THE DERIVE - NEWSLETTER #11

THE BULLETIN OF THE

DERIVE®

USER GROUP

Contents:

revised reprint 2006 **September 1993**

Turbo-Charging DERIVE XM Version 2.57

Several users have commented that the XM (extended memory) version of DERIVE ran significantly slower than the conventional memory version.

Our test showed that XM version 2.56 ran about 30% to 40% slower on average (on some problems it ran twice as slow) as regular DERIVE. However, on big problems, XM was many times faster (e.g. on computing 4000!) since it did not have to thrash memory like regular DERIVE.

I have spent several weeks hand-assembling some of XM´s critical routines to give the ultimate in performance. Thus I am happy to report that the just released DERIVE XM version 2.57 runs as fast or faster than the regular DERIVE on small problems and much faster on big ones!

Also new with XM version 2.57 is our ability to limit the amount of extended memory XM uses. This is done by means of an MS-DOS environment variable you can set before starting DERIVE XM. Users of multi-tasking environments as Windows or OS/2 may find this useful to prevent XM taking all memory resources.

Aloha,
Al Rich, Soft Warehouse, Inc.

Members of the DERIVE User Group are eligible for a free exchange of their DERIVE XM license to DERIVE XM version 2.57. Please send your original DERIVE XM diskette and a reference to this offer to one of the following addresses:

For Non-European members:
Soft Warehouse, Inc.
3660 Waialae Avenue, Suite 304
Honolulu, Hawaii 96816
USA

For European members:
Soft Warehouse Europe
Softwarepark
A-4232 Hagenberg
AUSTRIA

D E R I V E - B O O K - S H E L F

Es freut mich, Ihnen ein neues DERIVE Buch ankündigen zu können, das von DUG-Mitgliedern geschrieben wurde:

Mauve-Moos, Mathematik mit DERIVE, Ferd. Dümmler, Bonn (mit Diskette Dümmler Nr. 4588)

2nd Krems Conference on Mathematics Education, September 27 – 30

Ideas for Future Versions of DERIVE; Success and Failure in Mathematics; Number Theory with DERIVE; The Assessment of Mathematical Ability in the "Light" of DERIVE; Approximation Methods via DERIVE; From Harmony to Chaos; Deterministische und Stochastische Simulationen für die Schule; Learning Visually; Using DERIVE to Explore Meaningful Applications in Calculus; White Box-Black Box-Principle; A Mathematical Model of a Firebreak using DERIVE; The World System of Johannes Kepler in Stereo-Vision using DERIVE; An Experience in Algebra and Discrete Mathematics; Using DERIVE in the Calculus Classroom; Geometric Maximum and Minimum Problems; Using CAS in Teaching Mathematical Modelling; The Understanding of the Concept of a Derivative; The Use of Graphics Calculators and CAS; The Application of a CAS as a Tool in College Algebra; Symbolic-Computation-unterstützter Mathematikunterricht; The DERIVE Project in Petrópolis; DERIVE-centred Research at the University of Plymouth; Methodische und didaktische Bemerkungen zum Einsatz von DERIVE; DERIVE im Unterricht der 11. und 12. Schulstufe.

Liebe DERIVE Anwender!

Viele von Ihnen haben zum Teil recht ausführliche Rückmeldungen auf meine Fragen zur Gestaltung des DNL und der Arbeit der DUG zurück geschickt. Dafür möchte ich mich nochmals recht herzlich bedanken. Ich habe nun die Anregungen, die an mich herangetragen worden sind, in einer Wunschliste zusammengestellt. Einige Themen wurden mehrfach genannt. Die Liste ist ungeordnet. Manches wurde bereits in diesem Jahr und einiges wird in diesem neuen DNL behandelt. Einiges ist in Vorbereitung (Abbildungsgeometrie mit Matrizen, Arbeitsblätter, Anwendungen für die Elektrotechnik – H. Scheuermann, ...), aber viele interessante Ideen bleiben offen! Ich danke auf diesem Weg für die vielen zustimmenden Zuschriften und für ihre Beiträge. Auch in diesem Heft findet sich eine recht internationale Mischung: USA, UK, Portugal, Slowakei und die BRD.

Ende September findet in Krems, Niederösterreich, die 2. DERIVE Konferenz statt. Auf der linken Seite können Sie die für diese Konferenz angenommenen Vorträge finden. Wir sind sicher, dass auch von diesem Treffen wieder wertvolle Anregungen und Impulse ausgehen werden. In den nächsten DNLs werde ich darüber berichten.

Mit den besten Grüßen Ihr

Dear DERIVE Users,

Many of you have answered my questions concerning the development of the DNL and the further work of the DUG. Many thanks for that. I collected all your ideas and suggestions in a wishlist. There is no special order in this list. Some items were mentioned several times. Some contributions of 1993 are meeting your wishes, some answers are in preparation (Mappings with matrices, worksheets, applications for electrical engineering – H. Scheuermann, ...). But many interesting ideas are still open. I am very indebted for so many positive letters and for your contributions. I think that we have a very international mixture in this issue, too: USA, UK, Portugal, Slovakia and Germany.

By end of September the 2nd DERIVE Conference will be held in Krems, Lower Austria. We are sure that this meeting will bring many inspirations and impulses. I will report in the next DNLs.

On the left page you can find the lectures which have been accepted for this conference.

with my best regards

DERIVE USER GROUP Members' Wishlist

Teaching Algebra, Calculus, ODEs, Integration; Vektoranalysis; Economics; Statistics; Probability Theory; Ökonometrie; Good examples which require CAS for their solution; Astronomical Content; Celestial Mechanics; Chaos; Technical, Physical and Economic Applications; Approximation; Ingenieurwissenschaften; Advanced Stuff; Worksheets; Matrizen; Abbildungsgeometrie mit Matrizen; 2D- und 3D-Plots; Anwendungen zur analytischen Geometrie; Anregungen zu Printer & Plotter; Unterrichtskonzepte – Beispiele; Erfahrungsberichte aus der Schule; Erfahrungen über den Gebrauch von DERIVE bei wissenschaftlichen Arbeiten; Technische Problemlösungen (zB Elektrotechnik); Didactics; 3D-Graphing of Complex Numbers; Mathematical Games; Exporting Plots into other Programs; Recursive Programming; Differentialgleichungen – Differenzenverfahren; Background Knowledge to Implementations of DERIVE

The *Derive-News-Letter* is the Bulletin of the Derive-User Group. It is published at least three times a year with a contents of 30 pages minimum. The goals of the *D-N-L* are to enable the exchange of experiences made with Derive as well as to create a group of experts to discuss the possibilities of new methodical and didactic manners in teaching Mathematics.

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Contributions:

Please send all contributions to the below address. Non-English speakers are encouraged to write their contributions in English to reinforce the international touch of the *D-N-L*. It must be said, though, that non-English articles will be warmly welcomed nonetheless. Your contributions will be edited but not assessed. By submitting articles the author gives his consent for reprinting it in the *D-N-L*. The more contributions you will send to the Editor, the more lively and richer in contents the Derive-News-Letter will be.

Preview: (Contributions for the next issues)

Fluid Flow in DERIVE, Reuther a.o., BRA
Newton´s Chaos, Graphic Integration, J. Böhm, AUT
Computer Aided Mathematics in School, Karl-Heinz Keunecke, GER
DERIVE in Hawaiian Classrooms, E. Sawada, USA
Applications in Electrical Engineering, H. Scheuermann, GER
Stability of Systems of ODEs, A. Kozubik, Slovakia
Minimization of a „Flat Function“, C. Lopes, Portugal
Maximum and Minimum Values – A Tool for Teachers, E. Zott, AUT

(DNL#12 will be published December 1993)

Impressum:
Medieninhaber: *DERIVE* User Group, A-3042 Würmla, D’Lust 1, AUSTRIA
Richtung: Fachzeitschrift
Herausgeber: Mag.Josef Böhm
Herstellung: Selbstverlag

K. Herdt, Osnabrück, Germany

Den Erhalt der neuen DERIVE XM Version und des DNL#10 habe ich zum Anlass genommen, einen Zeitvergleich anzustellen.
Herr Mauve approximiert auf Seite 3 des DNL die Halbkreisfläche durch eine Untersumme von Rechtecken (wobei allerdings die Summation nur bis n–1 zu laufen braucht, da der letzte Summand ohnehin 0 wird). Lässt man die Summation stattdessen von 0 bis n–1 laufen, erhält man die Obersumme der Rechtecksflächen (anschaulich: die Rechteck des Herrn Mauve werden um ihre obere horizontale Kante geklappt, „unten“ kommt ein weiteres Rechteck dazu). Die Differenz zwischen Ober- und Untersumme beträgt dann bei gleichem n stets $2r^2/n$, also die Fläche des untersten – und somit größten – aller Rechtecke. Berechnet man nun beispielsweise die Obersumme für n = 1000, also $O(1000)$, so liefert DERIVE 2.52 das Resultat $1.57368r^2$ in 4,4 Sekunden. Das neue SUPER-DERIVE XM lässt sich demgegenüber 38 Sekunden Zeit! Mich interessiert nun, ob andere Anwender ähnliche Erfahrungen gemacht haben?

#1: $$\frac{2\cdot\sqrt{r^2-\left(\frac{(k-1)\cdot r}{n}\right)^2}\cdot r}{n}$$

#2: $$O(n) := \sum_{k=1}^{n} \frac{2\cdot\sqrt{r^2-\left(\frac{(k-1)\cdot r}{n}\right)^2}\cdot r}{n}$$

#3: $$U(n) := O(n) := \sum_{k=2}^{n} \frac{2\cdot\sqrt{r^2-\left(\frac{(k-1)\cdot r}{n}\right)^2}\cdot r}{n}$$

#4: $O(1000)$

#5: $1.57177\cdot r^2$

0.2 sec with DERIVE 6.10

DNL: *Betreffend der Arbeitsgeschwindigkeit von DERIVE XM lesen Sie bitte die Information auf der ersten Innenseite.*

Dr. W. Koepf, Berlin, Germany

to Mr. Eames' question in DNL#10, page 22:

The DERIVE function

```
MAP(f,x,v):=VECTOR(LIM(f,x,ELEMENT(v,k),k,1,DIMENSION(v))
```

applies the function f, given as a function of variable x, to the entries of the one-dimensional data vector v. For example the expression `MAP(LN(x),x,VECTOR(k,k,10)`

results in the vector

```
[0,LN(2),LN(3),LN(4),LN(5),LN(6),LN(7),LN(8),LN(9),LN(10)].
```

The function MAP may be changed in an obvious way if the vector *v* is an array of higher dimension.

You cannot use this MAP-function from 1993 now because there is a function MAP implemented in the recent DERIVE versions. Functions `MAP_LIST(u,x,c)` and `MAP_LIST(u,k,m,n,s)` are very useful – I recommend a glimpse into the Online-Help. `MAP(u,x,c)` returns the truth value true. (Josef 2006)

```
#1:   MAP_LIST(LN(x), x, VECTOR(k, k, 10))

#2:   [0, LN(2), LN(3), 2·LN(2), LN(5), LN(6), LN(7), 3·LN(2), 2·LN(3), LN(10)]

#3:   MAP(LN(x), x, VECTOR(k, k, 10))

#4:   true
```

Glynn D Williams, Gwynedd, UK

I regularly use the DERIVE program, but find it annoying that a different set of keys is needed to edit an algebraic expression than to select the expression or subexpression to be edited. The reason given in the manual for these inconsistencies is that certain combinations of keys which would be logical are not recognized; e.g. →, ←, HOME and END for editing an expression in the bottom editor line; or CTRL+↑ and CTRL+↓ for moving the cross in the plot window one large division. However, if **ANSI.SYS**. or a compatible public-domain or shareware driver is loaded, it is quite easy to patch the keyboard so that it behaves logically. The batch file below does this:

```
esc 0;71;17;19p    :    rem Home to give ^Q^S
esc 0;79;17;4p     :    rem End to give ^Q^D
esc 0;75;19p       :    rem Left to give ^S
esc 0;77;4p        :    rem Right to give ^D
esc 0;141;0;73p    :    rem Ctrl-Up to give PgUp
esc 0;145;0;81p    :    rem Ctr-Down to give PgDn
derive.exe %1
esc 0;71;0;71p
esc 0;79;0;79p
esc 0;75;0;75p
esc 0;77;0;77p
esc 0;141;0;141p
esc 0;145;0;145p
```

Six keys need to be patched: they are all in the extended code set, and they are all to be patched to give codes in the normal code set. Note that they must all be "unpatched" after exiting from DERIVE, as otherwise bizarre results may be caused especially if an enhanced command interpreter such as **4DOS** or **Doskey** (or a word-processor or editor) which use the arrow keys is in use.

`esc` refers to a small machine-code program **`esc.com`**, which outputs the **ESC** character followed by a [to standard output when it is activated., and then tags on the characters given as parameters. With this patch in operation one loses the ability to select a sub-expression of a highlighted expression while there is an expression in the edit window, but this is rarely needed; apart from this there is no ambiguity. If it is needed one can always isolate the sub-expression in a spare numbered "slot" (by highlighting it and then copying it using F3) before editing it; there is usually enough memory to do this.

