THE DERIVE - NEWSLETTER #5

THE BULLETIN OF THE

DERIVE®

USER GROUP

C o n t e n t s:

revised reprint

March 1992

CALL FOR PAPERS

The British NCET (National Council for Educational Technology) welcomes accounts of the way teachers use DERIVE and how it changes their mathematics curriculae. The work will be considered for publications in microMATH, the journal of the Association of teachers of Mathematics. Manuscripts should be sent to:

Ronnie Goldstein, NCET, Sir William Lyons Road, Science Park,
Coventry CV4 7EZ, UK.

CALL FOR DERIVE-AUTHORS

Chartwell-Bratt is looking for authors for English language workbooks and tutorial guides on using DERIVE at all levels of education.

Philip Yorke, Chartwell-Bratt Ltd, Old Orchard, Bickley Road,
Bromley, Kent, BR1 2NE, England.

The 'DERIVE' Song

(Nicola Cornforth & Alison Mingard, UK)

(Sing to 'She'll be coming round the mountain when she comes')

We will be doing long division till we're dead,
We will be integrating just like teacher said,
We will be getting indigestion,
Working out a question,
Taking all the reasoning as read.

Singing down with "Plug and Chug", let's use DERIVE,
Singing "white box" if you want to stay alive,
Singing down with instrumental,
It's always detrimental,
Understand this and your answers will arrive.

Now we'll be comprehending those techniques that stink,
So this will give us far more time to really think,
Of solving complex type equations,
And saving all the nations,
Then we'll all go down the pub and have a drink.

Singing down with "Plug and Chug", ...

As you can see, while *DERIVE* is doing the work the students are getting more time for the "beautiful arts".

Lieber Derive Anwender!

Vorerst recht herzlichen Dank dafür, dass Sie so zahlreich der DUG die Treue gehalten haben. Viele Grüße haben mich zu Beginn des 2. DUG-Jahres erreicht. Ich danke Ihnen allen auf diesem Weg für ihre Grüße, für Ihre Anregungen und für Ihre anerkennenden Worte. Da aber einige von Ihnen den Mitgliedsbeitrag für 1992 noch nicht entrichtet haben, lege ich als Erinnerung eine Rechnung für die Mitgliedschaft 1992 bei. Bitte begleichen Sie den Betrag, er dient der Finanzierung des DNL. Anderenfalls kündigen Sie bitte Ihre Mitgliedschaft, die Administrierung wird einfacher.

Im besonderen bedanke ich mich für die umfangreichen Beiträge von Frau Casteletti und Herrn Fuchs, und vor allem Mr. Stoutemyer von Soft Warehouse, Inc. Ich freue mich ganz besonders, einen Beitrag von einem der DERIVE-Väter veröffentlichen zu können, noch dazu weil er keinem rein mathematischen Thema gewidmet ist. Besonders interessant dürfte für den nächsten DNL ein Artikel von Prof Fuchs, Salzburg, A, werden. Er verwendet *DERIVE* für die Aussagenlogik. Herrn Appels Ausführungen in diesem Heft beschreiben die Verhältnisse in Deutschland. Wie sieht die Situation in anderen Ländern aus? Vielleicht können wir in einem der nächsten DNL Situationsberichte geben.

Herr Dr. Kayser, Düsseldorf wies in einem Brief auf Initiativen hin. Eine Sammlung von *DERIVE*-bezogenen Artikeln soll in Vorbereitung sein. Ein sehr gelungenes Exemplar finden Sie in der veröffentlichten Literaturliste (MBU 6/91).

Ich habe vor, dem nächsten DNL einen Fragebogen zur Gestaltung des DNL und zur Arbeit der DUG beizulegen. Im April findet das 1. DUG-Meeting in Nottingham statt. Es ist für die UK-Mitglieder gedacht, Gäste sind herzlich willkommen. Ein Anmeldeformular liegt bei. Für den Herbst könnte ich mir ein Treffen der österreichischen DUG-Mitglieder vorstellen. Was halten Sie davon?

Mit den besten Grüßen

Dear Derive User,

First of all I want to thank for your remaining loyalty to the DUG. Many greetings have reached me at the beginning of this 2nd DUG-year. I'd like to thank you for your good wishes, for your suggestions and for your appreciation. As some of you have not paid their membership fee due for 1992 please use the invoice enclosed as a reminder. Please pay the fee, it covers the DNL's costs. Otherwise be so kind as to cancel your membership, it simplifies (!) administration.

In particular I'd like to thank for the extensive contributions from Signora Casteletti, Mr. Fuchs, and specially Mr. Stoutemyer, Soft Warehouse, Inc. I'm delighted to be able to publish an article from one of DERIVE's fathers. It is an article not dedicated to pure mathematics. Mr. Fuchs' article for the next DNL will definitely be really interesting. He uses DERIVE for logic. Mr. Appel's comments in this issue are describe the situation in Germany. What is the situation like in other countries? Maybe we could give a report in one of the next DNLs.

Dr. Kayser calls our attention to initiatives in Germany. A collection of DERIVE concerning articles is in preparation. You will find a very good example in the booklist (page 6; MBU 6/91).

In the next DNL I plan to enclose a questionnaire about the future planning for DNL and DUG. In April the first DUG-meeting especially for UK-members will be held in Nottingham. Guests from outside Great Britain are warmly welcomed. An application form is enclosed. I'm thinking about organizing an Austrian DUG meeting in autumn. What's your opinion?

With my best regards

The Derive-News-Letter is the Bulletin of the Derive-User Group. It is published at least three times a year with a contents of 30 pages minimum. The goals of the D-N-L are to enable the exchange of experiences made with Derive as well as to create a group to discuss the possibilities of new methodical and didactic manners in teaching Mathematics.

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Contributions

Please send all contributions to the Editor. Non-English speakers are encouraged to write their contributions in English to reinforce the international touch of the *DNL*. It must be said, though, that non-English articles will be warmly welcomed nonetheless. Your contributions will be edited but not assessed. By submitting articles the author gives his consent for reprinting it in the *DNL*. The more contributions you will send, the more lively and richer in contents the *DERIVE - Newsletter* will be.

Preview: Contributions waiting to be published

Logic with DERIVE

DERIVE´s Impact on Mechanics

Probability Theory

DERIVE in Italy

will be published in June 1992

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Klaus Fischer, Darmstadt

... Eine Frage oder Anregung: Welche Bücher gibt es inzwischen zu *DERIVE* außer den im Manual aufgeführten und wie kann man diese beziehen - eine hiesige Buchhandlung sah sich außerstande, mir die Bücher aus Amerika zu besorgen. Und als letztes: welche der angegebenen Titel eignen sich für den Oberstufeneinsatz.

Dr. Bernhard Nuß, Bremerhaven

... Ich würde es begrüßen, wenn Sie im DNL eine Übersicht über sämtliche, *DERIVE* betreffende Publikationen (Bücher, Zeitschriften) abdrucken könnten. Die Liste sollte ständig aktualisiert werden. Die 205 Mitglieder in aller Welt könnten gut dazu mitbeitragen, diese Literaturliste immer auf dem aktuellen Stand zu halten. Mir selbst liegen drei Artikel aus didaktischen Fachzeitschriften vor:

D-N-L: Wir haben die Anregung der beiden Herren mit Freude aufgenommen und versuchen eine Literaturliste zu erstellen. Dazu bitten wir alle DUG-Mitglieder um tatkräftige Hilfe. Falls Sie einen entsprechenden Beitrag kennen, teilen Sie uns diesen bitte mit. (Wenn möglich mit Bezugsquelle, Preis und eventuell eine Minibeschreibung des Inhalts). Wir machen in dieser Ausgabe einen Anfang.

Peter Schmidt, Wiesloch

... Mein Jahreswunsch für den DNL wäre, daß auch technische Anwendungen mit *DERIVE* veröffentlicht werden und dafür wünsche ich Ihnen im Neuen Jahr viel Glück.

Dipl.-Ing. Klaus-Peter Daub, Halstenbek

Ich suche einen Gesprächs- bzw. Schreibpartner, der technische Statistik mit *DERIVE* bearbeitet zum Gedankenaustausch. Ansonsten wünsche ich mir, daß der DNL öfter erscheint. Dafür würde ich auch einen höheren Beitrag in Kauf nehmen.
(Anschrift: Luruper Weg 58, D-2083 Halstenbek)

Dr. Dietmar Merkert, Ludwigshafen

Die Anregungen und die Darstellung auf der Diskette finde ich gut, wenn ich auch momentan dies nicht in vollem Umfang ausnutzen kann. In der Schule wird bei uns an den Einsatz dieses Programms gedacht, vorläufig aber nur in demonstrativer Art, da das Geld für die Anschaffung auf jedem Schülerplatz fehlt.

Gerd Heidrich, Nohfelden

Die News-Letter sind phantastisch. Vielen Dank für die Übersendung der Diskette mit den Programmen.
Bei uns ist *DERIVE* weniger bekannt unbd wird an Schulen kaum eingesetzt. Mir persönlich gefällt es sehr gut und in meinem Leistungskurs Mathematik setze ich es sporadisch ein. Demnächst will ich es in meinem Grundkurs Informatik einführen.

Claus M.Brinckmann, Hamburg

... möchte ich Ihnen meinen besonderen Dank sagen, denn ich habe manch Anregung zu eigener Arbeit daraus entnommen.
Es gibt nach meiner Kenntnis in den USA einen DERIVE USER ELECTRONIC BULLETIN BOARD SERVICE, der von Hawaii aus gepflegt wird. Die nicht kontrollierbaren Telefonkosten über den großen Teich haben mich bisher davon abgehalten, mir die Hardware anzuschaffen, um damit Kontakt aufzunehmen. Sehen Sie eine Möglichkeit, die dortigen Informationen auch uns zugänglich zu machen? Können Sie das Soft Warehouse vielleicht dazu anregen, periodisch den DUG's die wesentlichen Erkenntnisse aus dem Service schriftlich zur Verfügung zu stellen? Das wäre als Anhang zu Ihrem NEWS LETTER eine willkommene Ergänzung.

D-N-L: Soft Warehouse Europe wird mit Albert Rich, Hawaii, Kontakt bezüglich des BULLETIN BOARD aufnehmen. Die Idee finde ich großartig und ein Erfolg wäre eine echte Bereicherung.

David R.Stoutemyer, Hawaii

Thanks for sending me the DERIVE Newsletter #3. What a wonderful issue -- so many good ideas. Thanks also for reminding me to send you the mechanical engineering article. The American Society of Mechanical Engineers permits free reprinting, provided the source is fully referenced, so I have also included a copy of the book title page. What you might to do is reprint a section whenever you are short of material.

D-N-L: The DNL is not short of material, but we are glad to receive a contribution from one of the DERIVE developers. We will start printing "Symbolic Computations and Their Impact on Mechanics" in the next issue. Many thanks Mr. Stoutemyer!

StD. Richard Schorn, Kaufbeuren

Im Artikel "With Iterates to the Chaos" scheint mir das Beispiel „Volterra-Zyklus“ unpassend. Das Differentialgleichungssystem läßt sich zwar nicht in geschlossener Form lösen, der in diesem Artikel eingeschlagene Weg zeigt aber nur, daß das verwendete numerische Verfahren völlig ungeeignet ist. Für das System gibt es aber numerisch „stabile“ Verfahren.

Seit ich vor etwa 20 Jahren vom Räuber-Beute-Modell las, habe ich dies Modell in mehreren "Sprachen" und auf vielen Rechnern simuliert. Von den „klassischen“ numerischen Methoden hat sich das Verfahren nach Runge-Kutta am besten bewährt.

Gestern habe ich mit Hilfe der Datei ODE_APPR.MTH (bei DERIVE mitgeliefert) das entsprechende Gleichungssystem mit approX „gelöst“:

$$\dot{x} = a \cdot x - b \cdot x \cdot y \qquad \text{(Hasen, } \textit{Rabbits}\text{)}$$
$$\dot{y} = -c \cdot y + d \cdot x \cdot y \qquad \text{(Füchse, } \textit{Foxes}\text{)}$$

Die Koeffizienten sind hier (aus historischen Gründen) anders gewählt als in dem Artikel. (Der dort verwendete Parameter $a = 1$, führt wegen $x = (1+a)\,x - \ldots$ zu einer Wachstumsrate von 200%).

