sei nur, daß für $3 < r < 3.4495$ zwei Attraktoren, für $3.4495 < r < 3.75$ vier und für $r > 3.75$ acht und mehr Attraktoren auftreten.

We will not explain here why for $r > 3$ the equilibrium $x_{fix} = (r–1)/r$ becomes unstable, because there is so much reference literature about CHAOS ("bifurcation", "Feigenbaum"). It shall be mentioned that for $3 < r < 3.4495$ two attractors are appearing, for $3.4495 < r$ 3.75 there are four and for $r > 3.75$ there are eight or more attractors.

2.4 Graphische Darstellung

Gibt man eine Matrix, wie z.B.

$$\begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 3 & 7 \end{bmatrix}$$

in DERIVE durch: `[[2,3],[4,5],[3,7]]` (oder später durch `[2,3;4,5;3,7]`) ein, so liefert der 2D-Plot ein Schaubild, in welchem die Punkte (2|3),(4|5) und (6|7) eingetragen sind. (Im Options State muss man Rectangular und Discrete wählen – *Derive for DOS!*). Um unsere Iterationen graphisch darzustellen, sollten wir aus dem Ergebnisvektor

```
v := [x0,f(x0),f(f(x0)),......]
```

eine Matrix "`Bild`" mit folgender Form herstellen:

```
       ┌ 0      x0      ┐          ┌ 0  ELEMENT(v, 1)   ┐          ┌ 0   v   ┐
       │ 1     f(x0)    │          │ 1  ELEMENT(v, 2)   │          │      1  │
Bild:= │ 2   f(f(x0))   │   Bild:= │ 2  ELEMENT(v, 3)   │          │ 1   v   │
       │ .       .      │          │ .        .         │          │      2  │
       │ .       .      │          │ .        .         │          │ 2   v   │
       └ n   f(f(f(..)) ┘          └ n  ELEMENT(v, n+1) ┘   Bild:= │      3  │
                                                                   │ .    .  │
                                                                   │ .    .  │
                                                                   │ n  v    │
                                                                   └    n+1  ┘
```

Mit dem `ELEMENT`-Befehl können wir die Matrix `Bild` aufbauen und mit Hilfe von `VECTOR` läßt sich diese Matrix über eine Schleife erzeugen:

```
BILD:=VECTOR([i,ELEMENT(v,i+1)],i,0,n]
```

oder mit Derive 5

```
BILD:=VECTOR([i, v SUB(i+1)],i,0,n]
```

Markiert man diese Zeile und führt **Simplify** aus, so erhält man genau solche Matrizen, die unsere Iterationsergebnisse geeignet für das PLOT-Fenster aufbereiten. Mit Hilfe der Tasten F3 und F4, sowie mittels **Substitute** lassen sich die Ergebnisse bequem zu einer Funktion "`BILD`" zusammenfassen. Der unnötige Rest wird gelöscht. Wir können z.B. ausschließlich folgende Zeilen beibehalten und in einem File speichern.

```
#1:   [r :=, x0 :=]
#2:   F(x, r) := r·x·(1 - x)
#3:   ergeb_vektor(r, x0, n) := ITERATES(F(x, r), x, x0, n)
#4:   BILD(r, x0, n) := VECTOR([i, (ergeb_vektor(r, x0, n))_(i + 1)], i, 0, n)
```

Geeignete Werte für (*r*,*x*0,*n*) sind z.B. (1.2,0.001,100), (1.2,0.4,100), (3.2,0.687,100), (3.5,0.714,100) oder (3.6,3.722,100). So liefert die Eingabe:

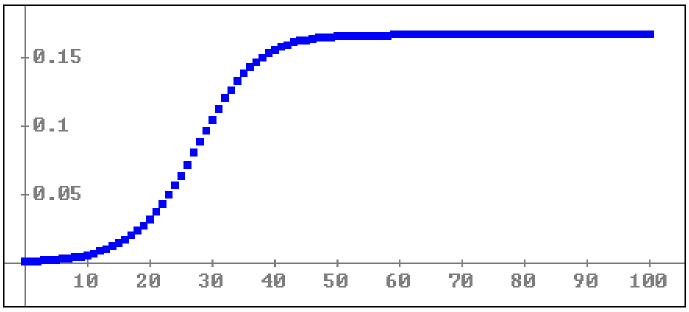
```
#5:   BILD(3.2, 0.687, 100)
```

nach **Simplify** die folgende Bildmatrix:

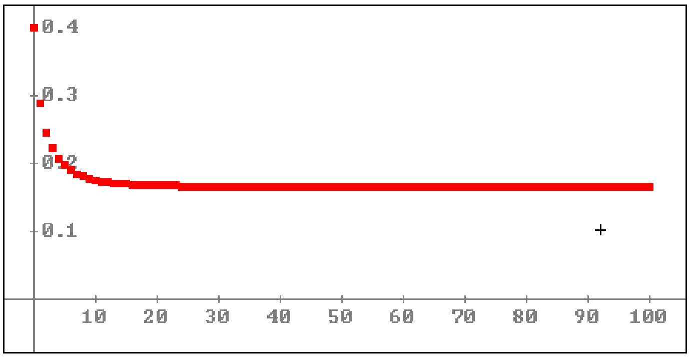
0	0.687
1	0.688
2	0.686
3	0.688
....	
....	
98	0.513
99	0.799
100	0.513

Mit Plot kann das entsprechende *t*-*x*-Diagramm ausgegeben werden.

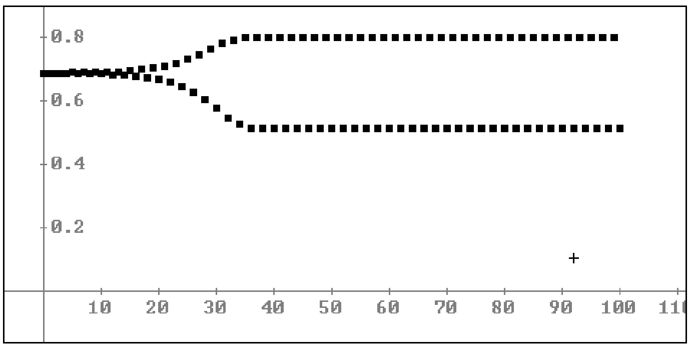
```
#7:   BILD(1.2, 0.001, 100)
```



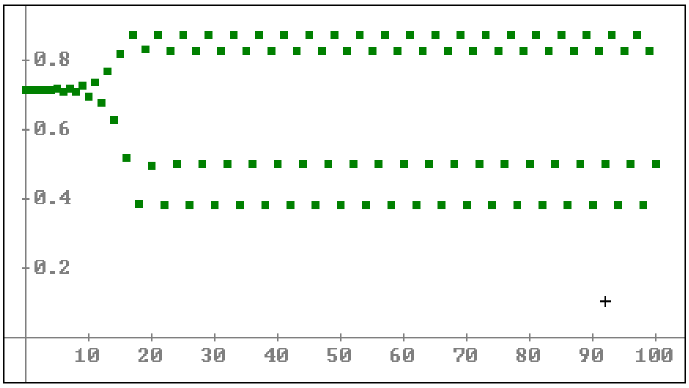

```
#8:   BILD(1.2, 0.4, 100)
```



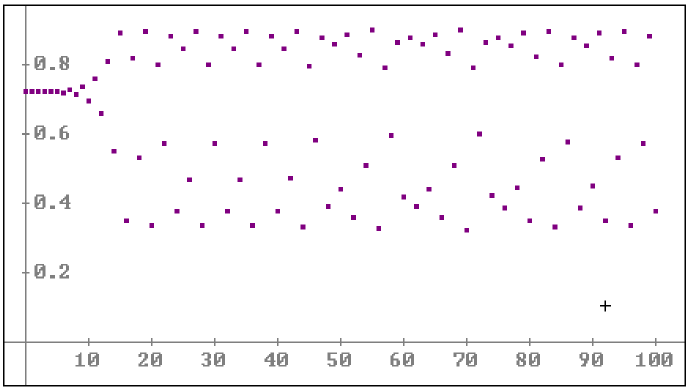

```
#9:   BILD(3.2, 0.687, 100)
```



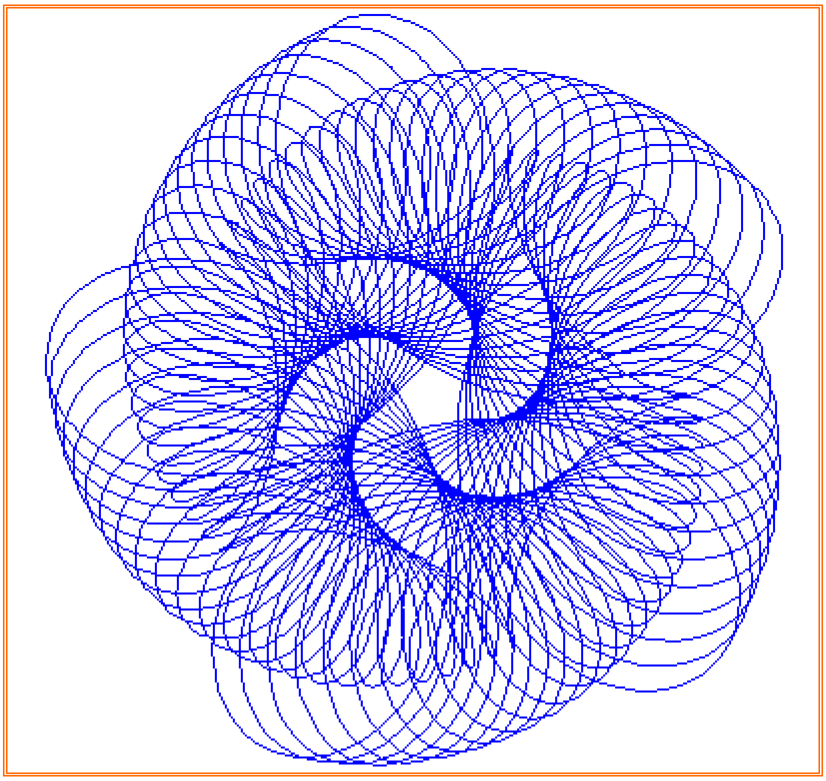
#10: BILD(3.5, 0.714, 100)



#11: BILD(3.6, 0.722, 100)



wird fortgesetzt *(to be continued)*.



A Family of Trochoids

Solving Odes Using *DERIVE*'s ODE1.MTH, Part 2

Josef Böhm, Würmla

I tried to extend the file ODE1.MTH in orderto be able to write general solutions with a parameter *c* instead of ($x0$,$y0$) used in ODE1.MTH. You will find a function to express the integrating factor and a function to solve the LAGRANGE-DE. The expressions #1 - #4 have been used in Part 1 and will be used now, too. I collect all the new functions and hints in the file ODE1_EXT.MTH. Having loaded ODE1, I merge (load) ODE1_EXT and start working.

(In later MATH-files ODE1.MTH general solutions using a constant *c* have been added.)