DNL: *In DERIVE 2.55 the F6-key enables comfortable editing now, but I am sure that there are a lot of our members who don´t own the latest version. I find it very interesting to see how to adapt a program for one's personal use.*

A.C. Robin, Colchester, UK

I had some difficulties trying to use DERIVE for solving some problems, and I wrote to Soft Warehouse about them. I am enclosing copies of the letters I sent together with a reply I had to the first difficulty. I thought that you or readers of the DNL might be interested:

#1: $ma4 := \begin{bmatrix} 0 & 1 \\ 1 & 2 \\ 2 & 4 \\ 3 & 8 \\ 4 & 16 \\ 5 & 31 \end{bmatrix}$

The FIT-function does not give back the correct quartic.

#2: $FIT([x, a\cdot x^4 + b\cdot x^3 + c\cdot x^2 + d\cdot x + e], ma4)$

#3: $\frac{338\cdot x^4}{8115} - \frac{787\cdot x^3}{9462} + \frac{2220\cdot x^2}{4849} + \frac{7699\cdot x}{13187} + \frac{35243}{35244}$

See David Stoutemyer´s answer:

The problem disappears if before simplifying the FIT example you choose

Options Precision 15

The explanation is that FIT uses the straightforward "normal equation" method together with appproximate mode, and this mode typically loses about half the working precision, which defaults to 6 digits. Also, using the monomial x^m as basis functions is notoriously subject to additional roundoff error.

Methods such as singular-value decomposition together with orthogonal polynomial bases are more accurate, but they require an impractical amount of code for a general purpose computer algebra system intended to run in as little as 512 kilobytes. Thus, given the arbitrary precision of DERIVE, it is more appropriate merely to increase the working precision as the desired polynomial degree increases.

Sincerely yours, ...

#4: $\frac{x^4}{24} - \frac{x^3}{12} + \frac{11\cdot x^2}{24} + \frac{7\cdot x}{12} + 1$

This is no problem with recent versions of DERIVE:

#5: PrecisionDigits := 3

#6: NotationDigits := 3

#7: $FIT([x, a\cdot x^4 + b\cdot x^3 + c\cdot x^2 + d\cdot x + e], ma4) = \frac{x^4}{24} - \frac{x^3}{12} + \frac{11\cdot x^2}{24} + \frac{7\cdot x}{12} + 1$

Mr Robin's 2nd problem:

I wonder why DERIVE 2.55 gives the following, have I done something wrong or is there a bug?

As you can see in the output below (left column, editor), I defined the `cube` function as `x^3`, and defined the `dcube`-function as its derivative, namely `3x^2`. When I asked DERIVE to integrate `x*dcube(x)` and `(x+u)*dcube(x)`, it gave sensible and correct answers (`u` was assumed constant as I wished). However when I asked it to find the integral of `x*dcube(x+u)` then a strange expression occurred, and this was also obtained when just `dcube(x+u)` was simplified. DERIVE does correctly simplify `dcube(z)` as `3z^2`, so why does not `dcube(x+u)` simplify to `3(x+u)^2`?

I came across this problem by trying to use a much more complicated function than x^3, so that I really needed to define the function and its derivatives in this way.

#1: $\text{CUBE}(x) := x^3$

#2: $\text{DCUBE}(x) := \frac{d}{dx}\text{CUBE}(x)$

#3: $\int x\cdot\text{DCUBE}(x)\,dx$

#4: $\frac{3\cdot x^4}{4}$

#5: $\int (x+u)\cdot\text{DCUBE}(x)\,dx$

#6: $\frac{x^3\cdot(3\cdot x+4\cdot u)}{4}$

#7: $\int x\cdot\text{DCUBE}(x+u)\,dx$

#8: $\int x\cdot\frac{d}{d\ x+u}\text{CUBE}(x+u)\,dx$

#9: $\text{DCUBE}(z)$

#10: $3\cdot z^2$

#11: $\text{DCUBE}(x+y)$

#12: $\frac{d}{d\ x+y}\text{CUBE}(x+y)$

In 1993 I overcame the problem introducing an auxiliary variable. In the meanwhile this is not necessary, because the later versions of DERIve behave as expected, Josef

#1: $\text{CUBE}(x) := x^3$

#2: $\text{DCUBE}(x) := \frac{d}{dx}\text{CUBE}(x)$

#3: $\int x\cdot\text{DCUBE}(x)\,dx = \frac{3\cdot x^4}{4}$

#4: $\int (x+u)\cdot\text{DCUBE}(x)\,dx = \frac{3\cdot x^4}{4} + u\cdot x^3$

#5: $\text{DCUBE}(x+u) = 3\cdot(x+u)^2$

#6: $3\cdot(x+u)^2\cdot x = 3\cdot x\cdot(x+u)^2$

#7: $\int 3\cdot x\cdot(x+u)^2\,dx = \frac{(x+u)^3\cdot(3\cdot x-u)}{4}$

#8: $\int x\cdot\text{DCUBE}(x+u)\,dx = \frac{(x+u)^3\cdot(3\cdot x-u)}{4}$

Message 3000 was entered on 5/21/93 at 2:06 PM
From SOFT WAREHOUSE to PUBLIC about CHAINE RULE DIFFERENTIATION

Recently there has been much discussion on this BBS about having DERIVE use the chain rule when simplifying partial derivatives of arbitrary functions. The most general case is derivatives of the form

$$\frac{d}{dx}F(G(x,y),H(x,y)).$$

Applying the chain rule to this yields an expression of the form

$$\left(\lim_{@1\to G(x,y)}\frac{d}{d@1}F(@1,H(x,y))\right)\cdot\frac{d}{dx}G(x,y)+\left(\lim_{@2\to H(x,y)}\frac{d}{d@2}F(G(x,y),@2)\right)\cdot\frac{d}{dx}H(x,y).$$

Of course if G and H are specified rather than arbitrary functions, simpler results would be produced. For example, DIF(F(sin(x)),x) would simplify to LIM(DIF(F(@1),@1),@1,SIN(X))*COS(X).

If DERIVE performed such transformations, would this help satisfy your differentiation needs?

Message 3021 was entered on 5/28/93 at 2:06 AM
From ROGER FOLSOM to SOFT WAREHOUSE about DERIVE SUGGESTIONS

CHAIN RULE DIFFERENTIATION. You ask whether it would meet user needs if Derive calculated

$\frac{d}{dx}F(G(x,y),H(x,y))$ as … (Please see message 3000.)

I think this **would** be useful for those working with specified functions. But since I work mostly with what Derive calls arbitrary, unspecified, functions (which I'm used to calling general functions – which phrasing, arbitrary or general, is more commonly used among mathematicians?), I fear that carrying around all those limit operations would chew up an enourmous amount of memory. Consequently, I **think** I'm better off defining F(@1,@2):=, G(x,y):=, H(x,y):=, and for the system

```
G(x,y) - @1 = 0
H(x,y) - @2 = 0
```

setting J:=JACOBIAN([G(x,y)-@1,H(x,y)-@2],[@1,@2,x,y]), noting that J . [d@1,d@2,dx,dy]=[0,0], and if JN is the first two columns of J and JX is the last two columns of J, then solving

```
            JN . [d@1,d@2] = - JX [dx,dy]
gives            [d@1,d@2] = - JN^(-1) . JX [dx,dy]
```

Substituting this last result into the total differential of F(@1,@2) that is into GRAD(F(@1,@2)[@1,@2]) . [d@1,d@2], gives the partial derivative of F(.) wrt x (if dy=0) or y (if dx=0).

But it would be wonderful if DERIVE automated this process. If @1 = G(x,y) and @2 = H(x,y), then the objective should be to end up with something at least as simple notationally as:

$$\frac{d}{dx}F(G(x,y),H(x,y))=\frac{d}{d@1}F(@1,@2)*\frac{dH}{dx}+\frac{d}{d@2}F(@1,@1)*\frac{dH}{dx}$$

Note that much memory would be saved (I think) and certainly much screen and paper real estate would be saved if DERIVE were willing to substitute the symbols F for F(@1,@2) and H for H(x,y).

Message 3034 was entered on 6/2/93 at 10:29 PM
From SOFT WAREHOUSE to ROGER FOLSOM about #3021/DERIVE SUGGESTIONS

CHAIN RULE DIFFERENTIATION. Your notation for the partial derivative of F(G(x,y),H(x,y)) wrt x appears compact only because of the hidden assumptions it makes. Specifically, the assignments

@1 := G(x,y) and @2 := H(x,y)

would have to be made as well as the declaration that G and H are functions of x. Note that there are a couple of typos in your example: the first dH/dx should be dG/dx and the last @1 should be @2.
Aloha, Al Rich, Soft Warehouse, Inc.

Message 3045 was entered on 6/6/93 at 4:41 PM
From ROGER FOLSOM to SOFT WAREHOUSE about DERIVE SUGGESTIONS

CHAIN RULE DIFFERENTIATION: you are quite right that in my notation for the partial derivative of F(G(x,y),H(x,y)) wrt x, namely

$$\frac{d}{dx}F(G(x,y),H(x,y)) = \frac{d}{dX}F(X,Y) * \frac{dG}{dx} + \frac{d}{dY}F(X,Y) * \frac{dH}{dx}$$

(where I hope I have fixed the typos in my earlier message), with F(X,Y):=, G(X,Y):=, H(x,y):=, I "forgot" to state the assignments

X := G(x,y) and Y := H(x,y)

Prior to today, my **previous** experience was that if I had included those last two assignments, Derive (2.08) would have refused to differentiate F(.) with respect to X and Y, which is why I simply kept those assignments "in my head". Today, however on checking this issue, I found that Derive **was** willing to differentiate with respect to X, Y, x and y, so I must have had some confusion in my earlier work (possibly an uncleared conflicting assignment). Or maybe I'm confused now.

Aside fom that issue, note that even if the approach I suggested requires as many characters or keystrokes as the limit notation you suggested, my result line is shorter (even if I need more lines for function assignments), and therefore easier to read, print, and manipulate.

(Your discussion used @1 for X and @2 for Y. Does @ have some special meaning? I don´t find it in the manual index, or in the manual itself.)

Message 3000 was entered on 5/21/93 at 2:11 PM
From GREG SMITH to JACKNOL about #2998 / MATH TOOLKITS FROM U OF AZ

We offer some of the Arizona toolkits for download here as a service to users of the BBS, but it would be too labor-intensive to offer them on disk. You may want to download one to get the address and then write or call the authors to see if they are available some other way.

Message 3014 was entered on 5/2693 at 11:17 AM
From DELAWARE to PUBLIC about UNIV. OF ARIZONA PROGRAMS

To all those interested in the U of AZ software: I recently bought them all, meaning I sent the UA Math Dept $28.00 for 14 diskettes which include both the toolkits (9 diskettes) and non-toolkits (5 diskettes). The latter include slide shows, simulations, Are You Ready quizzes, etc. I find many of them very useful, both as assignments and in class preparations. Descriptions come with the diskettes, in fair detail, but the on-line instructions are usually insufficient. And, since this is freeware, I´ve passed them around my region (KC MO). I recommend them all. Richard Delaware

Message 3036 was entered on 6/3/93 at 10:16 PM
From DELAWARE to PUBLIC about UNIV. OF AZ FREEWARE

To those interested in obtaining the UA freeware, the info is:
Dept of Mathematics, Univ. of Arizone, Tucson, AZ 85721
Phone: (602) 621-6893. Call and have them send you an information sheet which has all the cost, options, etc. data. Goodluc. RD

Message 3013 was entered on 5/26/93 at 7:09 AM
From JERRY GLYNN to BOOM-BOOM about S^N/(S^N+C^N)

I sent a note to Al about integral(sin(x)^n/(sin(x)^n+cos(x)^n),x,0,π/2). I noticed that it seemed to be = π/4 for all n>0 so how about if Derive would recognize this ... he said that would be too narrow for a separate rule. By the way Derive does better than Mathematica on the problem. Why is the answer π/4? It sounds like it's the area of a circle who´s radius is 1/2 but how does that connect?

Message 3023 was entered on 5/30/93 at 12:10 PM
From HADUD to JERRY GLYNN about COMMENT ON #3013

The integral in question illustrates what happens when you ignore the special symmetry properties of a problem and try to solve it with a general program like DERIVE. In the present case the integrand is complementary, i.e.

f(x) + f(π/2-x) = 1

Integrating this equation and substituting u for (π/2-x) in the second integral you find immediately

2 INT(f(x),x,0,π/2) = π/2.

This shows, incidentally, that the result holds for any finite n (positive, negative real, complex). DERIVE could in principle, test for complementarity and thus solve problems of this kind easily. Most people would agree, though, that this is too special a case to build into a general program. **The lesson to be learned here is that even a great symbolic program like DERIVE is no substitute for mathematical insight!**

Message 3024 was entered on 5/30/93 at 1:12 PM
From JERRY GLYNN to HADUD about #3023 / COMMENT ON #3013

Thanks very much for your comment. Where do I loo for more examples of mathematical insight? Can we do a few more examples here?