Nachfolgend der Inhalt der Datei RAEUBEUT.MTH:

```
values := RK([(0.07 - 0.0051·y)·x, (-0.08 + 0.0012·x)·y], [t, x, y], [0, 160, 10], 1, 20(
```

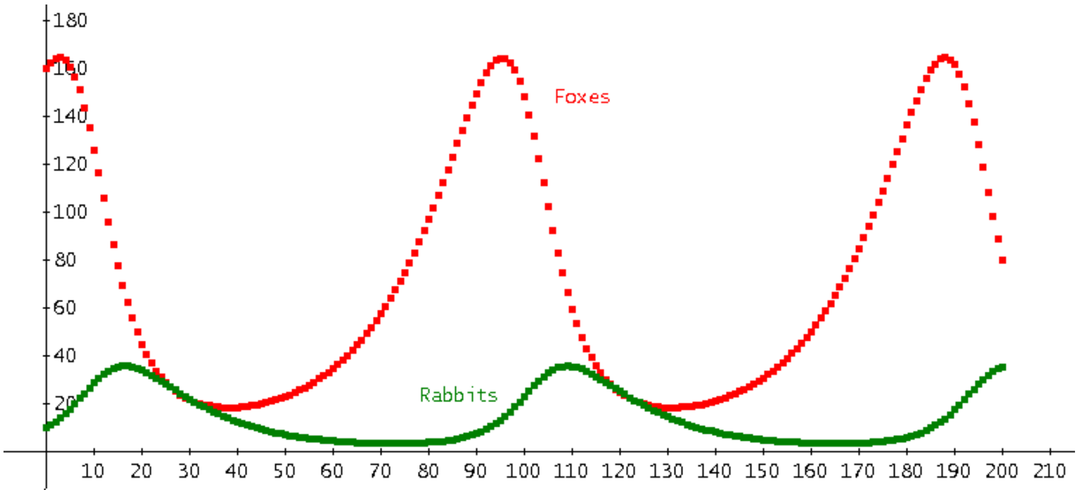
#1 gives the full table.
We extract pairs of columns to plot the development of the various populations.

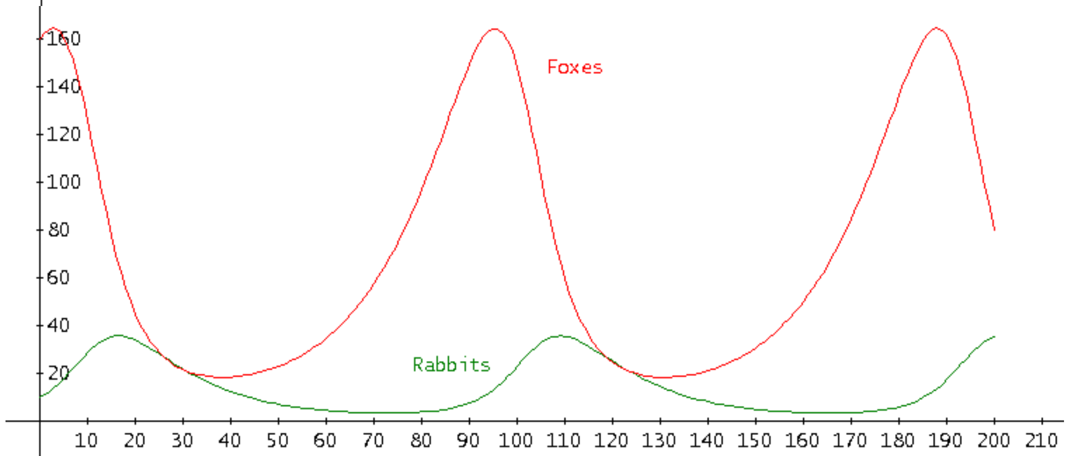
```
values↓↓[1, 2]

values↓↓[1, 3]
```

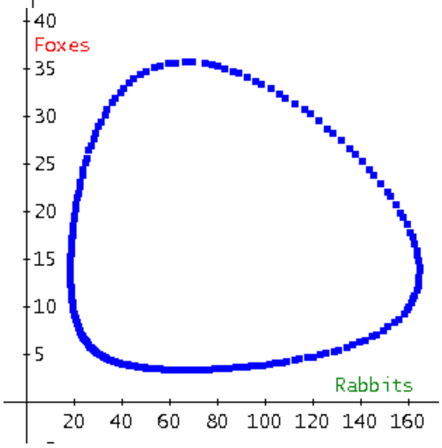
#2 represents the Foxpopulation (red)
#3 represents the Rabbitpopulation (green)

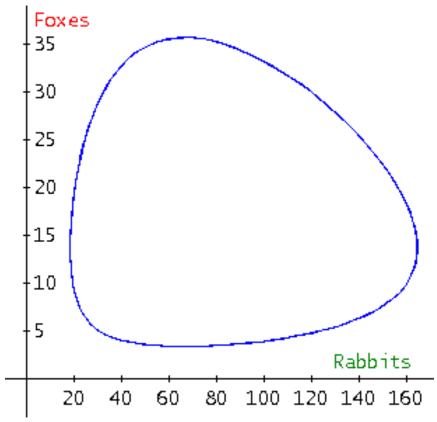
Extracting colums 2 and 3 leads to a phase diagram
which shows the connection between the two populations -
- but we loose information about the time

```
values↓↓[2, 3]
```









A lot of DUG-Members are looking for a collection of DERIVE related books and papers. Here is a list, so far I'm knowing the books and the source of supply.

Engel, Eine Vorstellung von DERIVE in Didaktik der Mathematik
(DdM) 18, Aachen 1990

Weigand u. Weth, Lösen von Abituraufgaben mit DERIVE,
BUS 20, bsv-Verlag München und MNU 44/3

Keil, Formeln, Gleichungen, Funktionen, Graphik mit dem Computer,
BUS 17, bsv-Verlag München

Schnegelberger u. Wynands, DERIVE für den Analysisunterricht in der
Sekundarstufe II, BzM, 1990

Winkelmann, Didaktische Beschreibung mathematischer Software am Beispiel DERIVE,
BzM, Bad Salzdetfurth, 1989

Schönwald, Zur Evaluation von DERIVE, DdM 19, Aachen 1991

Barzel, ANALYSIS-Taylorreihenentwicklung mit DERIVE,
MBU 6/91.1, Bergmoser+Höller, Aachen

Buchberger, Should Students learn Integration Rules?
RISC-Institut, A-4232 Schloß Hagenberg

Davenport-Siret-Tournier, Computer Algebra, Academic Press London 1988

Watkins, A Guide to using DERIVE, Polytechnic South West, Plymouth
Dpt. of Mathematics & Statistics

Horbatsch, Solving Physics Problems in the Classroom with DERIVE 2.0,
Computers in Physics (1990,1991)

The following books are obtainable at Chartwell-Bratt Ltd, Old Orchard, Bickley Road, Bromley,
Kent, BR1 2NE, England:

Arney, DERIVE labratory Manual for Differential Equations,BP15.95

Ellis-Lodi, A Tutorial Introduction to DERIVE, BP10.95

Gilligan-Marquardt, Calculus and the DERIVE Programm: 2nd Ed, BP 13.95

Glynn, Exploring Math from Algebra to Calculus with DERIVE, BP 13.95

Leinbach, Calculus Laboratories Using DERIVE, BP14.95

If anybody knows any other literature concerning DERIVE, then please let me know. So I am able to complete the list.

See also Mr. Appel's announcement on page 21.

One of Mr Setif's favourite items is the number theory. For this reason we are publishing some ideas dealing with prime numbers. I think that these investigations are nice because of a lot of examples for the use of the DERIVE's programming features. In the next D-N-L we will rejoice in a lot of Mr Setif's curve producing ideas.

```
#1:  NPR(n) := NEXT_PRIME(n)
#2:  NTHPR(n) := ITERATE(NPR(k),k,1,n)
#3:  LISPR(n) := ITERATES(NPR(k),k,2,n-1)
#4:  LISPR(10) = [2,3,5,7,11,13,17,19,23,29]
#5:  LISPRFROM(fst,n) := ITERATES(NPR(k),k,NPR(fst),n-1)
#6:  LISPRFROM(100,10) = [101,103,107,109,113,127,131,137,139,149]
```

1992 we could use PRIME as a function name which is not possible now, because prime?() is a DERIVE – function in DfW5 and Dfw6 (see the next two expressions!)

```
#7:  PRIME?(1991) = false
#8:  PRIME?(1993) = true
#9:  PRIM(n) := NPR(n-1) = n
#10: PRIM(1991) = false
#11: PRIM(1993) = true
```

DERIVE 5:

```
PRIM(1991) = (1993 = 1991)
PRIM(1993) = (1993 = 1993)
```

In 1992 we found

```
PRIME(1991) --> 1993 = 1991
PRIME(1993) -->  1993 = 1993
```

Give a list of each 10th prime starting with 1 doing 10 steps:

```
#12: LISPRSKIP(fst,n,nskip) := ITERATES(ITERATE(NPR(k),k,m,nskip),m,fst,n)
#13: LISPRSKIP(1,10,100)
#14: [1,541,1223,1987,2741,3571,4409,5279,6133,6997,7919]
```

Give a list of each second prime starting with 100, doing 10 steps:

```
#15: LISPRSKIP(100,10,2)
#16: [100,103,109,127,137,149,157,167,179,191,197]
```

Which is the prime number preceeding n?

```
     PRPR(n) :=
       If n > 2
          If PRIM(n-1)
#17:         n-1
             PRPR(n-1)
          ?
#18: PRPR(100) = 97
#19: PRPR(7919) = 7907
```

Give 10 prime numbers preceeding 30 in descending order:

```
#20: REVLISPR(start, n) := ITERATES(PRPR(k),k,PRPR(start),n-1)
#21: REVLISPR(30,10) = [29,23,19,17,13,11,7,5,3,2]
```

Give 100 prime numbers preceeding 542 in descending order:

```
#22: REVLISPR(542, 100)
#23: [541,523,521,509,503, ...,11,7,5,3,2]
```

```
        PRECPR(n) :=
          If n = 3
             2
             If n > 3
                If PRIM(n)
                   If PRIM(n - 2)
#24:                  n - 2
                      PRECPR(n - 2)
                   If PRIM(n - 1)
                      n - 1
                      PRECPR(n - 1)
                ?
```

```
#25: REVLISPR1(start, n) := ITERATES(PRECPR(k), k, PRECPR(start), n - 1)
```

Which is the position of a prime number in the ascending sequence of prime numbers?

```
        NBPR_AUX(n, fst) :=
          If n = fst
#26:         1
             1 + NBPR_AUX(n, NPR(fst))

        NBPR(n) :=
          If PRIM(n)
#27:         NBPR_AUX(n, 2)
             ?

#28: NBPR(23) = 9

#29: NBPR(997) = 168
```

997 is the 168th prime number. Check it!

How many primes are between 1 and 542 – excluding the boundaries?

```
#30: NBPRFROMTO(fst, last) := NBPR(PRPR(last)) - NBPR(NPR(fst)) + 1

#31: NBPRFROMTO(10, 20) = 4

#32: NBPRFROMTO(1, 542) = 100
```

Give the list of prime numbers between 1000 and 1050

```
#33: LISPRFROMTO(fst, last) := ITERATES(NPR(k), k, NPR(fst), NBPRFROMTO(fst,
last) - 1)

#34: LISPRFROMTO(1000, 1050)

#35: [1009, 1013, 1019, 1021, 1031, 1033, 1039, 1049]
```

The distance from prime number 1993 to the next is 4 numbers (=1997)

```
        LENPR(n) :=
          If PRIM(n)
#36:         NPR(n) - n
             LENPR(NPR(n))

#37: LENPR(1993) = 4
```

#38: LISPRGAPS(start,n) := ITERATES($\left[\text{NPR}(v_1),\ \text{NPR}(v_1) - v_1\right]$,v,[start,start],n)

```
#39: LISPRGAPS(2, 20)
```

#40: $\begin{bmatrix} 2 & 2 \\ 3 & 1 \\ 5 & 2 \\ \cdots & \cdots \\ \cdots & \cdots \\ 71 & 4 \\ 73 & 2 \end{bmatrix}$ *Give a list of the gaps between the first 20 prime numbers*