ODE1_EXT.MTH

#1: $\text{GEN\_SOL\_SEP}(p, q, x, y) := \int \frac{1}{q}\,dy - \int p\,dx = c$

#2: $\text{SPEC\_SOL\_SEP}(p, q, x0, y0, x, y) := \int_{y0}^{y} \frac{1}{q}\,dy = \int_{x0}^{x} p\,dx$

#3: $\text{DIR}(r, x, y, x0, y0) := \lim_{y\to y0} \lim_{x\to x0} \text{IF}\left(\frac{1}{r} = 0, [x, y + t], \left[x + \frac{t}{\sqrt{(1 + r^2)}}, y + \frac{t\cdot r}{\sqrt{(1 + r^2)}}\right]\right)$

#4: FELD(r, x, y, xl, xr, xs, yd, yu, ys) := VECTOR(VECTOR(DIR(r, x, y, x0, y0), x0, xl, xr, xs), y0, yd, yu, ys)

#5: $\text{LINEAR1}(p, q, x, y, x0, y0) := y = \frac{y0 + \int_{x0}^{x} q\cdot ê^{\text{INT}(p, x, x0, x)}\,dx}{ê^{\text{INT}(p, x, x0, x)}}$

#6: $\text{GEN\_LIN1}(p, q, x, y) := y = \frac{\int q\cdot ê^{\int p\,dx}\,dx + c}{ê^{\int p\,dx}}$

#7: $\text{GEN\_BERN}(p, q, k, x, y) := \text{GEN\_LIN1}((1 - k)\cdot p, (1 - k)\cdot q, x, y^{1 - k})$

#8: $\text{GEN\_SOL\_HOM}(r, x, y) := \lim_{y\to y/x} \text{GEN\_SOL\_SEP}\left(\frac{1}{x}, (\lim_{y\to x\cdot y} r) - y, x, y\right)$

#9: $\text{GEN\_SOL\_EX}(p, q, x, y) := \int p\,dx + \int \left(q - \frac{d}{dy}\int p\,dx\right) dy = c$

#10: GEN_INT_FCT(μ, p, q, x, y) := GEN_SOL_EX(μ·p, μ·q, x, y)

#11: $\text{INT\_FCT}(t, v) := ê^{\int t\,dv}$

#12: $\text{GEN\_FRAC\_AUX}(r, p, q, x, y) := \lim_{[y, x]\to[y - q, x - p]} \text{GEN\_SOL\_HOM}(\lim_{[y, x]\to[y + q, x + p]} r, x, y)$

#13: $\text{GEN\_LIN\_FRAC}(r, a, b, c, p, q, k, x, y) := \text{GEN\_FRAC\_AUX}\left(r, \frac{b\cdot k - c\cdot q}{a\cdot q - b\cdot p}, \frac{c\cdot p - a\cdot k}{a\cdot q - b\cdot p}, x, y\right)$

#14: $\text{AUX\_LAGR}(p, q, x, y) := \dfrac{\int q\cdot \hat{e}^{\int p\, dx}\, dx + c}{\hat{e}^{\int p\, dx}}$

#15: $\text{LAGR}(p, q, x, y, v, c1) := \left[y = \lim_{v\to c1} (x\cdot p + q), \lim_{v\to c1} p = c1, x = \text{AUX\_LAGR}\left(\frac{\frac{d}{dv} p}{p - v}, -\frac{\frac{d}{dv} q}{p - v}, v, x\right), y = \text{AUX\_LAGR}\left(\frac{\frac{d}{dv} p}{p - v}, -\frac{\frac{d}{dv} q}{p - v}, v, x\right)\cdot p + q\right]$

Example 7: Find the general solutions of the given differential equations and verify them. Find also the special solutions containing the given points.

a) $x\cdot y' - 4y + 2x^2 + 4 = 0;$ P(1|-3)

b) $y'\cdot \cos x + y\cdot \sin x = \tan x;$ P(π/6|1)

c) $(1 - x^2)\ y' = 1 + x\cdot y;$ P(2|2)

These linear monic differential equations must be brought into the form $y'+ p(x).y = q(x)$. Then use `LINEAR1` from `ODE1.MTH` or `GEN_LIN1(p,q,x,y)` from ODE1_EXT.MTH to find the general solution written with a single symbolic constant c.

Example 7a $\quad x\, y' - 4y + 2x^2 + 4 = 0 \iff y' - \frac{4}{x} y = -\frac{2x^2 + 4}{x}$

#16: $\text{GEN\_LIN1}\left(-\frac{4}{x}, -\frac{2\cdot x^2 + 4}{x}\right)$

#17: $y = c\cdot x^4 + x^2 + 1$

Check for correctness of the solution

#18: $x\cdot\frac{d}{dx}(c\cdot x^4 + x^2 + 1) - 4\cdot(c\cdot x^4 + x^2 + 1) + 2\cdot x^2 + 4 = 0$

#19: $-3 = c\cdot 1^4 + 1^2 + 1$

#20: $\text{SOLVE}(-3 = c\cdot 1^4 + 1^2 + 1, c, \text{Real})$

#21: $c = -5$

Special solution: $y(x) = -5x^4 + x^2 + 1$

#22: $\text{LINEAR1}\left(-\frac{4}{x}, -\frac{2\cdot x^2 + 4}{x}, x, y, 1, -3\right) = (y = -5\cdot x^4 + x^2 + 1)$

using LINEAR1 gives the special solution promptly.

Example 7b $\quad y'\cos x + y\sin x = \tan x \iff y' + \frac{\sin x}{\cos x}y = \frac{\tan x}{\cos x} \iff y' + \tan x \cdot y = \frac{\tan x}{\cos x}$

#23: $\text{LINEAR1}\left(\text{TAN}(x), \frac{\text{SIN}(x)}{\text{COS}(x)^2}\right)$

#24: $y = \text{COS}(x)\cdot\left(\frac{y0}{\text{COS}(x0)} - \frac{1}{2\cdot\text{COS}(x0)^2}\right) + \frac{1}{2\cdot\text{COS}(x)}$

#25: $\text{GEN\_LIN1}\left(\text{TAN}(x), \frac{\text{SIN}(x)}{\text{COS}(x)^2}\right)$

Which solution is more comfortable? #24 or #26?

#26: $y = c\cdot\text{COS}(x) + \frac{1}{2\cdot\text{COS}(x)}$

#27: $1 = \frac{\sqrt{3}\cdot c}{2} + \frac{\sqrt{3}}{3}$

#28: $\text{SOLVE}\left(1 = \frac{\sqrt{3}\cdot c}{2} + \frac{\sqrt{3}}{3}, c, \text{Real}\right)$

#29: $c = \frac{2\cdot\sqrt{3}}{3} - \frac{2}{3}$

#30: $y = \left(\frac{2\cdot\sqrt{3}}{3} - \frac{2}{3}\right)\cdot\text{COS}(x) + \frac{1}{2\cdot\text{COS}(x)}$

or directly with LINEAR1(...)

#31: $\text{LINEAR1}\left(\text{TAN}(x), \frac{\text{SIN}(x)}{\text{COS}(x)^2}, x, y, \frac{\pi}{6}, 1\right) = \left[y = \left(\frac{2\cdot\sqrt{3}}{3} - \frac{2}{3}\right)\cdot\text{COS}(x) + \frac{1}{2\cdot\text{COS}(x)}\right]$

Example 7c $\quad (1-x^2)y' = 1 + xy \iff y' + \left(-\frac{x}{1-x^2}\right)y = \frac{1}{1-x^2}$

#32: $\text{LINEAR1}\left(\frac{x}{x^2 - 1}, \frac{1}{1 - x^2}\right)$

#33: $y = -\frac{\text{LN}(\sqrt{(x^2 - 1)} + x) - \text{LN}(\sqrt{(x0^2 - 1)} + x0) - y0\cdot\sqrt{(x0^2 - 1)}}{\sqrt{(x^2 - 1)}}$

#34: $\text{GEN\_LIN1}\left(\frac{x}{x^2 - 1}, \frac{1}{1 - x^2}\right)$

#35: $y = \frac{c - \text{LN}(\sqrt{(x^2 - 1)} + x)}{\sqrt{(x^2 - 1)}}$

#36: $y(x) := \frac{c - \text{LN}(\sqrt{(x^2 - 1)} + x)}{\sqrt{(x^2 - 1)}}$

#37: $(1 - x^2)\cdot y'(x) - (1 + x\cdot y(x))$

#38: 0

#39: $(1 - x^2)\cdot y'(x) = 1 + x\cdot y(x)$

#40: $y(x) :=$

#41: $y :=$

For verfication of the result we define $y(x)$.
For proceeding in this session we "redo" this definition and go on.

In later Derive versions LINEAR1_GEN() was included in ODE1.MTH together with other _GEN()-functions in order to create general solutions using a symbolic constant (c by default).

$$\#42:\quad \text{LINEAR1\_GEN}\left(\frac{x}{x^2-1}, \frac{1}{1-x^2}\right) = \left[y = \frac{c - \text{LN}(\sqrt{x^2-1}+x)}{\sqrt{x^2-1}}\right]$$

and finally the requested special solution

$$\#43:\quad \text{LINEAR1}\left(\frac{x}{x^2-1}, \frac{1}{1-x^2}, x, y, 2, 2\right)$$

$$\#44:\quad y = \frac{2\cdot\sqrt{3} - \text{LN}((2-\sqrt{3})\cdot(\sqrt{x^2-1}+x))}{\sqrt{x^2-1}}$$

Both of the following equations are of the Bernoulli-type:

$$y' + p(x)\cdot y = q(x)\cdot y^k \qquad (k = \text{const.})$$

In `ODE1.MTH` we found `BERNOULLI(p,q,k,x,y,x0,y0)`. `GEN_BERN(p,q,k)` gives the general solution with constant c. In *DERIVE5*'s `ODE1.MTH` from today we can find `BERNOULLI_ODE()` and `BERNOULLI_ODE_GEN()`.

Example 8:

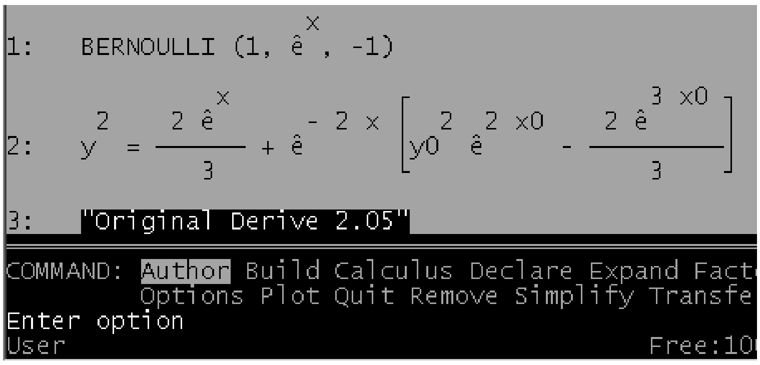
a) $y'\cdot y = e^x - y^2$; P(0|2)

b) $y'\cdot x^3 + y\cdot x^2 - (x^2+1)\cdot y^3$; verify the solution

BERNOULLI() now gives the BERNOULLI number!

$$\#45:\quad \text{BERNOULLI}(1, \hat{e}^x, -1) = -\frac{1}{2}$$

In 1991 `BERNOULLI()` *delivered the solution of the differential equation:*



Example 8a

This is *DERIVE*5 and ODE1_EXT.MTH

$$\#46:\quad \text{BERNOULLI\_ODE}(1, \hat{e}^x, -1)$$

$$\#47:\quad y^2 = \frac{2\cdot\hat{e}^x}{3} + \hat{e}^{-2\cdot x}\cdot\left(y0^2\cdot\hat{e}^{2\cdot x0} - \frac{2\cdot\hat{e}^{3\cdot x0}}{3}\right)$$

$$\#48:\quad \text{GEN\_BERN}(1, \hat{e}^x, -1)$$

$$\#49:\quad y^2 = \frac{2\cdot\hat{e}^x}{3} + c\cdot\hat{e}^{-2\cdot x}$$

#50: $\text{BERNOULLI\_ODE\_GEN}(1, ê^x, -1) = \left[y^2 = \frac{2 \cdot ê^x}{3} + c \cdot ê^{-2 \cdot x}\right]$

#51: $\text{BERNOULLI\_ODE}(1, ê^x, -1, x, y, 0, 2) = \left[y^2 = \frac{2 \cdot ê^x}{3} + \frac{10 \cdot ê^{-2 \cdot x}}{3}\right]$

#52: $2^2 = \frac{2 \cdot ê^0}{3} + c \cdot ê^{-2 \cdot 0}$

#53: $\text{SOLVE}\left(2^2 = \frac{2 \cdot ê^0}{3} + c \cdot ê^{-2 \cdot 0}, c, \text{Real}\right)$

#54: $c = \frac{10}{3}$

Compare with the Derive solution from above!