Message 3031 was entered on 6/2/93 at 10:02 PM
From HADUD to JERRY GLYNN about MORE ON COMPL. INTEGRANDS

Pursuant to our telephone conversation here are some more thoughts on evaluating definite integrals with complementary integrands. The general identity applicable in these cases is

INT(f(x),x,a,b) = INT(1/2((f(x)+f(a+b-x),x,a,b)   (1)

This holds for any f(x) but is useful only if the modified integrand on the right is simpler (i.e. integrable) than the original f(x). This is the case in your example with f(x)=sin(x)^n/(sin(x)^n+cos(x)^n), a=0 b=π/2. To apply this under DERIVE use the right-hand side of (1) to generate an expression (after declaring f(x):=). Then set f(x) equal to your function and simplify the expression with proper values for 'a' and 'b'. Further examples (all of which DERIVE gives up on) are:

f(x)	a	b
LN(x)/LN(x(1-x)	0	1
(3x^2-3x+1) LN(x)/LN(x(1-x))	0	1
SQRT(1-x^2)/(1+EXP(x))	-1	1
(1+x)^3 COS(x)/(1+3x^2)	-1	1
EXP(-x^2) SIN(x)	-c	c (c any real)
x^3 COS(x) SQRT(1-x^2)	-c	c

I was somewhat disappointed to find out that DERIVE does not recognize that the last two examples yield a zero result. The integrands are obviously odd functions of x. The class of odd integrands and symmetric limits is clearly very large, so this might be a good enough reason to incorporate the above procedure into DERIVE!

There is also the multiplicative equivalent to identity (1):

INT(f(x),x,a,b) = INT(1/2((f(x)+a b/x^2 f(ab/x),x,a,b) (2)

This one seems to be less useful except in the case (a=ε b=1/ ε, ε → 0). Then we have

INT(f(x),x,0,INF) = INT(1/2((f(x)+1/x^2f(1/x),x,0,INF) (3)

Examples: **f(x)**

1/((1+x^2)(1+x^n))
LN(x)/(1+x^2)
x ATAN(x)/((1+x^2)(1+x+x^2))

Message 3048 was entered on 6/7/93 at 12:42 PM
From HADUD to SOFT WAREHOUSE about COMPLEMENTARY INTEGRANDS

The evaluation of a definite integral is sometimes facilitated if the integrand exhibits certain symmetry properties. One such property is complementary symmetry.

1) HADUD's DEFINITION of COMPLEMENTARITY: The integrand f(x) of the definite integral INT(f(x),x,a,b) is complementary with respect to the transformation g(x) if DERIVE 2.55 integrates

   F(x) = (1-p) f(x) – p g'(x) f(g(x))

   more easily than f(x). Here

   g(x) : any continuous function that g(a) = b and g(b) = a

   p : any real constant with 0 < p < 1 (usually p = 1/2).

   We then have the identity (valid for any f(x))

   INT(f(x),x,a,b) = INT(F(x),x,a,b).

   Comment: The limit-swapping property of g(x) means that the final integral is computed by combining an integration from left to right and an integration from right to left. Hence the term "complementary" if F(x) turns out to be a constant or simple function.

2) CHOICE of TRANSFORMATION: The class of admissible transformations g(x) is clearly very large. For practical purposes only the simplest two cases are of any importance, however:

```
     Linear:      g(x) = a + b - x       p = 1/2
     Reciprocal:  g(x) = (a b)/x         p = 1/2
                  special case:  a -> 0, b -> inf, (a b) = 1

=>        INT(f(x),x,a,b) = INT(1/2(f(x)+f(a+b-x)),x,a,b)
          INT(f(x),x,a,b) = INT(1/2(f(x)+(a b)/x^2 f(a b)/x)),x,a,b)
```

3) IMPLEMENTATION in DERIVE: The functions INT1(f(x),x,a,b) and INT2(f(x),x,a,b) implement the above cases. Their syntax is the same as for INT. Thus if INT fails, before slashing your wrist try INT1 and then INT2.

Function definitions:

```
fF(x):=

INT1(fct,x,a,b) ≔ ELEMENT([fF(x):=fct,∫(1/2(fF(x)+fF(a+b-x)),x,a,b)],2)

INT2(fct,x,a,b) ≔
ELEMENT([fF(x):=fct,IF(a=0,IF(b=∞,∫(1/2(fF(x)+1/x^2fF(1/x)),0,∞),∫(fF(x), x, 0, b),
∫(1/2(fF(x)+1/x^2fF(1/x)),x,0,∞)),∫(1/2·(fF(x)+a·b/x^2fF(a·b/x)),x,a,b),
∫(1/2(fF(x)+a·b/x^2fF(a·b/x)),x,a,b))],2)
```

Comments:
a) Since you cannot have a function as an argument in a DERIVE function definition, the above definitions use 2-element vectors. The 1st element assigns the given integrand expression (fct) to an auxiliary function fF(x). The 2nd element then performs the integration and is returned as the result.

b) The definition of INT2 contains the special case of a=0 and b=inf, hence the IF functions.

(See the comments of 2006 at the end of this message, Josef)

4) EXAMPLES (in all these cases INT fails):

1. INT1(SIN(x)^n/(SIN(x)^n+cos(x)^n),x,0,pi/2)
2. INT1(LN(u)/LN(u(u-1)),u,0,1)
3. INT1((3x^2-3x+1) LN(x)/LN(x(1-x)),x,0,1)
4. INT1(x SIN(x)/(2+SIN(x)),x,0,pi)
5. INT1(SQRT(1-x^2)/(1+EXP(q x)),x,-1,1)
6. INT1((1+x)^3 COS(x)/(1+3x^2),x,-1,1)
7. INT1(EXP(-x^2) SIN(x),x,-c,c)
8. INT2(1/((1+x^2)(1+x^n)),x,0,inf)
9. INT2(LN(x)/(1+x^2),x,0,inf)
10. INT2(x ATAN(x)/((1+x^2)(1+x+x^2)),x,0,inf)

Comment: All the above cases are contrived examples and of little practical use. However INT1 does solve the general case where f(x) is odd and a=-b. DERIVE 2.55 does not (in general) recognize that this gives a zero result (see example 7.). This is a serious shortcoming and makes a strong case for incorporating INT1 into the INT function.

Comments of 2006:

First of all: Recent versions of DERIVE don´t have problems with most of the integrals mentioned above. DERIVE has learnt a lot between version 2.55 and later ones. Moreover using the wonderful "Stepwise Simplification" tool you can find the rules implemented as suggested in this email:

#14: $$\int_0^{\pi/2} \frac{\mathrm{SIN}(x)^n}{\mathrm{SIN}(x)^n + \mathrm{COS}(x)^n}\,dx$$

$$\int_a^b F(x)\,dx \rightarrow \frac{\int_a^b (F(x) + F(a + b - x))\,dx}{2}$$

#15: $$\int_0^{\pi/2} \frac{1}{2}\,dx$$

#16: $$\frac{\pi}{4}$$

#17: $$\int_{-c}^{c} e^{-x^2}\cdot \mathrm{SIN}(x)\,dx$$

$$\int_a^b F(x)\,dx \rightarrow \frac{\int_a^b (F(x) + F(a + b - x))\,dx}{2}$$

#18: $$\int_{-c}^{c} 0\,dx$$

#19: 0

Trying the examples using the multiplicative equivalent, we can still observe problems:

#20: $$\int_0^{\infty} \frac{\mathrm{LN}(x)}{1 + x^2}\,dx = 0$$

#21: $$\int_0^{\infty} \frac{1}{(1 + x^2)\cdot(1 + x^n)}\,dx = \int_0^{\infty} \frac{1}{x^n\cdot(x^2 + 1) + x^2 + 1}\,dx$$

#22: $$\int_0^{\infty} \frac{x\cdot \mathrm{ATAN}(x)}{(1 + x^2)\cdot(1 + x + x^2)}\,dx = \frac{2\cdot\sqrt{3}\cdot \int_0^{\infty} \frac{\mathrm{ATAN}\left(\frac{\sqrt{3}\cdot(2\cdot x + 1)}{3}\right)}{x^2 + 1}\,dx}{3} + \pi^2\cdot\left(\frac{1}{8} - \frac{\sqrt{3}}{6}\right)$$

Numeric integration of #22 gives: #24: 0.2839974238

HADUD tried something like programming with DERIVE 2.55, which worked in 1993. Recent version don´t accept this way of assigning auxiliary variables in the first element of a vector which should be processed in the second one. We could write a small program or we avoid the auxiliary variable, substituting for x in fct (without using SUBST, to remain in the style of version 2.55).

First attempt with the original INT1-function

#3: $$\mathrm{INT1}\left(\frac{\mathrm{SIN}(x)^n}{\mathrm{SIN}(x)^n+\mathrm{COS}(x)^n},\ x,\ 0,\ \frac{\pi}{2}\right)=\frac{\int_0^{\pi/2}\mathrm{fF}\left(\frac{\pi}{2}-x\right)\,dx+\int_0^{\pi/2}\mathrm{fF}(x)\,dx}{2}$$

I rewrite this function and use it twice:

#4: $$\mathrm{INT1}(fct,\ x,\ a,\ b):=\int_a^b\frac{1}{2}\cdot\left(fct+\lim_{x\to a+b-x}fct\right)dx$$

#5: $$\mathrm{INT1}\left(\frac{\mathrm{SIN}(x)^n}{\mathrm{SIN}(x)^n+\mathrm{COS}(x)^n},\ x,\ 0,\ \frac{\pi}{2}\right)=\frac{\pi}{4}$$

#6: $$\mathrm{INT1}\left(\frac{(1+x)^3\cdot\mathrm{COS}(x)}{1+3\cdot x^2},\ x,\ -1,\ 1\right)=2\cdot\mathrm{SIN}(1)$$

#7: $$\int_{-1}^{1}\frac{(1+x)^3\cdot\mathrm{COS}(x)}{1+3\cdot x^2}\,dx=2\cdot\mathrm{SIN}(1)$$

Same happens with INT2

INT2 from 1993 fails:

#9: $$\mathrm{INT2}\left(\frac{\mathrm{LN}(x)}{1+x^2},\ x,\ 0,\ \infty\right)=\frac{\int_0^{\infty}\frac{x^2\cdot\mathrm{fF}(x)+\mathrm{fF}\left(\frac{1}{x}\right)}{x^2}\,dx}{2}$$

I rewrite this function, too and try once more and then I try the two Integrals which DERIVE refused to evaluate:

```
INT2(fct, x, a, b) :=
  If a = 0
     If b = ∞
        ∫(1/2·(fct + 1/x^2·LIM(fct, x, 1/x)), x, 0, ∞)
#10:    ∫(fct, x, 0, b)
        ∫(1/2·(fct + 1/x^2·LIM(fct, x, 1/x)), x, 0, ∞)
     ∫(1/2·(fct + a·b/x^2·LIM(fct, x, a·b/x)), x, a, b)
     ∫(1/2·(fct + a·b/x^2·LIM(fct, x, a·b/x)), x, a, b)
```

#11: $\text{INT2}\left(\frac{\text{LN}(x)}{1 + x^2}, x, 0, \infty\right) = 0$

#12: $\text{INT2}\left(\frac{1}{(1 + x^2)\cdot(1 + x^n)}, x, 0, \infty\right) = \frac{\pi}{4}$

#13: $\int_0^\infty \frac{1}{(1 + x^2)\cdot(1 + x^{10})}\, dx = \frac{\pi}{4}$

#14: $\int_0^\infty \frac{1}{(1 + x^2)\cdot(1 + x^{11})}\, dx = \int_0^\infty \frac{1}{(x^2 + 1)\cdot(x^{11} + 1)}\, dx$

#15: $\text{INT2}\left(\frac{x\cdot\text{ATAN}(x)}{(1 + x^2)\cdot(1 + x + x^2)}, x, 0, \infty\right) = \pi^2\cdot\left(\frac{1}{8} - \frac{\sqrt{3}}{18}\right)$

#16: $\text{INT2}\left(\frac{x\cdot\text{ATAN}(x)}{(1 + x^2)\cdot(1 + x + x^2)}, x, 0, \infty\right) = 0.2839974238$

It is interesting that DERIVE integrates #13, but does not accept exponents >10 in the second factor of the denominator. INT2 resolves the general case. Approximating the result of #15 and comparing it with the result in the last line on page 12, we can see that HADUD´s functions are still working considering the adapted syntax.

The Bisection Method with DERIVE

David Dyer, Crownsvill, USA

This article discusses the bisection method for approximating roots of a polynomial, the method's accuracy of approximation, and a DERIVE algorithm for simulating the method. Further discussion concerns the considerations of the mathematical issues, as well as the DERIVE experiences surrounding possible discussions during a classroom demonstration of the algorithm.

The bisection method, simply put, locates a small interval which contains a zero of a continuous function, usually a polynomial. The Intermediate Value Theorem is repeatedly applied to a sequence of nested intervals, $I_j = [a_j, b_j]$ on which the function, $p(x)$, has different signs at the endpoints of each interval.