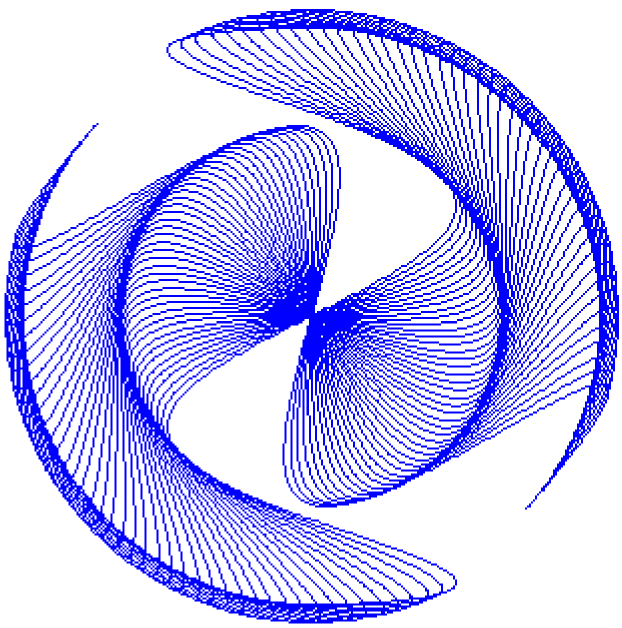
Which is the first pair of twin primes > 13, >500?

```
          TWINPR(n) :=
            If PRIME(n) ∧ PRIME(n + 2)
#41:           [n, n + 2]
               TWINPR(NPR(n))

#42: TWINPR(13) = [17, 19]

#43: TWINPR(500) = [521, 523]
```

Give the list of the next 30 twin primes greater 2

```
#44: LISTWINPR(start, n) := ITERATES(TWINPR(v ), v, TWINPR(start), n)
                                             2

#45: LISTWINPR(2, 30)

        ┌  3    5 ┐
        │  5    7 │
        │ 11   13 │
        │ 17   19 │
#46:     ..........

         ..........
        │ 617  619 │
        │ 641  643 │
        │ 659  661 │
        └ 809  811 ┘

LISPR1(fst, n) := ITERATES([1 + v , NPR(v )], v, [1, fst], n - 1)
                                 1        2

LISPR1(2, 20)

┌  1   2 ┐
│  2   3 │
│  3   5 │
 ........

 ........
│ 18  61 │
│ 19  67 │
└ 20  71 ┘
```

Give the list of the next 20 primes starting with 2



Module on Sequences and Series with *Derive*
Part 2

David E. Hodgkinson, Liverpool

GEOMETRIC SERIES

In the arithmetic sequence and series a common difference was added to each term. In a geometric sequence or series there is a common ratio between consecutive terms. One example is

$$u_{n+1} = 2u_n, \quad u_1 = 1$$

This is entered into *Derive* by

$$\text{ITERATES}([1 + v_1,\ 2 \cdot v_2,\ 2 \cdot v_2 + v_3],\ v,\ [1, 1, 1],\ 6)$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 3 \\ 3 & 4 & 7 \\ 4 & 8 & 15 \\ 5 & 16 & 31 \\ 6 & 32 & 63 \\ 7 & 64 & 127 \end{bmatrix}$$

This gives a sequence for n as

1, 2, 3, 4, 5, 6, 7,

for u as

1, 2, 4, 8, 16, 32, 64

and for s as

1, 3, 7, 15, 31, 63, 127.

Looking at the connections between the sequences it appears that the n^{th} term is 2^{n-1} and the sum to n terms is $2^n - 1$. Testing with

$$\text{VECTOR}(2^{n-1},\ n,\ 7)$$

$$[1, 2, 4, 8, 16, 32, 64]$$

$$\text{VECTOR}(2^{n} - 1,\ n,\ 7)$$

$$[1, 3, 7, 15, 31, 63, 127]$$

Another geometric series arises from the sequence

$$u_{n+1} = 3u_n, \quad u_1 = 1$$

$$\text{ITERATES}([1 + v_1,\ 3 \cdot v_2,\ 3 \cdot v_2 + v_3],\ v,\ [1, 1, 1],\ 6)$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 3 & 9 & 13 \\ 4 & 27 & 40 \\ 5 & 81 & 121 \\ 6 & 243 & 364 \\ 7 & 729 & 1093 \end{bmatrix}$$

This gives a sequence for n as

1, 2, 3, 4, 5, 6, 7,

for u as

1, 3, 9, 27, 81, 243, 729

and for s as

1, 4, 13, 40, 121, 363, 1093.

Again the formula for the n^{th} term is the obvious $u_n = 3^{n-1}$, but the formula for the sum to n i.e. s_n is not clear. Taking a clue from the previous example $s_n = 3^n$ seems worth a try.

$\text{VECTOR}(3^n, n, 7)$

$[3, 9, 27, 81, 243, 729, 2187]$

This sequence is compared with

$$\text{VECTOR}\left(\frac{3^n - 1}{2}, n, 7\right) = [1, 4, 13, 40, 121, 364, 1093]$$

and there does not seem to be a connection except that the terms in the second sequence are about half the terms in the first sequence. Doubling the terms in the second sequence

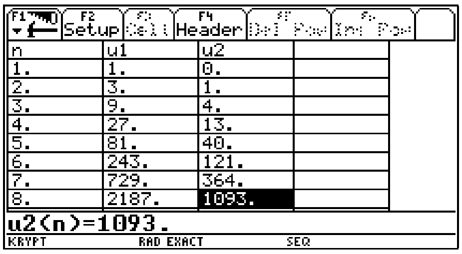
[2, 8, 26, 80, 242, 728, 2186]

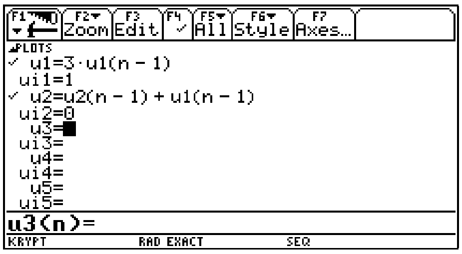
and they are now one less than their corresponding values in the 3^n seqence. This suggests, that $\frac{3^n - 1}{2}$ should give the sum to n terms of the sequence $u_{n+1} = 3u_n, \quad u_1 = 1$

Check with

$$\text{VECTOR}\left(\frac{3^n - 1}{2}, n, 7\right) = [1, 4, 13, 40, 121, 364, 1093]$$

which gives the correct match with the *s*-sequence.





It´s easy to transfer the investigations to the handheld.

Exercises:

1. Find the formula for the sum to n terms of

   (a) $u_{n+1} = \frac{u_n}{2}, \quad u_1 = 1$

   (b) $u_{n+1} = \frac{u_n}{3}, \quad u_1 = 1$

   (c) $u_{n+1} = 2u_n, \quad u_1 = -1$

   (d) $u_{n+1} = 3u_n, \quad u_1 = -2$

   (e) $u_{n+1} = -u_n, \quad u_1 = 1$

   (f) $u_{n+1} = 4u_n, \quad u_1 = 1$

   (g) $u_{n+1} = r \cdot u_n, \quad u_1 = 1$

2. Show that the general formula for the nth term of a geometric series

   $u_{n+1} = r \cdot u_n, \quad u_1 = a$

   with common ratio r and first term a is $u_n = a \cdot r^{n-1}$.

   Show that the general formula for the sum to n terms of the geometric series is $s_n = \frac{a(r^n - 1)}{r - 1}$.

GENERAL SERIES

Any sequence connecting u_{n+1} with u_n will give a series. The special cases of arithmetic and geometric have been considered. There are many other useful and common series, some of which will now be discussed. It is very easy to construct sequences giving series that have sums which are difficult to calculate with a general formula. It is usually much quicker and easier to ask the computer to do the summation to the required number of terms. The sequence

$$u_{n+1} = n \cdot u_n, \quad u_1 = 1$$

$$\text{ITERATES}\left(\left[1 + v_1,\ v_1 \cdot v_2,\ v_1 \cdot v_2 + v_3\right],\ v,\ [1,\ 1,\ 1],\ 6\right)$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \\ 3 & 2 & 4 \\ 4 & 6 & 10 \\ 5 & 24 & 34 \\ 6 & 120 & 154 \\ 7 & 720 & 874 \end{bmatrix}$$

gives a sequence whose nth term is $n!$, but the series formed by adding the factorials does not have an obvious formula for the sum to the nth term whereas the sequence

$$u_{n+1} = \frac{u_n}{n}, \quad u_1 = 1$$

produced by

$$\text{ITERATES}\left(\left[1 + v_1,\ \frac{v_2}{v_1},\ \frac{v_2}{v_1} + v_3\right],\ v,\ [1,\ 1,\ 1],\ 8\right)$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \\ 3 & 0.5 & 2.5 \\ 4 & 0.1666666666 & 2.666666666 \\ 5 & 0.04166666666 & 2.708333333 \\ 6 & 0.008333333333 & 2.716666666 \\ 7 & 0.001388888888 & 2.718055555 \\ 8 & 0.0001984126984 & 2.718253968 \\ 9 & 2.480158730 \cdot 10^{-5} & 2.718278769 \end{bmatrix}$$

gives a sequence whose nth term is $1/n!$, and the series formed by adding the reciprocals of the factorials does not have an obvious formula for the sum to the nth term, but does have a finite sum to infinity. The value of this sum is an irrational number that has a decimal representation that starts 2.718281729... it is a very famous number in mathematics and is called e.

In Derive you can use the special variable 2 + -e i.e. pressing 2 and e together when you want to use e or you can use #e. Similarly use 2 +p or #p for π - or simply type `pi`.

Exercises:

Use the above series for *e* and an iterative technique to find *e* to 4 decimal places. How many terms did you need? By considering 1/7! and 1/8! could you have predicted the number of terms needed for 4 decimal place accuracy. Give your reasons. How many terms would be needed to give *e* correct to 10 decimal places?

The sum of the integers was an arithmetic series, but the sums of the square of the integers is a new type of series.

$$1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2 + \ldots$$

Obviously this series is divergent, but there is a formula which gives the sum to *n* terms. The next trials are to produce this formula.

$$\mathrm{ITERATES}\left(\left[1 + v_1,\ (v_1 + 1)^2,\ (v_1 + 1)^2 + v_3\right],\ v,\ [1, 1, 1],\ 6\right)$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 5 \\ 3 & 9 & 14 \\ 4 & 16 & 30 \\ 5 & 25 & 55 \\ 6 & 36 & 91 \\ 7 & 49 & 140 \end{bmatrix}$$

These steps produce three sequences

(i) for *n*: 1, 2, 3, 4, 5, 6, 7

(ii) for *u*: 1, 4, 9, 16, 25, 36, 49

(iii) for *s*: 1, 5, 14, 30, 55, 91, 140

Finding a formula for the sum to *n* terms involves relating the corresponding values in the *n* sequence and the *s* sequence.

One of the ideas used in this search is to try and gain an impression of how "fast" the sequences are increasing. This means guessing the pattern the sequence is following and then comparing that pattern with a known sequence that is increasing (or decreasing) at a similar rate. Since *s* is increasing at a faster rate than *u* which follows a square law, guess that the sum to *n* terms has a cubic formula.

The general cubic is

$$a\,n^3 + b\,n^2 + c\,n + d$$

so the first four values of *s* are needed to find *a*, *b*, *c* and *d* then the predictions for the fifth, sixth and seventh sums can be tested.

Set up and solve the linear equations for *a*, *b*, *c* and *d*

$$\mathrm{cub}(n) := a \cdot n^3 + b \cdot n^2 + c \cdot n + d$$

$$\mathrm{SOLVE}(\mathrm{cub}(1) = 1 \wedge \mathrm{cub}(2) = 5 \wedge \mathrm{cub}(3) = 14 \wedge \mathrm{cub}(4) = 30,\ [a, b, c, d])$$

$$a = \frac{1}{3} \wedge b = \frac{1}{2} \wedge c = \frac{1}{6} \wedge d = 0$$

This is checked with

$$\text{VECTOR}\left(\frac{1}{3}\cdot n^3 + \frac{1}{2}\cdot n^2 + \frac{1}{6}\cdot n,\ n,\ 7\right)$$

[1, 5, 14, 30, 55, 91, 140]

Exercises:

Try to find a formula for the sum of the cubes of the integers

$$1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3 + 7^3 + \ldots$$

Not surprisingly Derive does have a command called **sum** which calculates many finite or infinite sums e.g. to test your answer to the exercise type **sum(i^3,i,1,n)=** and simplify this expression:

$$\sum_{i=1}^{n} i^3 = \frac{n^2\cdot(n+1)^2}{4}$$

From this example **sum** needs four arguments. The first one is the expression you wish to sum, it should contain the variable, that you wish to sum over. The second argument is the name of this variable and the fourth is the finishing value of your variable. Other examples are

sum(1, i, 1, 10)

since *i* does not appear in the first argument Derive just adds 10 ones.

sum(a, i, 1, 10)

Derive adds 10 *a*'s.

sum(i, i, 1, 10)

sums the integers from 1 to 10.

sum(i^2, i, 1, 10)

sums the squares of the integers from 1 to 10.

sum(i+i^2, i, 1, 10)

sums the integers plus their squares from 1 to 10.

sum(i+i^2, i, 1, n)

indefinite summation of the same expression i.e. the formula for the sum to *n* terms.