Example 8b

#55: $\text{GEN\_BERN}\left(\frac{1}{x}, \frac{x^2 + 1}{x^3}, 3\right)$

#56: $\frac{1}{y^2} = \frac{2 \cdot c \cdot x^4 + 2 \cdot x^2 + 1}{2 \cdot x^2}$

#57: $\frac{1}{\frac{1}{y^2} = \frac{2 \cdot c \cdot x^4 + 2 \cdot x^2 + 1}{2 \cdot x^2}}$

#58: $y(x) := \sqrt{\left(\frac{2 \cdot x^2}{2 \cdot c \cdot x^4 + 2 \cdot x^2 + 1}\right)}$

#59: $y'(x) \cdot x^3 + y(x) \cdot x^2 - (x^2 + 1) \cdot y(x)^3 = 0$

#60: $0 = 0$

#61: $y :=$

#62: $y = y$

Example 9: Test the following differential equations on homogeneity. Find two ways for obtaining the general solutions.

a) $y' = \frac{y + x}{y - x}$; P(1|2)

b) $y' = \frac{y\,e^{y/x} - x}{x\,e^{y/x}}$; P(1/2|-3/2) Verify the solution!

Derive 2.05 had a function `HOMOGENEOUS_TEST()`*, which disappeared in later versions.*
The equation is homogeneous if `HOMOGENEOUS_TEST()` *gives back an expression which is free of the independent variable. We can now reanimate this function or make a new one, which uses the definition of homogeneous expressions: substituting k x and k y for x and y should result in an equivalent expression.*

$$\text{HOMOGENEOUS\_TEST}(r, x, y) := \lim_{y \to x \cdot y} r$$

$$\text{HOMOGENEOUS\_TEST}\left(\frac{y + x}{y - x}\right) = \frac{y + 1}{y - 1}$$

$$\text{HOMOGENEOUS\_TEST}\left(\frac{y \cdot \hat{e}^{y/x} - x}{x \cdot \hat{e}^{y/x}}\right) = y - \hat{e}^{-y}$$

$$\text{HOMOGENEOUS\_TEST}\left(\frac{y \cdot \hat{e}^{y/x} - 1}{x \cdot \hat{e}^{y/x}}\right) = y - \frac{\hat{e}^{-y}}{x}$$

```
my_hom_test(r, x, y) :=
  If r - SUBST(r, [x, y], [k·x, k·y]) = 0
     "homogeneous"
     "non homogeneous"
     "non homogeneous"
```

$$\text{my\_hom\_test}\left(\frac{y + x}{y - x}\right) = \text{homogeneous}$$

$$\text{my\_hom\_test}\left(\frac{y \cdot \hat{e}^{y/x} - x}{x \cdot \hat{e}^{y/x}}\right) = \text{homogeneous}$$

$$\text{my\_hom\_test}\left(\frac{y \cdot \hat{e}^{y/x} - 1}{x \cdot \hat{e}^{y/x}}\right) = \text{non homogeneous}$$

Proceed with solving the equation:

$$\#63: \quad \text{HOMOGENEOUS}\left(\frac{y + x}{y - x}\right)$$

$$\#64: \quad \frac{\text{LN}(- x0^2 - y0 \cdot (2 \cdot x0 - y0))}{2} - \frac{\text{LN}\left(- \frac{x0^2 \cdot (x^2 + 2 \cdot x \cdot y - y^2)}{x^2}\right)}{2} = - \text{LN}\left(\frac{x0}{x}\right)$$

This was the 1991-recipe for manipulating this expression in order to find the solution:

```
Build #64, ENTER, *, 2, ENTER, E(xp), R(ecip), ^ENTER
Manage Substitute x0 = 1, y0 = 2, ^ENTER
```

$$x^2 + 2 \cdot x \cdot y - y^2 = 1$$

Let´s do the same without the "Building procedure" from Derive-for-DOS-times:

#64 * 2

$$\#65: \quad \left(\frac{\text{LN}(- x0^2 - y0 \cdot (2 \cdot x0 - y0))}{2} - \frac{\text{LN}\left(- \frac{x0^2 \cdot (x^2 + 2 \cdot x \cdot y - y^2)}{x^2}\right)}{2} = - \text{LN}\left(\frac{x0}{x}\right)\right) \cdot 2$$

$$\#66: \quad \text{LN}(- x0^2 - y0 \cdot (2 \cdot x0 - y0)) - \text{LN}\left(- \frac{x0^2 \cdot (x^2 + 2 \cdot x \cdot y - y^2)}{x^2}\right) = - 2 \cdot \text{LN}\left(\frac{x0}{x}\right)$$

e^(#66)

#67: $\hat{e}^{\mathrm{LN}(-x0^2-y0\cdot(2\cdot x0-y0))-\mathrm{LN}(-x0^2\cdot(x^2+2\cdot x\cdot y-y^2)/x^2)}=-2\cdot\mathrm{LN}(x0/x)$

#68: $\dfrac{x^2\cdot(x0^2+2\cdot x0\cdot y0-y0^2)}{x0^2\cdot(x^2+2\cdot x\cdot y-y^2)}=\dfrac{x^2}{x0^2}$

1/#68

#69: $\dfrac{1}{\dfrac{x^2\cdot(x0^2+2\cdot x0\cdot y0-y0^2)}{x0^2\cdot(x^2+2\cdot x\cdot y-y^2)}=\dfrac{x^2}{x0^2}}$

#70: $\dfrac{x0^2\cdot(x^2+2\cdot x\cdot y-y^2)}{x^2\cdot(x0^2+2\cdot x0\cdot y0-y0^2)}=\dfrac{x0^2}{x^2}$

#70/(x0^2/x^2)

#71: $\dfrac{\dfrac{x0^2\cdot(x^2+2\cdot x\cdot y-y^2)}{x^2\cdot(x0^2+2\cdot x0\cdot y0-y0^2)}=\dfrac{x0^2}{x^2}}{\dfrac{x0^2}{x^2}}$

#72: $\dfrac{x^2+2\cdot x\cdot y-y^2}{x0^2+2\cdot x0\cdot y0-y0^2}=1$

Substitute for x0 = 1, y0 = 2, leading to

#73: $x^2+2\cdot x\cdot y-y^2=1$

#74: $\mathrm{GEN\_SOL\_HOM}\left(\dfrac{y+x}{y-x},\ x,\ y\right)$

#75: $-\dfrac{\mathrm{LN}(-x^2-y\cdot(2\cdot x-y))}{2}=c$

#76: $-\dfrac{\pi\cdot\hat{\imath}}{2}=c$

#77: $-\dfrac{\mathrm{LN}(-x^2-y\cdot(2\cdot x-y))}{2}=-\dfrac{\pi\cdot\hat{\imath}}{2}$

#78: $\hat{e}^{(-\mathrm{LN}(-x^2-y\cdot(2\cdot x-y))/2}=-\pi\cdot\hat{\imath}/2)\cdot(-2)$

#79: $-x^2-y\cdot(2\cdot x-y)=-1$

If you exactly know what to do, then it is easy in *DERIVE*5, too:

`SUBST(1/ê^(2*#64)/(x0^2/x^2),[x0,y0],[1,2])` which results in $x^2+2\cdot x\cdot y-y^2=1$

#80: $\mathrm{HOMOGENEOUS\_GEN}\left(\dfrac{y+x}{y-x}\right)$

#81: $-\mathrm{LN}\left(\dfrac{\sqrt{(-x^2-y\cdot(2\cdot x-y))}}{x}\right)=\mathrm{LN}(x)+c$

#82: $\mathrm{HOMOGENEOUS}\left(\dfrac{y+x}{y-x},\ x,\ y,\ 1,\ 2\right)$

#83: $-\dfrac{\mathrm{LN}\left(-\dfrac{x^2+2\cdot x\cdot y-y^2}{x^2}\right)}{2}+\dfrac{\pi\cdot\hat{\imath}}{2}=\mathrm{LN}(x)$

Transform to find a nice reprecentation of the solution!

#84: $\text{HOMOGENEOUS}\left(\dfrac{y\cdot ê^{y/x} - x}{x\cdot ê^{y/x}}, x, y, \dfrac{1}{2}, -\dfrac{3}{2}\right)$

#85: $ê^{-3} - ê^{y/x} = \text{LN}(2\cdot x)$

#86: $\text{SOLVE}(ê^{-3} - ê^{y/x} = \text{LN}(2\cdot x), y, \text{Real})$

#87: $y = x\cdot \text{LN}(1 - ê^{3}\cdot \text{LN}(2\cdot x)) - 3\cdot x$

Or do it stepwise:

#88: $\text{HOMOGENEOUS\_GEN}\left(\dfrac{y\cdot ê^{y/x} - x}{x\cdot ê^{y/x}}\right)$

#89: $- ê^{y/x} = \text{LN}(x) + c$

#90: $\text{SOLVE}\left(\text{SUBST}\left(- ê^{y/x} = \text{LN}(x) + c, [x, y], \left[\dfrac{1}{2}, -\dfrac{3}{2}\right]\right), c\right)$

#91: $c = \text{LN}(2) - ê^{-3}$

#92: $\text{SOLVE}(- ê^{y/x} = \text{LN}(x) + \text{LN}(2) - ê^{-3}, y)$

#93: $y = x\cdot \text{LN}(1 - ê^{3}\cdot \text{LN}(2\cdot x)) - 3\cdot x - 2\cdot \pi\cdot î\cdot x \vee y = x\cdot \text{LN}(1 - ê^{3}\cdot \text{LN}(2\cdot x)) - 3\cdot x + 2\cdot \pi\cdot î\cdot x$

$\vee\ y = x\cdot \text{LN}(1 - ê^{3}\cdot \text{LN}(2\cdot x)) - 3\cdot x$

As we didn´t ask for the real solutions only, we have to select the real one:

#94: $y = x\cdot \text{LN}(1 - ê^{3}\cdot \text{LN}(2\cdot x)) - 3\cdot x$

Verification of the solution:

#95: $y := x\cdot \text{LN}(1 - ê^{3}\cdot \text{LN}(2\cdot x)) - 3\cdot x$

#96: $\dfrac{d}{dx}\, y - \dfrac{y\cdot ê^{y/x} - x}{x\cdot ê^{y/x}} = 0$

The counter example:

$\text{HOMOGENEOUS\_GEN}\left(\dfrac{y\cdot ê^{y/x} - 1}{x\cdot ê^{y/x}}\right) = \text{inapplicable}$

Example 10: Compare the integral curves of

$y' = \dfrac{y^2 - x^2}{2xy}$ and $y' = \dfrac{2xy}{y^2 - x^2}$ intersecting in (2|-1)

Plot both curves together with their tangents in P on the same axes.

#99: $\text{GEN\_SOL\_HOM}\left(\frac{y^2 - x^2}{2\cdot x\cdot y}\right)$

#100: $-\text{LN}\left(\frac{x^2 + y^2}{x}\right) = c$

#101: $\hat{e}^{-(-\text{LN}((x^2 + y^2)/x) = c)}$

#102: $\frac{x^2 + y^2}{x} = \hat{e}^{-c}$

#103: $\text{HOMOGENEOUS}\left(\frac{y^2 - x^2}{2\cdot x\cdot y}, x, y, 2, -1\right)$

#104: $-\text{LN}\left(\frac{4\cdot(x^2 + y^2)}{5\cdot x^2}\right) = \text{LN}\left(\frac{x}{2}\right)$

#105: $\text{SOLVE}\left(-\text{LN}\left(\frac{4\cdot(x^2 + y^2)}{5\cdot x^2}\right) = \text{LN}\left(\frac{x}{2}\right), y\right)$

#106: $y = -\frac{\sqrt{2}\cdot\sqrt{x}\cdot\sqrt{(5 - 2\cdot x)}}{2} \vee y = \frac{\sqrt{2}\cdot\sqrt{x}\cdot\sqrt{(5 - 2\cdot x)}}{2}$

This turns out to be a family of circles with centers on the *x*-axis. The solution containing point P is one of the semicircles from #106. The direction field for only one single point gives the tangent of the integral in this point. (Plot #106 - #109 and see the circle, the point and the tangent (-2 ≤ *t* ≤ 2).