In this paper, I discuss the method itself, its precision, and how a few simple lines in the DERIVE environment allow for simulation of the method. The DERIVE algorithm can be introduced to a variety of students operating at different achievement levels in mathematics. It can, with appropriate adjustments by the instructor, promote any of a number of educational objectives depending upon the level of the course, the students, and their experience with DERIVE.

The Method

The sequence of intervals is defined recursively as follows. Start with a closed interval, $[a, b]$, where $p(a)$ and $p(b)$ have opposite signs. Integer values of a and b are usually easy to observe. Set $a_1 = a$ and $b_1 = b$. To construct a subsequent interval, at any stage, evaluate $p(x)$ at the midpoint, m_j, of $[a_j, b_j]$. If the sign of $p(m_j)$ agrees with that of $p(a_j)$ then set $a_{j+1} = m_j$ and $b_{j+1} = b_j$. On the other hand, if $p(m_j)$ has the same sign as $p(b_j)$, then set $a_{j+1} = a_j$ and $b_{j+1} = m_j$. Inductively, this procedure defines a nested sequence of closed bounded intervals, $I_j = [a_j, b_j]$, $j = 1,2,3, \ldots$, with endpoints at which $p(x)$ has different signs. If eventually $p(m_j) = 0$ for some j, then an exact root will have been found. If there is no such m_j then the infinite intersection of the sequence is guaranteed, by a version of the Heine Borel Theorem, often referred as the Principle of Nested Intervals (Fulks, p.69), to be a singleton, $\{x^*\}$. It is a simple matter to verify that $p(x^*\} = 0$. In this case, the root, x^*, of $p(x)$ will never be found exactly. However, the procedure, when continued, will locate the root with any predetermined accuracy, ε.

The DERIVE Algorithm

With DERIVE, one can very simply apply the bisection method. Start with a clear DERIVE workspace and Author the following lines. (DERIVE will provide the line numbers.)

```
#1:   P(x) :=
#2:   t := ["a","m","b"]
#3:   v := [a, (a + b)/2, b]
#4:   pt := ["p(a)","p(m)","p(b)"]
#5:   pv := [P(a), P((a + b)/2), P(b)]
#6:   [t,v,pt,pv]
```

(These lines can be saved in a DERIVE file for future use. At that time, they can be loaded into DERIVE via the Load command.) Collectively lines #1 through #6 define a 4 × 3 matrix which identifies the interval [a, b], its midpoint, m, and the values of the function, $p(x)$, at a, m and b. Once $p(x)$ and the values of a and b have been defined, line #6, when Simplified, will display the data matrix.

To begin the bisection process, define the function, $p(x)$, and choose values for a and b. For example, one can approximate √3 by applying the algorithmus to the function $p(x) = x^2 – 3$. Since $p(1) = –2$ and $p(2) = 1$, start with $a = 1$ and $b = 2$. Append the following line to those above. (Again DERIVE will provide the line numbers.)

```
#7:   [P(x) := x^2 - 3,a := 1,b := 2]
```

Now Simplify line #6 (use the arrow keys to move the lightbar to line #6 and then execute the Simplify command). The display will be line #8 (see figure 1.)

Notice that $p(x)$ is negative at $x = 3/2$, the midpoint of [1, 2], and also at $a = 1$. Thus, the value of the left endpoint, a, should be reassigned to 3/2. That is, by changing the value of a to 3/2, the next interval $[a_2, b_2]$ = [3/2, 2] is defined. The procedure is then repeated, a new matrix is generated and the interval is subsequently refined. Author a new line (#9):

```
#9:   a := 3/2
```

and then Simplify line #6 again. The new matrix, line #10, is shown in figure 2. This time note that the new midpoint, m_2, is 7/4, and that the sign of $p(7/4)$ agrees with that of the right endpoint, $b = 2$. To reassign the value of b to 7/4, Author line #11:

```
#11:  b := 7/4
```

and then Simplify line #6 again. Shown in figure 3 is the matrix corresponding to the interval, [3/2, 7/4].

Line #6, will by this time, have risen out of the viewing screen. It can of course be recovered by using the arrow keys to move the lightbar up to line #6, but it is usually more convenient to have it in the visible screen. To do this move the lightbar to line #6, press "a" for Author and then the F3 key followed by "RETURN". (Pressing F3 does now the full job.) This procedure "pulls down" the highlighted line into the visible screen with a new line number. Now a matrix of values will be displayed whenever the new line is Simplified.

To display the data in decimal, rather than fractional form, move the lightbar to the most recently displayed matrix and press "x" for ApproXimate, follwed by "RETURN". The approximated matrix , line #13, is figure 4. (Press the ≈-button or Ctrl+G in recent DERIVE versions. Ctrl+G has the additional advantage of setting the accuracy.)

If you prefer to see only decimals as sequential matrices are generated, use the Options Precision command to set the mode to "approXimate" and set digits to the desired level. Shown here are six digits, the DERIVE default. While in approximate mode, the next matrix, that for [a, b] = = [1.625, 1.75], is shown in figure 5.

Although, in the example, the process will never arrive at the exact value of √3, the midpoint m_j, will eventually provide a very good approximation. Note that the length of the interval containing the root is halved at each stage. Therefore, if the midpoint, m_j, is used to approximate the root of $p(x)$ after the j^{th} step, the error of approximation will not exceed $(b_1 – a_1)/2^j$. In the example, $a_1 = 1$ and

$b_1 = 2$, thus the midpoint m_j, is never further than $1/2^j$ from $\sqrt{3}$. From the <u>next</u> matrix, we have $[a_4, b_4] = [1.625, 1.75]$ and thus $m_4 = 1.6875$, approximates $\sqrt{3}$ with a maximum error of $2^{-4} = 0{,}0625$.

To approximate a root to some predetermined accuracy, ε, find the first integer, n, such that $(b_1 - a_1)/2^n < \varepsilon$ and then determine the midpoint m_n, of $[a_n, b_n]$ which provides the desired approximation. In the example, if we set $\varepsilon = 0.001$ and note that $2^{-n} < 0.001$ for $n = 10$, then $m_{10} = 1.73241$ approximates $\sqrt{3}$ with the desired accuracy, 0.001.

$$\begin{bmatrix} a & m & b \\ 1 & \frac{3}{2} & 2 \\ p(a) & p(m) & p(b) \\ -2 & -\frac{3}{4} & 1 \end{bmatrix}$$

fig 1

$$\begin{bmatrix} a & m & b \\ \frac{3}{2} & \frac{7}{4} & 2 \\ p(a) & p(m) & p(b) \\ -\frac{3}{4} & \frac{1}{16} & 1 \end{bmatrix}$$

fig 2

$$\begin{bmatrix} a & m & b \\ \frac{3}{2} & \frac{13}{8} & \frac{7}{4} \\ p(a) & p(m) & p(b) \\ -\frac{3}{4} & -\frac{23}{64} & \frac{1}{16} \end{bmatrix}$$

fig 3

$$\begin{bmatrix} a & m & b \\ 1.5 & 1.625 & 1.75 \\ p(a) & p(m) & p(b) \\ -0.75 & -0.359375 & 0.0625 \end{bmatrix}$$

fig 4

$$\begin{bmatrix} a & m & b \\ 1.625 & 1.6875 & 1.75 \\ p(a) & p(m) & p(b) \\ -0.359375 & -0.152343 & 0.0625 \end{bmatrix}$$

fig 5

The DERIVE Algorithm in the Classroom

The DERIVE bisection algorithm described here requires little practice to be mastered. It is ideally suited to classroom demonstration. After a short chalkboard introduction to the method, it becomes clear that "hand by hand" calculations are prohibitive. At that point, engage the class in an interactive session with the microcomputer. At each stage of the process, have the class decide how to refine the interval which will be used to generate the next matrix. Later students can apply the method to other problems in the microcomputer lab. They will gain insight into the method, both how and why it works, as well as gain experience with DERIVE. There are a number of other opportunities which are made available by the algorithm. For example, have the students who understand the procedure and who are sufficiently experienced with DERIVE dissect and interpret the lines to compute the matrix at each stage of the process. Have them explore possible alternatives to the coding and possibly discover a more informative matrix. Ask more advanced students to research the version of the Heine-Borel Theorem which guarantees that an infinite nested sequence of closed intervals will have a non-empty intersection, $\{x^*\}$. Have them argue, if not prove, that the value of x^* must actually satisfy, $p(x^*) = 0$, and that $m_j \to x^*$ as $j \to \infty$. Challenge them to provide examples which demonstrate the necessity for the nested intervals to be both closed and bounded.

In general there are many opportunities which are provided by the method, especially in light of the simplicity with which DERIVE casts the calculations and ease of display.

Fulks, W (1969). *Advanced Calculus: An Introduction to Analysis (2nd ed.)*. New York: Wiley

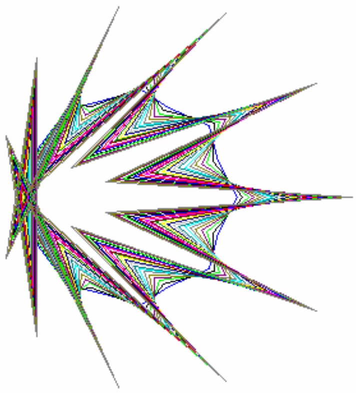
Another challenge could be to enclose the whole procedure in a “program”. Without introducing the programming features of Derive one can emphasize the recursive nature of the method by applying the ITERATES-command which cannot be offered too often to the students. I present one first approach including the given accuracy in the last column. This tool does not take into account any special case mentioned above (eg. f(x)=0 at one of the three positions)

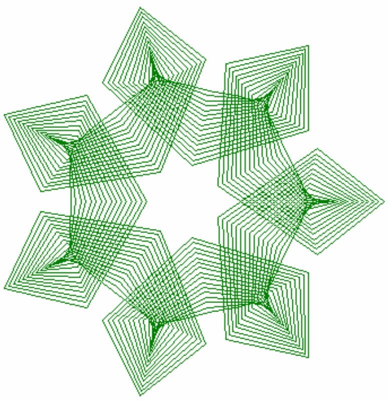
$$\text{bisect}(fct, a, b, n) := \text{ADJOIN}\left([a, m, b, max\_error], \text{ITERATES}\left(\text{IF}\left(\text{SIGN}\left(\lim_{x \to v\downarrow 1} fct\right) = \text{SIGN}\left(\lim_{x \to v\downarrow 2} fct\right), \left[v_2, \frac{v_2 + v_3}{2}, v_3, \frac{v_4}{2}\right], \left[v_1, \frac{v_1 + v_2}{2}, v_2, \frac{v_4}{2}\right]\right), v, \left[a, \frac{a+b}{2}, b, \frac{b-a}{2}\right], n\right)\right)$$

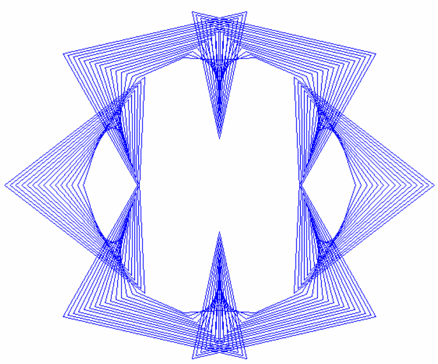
$$\text{bisect}(x^2 - 3, 1, 2, 6)$$

a	m	b	max_error
1	1.5	2	0.5
1.5	1.75	2	0.25
1.5	1.625	1.75	0.125
1.625	1.6875	1.75	0.0625
1.6875	1.71875	1.75	0.03125
1.71875	1.734375	1.75	0.015625
1.71875	1.7265625	1.734375	0.0078125

I created the pictures below working with epicycloids and hypocycloids. See more about this in one of the next DNLS. Josef







Digital Filter Design Using DERIVE

David Hood, Nottingham, UK

1. Introduction

The performance of a Finite Impulse Response Digital filter is determined by the values of its multipliers $\{a_{-m}, a_{-m+1}, ..., a_{-2}, a_{-1}, a_0, a_1, a_2, a_3, ..., a_n\}$. A Finite Impulse Response filter operates on a signal s(t), which has been sampled in time at intervals of T. The response, r(t), from such a filter is related to the input by:

$$r(jT) = a_{-m}\, s((j+m)T) + ... + a_0\, s(jT) + a_1\, s((j-1)T) + ... + a_n\, s((j-n)T).$$

For such a filter to operate in real time we require $a_{-m} = a_{-m+1} = ... = a_{-1} = 0$. If we are processing stored data, this restriction need not apply. The applications we will give will be to filters with $a_{-i} \neq 0$, but the approach can be extended to cover the case $a_{-i} = 0$, $m \leq i \leq 1$.

In many applications, we want to determine the multipliers of a filter with a specified frequency response. It can be shown that the multipliers of a finite impulse response filter are the coefficients in the *inverse discrete time Fourier transform* of the required frequency response. The inverse discrete time Fourier transform is defined as:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} F(\omega)e^{jn\omega}\, d\omega$$

where ω is the angular frequency measured in radians per sample.