1. What is the least nuber of terms of the geometric series

$$\frac{2}{3}+\frac{4}{9}+\frac{8}{27}+\cdots$$

which must be taken to make the sum exceed 1.9999. Use **sum** and then approximate.

2. If u_n denotes the nth term of a series and s_n denotes the sum of the first n terms of the series
   (a) find u_n if $s_n = n(2n^2 - 1)$
   (b) find s_n if $u_n = \dfrac{2}{(n+1)(n+3)}$

   and find the limit to which s_n tends in the latter series if n increases without limit. (Use **`inf`** as the upper limit in the sum.)

Given the formula $f(n)$ it is very easy to substitute integral values of n to get a sequence of values e.g. if $f(n) = n^2 + n + 41$ then substituting values from 1 to 7 gives the sequence

$$[43, 47, 53, 61, 71, 83, 97].$$

However, presented with the above sequence and no clues about how it was produced, the problem of deciding if it is based on a formula and then finding this formula is not as straightforward. In fact given a sequence of numbers there is no guarantee that they are based an a formula and even if they are it may be an arduous task to discover this formula if the sequence is not of a standard type.

There are various general methods for finding $f(n)$ for given sequences, but with Derive there is ample opportunity to try and discover them by trial and error. Usually the more terms that are given the more convincing a formula becomes, but sometimes quite involved mathematical techniques are needed to prove an "obvious" formula is the correct one.

The following exercises are to test your ability at finding general formula, they have all been generated by formula so there is a pattern. Use **`VECTOR`** to produce your sequences. The **`FACTOR`**-command is also useful. The last two are hard. The final one turned up in a research problem and took over a day to break so be warned!

(a) $[1, 4, 27, 256, 3125, \ldots]$

(b) $\left[1, -\dfrac{1}{6}, \dfrac{1}{120}, -\dfrac{1}{5040}, \dfrac{1}{362880}, \cdots\right]$

(c) $\left[1, \dfrac{1}{2}, -\dfrac{1}{8}, \dfrac{1}{16}, -\dfrac{5}{128}, \dfrac{7}{256}, -\dfrac{21}{1024}, \cdots\right]$

(d) $\left[1, \dfrac{9}{8}, \dfrac{625}{192}, \dfrac{117649}{9216}, \dfrac{4782969}{81920}, \dfrac{25937424601}{88473600}, \dfrac{23298085122481}{14863564800}, \cdots\right]$

(e) $\left[3, \dfrac{17}{3}, \dfrac{167}{20}, \dfrac{773}{70}, \dfrac{41555}{3024}, \dfrac{911693}{55440}, \dfrac{2628983}{137280}, \dfrac{54532559}{2494800}, \dfrac{24081124259}{980179200}, \dfrac{4689477731}{171908352}, \cdots\right]$

One of the geometric series above had the form

$1 + 2 + 4 + 8 + 16 + 32$, whilst another had the form

$1 + 1/2 + 1/4 + 1/8 + 1/16 + 1/32$.

Both these series were special cases of $1 + x + x^2 + x^3 + x^4 + \ldots$ with $x = 2$ and $x = 1/2$. Series involving powers of a variable are called power series.

POWER SERIES

The command to produce the power series of an expression if it exists is part of an application of calculus and was widely used by a man called Taylor so the command is

`TAYLOR(expr, var, point, n)`

which returns an *n*th degree truncated power series of **`expr`** with respect to the variable **`var`** expanded about **`point`**. For example the sixth degree Taylor expansion around the origin of e^x is

$$\text{TAYLOR}(e^x, x, 0, 6)$$

$$\frac{x^6}{720}+\frac{x^5}{120}+\frac{x^4}{24}+\frac{x^3}{6}+\frac{x^2}{2}+x+1$$

and a similar one for sin (x) is

$$\text{TAYLOR}(\text{SIN}(x), x, 0, 6)=\frac{x^5}{120}-\frac{x^3}{6}+x$$

When the point of expansion is not the origin then further simplification my be needed.

$$\text{TAYLOR}(\text{LN}(x), x, 1, 3)=\frac{2\cdot x^3-9\cdot x^2+18\cdot x-11}{6}$$

$$\text{TAYLOR}(\text{LN}(x), x, 1, 6)$$

$$-\frac{10\cdot x^6-72\cdot x^5+225\cdot x^4-400\cdot x^3+450\cdot x^2-360\cdot x+147}{60}$$

If you want an expansion in terms of x–1, do

$$\text{TAYLOR}(\text{LN}(u+1), u, 0, 6)$$

$$-\frac{u^6}{6}+\frac{u^5}{5}-\frac{u^4}{4}+\frac{u^3}{3}-\frac{u^2}{2}+u$$

$$-\frac{(x-1)^6}{6}+\frac{(x-1)^5}{5}-\frac{(x-1)^4}{4}+\frac{(x-1)^3}{3}-\frac{(x-1)^2}{2}+(x-1)$$

The Taylor function can be used to give the binomial expansion for fractional and negative exponents.

To find the first five terms of $\frac{1}{1+2x}$, $\left(1+\frac{x}{2}\right)^{\frac{2}{3}}$ use

$$\text{TAYLOR}\left(\frac{1}{1+2\cdot x}, x, 4\right)=16\cdot x^4-8\cdot x^3+4\cdot x^2-2\cdot x+1$$

$$\mathrm{TAYLOR}\left(\left(1+\frac{x}{2}\right)^{2/3},\ x,\ 4\right) = -\frac{7\cdot x^4}{3888}+\frac{x^3}{162}-\frac{x^2}{36}+\frac{x}{3}+1$$

Exercises:

Find the following

(a) the fourth term in the expansion of $(1+3x)^{-\frac{1}{2}}$

(b) the fifth term in the expansion of $(2-5x)^{-6}$

(c) the sixth term in the expansion of $\frac{1}{1-5x}$

Another facility is the use of Taylor series to find values of the elementary functions. The following example hints at the potential of Derive and the notion of convergence in infinite series. The problem is to find the value of cos(20) from its Taylor series, without angle reduction, where the 20 is in radian measure. Most calculators will fail at this value. The Taylor expansion to 10 terms about origin is generated and $x = 20$ is substituted.

$$\mathrm{TAYLOR}(\mathrm{COS}(x),\ x,\ 10)$$

$$-\frac{x^{10}}{3628800}+\frac{x^8}{40320}-\frac{x^6}{720}+\frac{x^4}{24}-\frac{x^2}{2}+1$$

$$-\frac{1286732833}{567}$$

Substituting for $x = 20$ gives

$$-2.269370075\cdot 10^{6}$$

Since -1 ≤ cos x ≤ 1 this is obviously incorrect. So more terms have to be generated. There is however a quicker way to generate power series, since the coefficients of the power series form a sequence which can be generated from the formula

$$\sum_{n=0}^{10}\frac{(-1)^n x^{2n}}{(2n)!}$$

$$\sum_{n=0}^{5}\frac{(-1)^n\cdot x^{2\cdot n}}{(2\cdot n)!} = -\frac{x^{10}}{3628800}+\frac{x^8}{40320}-\frac{x^6}{720}+\frac{x^4}{24}-\frac{x^2}{2}+1$$

hence up to x^{20}

$$\sum_{n=0}^{10}\frac{(-1)^n\cdot x^{2\cdot n}}{(2\cdot n)!} = \frac{x^{20}}{2432902008176640000}-\frac{x^{18}}{6402373705728000}+\frac{x^{16}}{20922789888000}-\frac{x^{14}}{87178291200}+\frac{x^{12}}{479001600}$$

$$-\frac{x^{10}}{3628800}+\frac{x^8}{40320}-\frac{x^6}{720}+\frac{x^4}{24}-\frac{x^2}{2}+1$$

$$2.09658\cdot 10^{7}$$

even worse so try up to x^{40}

$$\mathrm{SUBST}\left(\sum_{n=0}^{20} \frac{(-1)^n \cdot x^{2 \cdot n}}{(2 \cdot n)!}, x, 20\right)$$

$$2577.30$$

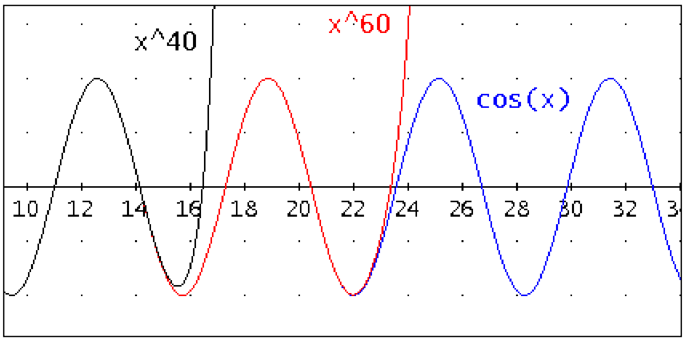
getting better so try up to x^{60}:

$$\mathrm{APPROX}\left(\mathrm{SUBST}\left(\sum_{n=0}^{30} \frac{(-1)^n \cdot x^{2 \cdot n}}{(2 \cdot n)!}, x, 20\right)\right) = 0.408075$$

well at least it is less than 1. There is still doubt whether it has converged. So try up to x^{70}, x^{80}, ... until successive results agree.

$$\mathrm{APPROX}\left(\mathrm{VECTOR}\left(\left[2 \cdot k, \mathrm{SUBST}\left(\sum_{n=0}^{k} \frac{(-1)^n \cdot x^{2 \cdot n}}{(2 \cdot n)!}, x, 20\right)\right], k, 26, 36, 2\right)\right)$$

52	0.477065
56	0.409162
60	0.408075
64	0.408061
68	0.408061
72	0.408061



Exercises

1. The Derive function **ATAN(X)** represents the inverse tangent.
   Find **ATAN(X)**'s power series and show that SUM provides a quicker way of generating it.

2. Find π correct to two decimal places using $\pi/4 = \arctan(1)$.
   Estimate the number of terms needed to calculate π to 20 decimal places.

3. Use **Machin´s formula**

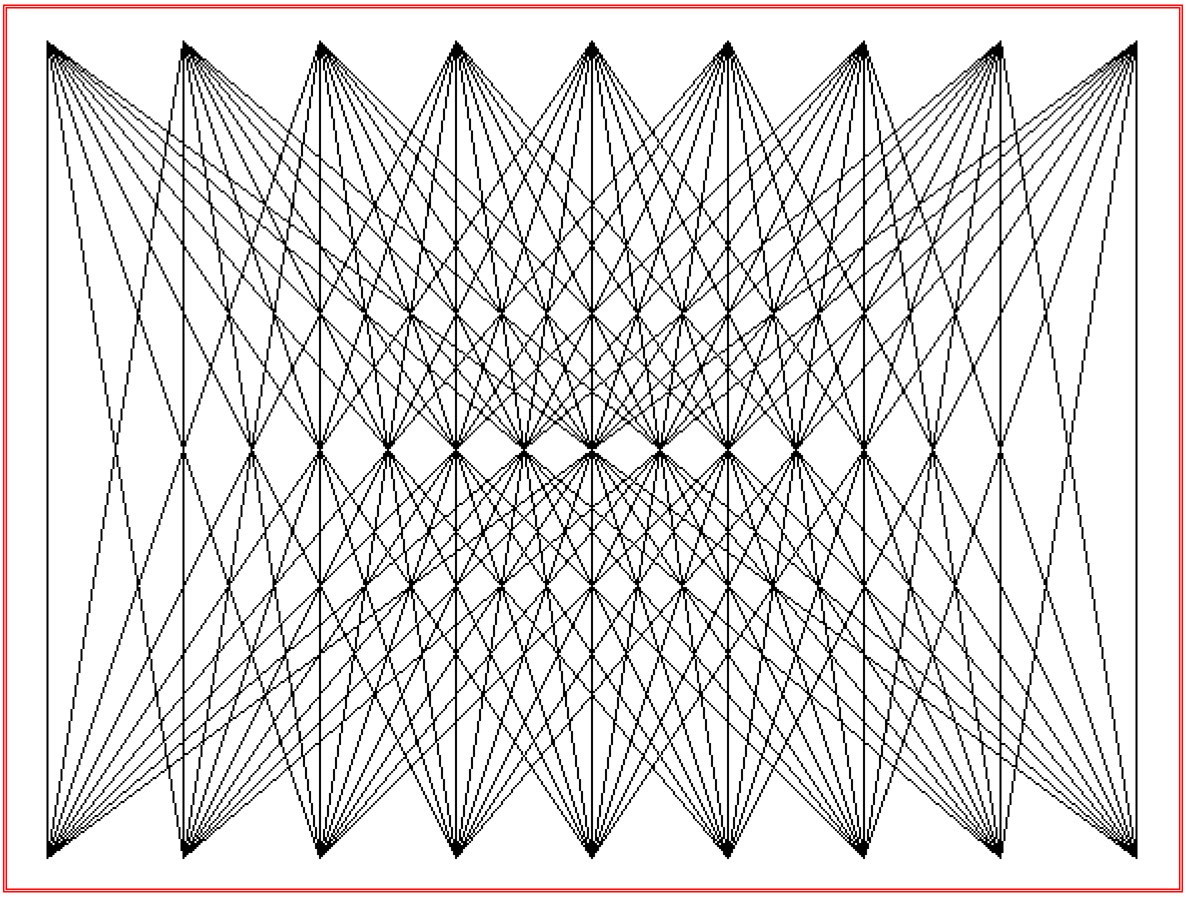
$$\frac{\pi}{4} = 4 \arctan\frac{1}{5} - \arctan\frac{1}{129}$$

   to calculate π to 20, 40 decimal places.