#107: $\text{FELD}\left(\frac{y^2 - x^2}{2\cdot x\cdot y}, x, y, 2, 2, 1, -1, -1, 1\right)$

#108: $\left[\left[\left[\frac{4\cdot t}{5} + 2, \frac{3\cdot t}{5} - 1\right]\right]\right]$

#109: [2, -1]

#110: Logarithm := Expand

#111: $\text{HOMOGENEOUS}\left(\frac{2\cdot x\cdot y}{y^2 - x^2}, x, y, 2, -1\right)$

#112: $-\frac{\text{LN}(y\cdot(y^2 - 3\cdot x^2))}{3} + \text{LN}(x) + \frac{\text{LN}(11)}{3} - \text{LN}(2) = \text{LN}(x) - \text{LN}(2)$

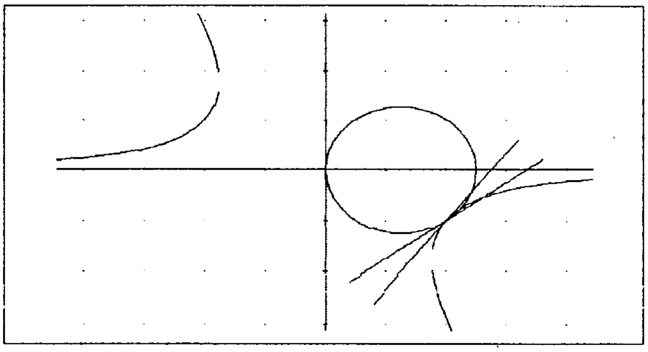
#113: $\hat{e}^{(-\text{LN}(y\cdot(y^2 - 3\cdot x^2))/3 + \text{LN}(x) + \text{LN}(11)/3 - \text{LN}(2) = \text{LN}(x) - \text{LN}(2))\cdot 3}$

#114: $\frac{11\cdot x^3}{8\cdot y\cdot(y^2 - 3\cdot x^2)} = \frac{x^3}{8}$

#115: $\left(\frac{11\cdot x^3}{8\cdot y\cdot(y^2 - 3\cdot x^2)} = \frac{x^3}{8}\right)\cdot\frac{8\cdot y\cdot(y^2 - 3\cdot x^2)}{x^3}$

#116: $11 = y\cdot(y^2 - 3\cdot x^2)$

In 1991 we couldn´t do implicit plots. Now we can. In DNL#3 we solved equation #116 for y and received three solutions (= three explicit branches of the integral cuve). This is the plot from DNL#1. There is a third part which is outside of this plot window.



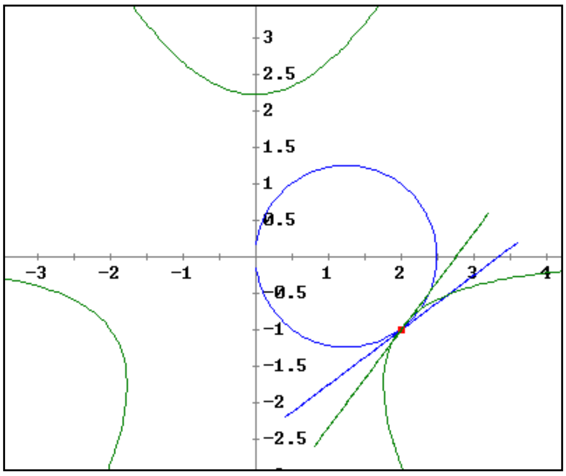
Without any doubt I believed Derive, but today I know a bit more:

Check if both points (2|-1) and (-2|1) really fulfill equation #116?

#117: $\text{FELD}\left(\frac{2\cdot x\cdot y}{y^2 - x^2}, x, y, 2, 2, 1, -1, -1, 1\right)$

#118: $\left[\left[\left[\frac{3\cdot t}{5} + 2, \frac{4\cdot t}{5} - 1\right]\right]\right]$

The final plot shows both curves together with the tangents. And we see that Derive 5 delivers another plot. Solving #116 now for y and plotting the three branches gives exact the same figure – and checking the points works, too.



Example 11: Find the curves which intersect the circles $x^2 + y^2 = r^2$ at angles of $\pi/4$.

Plot some circles and a family of curves representing the general solution of the underlying differential equation.

This problem leads to the homogeneous DE $y' = \frac{y-x}{y+x}$

We start working with rectangular coordinates and switch later to polar coordinates.
Compare my "do it yourself" GEN_SOL_HOM with professional HOMEGENEOUS_GEN:

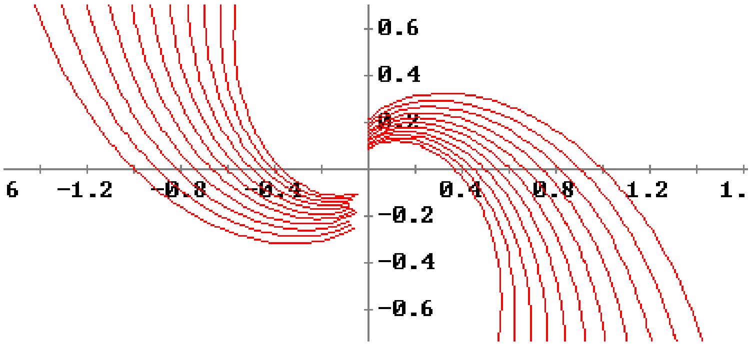
#122: $\text{GEN\_SOL\_HOM}\left(\frac{y - x}{y + x}\right)$

#123: $-\text{ATAN}\left(\frac{y}{x}\right) - \text{LN}(\sqrt{(x^2 + y^2)}) = c$

#124: $\text{HOMOGENEOUS\_GEN}\left(\frac{y - x}{y + x}\right)$

#125: $-\text{ATAN}\left(\frac{y}{x}\right) - \text{LN}\left(\frac{\sqrt{(x^2 + y^2)}}{x}\right) = \text{LN}(x) + c$

#126: $\text{VECTOR}\left(-\text{ATAN}\left(\frac{y}{x}\right)-\frac{\text{LN}(x^2+y^2)}{2}=c,\ c,\ 0,\ 1,\ 0.1\right)$



These partially plotted spirals are not really satisfying me. We switch to polar coordinates substituting for $x = r\cos(\varphi)$ and for $y = r\sin(\varphi)$.

#127: $r :\in \text{Real } (0, \infty)$

#128: $-\text{ATAN}\left(\frac{r\cdot\text{SIN}(\varphi)}{r\cdot\text{COS}(\varphi)}\right)-\frac{\text{LN}((r\cdot\text{COS}(\varphi))^2+(r\cdot\text{SIN}(\varphi))^2)}{2}=c$

#129: $\pi\cdot\text{FLOOR}\left(\frac{\varphi}{\pi}+\frac{1}{2}\right)-\text{LN}(r)-\varphi=c$

#130: $\pi\cdot\text{FLOOR}\left(\frac{\varphi}{\pi}+\frac{1}{2}\right)-\text{LN}(r)-\varphi=-\text{LN}(c)$

#131: $\text{SOLVE}\left(\pi\cdot\text{FLOOR}\left(\frac{\varphi}{\pi}+\frac{1}{2}\right)-\text{LN}(r)-\varphi=-\text{LN}(c),\ r\right)$

#132: $r = c\cdot\hat{e}^{\pi\cdot\text{FLOOR}(\varphi/\pi+1/2)-\varphi}$

#134: $-\text{LN}(r)-\varphi=c$

#135: $-\text{LN}(r)-\varphi=-\text{LN}(c)$

#136: $\text{SOLVE}(-\text{LN}(r)-\varphi=-\text{LN}(c),\ r)$

#137: $r = c\cdot\hat{e}^{-\varphi}$

#138: $\text{VECTOR}(c\cdot\hat{e}^{-\varphi},\ c,\ 0.5,\ 5,\ 0.5)$

#139: $\text{VECTOR}(c,\ c,\ 0.5,\ 5,\ 0.5)$

#140: 2

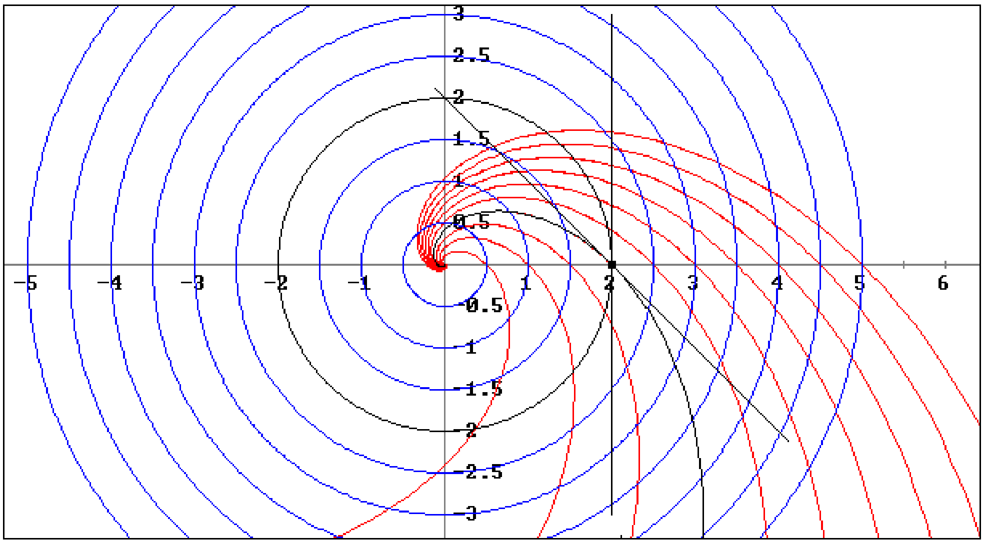
#141: $2\cdot\hat{e}^{-\varphi}$

#142: $[2,\ 0]$

#143: $\text{FELD}\left(\frac{y-x}{y+x},\ x,\ y,\ 2,\ 2,\ 1,\ 0,\ 0,\ 1\right)$

#144: $\left[\left[\left[\frac{\sqrt{2}\cdot t}{2}+2,\ -\frac{\sqrt{2}\cdot t}{2}\right]\right]\right]$

#145: $[2,\ t]$



A family of circles together with a family of integral curves. The tangents in point (2|0) are plotted

Example 12: A point moves on a curve in *x*-*y*-plane in such a way that the angle formed by the tangent of the curve and the *x*-axis is three times the angle between the radius vector and the *x*-axis. Find the Cartesian equation of the family of curves satisfying this condition.

Plot a family of curves, and plot the special curve containing point P(-3|-2).

Using $y' = \tan(\alpha) = \tan(3\beta)$ and $\tan(\beta) = y/x$ you find after some calculations – let Derive do this job – ,applying trigonometric identities the homogeneous differential equation $y' = \dfrac{3x^2y - y^3}{x^3 - 3xy^2}$.