In fact, the inverse discrete time Fourier transform of $F(\omega)$ is the set of coefficients in the complex form of the Fourier series for $F(\omega)$ over $-\pi$ to π. The digital filter generated in this manner will usually have an infinite number of multipliers, and hence not be realisable. A realisable filter can be created by using a finite number of the multipliers. This truncation of the multipliers will obviously affect the actual frequency response of the filter. Thus, after truncation, the actual frequency response will have to be compared with the required frequency response to assess the performance of the filter.

If the reqired frequency response is discontinuous, the truncation of the coefficients will lead to Gibbs phenomenon, casing large oscillations in the filters frequency response in the region of this discontinuity. Special window functions may be used to reduce these oscillations.

In the following sections we will describe the use of DERIVE to

a. calculate the multipliers from the specified frequency response,
b. reduce the number of multipliers in the filter by truncation and the use of standard window functions,
c. plot the actual frequency response of the filter.

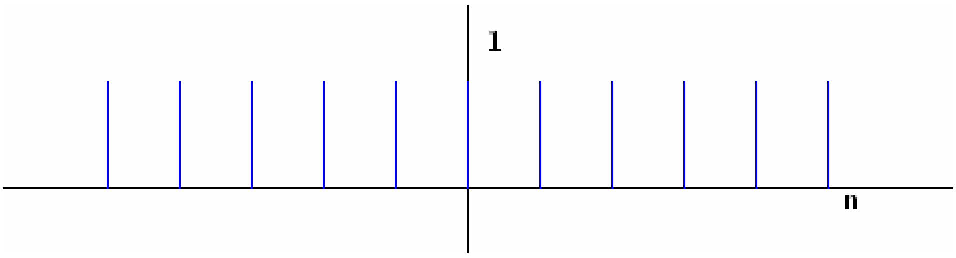
The design process outlined will be applied to the design of three different filters.

2. Window Functions

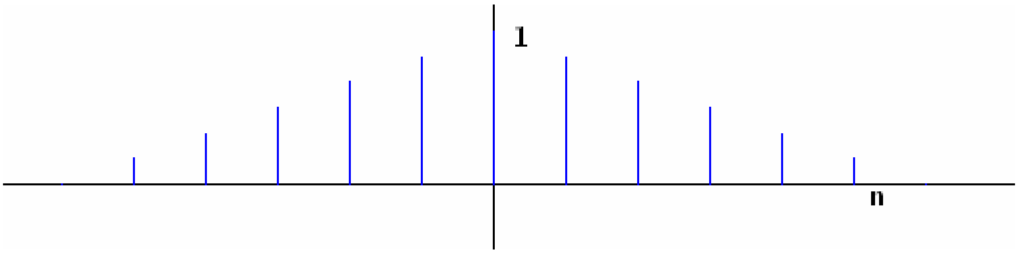
As already mentioned, the truncation of the impulse response of a filter with a discontinuous frequency response leads to a filter whose frequency response has large oscillations in the vicinity of the discontinuity. Truncation of the impulse response can be considered as term by term multiplication of the impulse response with the rectangular window function:

$$\{..., 0, 0, 1, ..., 1, 1, 1, 1, 1, 1, 1, 1, ..., 1, 0, 0, ...\}$$
$$\times$$
$$\{a_n, ..., a_2, a_1, a_0, a_{-1}, a_{-2}, ..., a_{-n}\}$$

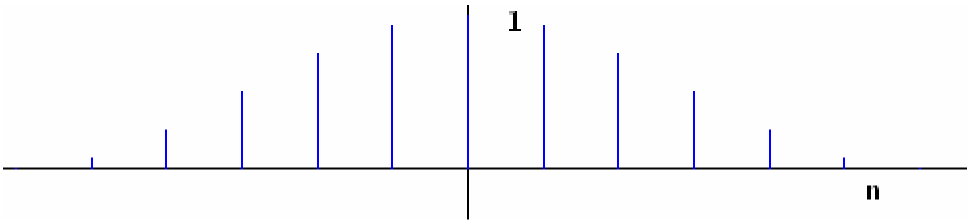
or pictorially



This interpretation of truncation lends itself to extension. We can remove the oscillations in the vicinity of discontinuities by truncating the impulse response less abruptly. For instance by term by term multiplication by the triangular window function:



or a raised cosine or Hamming window:

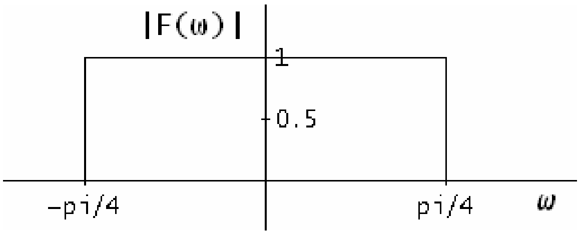


3. The required filters

We will design filters to perform 3 different tasks:

1. A filter that will use sampled values from a function of time (sampled at unit intervals) and approximate the first derivative of the sampled function.
2. A filter that will use sampled values from a function of time (sampled at unit intervals) and approximate the second derivative of the sampled function.

3. A low pass filter that will use sampled values from a function of time, and remove components with frequency higher than π/4 rads/sample, but allow all lower frequency terms to pass unaltered. The modulus of the frequency response (the amplitude response) for this filter is:



4. Some useful DERIVE functions

We will define some DERIVE functions which will facilitate the design of a filter with a specified frequency response. The symbol Ω is used to denote the angular frequency (radians per sample). The functions are:

$$\#1: \quad \text{MEAN}(f, \Omega) := \frac{1}{2\cdot\pi}\cdot\int_{-\pi}^{\pi} f \, d\Omega$$

$$\#2: \quad \text{COFOUR}(f, \Omega, n) := \frac{1}{2\cdot\pi}\cdot\int_{-\pi}^{\pi} f\cdot e^{i\cdot\Omega\cdot n} \, d\Omega$$

`MEAN(f,Ω)` is the value of a_0, and `COFOUR(f,Ω,-π,π)` is the value of a_n in the inverse Fourier transform of f(Ω).

$$\#3: \quad \text{WR}(k, n) := \text{STEP}(k + n + 0.5) - \text{STEP}(k - n - 0.5)$$

$$\#4: \quad \text{HAM}(k, n) := \left(0.46\cdot\text{COS}\left(\frac{k\cdot\pi}{n}\right) + 0.54\right)\cdot\text{WR}(k, n)$$

$$\#5: \quad \text{TRIW}(k, n) := \left(\frac{|k + n + 1|}{2} + \frac{|k - n - 1|}{2} - |k|\right)\cdot\frac{\text{WR}(k, n)}{n + 1}$$

These are the rectangular, Hamming and triangular window functions over the samples –n to n. These are only required for integer k and n.

$$\#6: \quad \text{PSFOUR}(f, \Omega, n) := \text{MEAN}(f, \Omega) + \left(\sum_{k=1}^{n} \text{WR}(k, n)\cdot\text{COFOUR}(f, \Omega, k)\cdot\text{EXP}(-i\cdot k\cdot\Omega)\right) + \sum_{k=1}^{n} \text{WR}(k, n)\cdot\text{COFOUR}(f, \Omega, -k)\cdot\text{EXP}(i\cdot k\cdot\Omega)$$

$$\#7: \quad \text{PSFHMG}(f, \Omega, n) := \text{MEAN}(f, \Omega) + \left(\sum_{k=1}^{n} \text{HAM}(k, n)\cdot\text{COFOUR}(f, \Omega, k)\cdot\text{EXP}(-i\cdot k\cdot\Omega)\right) + \sum_{k=1}^{n} \text{HAM}(k, n)\cdot\text{COFOUR}(f, \Omega, -k)\cdot\text{EXP}(i\cdot k\cdot\Omega)$$

$$\#8:\quad PSFTRI(f, \Omega, n) := MEAN(f, \Omega) + \left(\sum_{k=1}^{n} TRIW(k, n)\cdot COFOUR(f, \Omega, k)\cdot EXP(-\hat{\imath}\cdot k\cdot \Omega)\right) + \sum_{k=1}^{n} TRIW(k, n)\cdot COFOUR(f, \Omega, -k)\cdot EXP(\hat{\imath}\cdot k\cdot \Omega)$$

These functions give the frequency response of the filter found by truncation of the inverse Fourier transform by the rectangular, Hamming and triangular window functions of length n. In other words `PSFOUR` represents the actual frequency response obtained by the FIR (= Finite Impulse Response) filter with multipliers {···. 0, 0, a_{-n}, ···, a_n, 0, 0, ···}, and `PSFHMG` and `PSFTRI` represent the actual frequency responses when obtained by weighting the coefficients with the Hamming and triangular window functions.

```
#9:   MULTRECT(f, Ω, n) := VECTOR(IF(k = 0, MEAN(f, Ω), COFOUR(f, Ω, k)), k, -n, n)
#10:  MULTHMG(f, Ω, n) := VECTOR(IF(k = 0, MEAN(f, Ω), COFOUR(f, Ω, k)·HAM(k, n)), k, -n, n)
#11:  MULTTRI(f, Ω, n) := VECTOR(IF(k = 0, MEAN(f, Ω), COFOUR(f, Ω, k)·TRIW(k, n)), k, -n, n)
```

These functions give the multipliers {a_{-n}, ···, a_n} of the filter, using the rectangular, Hamming and triangular window functions respectively.

5. The Solution Using DERIVE

To use the frequency response method outlined above to calculate the coefficients of the filters which are to perform numerical first and second order differentiation, we first need to find the frequency response of the analytic operations of first and second order differentiation. The standard result from the theory of analogue control systems is that frequency response of a linear system may be determined by substituting s = $j\omega$ into the transfer function for the system. The transfer functions for first and second order differentiation are s and s^2 respectively, so the required frequency responses are $j\omega$ and $-\omega^2$. It should be pointed out that the units of ω are radians per second, in contrast to the units of radians per sample given for the digital filters. To simplify matters, we will choose a sampling rate of one sample per second, in which case the two measures of frequency will be equal.

Thus, to design a digital filter to approximate the first order derivative, *Author* the reqired frequency response:

`f := ι·Ω`

To calculate the frequency response of the filter having 6 multipliers either side of a_0, *Author* and *Simplify* the expressions:

`PSFOUR(f, Ω, 6)`,

`PSFTRI(f, Ω, 6)` and

`PSFHMG(f, Ω, 6)`.

The frequnecy response is periodic, period 2π, and so can only match the required frequency response over the interval $[-\pi, \pi]$. All of these frequency responses are imaginary, so having simplified the above three expressions, take the imaginary parts of the frequency responses (using the DERIVE function `IM`), and plot them.

The periodic repetition of the required frequency response is discontinuous at odd integer multiples of π, and so the frequency response `PSFOUR(f,Ω,6)`, will be adversely affected by Gibbs phenomenon. From the plots, it is apparent that the filter designed by using the Hamming window is the most accurate, and is effective to $-2 < \Omega < 2$.

We have yet to find the coefficients, but this may be done by *Authoring* and *Simplifying*.

`MULTRECT(f,Ω,6).`

To calculate the coefficients for a FIR filter that will approximate the second derivative we apply the same process to:

`g := -Ω^2` (1)

The periodic extension of g is continuous, and so we would expect reasonable performance from the filter truncated by the rectangular window function. We will see that it is this filter that produces the best results.

To calculate the frequency response of the filter having 6 multipliers either side of a_0, *Author* and *Simplify* the expressions:

`PSFOUR(g, Ω, 6),` (2)

`PSFHMG(g, Ω, 6)` and (3)

`PSFTRI(g, Ω, 6).` (4)

Plotting the expressions labelled (1) to (4) above shows that the rectangular window function filter is accurate for frequencies of up to 3 radians per second.

The coefficients can be calculated by simplifying `MULTRECT(g,Ω,6)`. Because the rectangular window function performs best for this frequency response, the coefficients of a shorter filter are easily calculated by further truncation of the coefficients. The coefficients in the filter we have calculated are:

$$\left[\mathrm{VECTOR}(a_n, n, -6, 6), \mathrm{MULTRECT}(g, \Omega, 6)\right]$$

$$\begin{bmatrix} a_{-6} & a_{-5} & a_{-4} & a_{-3} & a_{-2} & a_{-1} & a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ -\frac{1}{18} & \frac{2}{25} & -\frac{1}{8} & \frac{2}{9} & -\frac{1}{2} & 2 & -\frac{\pi^2}{3} & 2 & -\frac{1}{2} & \frac{2}{9} & -\frac{1}{8} & \frac{2}{25} & -\frac{1}{18} \end{bmatrix}$$

Truncating the filter to three coefficients either side of a_0 produces a less accurate filter, which is still reasonable to $\omega = 2.8$ radians per sample.

The third (low pass) filter has only its amplitude response specified – the phase response is of less concern in this application. If we assume a phase response which is identically zero, the frequency response and amplitude response coincide, and the filter coefficients can be constructed by the same method as above. We define the desired frequency response using the DERIVE function `STEP(t)`:

`f := STEP(Ω+π/4)−STEP(Ω−π/4)` (5)

We then decide on the number of coefficients in the filter and author the appropriate expressions to calculate the frequency response of the filters. For example, for ten multipiers either side of the central multiplier, we Author:

`PSFOUR(f, Ω, 10),` (6)

`PSFHMG(f, Ω, 10)` and (7)

`PSFTRI(f, Ω, 10).` (8)

Simplify these expressions, and plot them. The required frequency response is discontinuous, so we would expect the Hamming window to give us the best coefficients. This should be verified by the plots produced. The coefficients in the Hamming filter can be calculated by simplifying `MULTHMG(f,Ω,10)`.