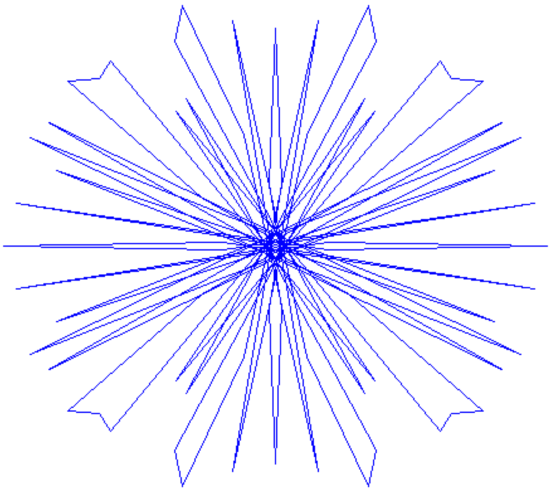
4. Use **Störmer´s formula**

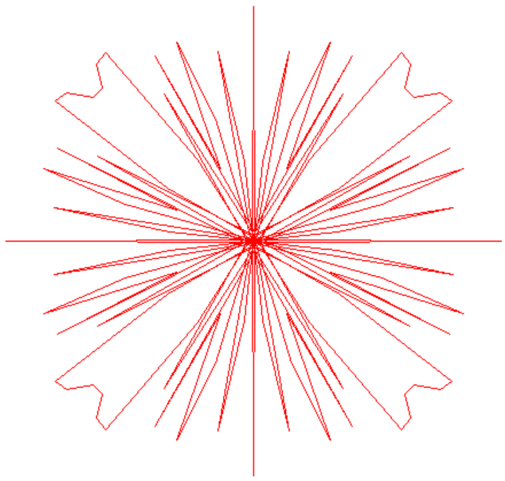
$$\pi = 24 \arctan\frac{1}{8} + 8 \arctan\frac{1}{57} + 4 \arctan\frac{1}{239}.$$

5. Knowing π to more than six decimal places, improve the previous caculation for cos(20).



$$\mathrm{VECTOR}\left(\mathrm{VECTOR}\left(\begin{bmatrix} -4 + k & 3 \\ -4 + j & -3 \end{bmatrix}, j, 0, 8\right), k, 0, 8\right)$$





$$P(r, s, c, n) := \left[r \cdot \mathrm{COS}\left(s \cdot \mathrm{SIN}\left(\frac{2 \cdot \pi \cdot c \cdot k}{n}\right)\right) \cdot \mathrm{COS}\left(\frac{2 \cdot \pi \cdot k}{n}\right), r \cdot \mathrm{COS}\left(s \cdot \mathrm{SIN}\left(\frac{2 \cdot \pi \cdot c \cdot k}{n}\right)\right) \cdot \mathrm{SIN}\left(\frac{2 \cdot \pi \cdot k}{n}\right)\right]$$

STAR(r, s, c, n) := VECTOR(P(r, s, c, n), k, 0, n)

STAR(3, 200.5, 35, 72)

STAR(3, 175, 40, 108)

Zur Problematik des Einsatzes von *DERIVE* im Mathematikunterricht.

Herbert Appel, Schweinfurt

DERIVE ist ein ungewöhnlich leistungsfähiges Programm, darüber sind wir uns vermutlich alle einig. Trotz der großen Vielfalt der Möglichkeiten ist es auch schnell und einfach zu bedienen. In seinen sonstigen Fähigkeiten erfüllt es weitgehend die Forderungen, die Herr A. Engel (in DM 1990/3 S. 165-244) an solche Programme gestellt hat.
Der Benutzer kann sich auf den Problemlöseprozess konzentrieren und wird vom „Handwerk Mathematik" weitestgehend befreit. Es gibt kaum Zweifel daran, daß diese Art von Programmen, ähnlich dem Taschenrechner, in nicht allzu ferner Zukunft ihren festen Platz auch in der Hand des Schülers finden wird.

Die Wirklichkeit sieht jedoch momentan noch etwas düster aus. Wir sind weit davon entfernt, daß jeder Schüler an seinem Arbeitsplatz in der Schule oder zu Hause einen Computer stehen hat und beliebig darauf zugreifen kann. Der Einsatz von *DERIVE* wird sich also auf einzelne, ausgewählte Stunden beschränken. Hierbei sehe ich zur Zeit zwei Möglichkeiten:

1. Die Schule ist im Besitz eines Panels und der Lehrer trägt Computer und Panel von Klassenzimmer zu Klassenzimmer, um dort mit den Schülern das Programm zum Lösen von Übungsaufgaben sinnvoll einzusetzen und daraus den gewollten Nutzen zu ziehen oder
2. der Lehrer „wandert" mit seiner Klasse in den Computerraum, opfert eine gewisse Zahl von Stunden, um die Schüler in den Gebrauch des Programms einzuweisen und läßt die Schüler anschließend in Gruppen Aufgaben (damit sind nicht einfache Termumformungen gemeint) lösen.

Ganz abgesehen von den oranisatorischen Problemen, die die beiden Verfahrensweisen mit sich bringen, bietet der Lehrplan der S I bzw. S II nicht gerade viele Ansatzpunkte für den Einsatz von *DERIVE*. Eine Möglichkeit, die momentane „Saure-Gurken-Zeit" zu überbrücken, ist die Einrichtung von speziellen Kursen (Wahlfach - Praktische Mathematik), in denen „Ausgewählte Kapitel" der Mathematik besprochen werden.

In diesem Zusammenhang verweise ich auf die Anlage 1 zum KMS vom M„rz 1990 Nr. VII/3-S6400/7-10/21 221, in der Herr StD. Steiner und Herr StD. Kappl vom Staatsinstitut für die Ausbildung von Lehrern an Realschulen Themenvorschläge für das Wahlfach bzw. Wahlpflichtfach Mathematik machen.

„Die vorgeschlagenen Themen berücksichtigen einerseits Inhalte auf der Basis der geltenden Lehrpläne, beinhalten aber andererseits auch Inhalte, die als Ergänzung oder Erweiterung zu den traditionellen Lerninhalten über den Lehrplan hinausgehen, deren Behandlung in der Realschule (Anm. d. Autors: gilt sicher auch für S I und S II) aber möglich und sinnvoll erscheint." *Anlage 1 zum KMS*

Es folgt eine Liste mit ca. 30 nach Jahrgangsstufen geordneten Themenvorschlägen, auf die ich jetzt im Einzelnen nicht eingehen möchte. Herr Steiner und Herr Kappl legen jedoch größten Wert darauf, daß der Rechner nicht im Vordergrund steht, sondern lediglich ein Hilfsmittel darstellt. Der Problemlöseprozeß wird von beiden in den Vordergrund gestellt. Als sehr wichtig erscheint mir jedoch der Passus 6 (Jgst. 10):

„Arbeiten mit je einem ausgewählten Standard-Programm für algebraisch-analytische bzw. für geometrisch-graphische Problemstellungen. Dem Schüler sollen dabei die Leistungsfähigkeit, Anwendungsmöglichkeiten, aber auch Grenzen dieser Programme bewußt werden.“ *Anlage 1 zum KMS*

Vielleicht bietet sich durch die Wahlfächer die Möglichkeit, *DERIVE* durch die Hintertür in den „normalen“ Unericht zu bringen. Die Ausgestaltung der Themen ist nur wenig reglementiert und lässt dem Lehrer (und den Schülern) viele Freiheiten.

Da ich glaube, mit der *DUG* bzw. *D-N-L* ein Forum gefunden zu haben, in dem sich Pioniere, Enthusiasten und auch Realisten vereinigen, sollen in der Zukunft des öfteren Artikel zu Themen erscheinen, die in der o.g. Anlage zum KMS enthalten sind und die an unserer Schule in einem Wahlfach behandelt wurden. Vielleicht wird doch der eine oder andere Lehrer ermuntert, den ersten Schritt zu tun.

Zu guterletzt sei noch ein Appell an den Hersteller von *DERIVE* gerichtet.

Möglicherweise ist es noch nicht ganz durchgedrungen, aber die Staatskassen und die Kassen der Schulaufwandsträger sind leer. Es bereitet unendlich viel Mühe, die Schulleitung einer Schule davon zu überzeugen, daß die Anschaffung einer 10er Version von *DERIVE* für ca. 1600 DM eine sinnvolle Ausgabe darstellt. Die Schulaufwandsträger stellen sich auf den Standpunkt, daß ein Programm für eine Schule nicht teurer sein darf als 20-30 DM.

Eine mögliche Weltfremdheit ist zwar nicht ganz abzustreiten, aber das Beispiel „Textmaker“ zeigt, daß alles möglich ist, wenn man nur will. Der Hersteller bietet Schulen die Möglichkeit, das Programm (Normalpreis ca 300 DM) zum Preis von 1500 DM für etwa 30 Rechner zu beziehen. Eingeschlossen ist eine Lehrerlizenz; das heißt, daß alle an der Schule tätigen Lehrer das Programm auch zu Hause benutzen dürfen. Umgerechnet macht das wirklich nur 30 DM pro Lizenz.

Aber sehen Sie es bitte von einer anderen Seite; wir Lehrer dienen durch die Benutzung des Programms im Unterricht als Multiplikatoren, und die Schüler, die das Programm kennen und schätzen gelernt haben, sitzen heute zwar noch auf der Schulbank, aber morgen schon in Führungspositionen von Firmen und entscheiden vielleicht auch über die Anschaffung von Programmen wie *DERIVE*.

****

Herr Appel bat mich, im D-N-L folgendes zu veröffentlichen:

An der Realschule Forchheim wurden im Schuljahr 1991/92 Unterrichtssequenzen zum Thema „Möglichkeiten des Einsatzes des Programmes DERIVE im Mathematikunterricht“ durchgeführt. Die Lerninhalte entstammen dem Lehrplan der 9. und 10. Jahrgangsstufen. Die aus den Unterrichtsversuchen gewonnenen Erkenntnisse wurden in einem ca. 50 Seiten umfassenden Bericht festgehalten. Weiterhin wurden sowohl methodisch-didaktische als auch softwareergonomische Gesichtspunkte berücksichtigt. Die Ergebnisse der Untersuchung dürften auch auf das Gymnasium übertragbar sein. Interessierte können den Bericht bei mir zum Preis von DM 30.- erwerben.

(H.Appel, Hauptstr. 27, D-8720 Schweinfurt; Ktonr.654129 bei der Städt. Sprakasse Schweinfurt, BLZ 793 500 00.)

.... In issue number 1 you say 'We are interested in the application of *DERIVE* on Mathematics teaching methods'. Because of *DERIVE* I have altered my approach to the start of the A level course in mathematics for 17 and 18 year old pupils, and a possible contribution to your magazine is enclosed.

Also enclosed are teaching notes I wrote for myself as a guide, but I think (or hope) that the methods are explained in the contribution. .....

.... It is an excellent idea to have a news letter and I hope you get many contributions.

Finding a Gradient

R. W. Southward

This article describes the use of *DERIVE* with a small group of students starting an A level course in England.

A typical textbook will have chapters which include the expansion of $(a+b)^k$ for $k \in$ N, the use of fractional and negative indices, the equation of a line and its gradient, and conditions for real roots of a quadratic. These topics will be placed before differentiation. In some examination syllabuses the pupil need not have to show how to differentiate, for example x^3, by considering

$$\lim_{\delta \to 0} \frac{(x+\delta)^3 - x^3}{(x+\delta) - x},$$

but most teachers would like to introduce some explanation of this kind. But pupils tend to have difficulty with algebraic manipulation, less of which is now required in the GCSE examination they take at age 16.