#185: TAN(3·β)

#186: Trigonometry := Expand

#187: $$\frac{4\cdot \mathrm{SIN}(\beta)\cdot \mathrm{COS}(\beta)}{1 - 4\cdot \mathrm{SIN}(\beta)^2} + \frac{\mathrm{SIN}(\beta)}{\mathrm{COS}(\beta)\cdot(4\cdot \mathrm{SIN}(\beta)^2 - 1)}$$

#188: $$\frac{4\cdot \mathrm{SIN}\left(\mathrm{ATAN}\left(\frac{y}{x}\right)\right)\cdot \mathrm{COS}\left(\mathrm{ATAN}\left(\frac{y}{x}\right)\right)}{1 - 4\cdot \mathrm{SIN}\left(\mathrm{ATAN}\left(\frac{y}{x}\right)\right)^2} + \frac{\mathrm{SIN}\left(\mathrm{ATAN}\left(\frac{y}{x}\right)\right)}{\mathrm{COS}\left(\mathrm{ATAN}\left(\frac{y}{x}\right)\right)\cdot\left(4\cdot \mathrm{SIN}\left(\mathrm{ATAN}\left(\frac{y}{x}\right)\right)^2 - 1\right)}$$

#189: $$\frac{y\cdot(3\cdot x^2 - y^2)}{x\cdot(x^2 - 3\cdot y^2)}$$

This is the original file from 1991:

#16: $\text{GEN\_SOL\_HOM}\left(\frac{3\cdot x^2\cdot y - y^3}{x^3 - 3\cdot x\cdot y^2}\right)$

#17: $-\text{LN}\left(\frac{x^2 + y^2}{\sqrt{x}\cdot\sqrt{y}}\right) = c$

#18: $-\text{LN}\left(\frac{x^2 + y^2}{\sqrt{x}\cdot\sqrt{y}}\right) = -\text{LN}(c)$

#19: $\frac{x^2 + y^2}{\sqrt{x}\cdot\sqrt{y}} = c$

#20: We cannot do implicit plots!

#21: So we use Polar coordinates!

#22: $\frac{(r\cdot\text{COS}(\varphi))^2 + (r\cdot\text{SIN}(\varphi))^2}{\sqrt{(r\cdot\text{COS}(\varphi))}\cdot\sqrt{(r\cdot\text{SIN}(\varphi))}} = c$

#23: $\frac{r}{\sqrt{(\text{COS}(\varphi))}\cdot\sqrt{(\text{SIN}(\varphi))}} = c$

#24: $\frac{r^2}{\text{COS}(\varphi)\cdot\text{SIN}(\varphi)} = c^2$

#25: $\frac{2\cdot r^2}{\text{SIN}(2\cdot\varphi)} = c^2$

This is Bernoulli´s Leminscate!

#26: $r = -\frac{\sqrt{2}\cdot c\cdot\sqrt{(\text{SIN}(2\cdot\varphi))}}{2}$

#27: $r = \frac{\sqrt{2}\cdot c\cdot\sqrt{(\text{SIN}(2\cdot\varphi))}}{2}$

#28: $\frac{(-3)^2 + (-2)^2}{\sqrt{(-3)}\cdot\sqrt{(-2)}} = c$

#29: $-\frac{13\cdot\sqrt{6}}{6} = c$

#30: $r = \frac{13\cdot\sqrt{3}\cdot\sqrt{(\text{SIN}(2\cdot\varphi))}}{6}$

The following commands are the base for the plot. It is important to switch between rectangular (#31, #32) and polar coordinates (#33).

It is one of Derive´s unique features that one is able to have both coordinates systems available in the same plot.

For plotting the family of lemniscates set $-\pi/2 \le \varphi \le \pi/2$.

#31: $\text{FELD}\left(\frac{3\cdot x^2\cdot y - y^3}{x^3 - 3\cdot x\cdot y^2}, x, y, -3, -3, 1, -2, -2, 1\right)$

#32: $\left[\left[\left[\frac{9\cdot\sqrt{13}\cdot t}{169} - 3, -\frac{46\cdot\sqrt{13}\cdot t}{169} - 2\right]\right]\right]$

#33: $\text{VECTOR}\left(\frac{\sqrt{2}\cdot c\cdot\sqrt{(\text{SIN}(2\cdot\varphi))}}{2}, c, 0.5, 5, 0.5\right)$

#1: $\text{HOMOGENEOUS}\left(\frac{3\cdot x^2\cdot y - y^3}{x^3 - 3\cdot x\cdot y^2}, x, y, -3, -2\right)$

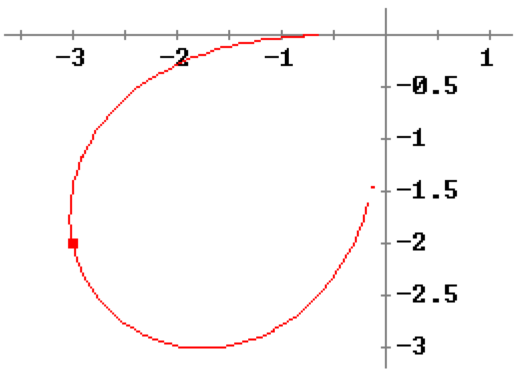
#2: $-\frac{\text{LN}\left(\frac{54\cdot(x^2 + y^2)^2}{169\cdot x^3\cdot y}\right)}{2} = \text{LN}\left(\frac{x}{3}\right) - \pi\cdot\hat{\imath}$

e^(#2):

#3: $\frac{13\cdot\sqrt{6}}{18\cdot(x^2 + y^2)\cdot\sqrt{\left(\frac{1}{x^3\cdot y}\right)}} = -\frac{x}{3}$

#4: [-3, -2]

This cannot be plotted in DERIVE *5, but interestingly enough it can be plotted in* DERIVE *6. And here we find one part of the lemniscate.*



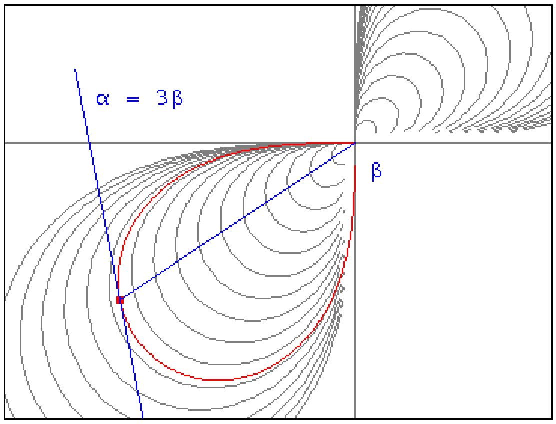
Finally we plot a family of the complete lemniscates together with the tangent in P(-3|-2) and the ray from 0 to P. We will show that the relation between the angles given in the task is true.

$$\beta := \pi + \text{ATAN}\left(-\frac{2}{-3}\right)$$

$$\alpha := \text{ATAN}\left(\text{SUBST}\left(\frac{3\cdot x^2\cdot y - y^3}{x^3 - 3\cdot x\cdot y^2}, [x, y], [-3, -2]\right)\right)$$

$$\alpha - 3\cdot\beta = -4\cdot\pi$$

$$3\cdot\beta - \alpha = 4\cdot\pi$$



Example 13: Show that the given differential equations are exact, give their general solutions and if there is a point given find also the respective special solution.

a) $2x^2 - y^2 + y - y'(2xy - x + 4y) = 0$; P(1/-1.5)

b) $2x + e^x \ln y + (e^x y')/y = 0$

c) $2xy - y'(2x^2 + y) = 0$; Q(-2|3)

DEs of form $p(x,y) + q(x,y)\,y' = 0$ may be exact. Exactness seldom is obvious, so `EXACT_TEST(p,q,x,y)` or `EXACT_IF_0(p,q,x,y)` in earlier versions will test this attribute.

EXACT_TEST is not included in recent ODE1.MTH. The test has been included into the EXACT()-functions for solving the DEs. For didactical reasons it might be useful to ask students to produce a selfmade EXACT_TEST-function. In the following I give the original EXACT_TEST from 1991 and an updated form for DERIVE 5 (and 6, of course).

$$\#246:\ \text{EXACT\_TEST}(p, q, x, y) := \frac{d}{dy}\,p - \frac{d}{dx}\,q$$

```
         EXACT_TEST_N(p, q, x, y) :=
           If DIF(p, y) - DIF(q, x) = 0
#247:         "exact"
              "not exact"
              "not exact"
```

$$\text{EXACT\_TEST}(2\cdot x^2 - y^2 + y, -(2\cdot x\cdot y - x + 4\cdot y)) = 0$$

$$\text{EXACT\_TEST\_N}(2\cdot x^2 - y^2 + y, -(2\cdot x\cdot y - x + 4\cdot y)) = \text{exact}$$

$$\text{EXACT\_TEST}\left(2\cdot x + ê^x\cdot \text{LN}(y), \frac{ê^x}{y}\right) = 0$$

$$\text{EXACT\_TEST\_N}\left(2\cdot x + ê^x\cdot \text{LN}(y), \frac{ê^x}{y}\right) = \text{exact}$$

$$\text{EXACT\_TEST}(2\cdot x\cdot y, -2\cdot x^2 - y) = 0$$

$$\text{EXACT\_TEST\_N}(2\cdot x\cdot y, -2\cdot x^2 - y) = \text{not exact}$$

Example 13 a

#254: $\text{EXACT}(2\cdot x^2 - y^2 + y,\ -(2\cdot x\cdot y - x + 4\cdot y))$

#255: $\dfrac{2\cdot x^3}{3} + x\cdot y\cdot(1 - y) - \dfrac{6\cdot y^2 + 2\cdot x0^3 + 3\cdot x0\cdot y0\cdot(1 - y0) - 6\cdot y0^2}{3} = 0$

#256: $\text{EXACT}\left(2\cdot x^2 - y^2 + y,\ -(2\cdot x\cdot y - x + 4\cdot y),\ x,\ y,\ 1,\ -\dfrac{3}{2}\right)$

#257: $\dfrac{2\cdot x^3}{3} + x\cdot y\cdot(1 - y) - \dfrac{24\cdot y^2 - 91}{12} = 0$

#258: $8\cdot x^3 + 12\cdot x\cdot y\cdot(1 - y) - 24\cdot y^2 + 91 = 0$

#259: $\text{EXACT\_GEN}(2\cdot x^2 - y^2 + y,\ -(2\cdot x\cdot y - x + 4\cdot y))$

#260: $\dfrac{2\cdot x^3}{3} + x\cdot y\cdot(1 - y) - 2\cdot y^2 = c$

#261: $-\dfrac{91}{12} = c$

leads to the same solution (#258)

Example 13 b

#262: $\text{EXACT\_GEN}\left(2\cdot x + ê^x\cdot \text{LN}(y),\ \dfrac{ê^x}{y}\right)$

#263: $ê^x\cdot \text{LN}(y) + x^2 = c$

#264: $\text{SOLVE}(ê^x\cdot \text{LN}(y) + x^2 = c,\ y)$

#265: $y = ê^{ê^{-x}\cdot(c - x^2)}$

#266: $\text{SOLVE}\left(\text{EXACT}\left(2\cdot x + ê^x\cdot \text{LN}(y),\ \dfrac{ê^x}{y}\right),\ y\right)$

#267: $y = \text{IF}\left(ê^{x0 - x} \le 1 \vee y0 > 0,\ ê^{ê^{-x}\cdot(ê^{x0}\cdot \text{LN}(y0) - x^2 + x0^2)}\right)$

Example 13 c

This equation is obviously not exact (see above). We assume, that we know an integrating factor $\mu = 1/y^3$. Now we are able to use `USE_INTEG_FCTR(μ,p,q,x,y,x0,y0)`. (*This function is not available in later Derive versions.*)

#268: USE_INTEG_FCTR(μ, p, q, x, y, x0, y0) := EXACT(μ·p, μ·q, x, y, x0, y0)

#269: $\text{USE\_INTEG\_FCTR}\left(\frac{1}{y^3}, 2\cdot x\cdot y, -2\cdot x^2 - y\right)$

#270: $\frac{y0^2\cdot x^2 - y\cdot(y\cdot(x0^2 + y0) - y0^2)}{y0^2\cdot y^2} = 0$

#271: $\text{USE\_INTEG\_FCTR}\left(\frac{1}{y^3}, 2\cdot x\cdot y, -2\cdot x^2 - y, x, y, -2, 3\right)$

#272: $\frac{9\cdot x^2 - y\cdot(7\cdot y - 9)}{9\cdot y^2} = 0$

#273: $\text{GEN\_INT\_FCT}\left(\frac{1}{y^3}, 2\cdot x\cdot y, -2\cdot x^2 - y\right)$

#274: $\frac{x^2}{y^2} + \frac{1}{y} = c$

#275: $\frac{7}{9} = c$

Solution: $\frac{x^2}{y^2} + \frac{1}{y} = \frac{7}{9}$

In times of Derive 5 and higher all the auxiliary functions (see later in example 14) are implemented in on function to solve immediately differential equations, which are supposed to be soluble by applying an integrating factor: I recommend to compare the recent ODE1.MTH file with the auxiliary procedures which where necessary in 1991. I can imagine that it might be useful for students studying differential equations to follow the old "recipes" (which can support understanding the solving process).