The values of the coefficients are:

$$\begin{bmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ \frac{1}{4} & \frac{23\cdot\sqrt{\sqrt{5}+5}}{200\cdot\pi}+\frac{27\cdot\sqrt{2}}{100\cdot\pi} & \frac{23\cdot\sqrt{5}}{400\cdot\pi}+\frac{131}{400\cdot\pi} & \frac{23\cdot\sqrt{5-\sqrt{5}}}{600\cdot\pi}+\frac{9\cdot\sqrt{2}}{100\cdot\pi} & 0 & -\frac{27\cdot\sqrt{2}}{500\cdot\pi} & \frac{23\cdot\sqrt{5}}{1200\cdot\pi}-\frac{131}{1200\cdot\pi} \end{bmatrix}$$

$$\begin{bmatrix} a_7 & a_8 & a_9 & a_{10} \\ \frac{23\cdot\sqrt{5-\sqrt{5}}}{1400\cdot\pi}-\frac{27\cdot\sqrt{2}}{700\cdot\pi} & 0 & \frac{3\cdot\sqrt{2}}{100\cdot\pi}-\frac{23\cdot\sqrt{\sqrt{5}+5}}{1800\cdot\pi} & \frac{1}{125\cdot\pi} \end{bmatrix}$$

$$\begin{bmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 \\ 0.25 & 0.2200116480 & 0.1451728339 & 0.06079995259 & 0 & -0.02430854053 & -0.02110671382 & -0.008669363940 & 0 \end{bmatrix}$$

$$\begin{bmatrix} a_9 & a_{10} \\ 0.002563750808 & 0.002546479089 \end{bmatrix}$$

The values of a_{-10} to a_{-1} are the same as a_{10} to a_1.

6. A DERIVE Session

The following is a complete DERIVE session in which the coefficients for both a first order and second order differentiator are determined.

We don´t repeat the first 11 expressions which are given on pages 21 and 22 and proceed with defining the required frequency response.

#12: Define the required frequency response

#13: $f := i\cdot\Omega$

#14: Calculate the coefficients

#15: MULTRECT(f, Ω, 6)

#16: $\left[-\frac{1}{6}, \frac{1}{5}, -\frac{1}{4}, \frac{1}{3}, -\frac{1}{2}, 1, 0, -1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{5}, \frac{1}{6}\right]$

#17: Now calculate the frequency response of this filter

#18: PSFOUR(f, Ω, 6)

#19: $-i\cdot\left(\frac{SIN(6\cdot\Omega)}{3}-\frac{2\cdot SIN(5\cdot\Omega)}{5}+\frac{SIN(4\cdot\Omega)}{2}-\frac{2\cdot SIN(3\cdot\Omega)}{3}+SIN(2\cdot\Omega)-2\cdot SIN(\Omega)\right)$

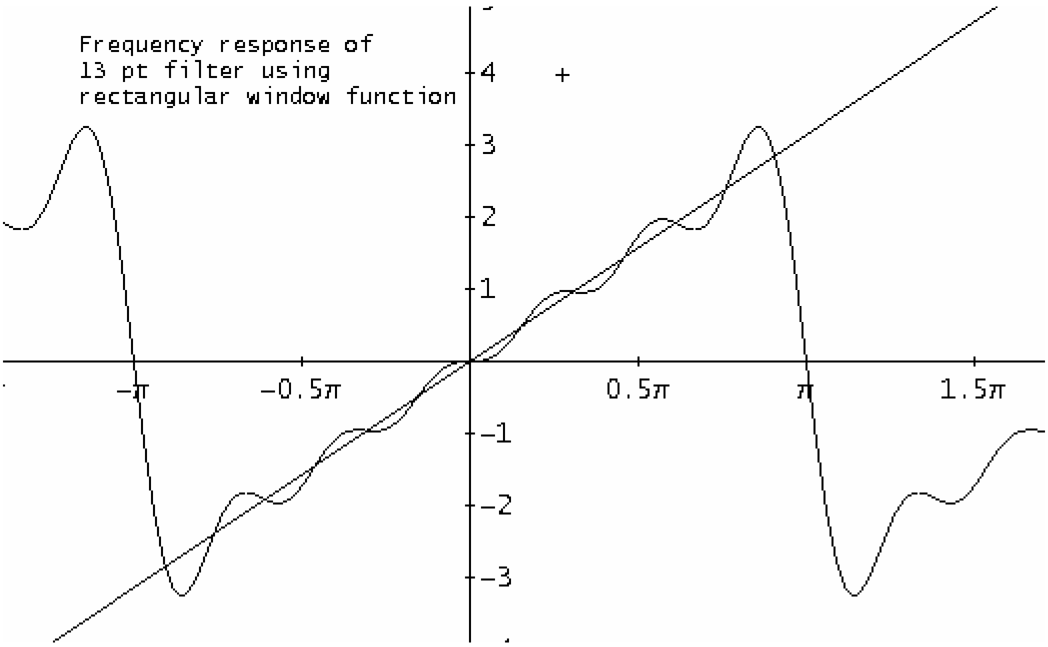
#20: Purely imaginary, so plot imaginary part

#21: $IM\left(-i\cdot\left(\frac{SIN(6\cdot\Omega)}{3}-\frac{2\cdot SIN(5\cdot\Omega)}{5}+\frac{SIN(4\cdot\Omega)}{2}-\frac{2\cdot SIN(3\cdot\Omega)}{3}+SIN(2\cdot\Omega)-2\cdot SIN(\Omega)\right)\right)$

#22: $-\frac{SIN(6\cdot\Omega)}{3}+\frac{2\cdot SIN(5\cdot\Omega)}{5}-\frac{SIN(4\cdot\Omega)}{2}+\frac{2\cdot SIN(3\cdot\Omega)}{3}-SIN(2\cdot\Omega)+2\cdot SIN(\Omega)$

#23: and plot imaginary part of f on same axes

#24: IM(f)



#25: The performance is poor due to Gibbs phenomena – use Hamming window

#26: MULTHMG(f, Ω, 6)

#27: $\left[-\frac{1}{75}, \frac{27}{250}-\frac{23\cdot\sqrt{3}}{500}, -\frac{31}{400}, \frac{9}{50}, -\frac{77}{200}, \frac{23\cdot\sqrt{3}}{100}+\frac{27}{50}, 0, -\frac{23\cdot\sqrt{3}}{100}-\frac{27}{50}, \frac{77}{200}, -\frac{9}{50}, \frac{31}{400}, \frac{23\cdot\sqrt{3}}{500}-\frac{27}{250}, \frac{1}{75}\right]$

#28: PSFHMG(f, Ω, 6)

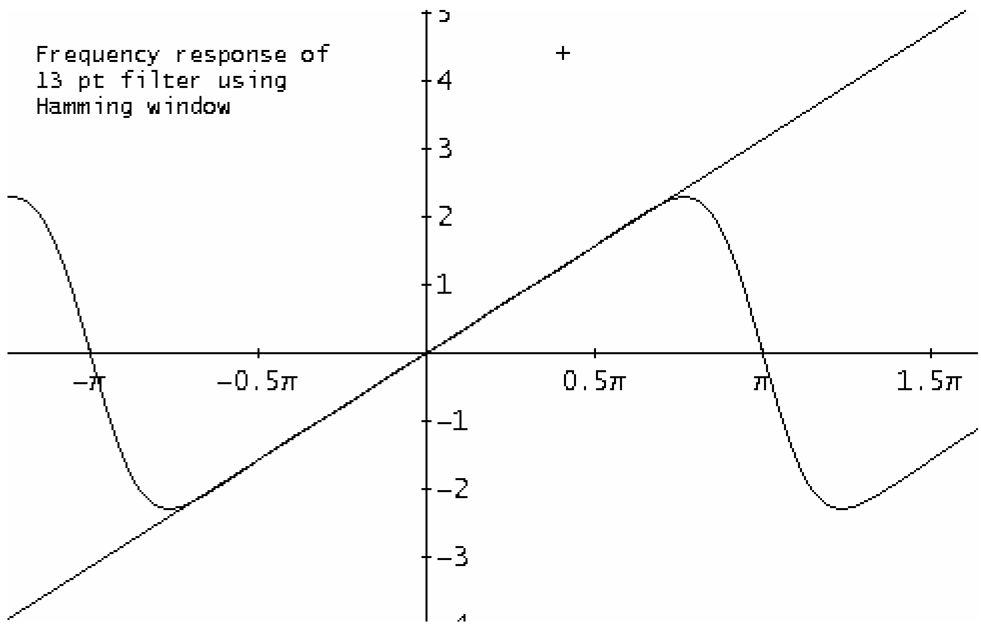
#29: $$-i\cdot\left(\frac{2\cdot \mathrm{SIN}(6\cdot\Omega)}{75}+\left(\frac{23\cdot\sqrt{3}}{250}-\frac{27}{125}\right)\cdot \mathrm{SIN}(5\cdot\Omega)+\frac{31\cdot \mathrm{SIN}(4\cdot\Omega)}{200}-\frac{9\cdot \mathrm{SIN}(3\cdot\Omega)}{25}+\frac{77\cdot \mathrm{SIN}(2\cdot\Omega)}{100}-\left(\frac{23\cdot\sqrt{3}}{50}+\frac{27}{25}\right)\cdot \mathrm{SIN}(\Omega)\right)$$

#30: Again purely imaginary, so plot imaginary part

#31: IM(PSFHMG(f, Ω, 6))

#32: and plot imaginary part of f on same axes

#33: IM(f)



#34: For a second order derivative, author appropriate frequency response

#35: $g := -\Omega^2$

#36: MULTRECT(g, Ω, 6)

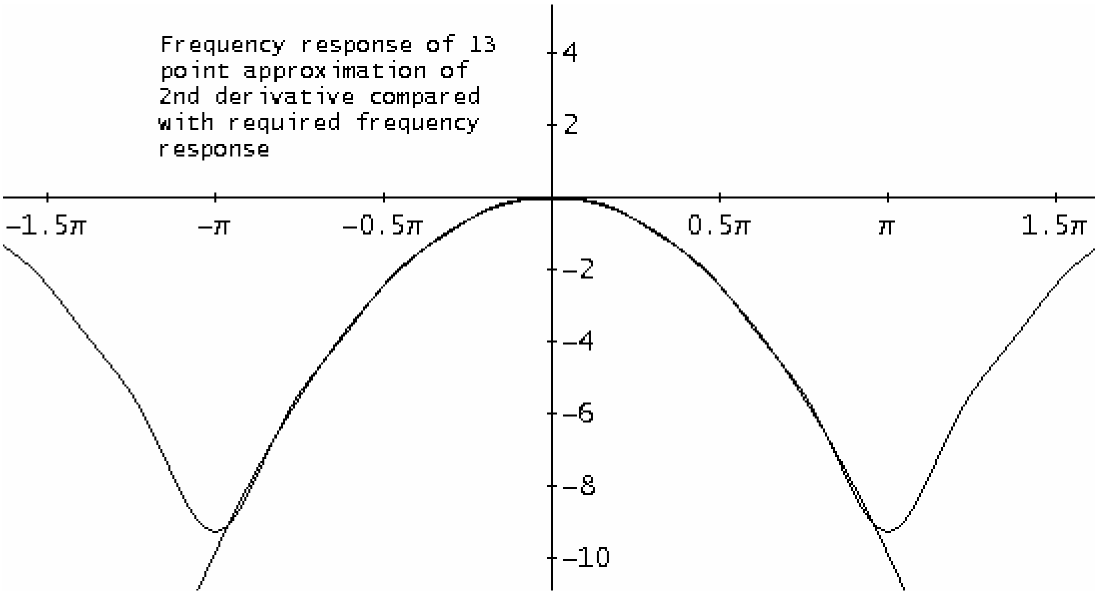
#37: Plot and compare required and actual frequency response

#38: $$\left[-\frac{1}{18},\ \frac{2}{25},\ -\frac{1}{8},\ \frac{2}{9},\ -\frac{1}{2},\ 2,\ -\frac{\pi^2}{3},\ 2,\ -\frac{1}{2},\ \frac{2}{9},\ -\frac{1}{8},\ \frac{2}{25},\ -\frac{1}{18}\right]$$

#39: PSFOUR(g, Ω, 6)

#40: $$-\frac{\mathrm{COS}(6\cdot\Omega)}{9}+\frac{4\cdot \mathrm{COS}(5\cdot\Omega)}{25}-\frac{\mathrm{COS}(4\cdot\Omega)}{4}+\frac{4\cdot \mathrm{COS}(3\cdot\Omega)}{9}-\mathrm{COS}(2\cdot\Omega)+4\cdot \mathrm{COS}(\Omega)-\frac{\pi^2}{3}$$