The pupils have however had experience of drawing tangents and estimating gradients of curves, and of finding maximum values of expressions by calculating values and drawing graphs. In the latter case a problem could be set for example an finding the maximum area given a shape with a fixed perimeter.

So we start these problems. With *DERIVE* we can reverse the order in which we supply to pupils the mathematical techniques to solve a problem. We can start with the problem of finding a gradient, obtain a solution and use it, and then consider why it works. When we do this we raise other problems.

Thus using *DERIVE* numerical approximations to the gradient are found at particular points on, for example, $y = x^3$. Then for a particular point such as (5,125) the expression $\frac{(5+\delta)^3 - 5^3}{(5+\delta) - 5}$ is calculated and the limit found.

Pupils can be asked to suggest a rule for calculating the gradient.

Finally, $\frac{(k+\delta)^3 - k^3}{(k+\delta) - k}$ is calculated. At each stage pupils can do the calculations using *DERIVE*,and the rules for differentiating are obtained before any algebra is done maually. After all, the rule "$n\,x^{n-1}$ is obtained when differentiating x^n ($n \in$ N)" is easy to remember.

Using this knowledge we can tackle some problems involving maximum and minimum values. With *DERIVE* the graph can be drawn, we can zoom in and see the gradient is zero, and any equations can be solved.

Now questions can be asked, and the computer does provoke enquiry. Perhaps pupils realize that they have to do what DERIVE does in order to pass an examination, but, perhaps surprisingly, they don't say 'Why do it if the computer can?' but rather 'How, why does it do that?'
Questions can be asked, by pupil or teacher, such as

- How do you work out $\dfrac{(k+\delta)^3 - k^3}{(k+\delta)-k}$?
- How do you expand $(k+\delta)^3$?
- What is the general result?
- How do you differentiate $\dfrac{1}{x}$? Is there a rule?
- Can you plot te tangent? and What is its equation?

This approach would not be possible without *DERIVE*. It enables pupils to have an overall view without being confused by detail. The teacher can start with a problem and its solution and work backwards finding out about steps carried out by *DERIVE*. Other topics may be capable of being taught by this method rather than first teaching preliminary technique, but this is of course a matter for the judgement of the teacher faced with a particular class at a particular time. At least with *DERIVE* you have a choice.

[Mr. Southward's teaching notes are included, because I think it is interesting to see how it is done (editor).]

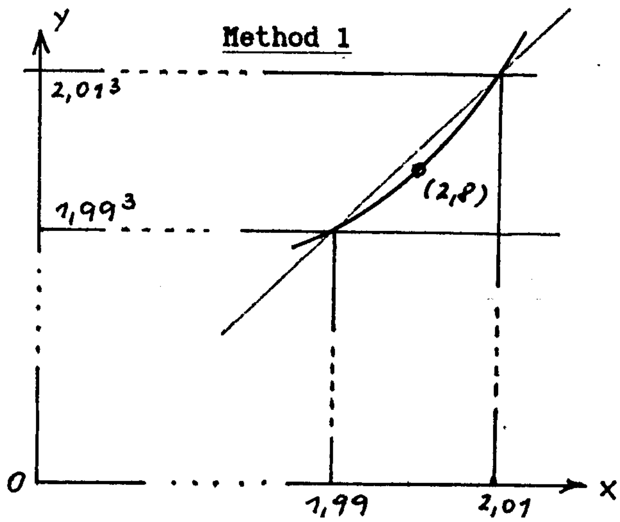
Topic:	**Gradient Function**
User Manual:	**Pages 111 – 113**
Textbook:	

How can we calculate the gradient of a tangent to a curve?

We could estimate it by drawing the curve, drawing the tangent, and finding the gradient of this tangent. How can we be more precise?

We will start with $y = x^3$

and with the point $x = 2$, $y = 8$.



The gradient of the tangent at (2,8) will be approximately the gradient of the chord joining $(1.99, 1.99^3)$ to $(2.01, 2.01^3)$.

The gradient of the chord is $\dfrac{2.01^3 - 1.99^3}{2.01 - 1.99}$.

```
(2.01^3-1.99^3)/0.02
```

$$\#1: \quad \frac{2.01^3 - 1.99^3}{0.02}$$

$$\#2: \quad 12.00009999$$

Try this method for the gradient of the tangent to $y = x^3$ at (4, 64).

You need to calculate e.g. $\frac{4.01^3 - 3.99^3}{4.01 - 3.99}$.

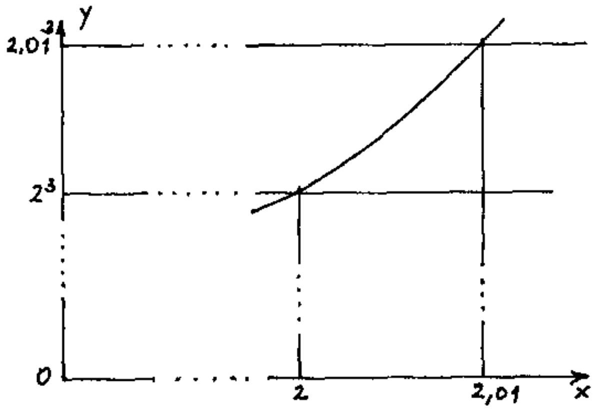
Gradients can be estimated this way for other curves such as $y = x^2$, $y = x^4$, $y = \sin x$ (but $y = x^2$ is special).

Find/Estimate the gradients for these when $x = 5$.
(the smaller the interval on the x-axis, then the better the estimation (?)).

Method 2 is similar.

For the example above.

Estimate of gradient is $\frac{2.01^3 - 2^3}{2.01 - 2}$.



For the curves we consider you can use either method.

1. Find a formula for the gradient of the tangent to $y = x^2$ at the point where $x = k$. (We have already found that if $y = 5$, gradient is 10).
2. Find a formula for the gradient of $y = x^3$ at the point $x = k$.
   Can you find other formulae?
3. We will show that the gradient of $y = x^4$ at $x = 7$ is 1372.

   Our estimate is $\frac{(7+\delta)^4 - (7-\delta)^4}{(7+\delta)-(7-\delta)}$ where δ is a diddly bit.

$$\frac{(7 + \delta)^4 - (7 - \delta)^4}{(7 + \delta) - (7 - \delta)} = 28\cdot(\delta^2 + 49)$$

   As δ gets smaller, so does δ^2, and the gradient gets nearer to 28 (0 + 49) = 1372.

   We can put this one expression by `lim(28·(δ^2 + 49),δ,0)=` which gives the limit, as δ tends to 0. Notice the way it is displayed.

$$\lim_{\delta \to 0} 28\cdot(\delta^2 + 49) = 1372$$

4. But why choose $x = 7$? Let us choose $x = k$.

   The gradient will be approximately $\dfrac{(k+\delta)^4-(k-\delta)^4}{(k+\delta)-(k-\delta)}$.

   Author this expression which will be highlighted and then using the F3-key insert it into the lim-expression : `lim(F3–Key,δ,0)` and then simplify to get the formula $4k^3$ for the gradient of x^4 at the point $x = k$.

$$\frac{(k+\delta)^4-(k-\delta)^4}{(k+\delta)-(k-\delta)}$$

$$\lim_{\delta\to 0}\frac{(k+\delta)^4-(k-\delta)^4}{(k+\delta)-(k-\delta)}$$

$$4\cdot k^3$$

5. But why choose $y = x^4$ as the curve?.
   Why not $y = x^6$?

   Find a formula for the gradient of $y = x^6$ at $x = k$.

6. Find a formula for the gradient at a point on sin x.

   You need to find $\lim\limits_{\delta\to 0}\dfrac{\sin(k+\delta)-\sin(k-\delta)}{(k+\delta)-(k-\delta)}$.

7. For a function f, to find the gradient at a point k, we need to calculate

$$\lim_{\delta\to 0}\frac{f(k+\delta)-f(k-\delta)}{(k+\delta)-(k-\delta)} \qquad (*)$$

   First make sure f is a function, then author (∗), define the function and finally highlight expression (∗) and simplify to obtain $3k^2 + 1$.

$$f(x) :=$$

$$\lim_{\delta\to 0}\frac{f(k+\delta)-f(k-\delta)}{(k+\delta)-(k-\delta)}$$

$$f(x) := x^3 + x$$

$$3\cdot k^2 + 1$$

Try other functions.

Write down rules for finding gradient functions without *DERIVE*.

Problem Solving with DERIVE

N.Pitcher & M.Johnson, Paisley

1. Introduction

The main driving force behind the development of DERIVE is to make mathematics more enjoyable by eliminating the drudgery of long mathematical calculations, so that the user can be free to look at interesting problems.

As soon as any mathematical educator acquires DERIVE the question arises, how best, and at what level, to use it among students? It is contended in this article that DERIVE can be used within basic Algebra and Calculus courses, both to give insight into concepts and to widen the student's perception of mathematics in its applicability to real world problems.

In this sense the availability of DERIVE raises a whole new agenda of finding problems and new examples for use among students who are DERIVE-literate. It is likely to take some time for such problems to be identified and used effectively. We should, however, take this as an opportunity to move on from some of the more artificial problems in traditional Calculus courses (such as farmers anxious to enclose maximum areas within perimeter fences - has anyone ever met such a farmer?) and to include problems which are a little nearer to reality.

In what follows a description is given of a problem in applied geometry which can be given as an exercise to students, in which DERIVE can be used to carry out all the computational tasks.

2. The Problem

Imagine two underground tunnels, located within a Cartesian coordinate system, where tunnel A has a circular cross-section which passes through the points

a = (1.268, 4.732, -10.732), b = (3.424, 3, -11.97), c = (3, 2.628, -6.023);

tunnel B is a horizontal cylinder of radius 2 with axis passing through the point p = (-12, -5, -6), parallel to the y-axis. The land surface is horizontal and this is the x-y plane, and the z-axis is vertically upwards.

The first part of the problem is to show that the tunnels do not intersect, and the second is to determine a new position for tunnel B, still parallel to the y-axis, so that the axes of the two cylinders do intersect. A solution is required which corresponds to minimum displacement of tunnel B from its original position. Finally it is required to describe and locate the curve on the land surface where tunnel A intersects. We shall demonstrate that this entire problem can be solved using DERIVE, including all calculations and a sketch of the curve of intersection with the land surface.

3. The Solution

As a first step it is necessary to obtain, for tunnel A its radius, the direction of its axis and a point on its axis. This is done, first by calculating the normalised cross product

$$\vec{u} = \frac{(\vec{b}-\vec{a})\times(\vec{c}-\vec{b})}{\left|(\vec{b}-\vec{a})\times(\vec{c}-\vec{b})\right|}.$$

```
#1:   a := [1.268, 4.732, -10.732]

#2:   b := [3.424, 3, -11.97]

#3:   c := [3, 2.628, -6.023]

#4:   p := [-12, -5, -6]

            CROSS(b - a, c - b)
#5:   u := ---------------------
            |CROSS(b - a, c - b)|

#6:   u = [-0.655648, -0.749241, -0.0936123]
```

For this we Author the points as #1, #2, #3 and calculate the above expression to give the direction vector of the axis of tunnel A u = (-0.6556, -0.7492, -0.0936). Then, to locate the centre $r = r0$ as a point on the axis, we equate

$$(\vec{r}-\vec{a})\cdot(\vec{r}-\vec{a}) = (\vec{r}-\vec{b})\cdot(\vec{r}-\vec{b}) = (\vec{r}-\vec{c})\cdot(\vec{r}-\vec{c}) \qquad (1)$$

$$(\vec{r}-\vec{c})\cdot\vec{u} = 0 \qquad (2)$$

If we Author the vector r as (x,y,z), then we form a set of three linear equations, which we solve to give

$$r0 = (3.00014, 2.99982, -8.99998)$$

The radius is then calculated as 3.00010 as an absolute value of $r0$ or using a dotproduct.

```
#7:   r := [x, y, z]

#8:   eq1 := (r - a)·(r - a) = (r - b)·(r - b)

#9:   eq2 := (r - b)·(r - b) = (r - c)·(r - c)

#10:  eq3 := (r - c)·u = 0

#11:  SOLVE(eq1 ∧ eq2 ∧ eq3, [x, y, z])

#12:  x = 3.00014 ∧ y = 2.99982 ∧ z = -8.99998

#13:  r0 := [3.00014, 2.99982, -8.99998]

#14:  |a - r0| = 3.00010

#15:  √((a - r0)·(a - r0)) = 3.00010
```

An interesting pedagogical point arising from this part of the problem is that, if the student mistakenly forms three linear equations from the two equations (1), without equation (2), a singular system results, so that any answer obtained by the computer is suspect. This allows the instructor to motivate the analysis of numerical methods applied to linear systems, and also to bring out the geometrical insight that the locus of solutions to the singular system obtained is actually the infinite set of points along the axis of the cylinder, providing an intuitive account of the nature of solutions of undetermined linear systems. The fact that we can often learn a lot from our mistakes in mathematics comes as a revelation to many students, and is an education in itself.