#277: $\text{INTEGRATING\_FACTOR}(2\cdot x\cdot y, -2\cdot x^2 - y, x, y, -2, 3)$

#278: $\frac{9\cdot x^2 - y\cdot(7\cdot y - 9)}{9\cdot y^2} = 0$

Compare with #275

File ODE1_1991.dfw is the original ODE1.MTH from Derive 2.05 together with ODE1_EXT.MTH from DNL#3.

Example 14: Try to find a solution of the DEs given below using the auxil iary functions of ODE1.MTH and ODE1_EXT.MTH concerning the application of integrating factors. Give the integrating fac tors.

a) $2xy - y'(2x^2 + y) = 0$

b) $(x^2 + y^2)(x\,dy - y\,dx) = (a + x)\,x^4\,dx$

c) $(x\cos y - y\sin y)dy + (x\sin y + y\cos y)dx = 0;\ P(1|\pi/2)$

d) $(2x^3y^2 - y)dx = (x - 2x^2y^3)dy$

The DEs must be brought into the form $p(x,y) + q(x,y)\,y' = 0$.
DERIVE can be helpful to find an integrating factor μ.

Preloaded is File ODE1.MTH, copyright (c) 1990 by Soft Warehouse, Inc.
& ODE1_EXT from Derive Newsletter #3

#41: $\text{FREE\_OF\_INDEPENDENT\_TEST}(2\cdot x\cdot y, -2\cdot x^2 - y) = -\frac{3}{y}$

simplifies to an expression free of x. Hence we get the solution by means of the next function:

#42: $\text{FREE\_OF\_INDEPENDENT}(2\cdot x\cdot y, -2\cdot x^2 - y)$

#43: $\frac{y0^2\cdot x^2 - y\cdot(y\cdot(x0^2 + y0) - y0^2)}{y0^2\cdot y^2} = 0$

#44: $\text{INT\_FCT}\left(-\frac{3}{y}, y\right) = \frac{1}{y^3}$

Follow example 13c from above

Example 14 b

$$(x^2+y^2)\,(x\,y'-y) = (a+x)x^4$$

The given equation must be rewritten as

$$-x^2y - y^3 - a\,x^4 - x^5 + (x^3 + x\,y^2)\,y' = 0$$
$$x^2y + y^3 + a\,x^4 + x^5 + (-x^3 - x\,y^2)\,y' = 0$$

#45: $\text{FREE\_OF\_INDEPENDENT\_TEST}(x^5 + a\cdot x^4 + x^2\cdot y + y^3, -x^3 - x\cdot y^2)$

#46: $-\frac{4\cdot(x^2+y^2)}{x^5 + a\cdot x^4 + x^2\cdot y + y^3}$

which is not free of independent x, but

#47: $\text{FREE\_OF\_DEPENDENT\_TEST}(x^5 + a\cdot x^4 + x^2\cdot y + y^3, -x^3 - x\cdot y^2)$

#48: $-\frac{4}{x}$

which is free of dependent y. This leads to

#49: $\text{FREE\_OF\_DEPENDENT}(x^5 + a\cdot x^4 + x^2\cdot y + y^3, -x^3 - x\cdot y^2)$

#50: $\frac{x^2}{2} + a\cdot x - \frac{y}{x} - \frac{y^3}{3\cdot x^3} - a\cdot x0 - \frac{3\cdot x0^5 - 6\cdot x0^2\cdot y0 - 2\cdot y0^3}{6\cdot x0^3} = 0$

#51: $\text{INT\_FCT}\left(-\frac{4}{x}, x\right) = \frac{1}{x^4}$ integrating factor!!

#52: $\text{GEN\_INT\_FCT}\left(\frac{1}{x^4}, x^5 + a\cdot x^4 + x^2\cdot y + y^3, -x^3 - x\cdot y^2\right)$

#53: $\frac{x^2}{2} + a\cdot x - \frac{y}{x} - \frac{y^3}{3\cdot x^3} = c$

Now in DERIVE5:

#279: $\text{INTEGRATING\_FACTOR}(x^5 + a\cdot x^4 + x^2\cdot y + y^3, - x^3 - x\cdot y^2)$

#280: $\dfrac{x^2}{2} + a\cdot x - \dfrac{y}{x} - \dfrac{y^3}{3\cdot x^3} - a\cdot x0 - \dfrac{3\cdot x0^5 - 6\cdot x0^2\cdot y0 - 2\cdot y0^3}{6\cdot x0^3} = 0$

Example 14 c

After rewriting we proceed:

#54: $\text{FREE\_OF\_DEPENDENT\_TEST}(x\cdot \text{SIN}(y) + y\cdot \text{COS}(y), x\cdot \text{COS}(y) - y\cdot \text{SIN}(y))$

#55: 1

#56: $\text{FREE\_OF\_DEPENDENT}\left(x\cdot \text{SIN}(y) + y\cdot \text{COS}(y), x\cdot \text{COS}(y) - y\cdot \text{SIN}(y), x, y, 1, \dfrac{\pi}{2}\right)$

#57: $\hat{e}^x\cdot(y\cdot \text{COS}(y) + (x - 1)\cdot \text{SIN}(y)) = 0$

#58: $\text{INT\_FCT}(1, x) = \hat{e}^x$

#59: $\text{GEN\_INT\_FCT}(\hat{e}^x, x\cdot \text{SIN}(y) + y\cdot \text{COS}(y), x\cdot \text{COS}(y) - y\cdot \text{SIN}(y))$

#60: $\hat{e}^x\cdot(y\cdot \text{COS}(y) + (x - 1)\cdot \text{SIN}(y)) = c$ substitute for x and y

#61: $0 = c$

Please check the result with the respective DERIVE5 function!

Example 14 d

#62: $\text{FREE\_OF\_INDEPENDENT\_TEST}(2\cdot x^3\cdot y^2 - y, 2\cdot x^2\cdot y^3 - x) = \dfrac{4\cdot x\cdot (y^2 - x^2)}{2\cdot x^3\cdot y - 1}$

#63: $\text{FREE\_OF\_DEPENDENT\_TEST}(2\cdot x^3\cdot y^2 - y, 2\cdot x^2\cdot y^3 - x) = \dfrac{4\cdot y\cdot (x^2 - y^2)}{2\cdot x\cdot y^3 - 1}$

neither free of x nor of y; we test other possibilities for the integrating factor:

We proceed in DNL#3 (1991) – style:

#64: $\text{EXACT\_TEST}(x\cdot y\cdot (2\cdot x^3\cdot y^2 - y), x\cdot y\cdot (2\cdot x^2\cdot y^3 - x)) = 6\cdot x^2\cdot y^2\cdot (x^2 - y^2)$

x y is no integrating factor

#65: $\text{EXACT\_TEST}((x + y)\cdot(2\cdot x^3\cdot y^2 - y), (x + y)\cdot(2\cdot x^2\cdot y^3 - x)) = 4\cdot x^4\cdot y + 6\cdot x^3\cdot y^2 - 6\cdot x^2\cdot y^3 + x\cdot(1 - 4\cdot y^4) - y$

no luck with (x + y)

#66: $\text{EXACT\_TEST}(x^2\cdot y^2\cdot(2\cdot x^3\cdot y^2 - y), x^2\cdot y^2\cdot(2\cdot x^2\cdot y^3 - x)) = 6\cdot x^2\cdot y^2\cdot(x^2 - y^2)$

no success with x^2 y^2

#67: $\text{EXACT\_TEST}\left(\frac{1}{x^2\cdot y^2}\cdot(2\cdot x^3\cdot y^2 - y), \frac{1}{x^2\cdot y^2}\cdot(2\cdot x^2\cdot y^3 - x)\right) = 0$

This was the right guess! Let´s finish the solution

#68: $\text{GEN\_INT\_FCT}\left(\frac{1}{x^2\cdot y^2}, 2\cdot x^3\cdot y^2 - y, 2\cdot x^2\cdot y^3 - x\right)$

#69: $\frac{x^3\cdot y + x\cdot y^3 + 1}{x\cdot y} = c$

with `MONOMIAL_TEST(p,q,x,y)` *DERIVE* can find an integrating factor of the form $x^m y^n$ if there exists one.

This is the way we do it now:

#281: $\text{INTEGRATING\_FACTOR\_GEN}(2\cdot x^3\cdot y^2 - y, 2\cdot x^2\cdot y^3 - x)$

#282: **inapplicable**

#283: $\text{MONOMIAL\_TEST}(2\cdot x^3\cdot y^2 - y, 2\cdot x^2\cdot y^3 - x) = \frac{1}{x^2\cdot y^2}$

#284: $\text{INTEGRATING\_FACTOR\_GEN}\left(\frac{1}{x^2\cdot y^2}\cdot(2\cdot x^3\cdot y^2 - y), \frac{1}{x^2\cdot y^2}\cdot(2\cdot x^2\cdot y^3 - x)\right)$

#285: $\frac{x^3\cdot y + x\cdot y^3 + 1}{x\cdot y} = c$

Example 15: (x – 2y + 5)dx + (2x – y + 4)dy = 0

a) Find the general solution using both (x0|y0) and constant c.
b) Find the special solution with x = 1, y = 1.
c) Sketch the integral curve and the direction field.

`LIN_FRAC(r, a, b, d, p, q, k, x, y, x0, y0)` simplifies to an implicit solution of a linear fractional equation $y' = r\cdot((a\cdot x+b\cdot y+d)/(p\cdot x+q\cdot y+k))$. However if q·a – p·b = 0, instead use `FUN_LIN_CCF` described above. (Online – Help).

We will tackle this problem in the DERIVE5/6 mode only and add the 1991 plot for c)

#286: $\text{LIN\_FRAC}\left(\frac{-x + 2\cdot y - 5}{2\cdot x - y + 4}, -1, 2, -5, 2, -1, 4\right)$

#287: $\text{LN}\left(\left|\frac{x + 1}{x0 + 1}\right|\cdot\sqrt{\left(-\frac{x - y + 3}{(x + y - 1)^3}\right)}\right) - \text{LN}\left(\sqrt{\left(-\frac{x0 - y0 + 3}{(x0 + y0 - 1)^3}\right)}\right) = \text{LN}(x + 1) - \text{LN}(x0 + 1)$

$$\#288:\ \text{LIN\_FRAC\_GEN}\left(\frac{-x + 2\cdot y - 5}{2\cdot x - y + 4}, -1, 2, -5, 2, -1, 4\right)$$

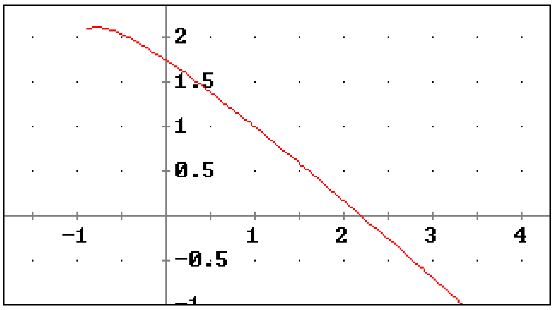
$$\#289:\ \frac{\text{LN}\left(-\dfrac{(x + 1)^2\cdot(x - y + 3)}{(x + y - 1)^3}\right)}{2} = \text{LN}(x + 1) + c$$

$$\#290:\ |x + 1|\cdot\sqrt{\left(-\frac{x - y + 3}{(x + y - 1)^3}\right)} = \hat{e}^c\cdot(x + 1)$$

$$\#291:\ \text{LIN\_FRAC}\left(\frac{-x + 2\cdot y - 5}{2\cdot x - y + 4}, -1, 2, -5, 2, -1, 4, x, y, 1, 1\right)$$

$$\#292:\ \frac{\text{LN}\left(-\dfrac{(x + 1)^2\cdot(x - y + 3)}{12\cdot(x + y - 1)^3}\right)}{2} - \frac{\pi\cdot\hat{\imath}}{2} = \text{LN}\left(\frac{x + 1}{2}\right)$$

$$\#293:\ \frac{\sqrt{3}\cdot|x + 1|\cdot\sqrt{\left(\dfrac{x - y + 3}{(x + y - 1)^3}\right)}\cdot\text{SIGN}((x + y - 1)\cdot(x - y + 3))}{6} = \frac{x + 1}{2}$$