#41: $-\Omega^2$



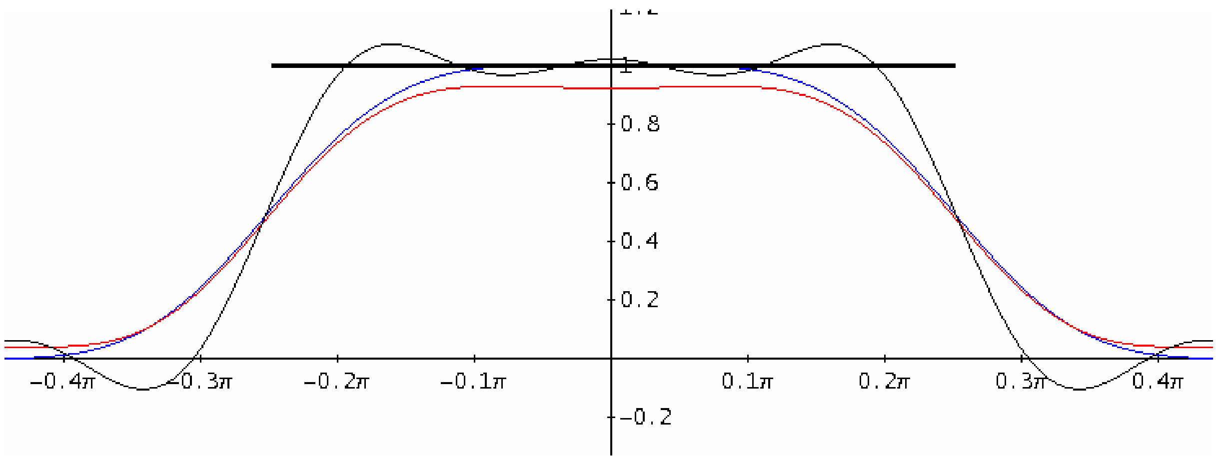

```
TABLE(f, Ω, -π/4, π/4, 0.01)

PSFOUR(f, Ω, 10)

PSFTRI(f, Ω, 10)

PSFHMG(f, Ω, 10)
```

PSFOUR (black), PSFTRI (red), PSFHMG (blue)



Dependent Repeated Experiments

Ales Kozubík Bratislava, Slovakia

In my paper (DNL#10) I dealt with using DERIVE in teaching Probability Theory to solve problems from independent repeated experiments. The following DERIVE code was used there:

```
#1:   pnom(k, n) := n! / (∏_{i=1}^{DIMENSION(k)} k_i!)

#2:   ir(k, n, p) := COMB(n, k)·p^k·(1 - p)^(n - k)

      mlc(n, p) :=
        If (n + 1)·p - 1 = FLOOR((n + 1)·p - 1)
#3:        [(n + 1)·p - 1, (n + 1)·p]
           FLOOR((n + 1)·p)

      lnr(α, p) :=
        If LN(1 - α)/LN(1 - p) = FLOOR(LN(1 - α)/LN(1 - p))
#4:        LN(1 - α)/LN(1 - p)
           FLOOR(LN(1 - α)/LN(1 - p)) + 1

#5:   af(i, p) := p·(1 - p)^(i - 1)

#6:   irs(k, n, p) := PNOM(k, n)·IF(DIM(p) = DIM(k), ∏_{i=1}^{DIM(k)} p_i^(k_i), 0)
```

In this paper I would like to present the continuation of that programme to complete it for solving the problems about dependent repeated experiments and the repeated eXperiments when the probability of possible results is changed at each experiment independently on the result of the previous experiments.

The case of dependent repeated experiments with two possible results can be modeled by the following scheme: n things are in a box and k of them have a given quality and the others have not. If we draw m things from that box what is the probability that j of the drawn things will have the given quality? The answer is given by the hypergeometric formula

$$\frac{\binom{k}{j}\cdot\binom{n-k}{m-j}}{\binom{n}{m}}$$ which is realized in the function `DRE(j,m,n)`:

```
#7:   DRE(j, k, m, n) := COMB(k, j)·COMB(n - k, m - j) / COMB(n, m)

#8:   HYPERGEOMETRIC_DENSITY(j, k, m, n)
```

HYPERGEOMETRIC_DENSITY(j,k,m,n) is implemented

```
#9:   DRE(3, 10, 10, 20) = 3600/46189

#10:  HYPERGEOMETRIC_DENSITY(3, 10, 10, 20) = 3600/46189
```

The hypergeometric formula can be generalized for the case of a set of n things containing n_1 things of the first kind, n_2 things of the second kind, …, n_s things of the s-th kind. We choose k things from this set. What is the probability that there will be k_1 things of the first kind, k_2 things of the second kind, …, k_s things of the s-th kind in the chosen set of things? The answer is given by the function

$$\#11:\quad \mathrm{DRES(k,\ n)} := \frac{\prod_{i=1}^{\mathrm{DIMENSION(n)}} \mathrm{COMB(ELEMENT(n,\ i),\ ELEMENT(k,\ i))}}{\mathrm{COMB}\left(\sum_{i=1}^{\mathrm{DIMENSION(n)}} \mathrm{ELEMENT(n,\ i)},\ \sum_{i=1}^{\mathrm{DIMENSION(k)}} \mathrm{ELEMENT(k,\ i)}\right)}$$

which is based on the formula: $\dfrac{\binom{n_1}{k_1}\cdot\binom{n_2}{k_2}\cdot\ldots\cdot\binom{n_s}{k_s}}{\binom{n_1+n_2+\cdots+n_s}{k_1+k_2+\cdots+k_s}}$.

In the function `DRES(k,n)` k and n are vectors $k = [k_1, k_2, \ldots, k_s]$ and $n = [n_1, n_2, \ldots, n_s]$.

One of the questions we answered in case of independent repeated experiments was: What is the probability that the success will be set in at first time in the i-th repetition. Likewise we can ask in case of dependent repeated experiments: What is the probability a thing with given quality will be drawn at first time in the k-th attempt? This problem is solved by the formula:

$$\frac{m}{n-k+1}\prod_{j=1}^{k-1}\left(1-\frac{m}{n-j+1}\right)$$

which is the base for the function

$$\#12:\quad \mathrm{AFT(k,\ m,\ n)} := \frac{m}{n-k+1}\cdot\prod_{j=1}^{k-1}\left(1-\frac{m}{n-j+1}\right)$$

The meaning of m and n is the same as in function `DRE(j,k,m,n)`.

The last problem which is solved by the programme is the problem of repeated experiments if the probability of the results is changed in single experiments but independently on the results of the previous experiments. Let p be a vector $[p_1, p_2, \ldots, p_n]$ where p_i is the probability of the success in i-th experiment. In order to compute the probability that in these n repetitions the success will be set k-times we have to construct the generating function in the form

$$\prod_{i=1}^{n}\left(q_i + p_i\cdot x\right);\quad q_i = 1 - p_i.$$

The unknown probability is equal to the coefficient of the k-th power of x in the generating function. The following function is taken from `MiscellaneousFunctions.mth` and can be used immediately:

$$\#13:\quad \mathrm{POLY\_COEFF(u,\ x,\ n)} := \frac{1}{n!}\cdot\lim_{x\to 0}\left(\frac{d}{dx}\right)^{n} u$$

We obtain the searched probability using function `IRC(k,p)`:

```
#14:  IRC(k, p) := POLY_COEFF( ∏_{i=1}^{DIMENSION(p)} (ELEMENT(p, i)·x + (1 - ELEMENT(p, i))), x, k)
```

Two worked problems:

Problem 1. A box contains fifteen balls. Five of them are white four are red, three are blue, two are green and one is black. We draw at random five of them. What is the probability that

a) two of the five balls will be white?

b) no pair of the five balls have the same colour?

c) the first white ball will be drawn in the fifth attempt?

```
#15:  Solution of problem 1, question a)
#16:  DRE(2, 5, 5, 15)
#17:  400/1001
#18:  questions b) and c)
#19:  DRES([1, 1, 1, 1, 1], [5, 4, 3, 2, 1])
#20:  40/1001
#21:  AFT(5, 5, 15)
#22:  10/143
```

The classical solutions:

a) $$\frac{\binom{5}{2}\cdot\binom{10}{3}}{\binom{15}{5}}=\frac{400}{1001}$$

b) $$\frac{5\cdot4\cdot3\cdot2\cdot1}{\binom{15}{5}}=\frac{40}{1001}$$

c) $$\frac{10}{15}\cdot\frac{9}{14}\cdot\frac{8}{13}\cdot\frac{7}{12}\cdot\frac{5}{11}=\frac{10}{143}$$

Problem 2. There are four guns in the shooting gallery. Probabilities of hitting the target are 0.1, 0.2, 0.3 and 0.4 for the single guns. One bullet is shot from each of them. What is the probability that

a) the target will be hit twice?

b) the target will be hit at least twice?

The classical solutions:

```
#24:  IRC(2, [0.1, 0.2, 0.3, 0.4])
#25:  134/625
#26:  IRC(2, [0.1, 0.2, 0.3, 0.4]) + IRC(3, [0.1, 0.2,
        0.3, 0.4]) + IRC(4, [0.1, 0.2, 0.3, 0.4])
#27:  643/2500
```

a) $$0.1\cdot0.2\cdot0.7\cdot0.6+0.1\cdot0.3\cdot0.8\cdot0.6+$$
$$+0.1\cdot0.4\cdot0.8\cdot0.7+0.2\cdot0.3\cdot0.9\cdot0.6+$$
$$+0.2\cdot0.4\cdot0.9\cdot0.7+0.3\cdot0.4\cdot0.9\cdot0.8=$$
$$=\frac{134}{625}$$

b) $$\frac{134}{625}+0.1\cdot0.2\cdot0.3\cdot0.6+$$
$$+0.1\cdot0.2\cdot0.4\cdot0.7+0.2\cdot0.3\cdot0.4\cdot0.9+$$
$$+0.1\cdot0.2\cdot0.3\cdot0.4=\frac{643}{2500}$$

It should be no problem to transfer the functions to the TI-92/V 200, Josef.

Word Processing and DERIVE Revised

J.M.M.C. Lopes & J. C. Tamames, Porto, Portugal

In DNL#2 F. Schumm and J. Boehm discuss “Word Processing and DERIVE”. Both authors suggest the use of CAPTURE.COM to capture graphics and insert then in the text of a report.

Later in DNL#6 D.A. Sjoestrand describes a method based on the capabilities of the Windows 3.1 Enhanced Mode.

To insert a DERIVE full screen in the text there is an easiest way: we only have to run DERIVE under Windows until we reach the screen we want and then press Print Screen (PrtSc). One image of the screen is passed to the clipboard, and, once in WORD, we can insert this image in the text (WORD command Edit/Paste). We can still complete the figure (legends, etc) with Microsoft Draw, a standard facility in WORD.

The capture of the screen through these methods is extremely rapid however it has the inconvenience of allowing only images with the original colours (at the Graphics Mode), which aren´t always the most appropriate. Of course, we can always choose the DERIVE colours as to present characters and graphics in black with a white background. But I prefer to apply PSP (Paint Shop Pro, JASC Inc, 1992) which has the advantage of allowing capturing an area or the full screen and adjusting the colours in the captured image. The steps to perform are:

1 – Start Windows; open WORD and minimize then;

2 – Open PSP and minimize then;

3 – Start and run DERIVE under Windows until you reach the desired screen and then pass to the 386 Enhanced Mode (ALT+ENTER): DERIVE passes running (at a low speed …) in a window occupying about 2/3 of the screen. Now you can access, in the inferior part of the screen, the icons of WORD and PSP, making it easy to pass from one program to another (see Fig. 1);

4 – Start PSP (click on the PSP icon) and then:

Capture (Area or Full Screen)

Make it an Icon

(the DERIVE window reappears with the PSP cursor (a double cross); next you have to define the captured area with the mouse)

Colours (Negative Image/Grey Scale/Gamma Correction 0.1)

Edit Copy

(one image of the desired area is copied now to the clipboard);

5 – Close ore minimize PSP and DERIVE (ALT+ENTER to leave 386 Enhanced Mode, Quit if you want to leave DERIVE, else minimize DERIVE) and return to WORD (a click on the WORD icon). Afterwards you have to insert the contents of the clipboard in the desired position (Edit/Paste)