Moving on with the solution of the problem, we use a scalar projection to calculate the minimum distance between the axes u of the tunnel A und j = (0,1,0) of tunnel B from

$$\frac{(\vec{p}-\vec{r}_0)\cdot(\vec{u}\times\vec{j})}{\left|(\vec{u}\times\vec{j})\right|}$$

The result 5.09, exceeding the sum of the radii of the cylinders, proves that the tunnels do not intersect.

#16: $j := [0, 1, 0]$

#17: $\dfrac{(p - r0)\cdot \mathrm{CROSS}(u, j)}{|\mathrm{CROSS}(u, j)|}$

#18: -5.09005

The second part of the problem is solved by moving the tunnel B, so that the point p is relocated to

$$\vec{p}+\frac{5.09\vec{u}\times\vec{j}}{\left|\vec{u}\times\vec{j}\right|}$$

for this DERIVE calculates the new coordinates (-11.2805, -5, -11.0389).

#19: $p + \dfrac{\mathrm{CROSS}(5.09005\cdot u, j)}{|\mathrm{CROSS}(u, j)|}$

#20: $[-11.2805, -5, -11.0389]$

Finally the equation of the surface of the cylinder representing the tunnel A is

$$(\vec{u}\times(\vec{r}-\vec{r}_0))\cdot(\vec{u}\times(\vec{r}-\vec{r}_0))=3.0001^2$$

Expressing this using variables $r = (x,y,z)$ and using Substitute to put $z = 0$, enables us to obtain the equation of the ellipse of intersection of the cylinder with the plane $z = 0$, and then solving for y gives two branches, from which the plot in Figure 1 is obtained.

$$|\mathrm{CROSS}(u, r - r0)|^2 = 3.0001^2$$

$$\mathrm{CROSS}(u, r - r0)\cdot \mathrm{CROSS}(u, r - r0) = 3.0001^2$$

$$0.570125\cdot x^2 - x\cdot(0.982476\cdot y + 0.122753\cdot z + 1.57843) + 0.438637\cdot y^2 - y\cdot(0.140276\cdot z + 0.946584) + 0.991236\cdot z^2 + 18.6312\cdot z + 87.6282 = 9.00060$$

$$0.570121\cdot x^2 - 0.982475\cdot x\cdot y - 1.57843\cdot x + 0.438637\cdot y^2 - 0.94658\cdot y = -78.6271$$

Figure 1 (next page) was the end – and highlight – of this contribution in 1992. At this time (1992) Neil & Co had to solve this equation for y because implicit plotting was not possible. Figure 1 contains an individual scaling created following the instructions given in the next contribution.We had only very limited possibilities to represent 3D-objects. It had been possible to create an isometric projection of the wire grids of the two cylinders. Now in 2005 we cannot only apply implicit plotting but also add a nice 3D-representation of the problem using functions f(x,y,z) = 0 and / or parameter forms f(s,t).

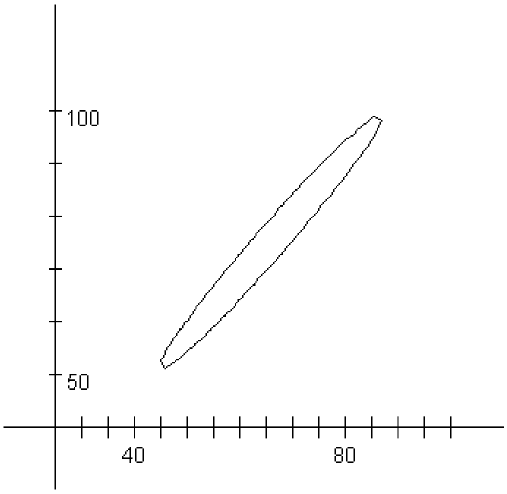
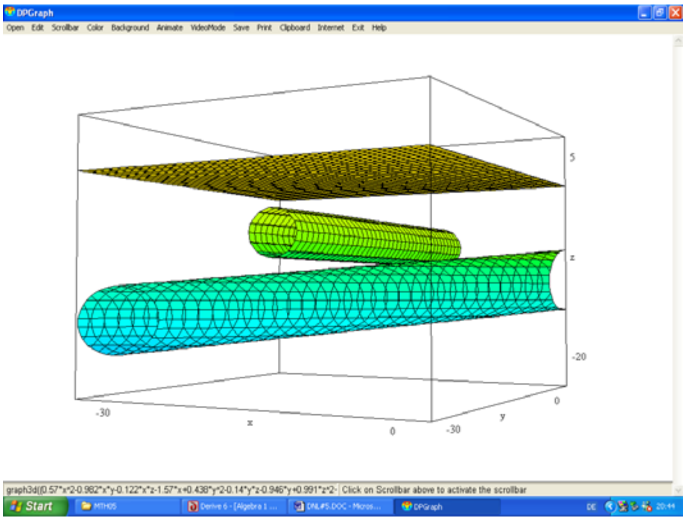


Figure 1

It is very easy, informative and motivating to transfer the equations of the two cylinders and the plane z = 0 to *DPGraph* and show the tunnels. Equation for tunnel B: $(x+12)^2+(z+6)^2 = 4$.



In the following you can find the important lines in the DPGraph-settings:

```
graph3d.minimumx  := -30
graph3d.maximumx  := 0
graph3d.minimumy  := -30
graph3d.maximumy  :=-0
graph3d.minimumz  := -20
graph3d.maximumz  := 5
graph3d((0.57*x^2-0.982*x*y-0.122*x*z-1.57*x+0.438*y^2-0.14*y*z-
0.946*y+0.991*z^2+18.6*z+87.6=9,(x+12)^2+(z+6)^2=4,z=0))
```

Derive is still unable to produce implicit 3D-plots. I wanted to have also a Derive 3D-plot, so I tried to express the two – and finally three – cylinders in parameter form. This was not so difficult for the horizontal tunnel B. (Given point P on its axis, the direction vector of the axis and the cylinder radius = 2.)

```
[-12, -5, -6] + t·[0, 1, 0] + [2·COS(s), 0, 2·SIN(s)]
```

I never taught analytic geometry in space, so it might be that my way to find the cylinder which represents tunnel A in parameter form is not the standard one.

#26 is the axis of tunnel A. I try to find two normalized vectors lying in a plane which is perpendicular wrt the axis.

#26: r0 + t·u

#27: [3.00014 − 0.655648·t, 2.99982 − 0.749241·t, − 0.0936123·t − 8.99998]

n1 is one of the two vectors and then using the cross product I obtain the second vector n2.

$u\cdot[0,\ 1,\ z] = 0$

$z = -8.00365$

$$n1 := \frac{[0,\ 1,\ -8.00365]}{|[0,\ 1,\ -8.00365]|}$$

$n1 = [0,\ 0.123979,\ -0.992284]$

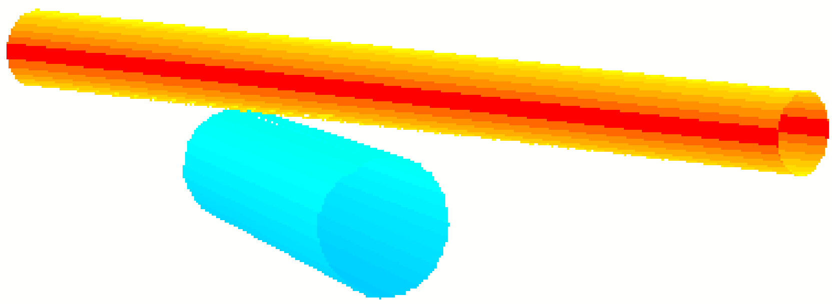
$$\frac{CROSS(u,\ n1)}{|CROSS(u,\ n1)|}$$

$$n2 := \frac{CROSS(u,\ n1)}{|CROSS(u,\ n1)|}$$

$n2 = [0.755066,\ -0.650589,\ -0.0812866]$

Putting one vector to the other leads to the requested parameter representation of tunnel A.

$r0 + t\cdot u + 3.0001\cdot n1\cdot COS(s) + 3.0001\cdot n2\cdot SIN(s)$

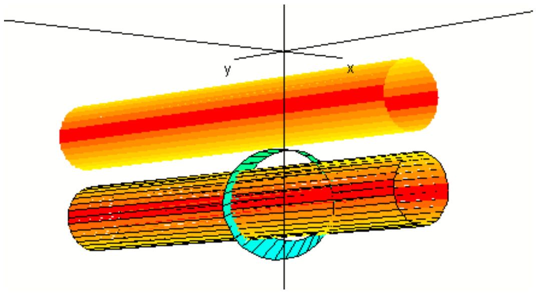


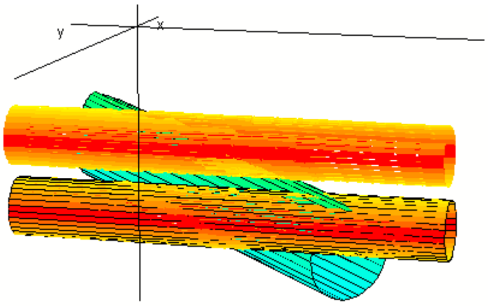
The next figure shows tunnel A and the shifted tunnel B (solution of problem 2!)

$[-11.2805,\ -5,\ -11.0389] + t\cdot[0,\ 1,\ 0] + [2\cdot COS(s),\ 0,\ 2\cdot SIN(s)]$

Tunnels with intersecting axes.

On the next page you can find tunnel A, tunnel B and the shifted tunnel.





The last picture shows tunnel B, the plane *z* = 0 and the intersection curve (Figure 1 in 3D!).

$$r0 + t\cdot u + 3.0001\cdot n1\cdot COS(s) + 3.0001\cdot n2\cdot SIN(s)$$

$$[2.26527\cdot SIN(s) - 0.655648\cdot t + 3.00014,\ 0.371949\cdot COS(s) - 1.95183\cdot SIN(s) - 0.749241\cdot t + 2.99982,\ - 2.97695\cdot COS(s) - 0.243867\cdot SIN(s) - 0.0936123\cdot t - 8.99998]$$

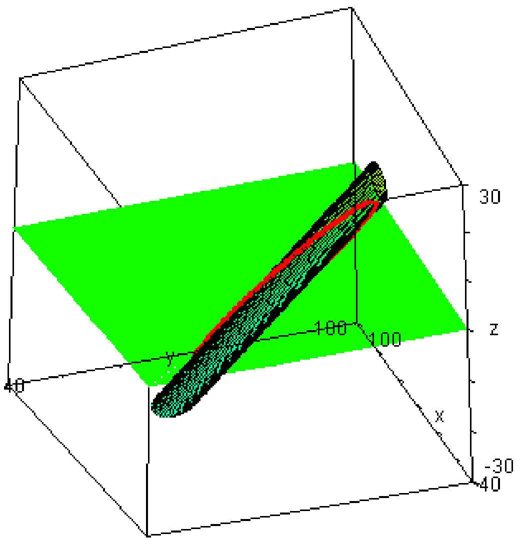
$$SOLVE(- 2.97695\cdot COS(s) - 0.243867\cdot SIN(s) - 0.0936123\cdot t - 8.99998,\ t)$$

$$t = - 31.8008\cdot COS(s) - 2.60507\cdot SIN(s) - 96.1409$$

Substituting for t results in the intersection curve. VECTOR gives the points to plot (Size Medium)

$$\left[20.8501\cdot COS(s) + 3.97327\cdot SIN(s) + 66.0347,\ 24.1984\cdot COS(s) - 4.74812\cdot 10^{-6}\cdot SIN(s) + 75.0325,\ 0\right]$$

$$VECTOR\left(\left[\left[20.8501\cdot COS(s) + 3.97327\cdot SIN(s) + 66.0347,\ 24.1984\cdot COS(s) - 4.74812\cdot 10^{-6}\cdot SIN(s) + 75.0325,\ 0\right]\right],\ s,\ 0,\ 2\cdot \pi,\ \frac{\pi}{100}\right)$$



Selfmade Scales

Josef Böhm, Würmla

Da in dieser Ausgabe des D-N-L schon so viele englischsprachige Artikel erschienen sind, will ich diesen kleinen Beitrag für unsere deutschsprachigen Mitglieder in Deutsch belassen. Die Grafiken und Listings sind sicher für alle leicht verständlich.