This is the graph of #293. Building e^(2#292) leads to expression #294 which can be plotted together with the direction field. As you can see a discontinuity seems to appear at (-1|2).*

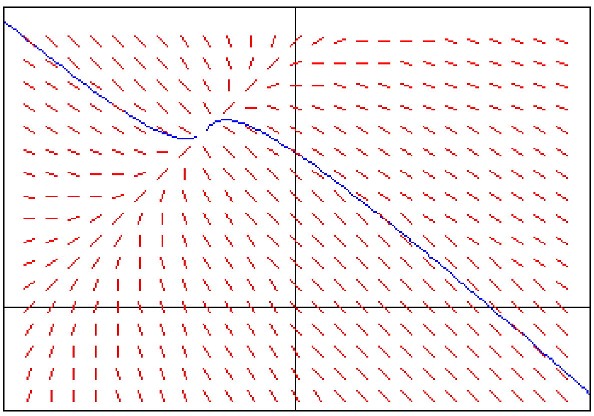
(Let students deduce this point from the task!)

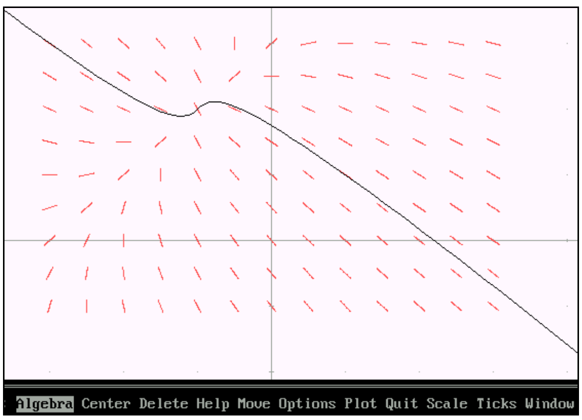
In 1991 we received the same solution – fortunately – but we couldn´t do the implicit plot. So I solved #294 for y and plotted the solutions together with the results of my FELD-function.

$$\#294:\ \frac{x - y + 3}{3\cdot(x + y - 1)^3} = 1$$

$$\#295:\ \text{DIRECTION\_FIELD}\left(\frac{-x + 2\cdot y - 5}{2\cdot x - y + 4}, x, -3, 3, 24, y, -2, 3, 20\right)$$

The right graph is "Derive for DOS made".





Example 16: Clairaut equation $y = x y' + y' = y'^2$

Find the general solution, obtain the singular solution, plot the graph.

Give the special solution(s) containing P(2|1).

We bring the equation into the form $p(x\,v - y) = q(v)$ and then use CLAIRAUT(p,q,x,y,v,c1) taking v as a variable instead of y'.

#296: $y = x\cdot v + v - v^2$

#297: $x\cdot v - y = v^2 - v$

#298: $\text{CLAIRAUT}(x\cdot v - y,\ v^2 - v)$

#299: $\left[c\cdot x - y - c^2 + c = 0,\ x - 2\cdot v + 1 = 0\right]$

#300: $\text{SOLVE}(x - 2\cdot v + 1 = 0,\ v) = \left[v = \frac{x + 1}{2}\right]$

#301: $\text{SUBST}\left(y = x\cdot v + v - v^2,\ v,\ \frac{x + 1}{2}\right)$

#302: $y = \frac{(x + 1)^2}{4}$

#303: $\text{SOLVE}(c\cdot x - y - c^2 + c = 0,\ y)$

#304: $y = c\cdot(x - c + 1)$

#305: $\text{VECTOR}(y = c\cdot(x - c + 1),\ c,\ -3,\ 3,\ 0.5)$

#306: $[2,\ 1]$

#307: $\text{SOLUTIONS}(\text{SUBST}(y = c\cdot(x - c + 1),\ [x,\ y],\ [2,\ 1]),\ c)$

#308: $\left[\frac{\sqrt{5}}{2} + \frac{3}{2},\ \frac{3}{2} - \frac{\sqrt{5}}{2}\right]$

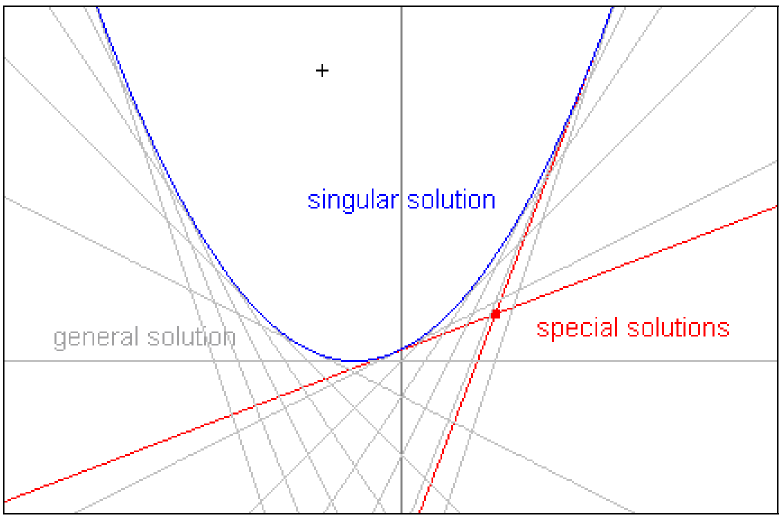
#309: $\text{VECTOR}\left(y = c\cdot(x - c + 1),\ c,\ \left[\frac{\sqrt{5}}{2} + \frac{3}{2},\ \frac{3}{2} - \frac{\sqrt{5}}{2}\right]\right)$

#310: $\left[y = \frac{(\sqrt{5} + 3)\cdot(2\cdot x - \sqrt{5} - 1)}{4},\ y = \frac{(3 - \sqrt{5})\cdot(2\cdot x + \sqrt{5} - 1)}{4}\right]$

In order to find the singular solution we solve the 2nd component of #299 for v and then substitute for v in the given differential equation (#296).

This parabola is the singular solution.

The 1st component of #299 gives the general solution.



The special solution(s) are the two tangents from point P to the singular solution.

Example 17: Find a curve with its tangent´s intercept between the axes having constant length a = 2.
Give both general and singular solutions.

The problem leads to the differential equation $y = x\,y' \pm \frac{2y'}{\sqrt{(1+y'^2)}}$, which is of Clairaut-form.

$$\#311:\quad cl := x\cdot v - y = \frac{2\cdot v}{\sqrt{(1 + v^2)}}$$

$$\#312:\quad \text{CLAIRAUT}\left(x\cdot v - y,\ \frac{2\cdot v}{\sqrt{(1 + v^2)}}\right)$$

$$\#313:\quad \left[\frac{c\cdot x\cdot\sqrt{(c^2 + 1)} - y\cdot\sqrt{(c^2 + 1)} - 2\cdot c}{\sqrt{(c^2 + 1)}} = 0,\ x - \frac{2}{(v^2 + 1)^{3/2}} = 0\right]$$

$$\#314:\quad \text{SOLUTIONS}\left(x - \frac{2}{(v^2 + 1)^{3/2}} = 0,\ v\right)$$

$$\#315:\quad \left[\sqrt{\left(\frac{2^{2/3} - x^{2/3}}{x^{2/3}}\right)},\ -\sqrt{\left(\frac{2^{2/3} - x^{2/3}}{x^{2/3}}\right)}\right]$$

$$\#316:\quad \text{VECTOR}\left(cl,\ v,\ \left[\sqrt{\left(\frac{2^{2/3} - x^{2/3}}{x^{2/3}}\right)},\ -\sqrt{\left(\frac{2^{2/3} - x^{2/3}}{x^{2/3}}\right)}\right]\right)$$

$$\#317:\quad \left[x\cdot\sqrt{\left(\frac{2^{2/3} - x^{2/3}}{x^{2/3}}\right)} - y = 2^{2/3}\cdot x^{1/3}\cdot\sqrt{\left(\frac{2^{2/3} - x^{2/3}}{x^{2/3}}\right)},\ -x\cdot\sqrt{\left(\frac{2^{2/3} - x^{2/3}}{x^{2/3}}\right)} - y = -2^{2/3}\cdot x^{1/3}\cdot\sqrt{\left(\frac{2^{2/3} - x^{2/3}}{x^{2/3}}\right)}\right]$$

$$\#318:\quad y^2 = (2^{2/3} - x^{2/3})^3$$

DERIVE2 was unable to solve #5 for v (see below), so I tried to find another way:

$$\#5:\quad \frac{x\cdot(v^2 + 1)^{3/2} - 2}{(v^2 + 1)^{3/2}} = 0$$

Substitute for v in the second component and in the equation: v = tan(φ).
Solve for x and y to obtain a parameter representation of the singular solution.

$$\#6:\quad x - 2\cdot\text{COS}(\varphi)^2\cdot|\text{COS}(\varphi)| = 0$$

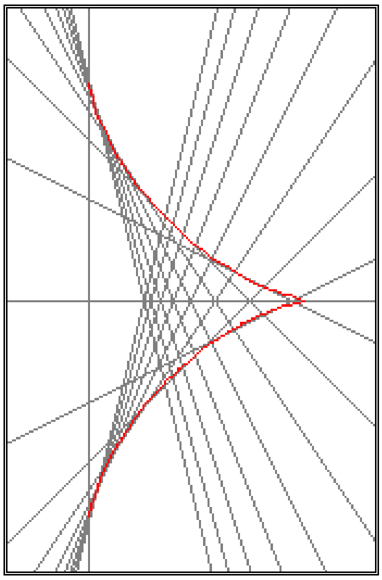
$$\#7:\quad x = 2\cdot\text{COS}(\varphi)^2\cdot|\text{COS}(\varphi)|$$

$$\#8:\quad 2\cdot\text{COS}(\varphi)^2\cdot|\text{COS}(\varphi)|\cdot\text{TAN}(\varphi) - y = \frac{2\cdot\text{TAN}(\varphi)}{\sqrt{(1 + \text{TAN}(\varphi)^2)}}$$

$$\#9:\quad y = -2\cdot\text{SIN}(\varphi)^3\cdot\text{SIGN}(\text{COS}(\varphi))$$

$$\#10:\quad \left[2\cdot\text{COS}(\varphi)^2\cdot|\text{COS}(\varphi)|,\ -2\cdot\text{SIN}(\varphi)^3\cdot\text{SIGN}(\text{COS}(\varphi))\right]$$

#318 (*DERIVE*5) and #10 (*DERIVE*2) give an astroid and the general solution is a family of lines (tangents of the astroid).



$$\#319:\ \mathrm{SOLVE}\left(\frac{c\cdot x\cdot\sqrt{c^2+1}-y\cdot\sqrt{c^2+1}-2\cdot c}{\sqrt{c^2+1}}=0,\ y\right)$$

$$\#320:\ y=\frac{c\cdot(x\cdot\sqrt{c^2+1}-2)}{\sqrt{c^2+1}}$$

$$\#321:\ \mathrm{VECTOR}\left(y=\frac{c\cdot(x\cdot\sqrt{c^2+1}-2)}{\sqrt{c^2+1}},\ c,\ -4,\ 4,\ 0.5\right)$$

Example 18: The following equations of form y = x p(y') + q(y') are called LAGRANGE differential equations.
Use LAGR(p,q,x,y,v,c) to solve the following equations:

a) $y = x\,(1 + y') + y'^2$

b) $y'^3 - 3y' = y - x$

c) $(x\,y' + y)^2 = y^2\,y'$

LAGR(p,q,x,y,v,c) with v instead of y' simplifies to a vector of four components with the last two of them giving the parameter form of the general solution. If one is able to solve the 2nd component for $c1$ one will obtain the singular solution by substituting for $c1$ in the 1st component. LAGR() is not part of ODE1.MTH but of ODE1_EXT.MTH. Look at the examples:

Example 18a

$$\#323:\ \left[y=x\cdot(c1+1)+c1^2,\ c1+1=c1,\ x=c\cdot\hat{e}^{-v}-2\cdot(v-1),\ y=c\cdot\hat{e}^{-v}\cdot(v+1)-v^2+2\right]$$

No singular solution because of $c1 + 1 = c1$.