Für viele Darstellungen von Funktionsgraphen oder statistische Grafiken wie etwa Regressionslinien (wird demnächst im D-N-L behandelt) ist ein speziell skaliertes Koordinatensystem mit Hervorhebung einzelner Skalenwerte nützlich, weil die Grafiken leichter lesbar werden. Koordinatentransformation der echten Daten ist eine andere Möglichkeit. Hier soll das Koordinatensystem der Realität angepasst werden.

Ich gehe zuerst davon aus, dass die Träger der Skalen die x- und y-Achse sind. Dann verlege ich den Schnittpunkt der Skalenträger in den Punkt $O' = (x0, y0)$.

For many presentations of function graphs and statistical data it would be useful to have an adapted system of coordinates with appropriate positions of scale axes together with accentuating special labels.

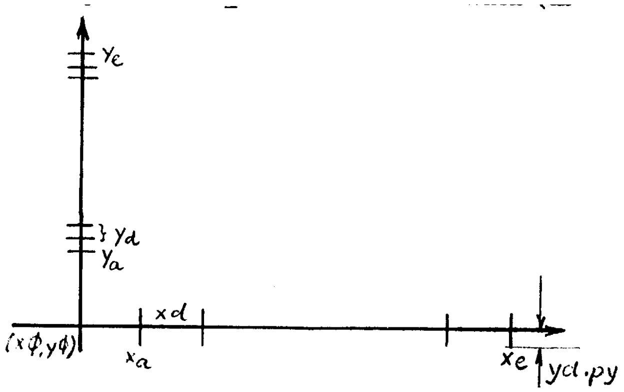
Wir benötigen eine Skala für:

$xa \le x \le xe$ mit Inkrement xd

$ya \le y \le ye$ mit Inkrement yd

px, py sind Stauchfaktoren,für die Länge der Skalenstriche;

$x0$, $y0$ sind die Koordinaten des Schnittpunkts der Skalengeraden.



We would like to plot a scale for $xa \le x \le xe$ and $ya \le y \le ye$ with increments xd and yd.

px and py are factors to control the lengths of the scaling segments.

$x0$ and $y0$ are the coordinates of the intersection point of the scale lines

In der Hilfsdatei SCALE.MTH habe ich alle nötigen Befehle vorbereitet: (neu: SCALE2005.MTH)

Utility file SCALE2005.MTH provides all necessary functions:

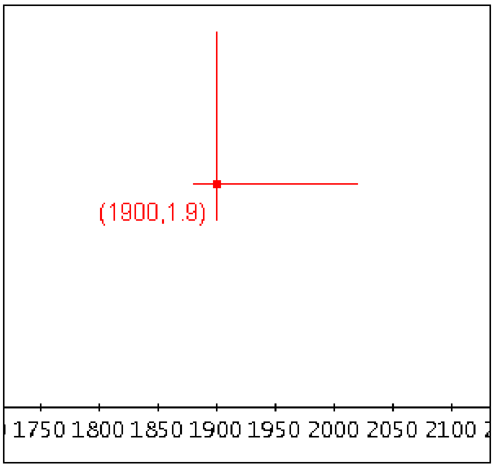
$$nx(xa, xe, xd) := \frac{xe - xa}{xd} + 1$$

$$ny(ya, ye, yd) := \frac{ye - ya}{yd} + 1$$

$$axes(xa, xe, xd, ya, ye, yd, x0, y0) := \left[\begin{bmatrix} xa - 2\cdot xd & y0 \\ 2\cdot xd + xe & y0 \end{bmatrix}, \begin{bmatrix} x0 & ya - 2\cdot yd \\ x0 & 2\cdot yd + ye \end{bmatrix}\right]$$

`axes(.....)` erzeugt die Achsen durch (*x*0,*y*0) von *xa* bis *xe* und *ya* bis *ye* in Skalierungen *xd* und *yd* mit einer kleinen Reserve nach allen Seiten hin.

`axes(.....)` plots axes through (*x*0,*y*0) from *xa* to *xe* and from *ya* to *ye* with scalings *xd* and *yd* with a small reserve in all directions.




```
axes(1900, 2000, 10, 1.8, 3, 0.1, 1900, 1.9)

[1900, 1.9]
```

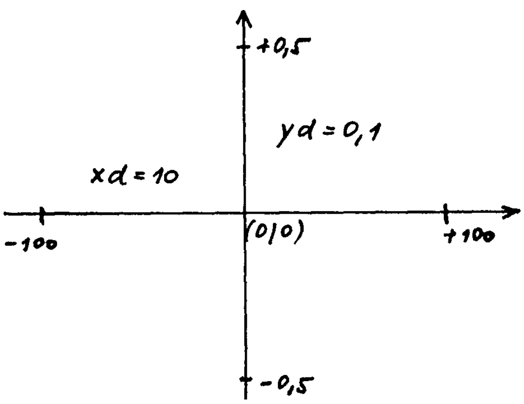
$$\text{axes}(xa,\ xe,\ xd,\ ya,\ ye,\ yd,\ x0 := 0,\ y0 := 0) := \left[\begin{bmatrix} xa - 2\cdot xd & y0 \\ 2\cdot xd + xe & y0 \end{bmatrix}, \begin{bmatrix} x0 & ya - 2\cdot yd \\ x0 & 2\cdot yd + ye \end{bmatrix}\right]$$

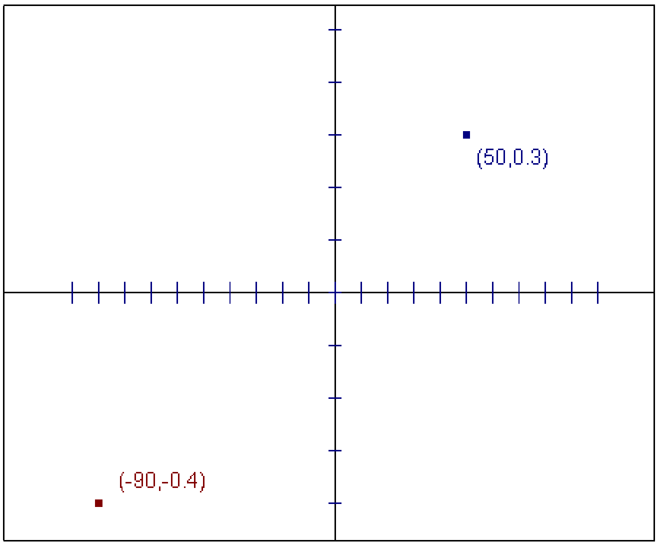
$$\text{sc\_x}(xa,\ xe,\ xd,\ py,\ yd,\ y0 := 0) := \text{VECTOR}\left(\begin{bmatrix} xa + (n-1)\cdot xd & y0 - py\cdot yd \\ xa + (n-1)\cdot xd & y0 + py\cdot yd \end{bmatrix},\ n,\ 1,\ nx(xa,\ xe,\ xd)\right)$$

$$\text{sc\_y}(ya,\ ye,\ yd,\ px,\ xd,\ x0 := 0) := \text{VECTOR}\left(\begin{bmatrix} x0 - px\cdot xd & ya + (n-1)\cdot yd \\ x0 + px\cdot xd & ya + (n-1)\cdot yd \end{bmatrix},\ n,\ 1,\ ny(ya,\ ye,\ yd)\right)$$

```
scale(xa, xe, xd, py, ya, ye, yd, px, x0 := 0, y0 := 0) := APPEND(sc_x(xa, xe, xd, py, yd, y0), sc_y(ya, ye, yd, px,
  xd, x0))

scale_ext(xa, xe, xd, py, ya, ye, yd, px, x0 := 0, y0 := 0) := APPEND(scale(xa, xe, xd, py, ya, ye, yd, px, x0, y0),
  axes(xa, xe, xd, ya, ye, yd, x0, y0))
```

Aufgabe 1 / Problem 1

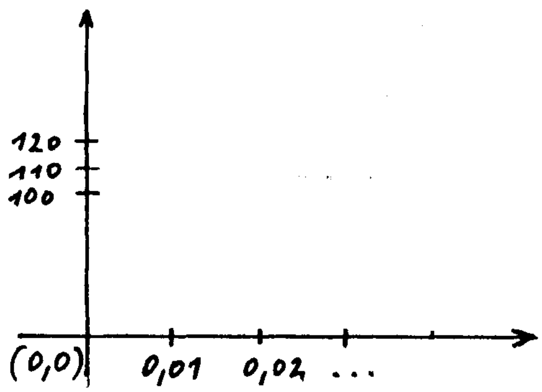


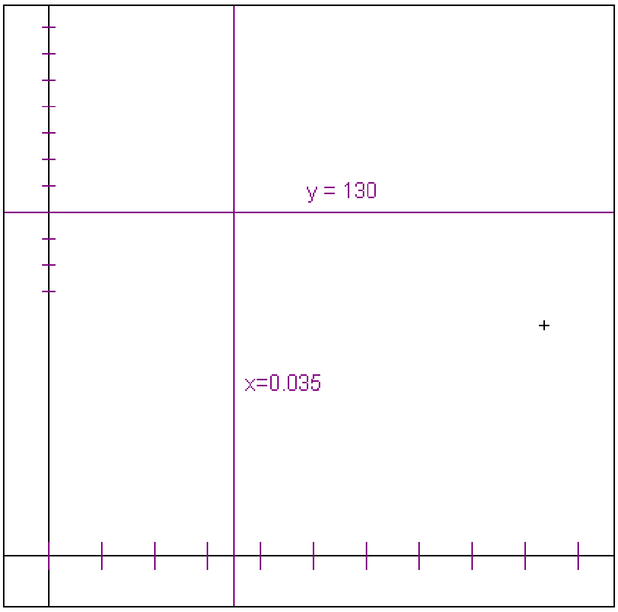



```
scale(-100, 100, 10, 0.2, -0.5, 0.5, 0.1, 0.2)
```

Wie Sie an der Hardcopy erkennen, muss ich mit Options Scale die Skalierung anpassen; in Options State stellen ich Connected und Small ein; in Options Color stelle ich Auto auf No und in Plot wähle ich eine geeignete Farbe für die Markierungen. (Comment from 1992!!)

Aufgabe 2 / Problem 2

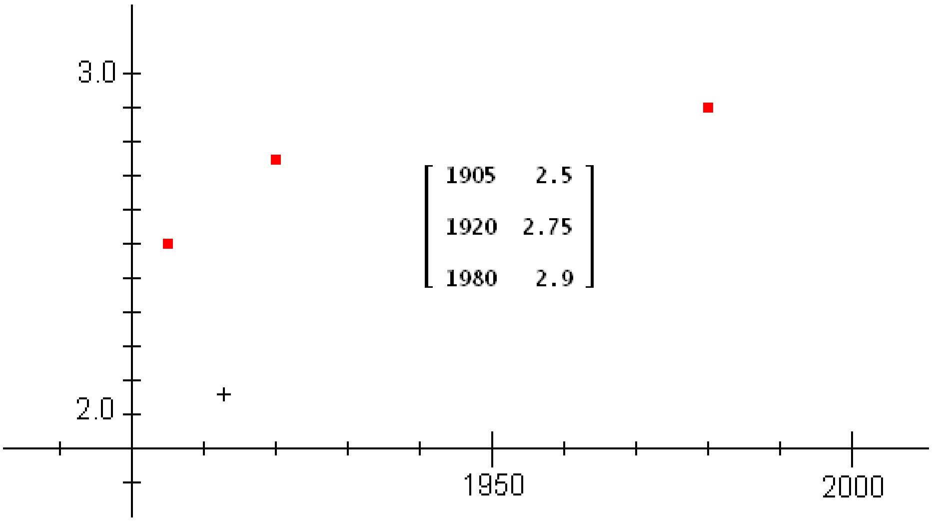





```
scale(0, 0.1, 0.01, 0.5, 100, 200, 10, 0.1)
```

Als drittes Beispiel will ich eine Skalierung mit dem Zentrum in ($x0$ = 1900 | $y0$ = 1.9) erzeugen, so dass das Zentrum in der linken unteren Ecke des Bildschirms zu liegen kommt. Zusätzlich sollen die 50er auf der horizontalen Achse hervorgehoben werden.

In my third example I create a scale with its center in (1900|1.9) in the bottom left corner of the screen together with an additional accentuating of the 50ties on the horizontal axis




```
scale_ext(1890, 2000, 10, 0.2, 1.8, 3, 0.1, 0.1, 1900, 1.9)
sc_x(1900, 2000, 50, 0.5, 0.1, 1.9)
```

`scale` zeichnet nur die Skalierung ohne Skalengeraden.
`scale_ext` zeichnet die Skalengeraden mit der Skalierung.

`scale` plots the scaling without the axes.
`scale_ext` plots the scaling together with the axes.