Integral curves + direction field:

$$\#324:\ \left[c\cdot\hat{e}^{-v}-2\cdot(v-1),\ c\cdot\hat{e}^{-v}\cdot(v+1)-v^2+2\right]$$

$$\#325:\ \mathrm{VECTOR}\left(\left[c\cdot\hat{e}^{-v}-2\cdot(v-1),\ c\cdot\hat{e}^{-v}\cdot(v+1)-v^2+2\right],\ c,\ -5,\ 5,\ 0.5\right)$$

$$\#326:\ \mathrm{SOLVE}(y=x\cdot(1+v)+v^2,\ v)=\left[v=\frac{\sqrt{x^2-4\cdot x+4\cdot y}-x}{2}\ \vee\ v=-\frac{\sqrt{x^2-4\cdot x+4\cdot y}+x}{2}\right]$$

$$\#327:\ \mathrm{DIRECTION\_FIELD}\left(\frac{\sqrt{x^2-4\cdot x+4\cdot y}-x}{2},\ x,\ -5,\ 5,\ 10,\ y,\ -4,\ 4,\ 8\right)$$

$$\#328:\ \mathrm{DIRECTION\_FIELD}\left(\frac{\sqrt{x^2-4\cdot x+4\cdot y}+x}{2},\ x,\ -5,\ 5,\ 10,\ y,\ -4,\ 4,\ 8\right)$$

Example 18b

#329: $\text{LAGR}(1, v^3 - 3\cdot v)$

#330: $\left[y = x + c1^3 - 3\cdot c1,\ 1 = c1,\ x = c + \dfrac{3\cdot v\cdot (v + 2)}{2},\ y = c + \dfrac{v^2\cdot (2\cdot v + 3)}{2}\right]$

#331: $\text{SUBST}(y = x + c1^3 - 3\cdot c1, c1, 1) = (y = x - 2)$ singular solution!!

#332: $\text{VECTOR}\left(\left[c + \dfrac{3\cdot v\cdot (v + 2)}{2},\ c + \dfrac{v^2\cdot (2\cdot v + 3)}{2}\right], c, -3, 3, 0.5\right)$

Example 18c

#333: $\text{SOLVE}((x\cdot v + y)^2 = y^2\cdot v, y) = \left(y = \dfrac{v\cdot x}{\sqrt{v} - 1} \vee y = -\dfrac{v\cdot x}{\sqrt{v} + 1}\right)$

Let´s take the first solution:

#334: $\text{LAGR}\left(\dfrac{v}{\sqrt{v} - 1}, 0\right)$

#335: $\left[y = \dfrac{c1\cdot x}{\sqrt{c1} - 1},\ \dfrac{c1}{\sqrt{c1} - 1} = c1,\ x = \dfrac{c\cdot (\sqrt{v} - 1)}{\sqrt{v}},\ y = c\cdot \sqrt{v}\right]$

#336: $x = \dfrac{c\cdot \left(\dfrac{y}{c} - 1\right)}{\dfrac{y}{c}}$

#337: $\text{SOLVE}\left(x = \dfrac{c\cdot \left(\dfrac{y}{c} - 1\right)}{\dfrac{y}{c}}, y\right)$

Example 18c

After eliminating parameter *v* we find a family of hyperbolas as general solution.

We have two singular solutions (lines).

The result of the LAGRANGE DE derived from the second solution for *y* in #333 can be found on the next page.

#338: $y = \dfrac{c^2}{c - x}$

#339: $\text{VECTOR}\left(y = \dfrac{c^2}{c - x}, c, -3, 3, 0.5\right)$

#340: $\text{SOLUTIONS}\left(\dfrac{c1}{\sqrt{c1} - 1} = c1, c1\right) = [0, 4]$

#341: $\text{VECTOR}\left(y = \dfrac{c1\cdot x}{\sqrt{c1} - 1}, c1, [0, 4]\right)$

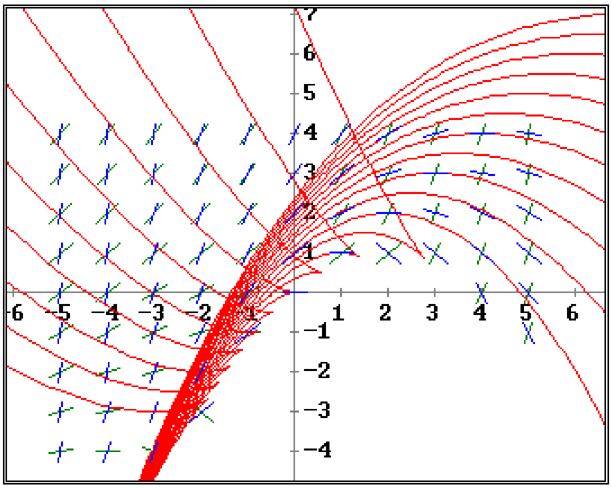
#342: $[y = 0, y = 4\cdot x]$

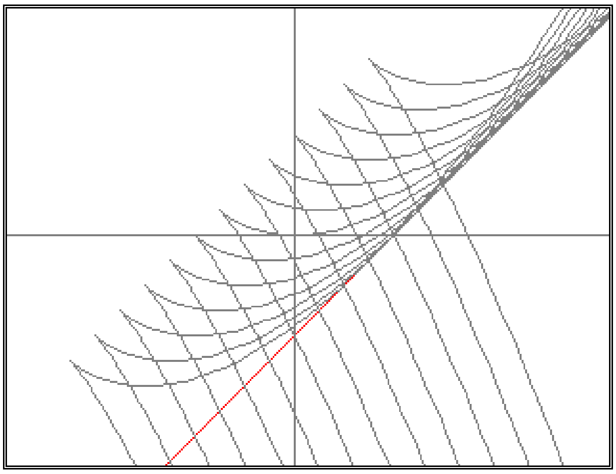
#343: $\text{LAGR}\left(\dfrac{v}{\sqrt{v} + 1}, 0\right)$

#344: $\left[y = \dfrac{c1\cdot x}{\sqrt{c1} + 1},\ \dfrac{c1}{\sqrt{c1} + 1} = c1,\ x = \dfrac{c\cdot ê^{-2/\sqrt{v}}\cdot (\sqrt{v} + 1)}{\sqrt{v}},\ y = c\cdot \sqrt{v}\cdot ê^{-2/\sqrt{v}}\right]$

Example 18a

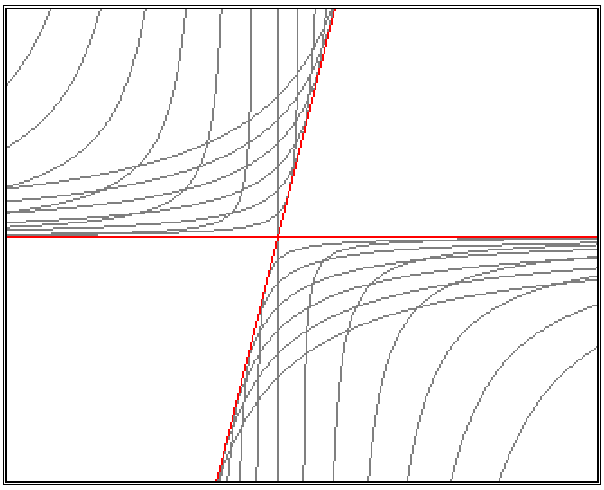
Yo see a part of the direction field and a family of integral curves

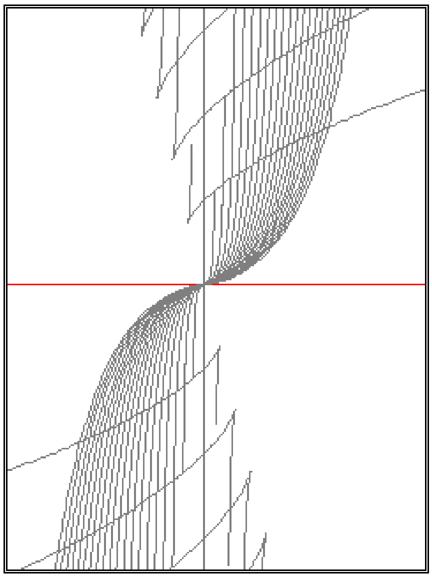




Example 18 b (left)

Example 18 c (below)
the hyperbolas and the other family of solution curves.





References:

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[2] The Calculus Problem Solver, Research & Education Association 1985

[3] Büktas, Aufgabensammlung 2, Diesterweg 1978

129/91 Wien, A
130/91 Korneuburg, A
131/91 Wittlich, D
132/91 Aalen, D
133/91 Edinburgh, GB
134/91 Hofheim/Ts, D
135/91 Bern, CH
136/91 Perg, A
137/91 Remscheid, D
138/91 Schwechat, A
139/91 Ispra, I
140/91 Berlin, D
141/91 Hagfors, S
142/91 München, D
143/91 Lysekil, S
144/91 Dirinon, F
145/91 Wien, A
146/91 Graz, A
147/91 Preetz, A
148/91 Herrmansburg, D
149/91 Heidelberg, D
150/91 Den Haag, NL
151/91 Lüdinghausen, D
152/91 Marseille, F
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158/91 Wartmannstetten, A
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180/91 Petropolis, BRA
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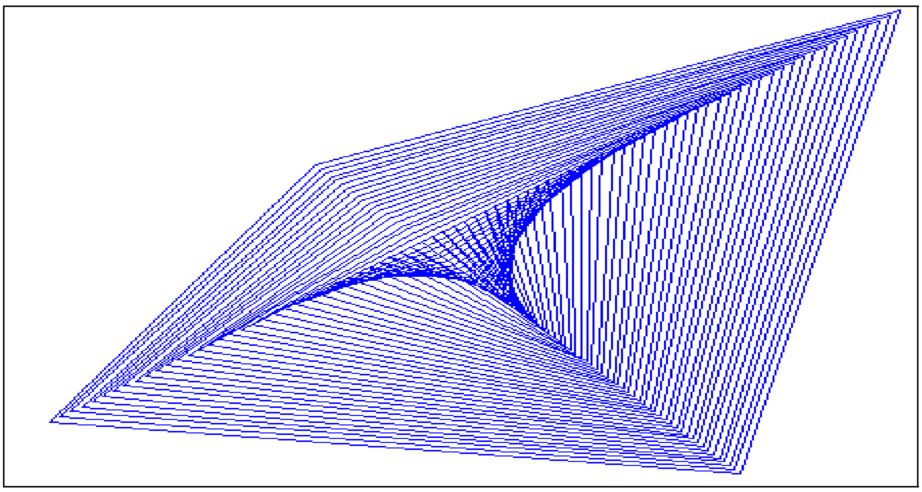
182/91 Burgdorf, CH



$$\mathrm{VECTOR}\left(\begin{bmatrix} t & t+1 \\ -\dfrac{t}{2}+\dfrac{1}{2} & \dfrac{3\cdot t}{4}-1 \\ -t & 1-t \end{bmatrix}, t, -4, 4, 0.1\right)$$