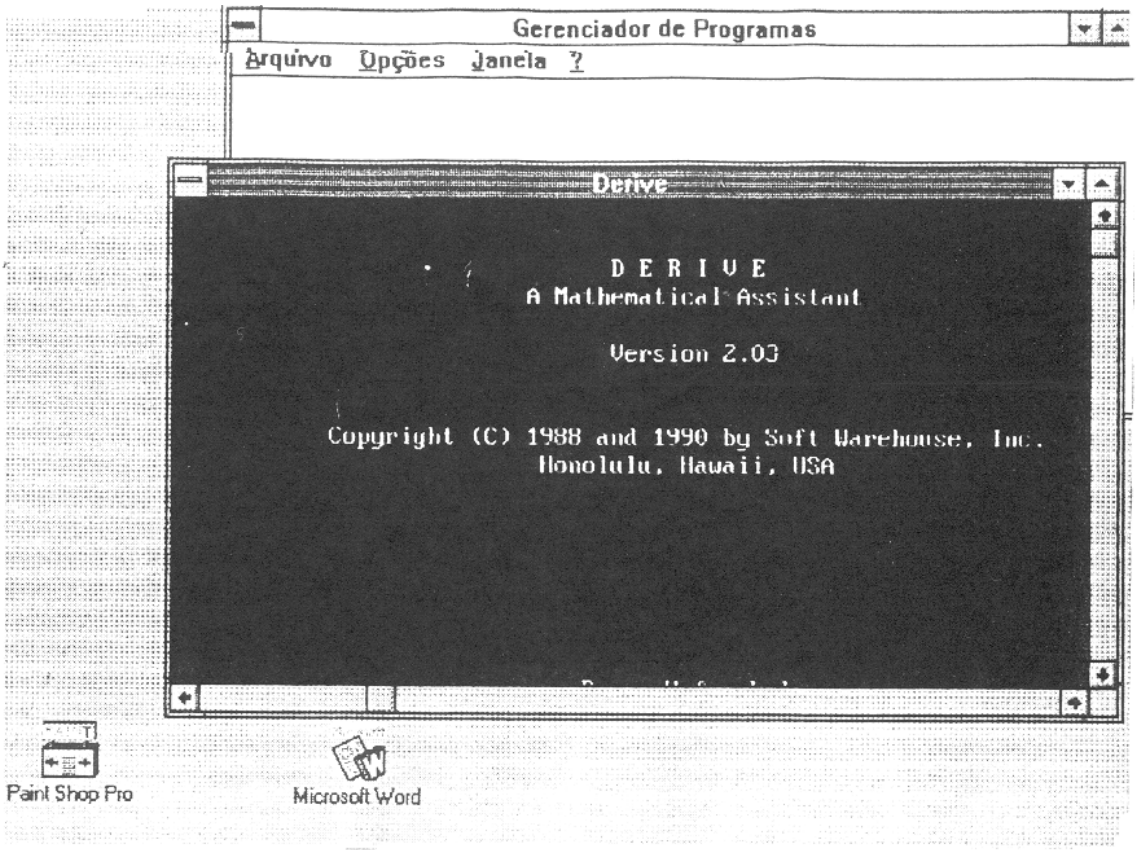


Fig.1 – The DERIVE window in 386 Enhanced Mode and the minimized icons of PSP and WORD. This figure is passed to the clipboard with the command Print Screen (with the original colours).

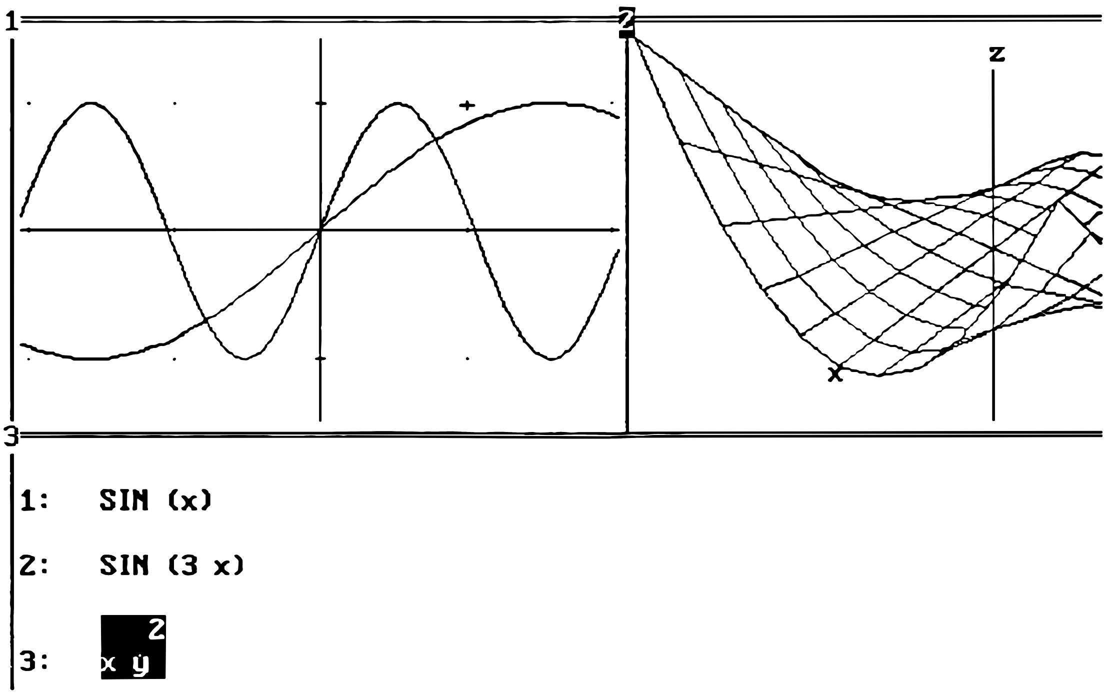


Fig.2 – An area screen captured with PSP, as described in the text (with the colours changed to black and white).

Ebene Algebraische und Transzendente Kurven (1)
Die Kissoide (auch Zissoide) – The Cissoid of Diokles

Thomas Weth, Würzburg, Germany

Die Theorie der ebenen Kurven ist seit alters her ein zentrales Thema der Geometrie. Durch die Methoden der Algebra der analytischen Geometrie und der Differentialgeometrie gelangte die Kurventheorie im 18. und 19. Jahrhundert zu ihrer Blütezeit. Durch die heute zur Verfügung stehenden Möglicheiten auf geometrischem (GEOLOG, Cabri-Géometre, Felix) und algebraisch-analytischem Gebiet (DERIVE) gewinnen Kurven auch wieder an Bedeutung für den Unterricht. Sie gestatten wie kaum ein anderes Themengebiet eine enge Verbindung von geometrischen und analytischen Methoden wie sie von Felix Klein zu Beginn des Jahrhunderts als „Basis" für den gesamten Mathematikunterricht gefordert wurde.

Da sich vor allem DERIVE dazu eignet, die Parameterdarstellung von Kurven graphisch darzustellen, die Kurven analytisch zu untersuchen, wenn möglich die algebraischen Gleichungen zu erhalten u.s.w. werden in den folgenden Nummern des DERIVE Newsletters jeweils einige der bekanntesten algebraischen und transzendenten ebenen Kurven vorgestellt. Als Literaturquelle werde ich im Wesentlichen die hervorragenden Bücher von Loria (Algebraische und transcendente ebene Kurven, Leipzig, Teubner, 1902) und von H. Schmidt (Ausgewählte höhere Kurven, Wiesbaden, Kesselring, 1949) zu Grunde legen (diese Werke werden im Folgenden in den Literaturverzeichnissen nicht mehr aufgeführt.)

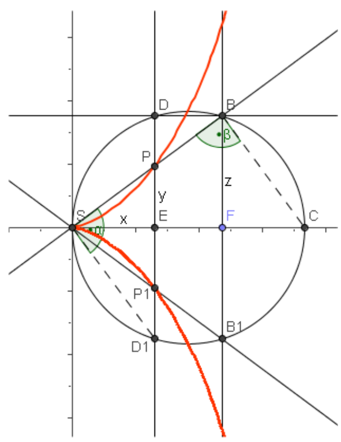
Die Kissoide (auch: Zissoide) – The Cissoid

Zur Lösung des Delischen Problems (Würfelverdoppelung) (vgl. Breidenbach, 1953) wurde die Kissoide des Diokles (ca. 200 v. Chr.) erfunden. Der Name beschreibt die Ähnlichkeit der Kurve mit der Spitze eines Efeublattes (*κισσόζ*, Efeu). Im 17. Jhdt beschäftigten sich u.a. Newton, Roberval, Fermat, Huygens und Sluse (Domherr zu Lüttich) mit dieser Kurve. Huygens schlug vor, die Kurve *Slusianische Kissoide* zu nennen, in der Literatur hielt sich aber die Verbindung des Namens zu *Diokles*.

The Cissoid of Diokles (200 bC) was invented to solve the ancient problem of doubling a die. Its name describes the similarity of the curve with an ivy leaf (*κισσόζ*, ivy). In the 17th century among others Newton, Roberval, Fermat, Huygens and Sluse (canon in Liége) dealt with this curve. Huygens proposed to name the curve *Slusian Cissoid* but in literature the connection with *Diokles* was kept.

Die usprüngliche Konstruktion der Kurvenpunkte:
Gegeben ist ein Kreis mit dem Durchmesser |SC| = *a*. Senkrecht zu SC werden zwei zum Mittelpunkt symmetrisch liegende Sehnen BB_1 und DD_1 konstruiert. Die Schnittpunkte von SB und SB_1 mit DD_1 ergeben zwei Kissoidenpunkte P und P_1.

Given is a circle with diameter |SC| = *a*. Two chords BB_1 and DD_1 perpendicular to the diameter and symmetric to the center of the circle are drawn. The intersection points of SB and SB_1 with DD_1 give two points of the cissoid P and P_1. (See the GeoGebra realisation →)



Die Koordinaten der Kissoidenpunkte (und damit die Parameterdarstellung der Kurve) leitet man folgendermaßen her: Mit SC als x-Achse und der Kreistangente in S als y-Achse gilt für die Höhe im rechtwinkligen Dreieck SBC: $\mathrm{BF} = \sqrt{x(a-x)}$. Mit dem Verhältnis $y : x = \mathrm{BF} : (a - x)$ erhält man:

$y = \dfrac{x\sqrt{x(a-x)}}{a-x}$ und daraus die algebraische Gleichung für die Kissoide: $y^2\,(a-x) - x^3 = 0$.

Mit $x = r \cos \varphi$ und $y = r \sin \varphi$ erhält man die Polardarstellung: $r = a \sin \varphi \tan \varphi$.

Eine Parametrisierung in kartesischen Koordinaten, welche die Kurve über den Kreis hinaus erweitert, und die neben der Polardarstellung für DERIVE die geeignetste ist, lautet:

$$x = \frac{a\,t^2}{1+t^2},\ y = \frac{a\,t^3}{1+t^2}.$$

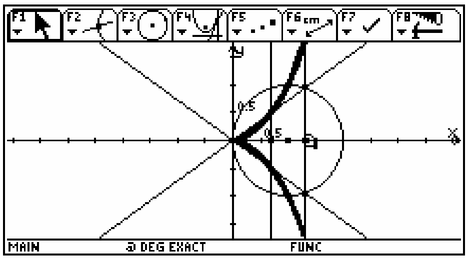
How to derive implicit, polar and parameter form of the curve: Take SC as x-axis and the tangent of the circle in point S as y-axis. Then altitude BF in triangle SBC is given by $\mathrm{BF} = \sqrt{x(a-x)}$. Using the ratio x = BF : (a – x) results in $y = \dfrac{x\sqrt{x(a-x)}}{a-x}$ which leads to the implicit form $y^2\,(a-x) - x^3 = 0$.

Substituting $x = r \cos \varphi$ and $y = r \sin \varphi$ gives the polar form: $r = a \sin \varphi \tan \varphi$.

Parameterisation in Cartesian coordinates gives a parameter form:

$$x = \frac{a\,t^2}{1+t^2},\ y = \frac{a\,t^3}{1+t^2}.$$

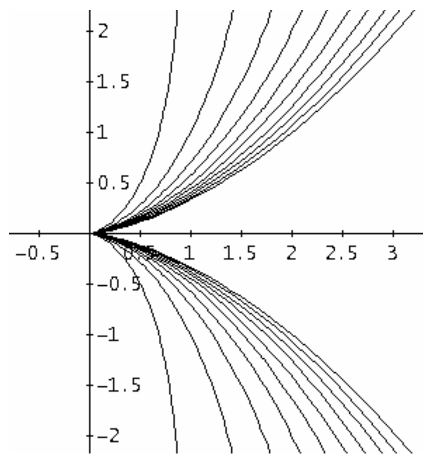
It might be a challenge for the students to find the given parameter form – or another one – and check if is “compatible” with the implicit form. (eg $x = t^2,\ y = \dfrac{t^3}{\sqrt{a-t^2}}$)

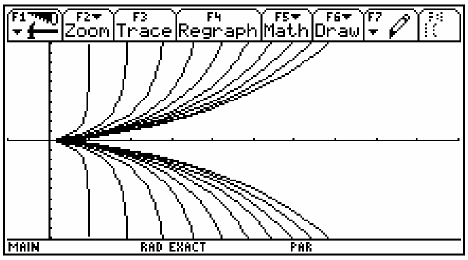


$$\mathrm{VECTOR}\left(\left[\frac{a\cdot t^2}{1+t^2},\ \frac{a\cdot t^3}{1+t^2}\right],\ a,\ 10\right)$$

$\checkmark \mathrm{xt1} = \dfrac{a\cdot t^2}{1+t^2} \mid a = \mathrm{seq}(k, k, 1, 10)$

$\checkmark \mathrm{yt1} = \dfrac{a\cdot t^3}{1+t^2} \mid a = \mathrm{seq}(k, k, 1, 10)$





Breidenbach, W. Das Delische Problem, Stuttgart (Teubner) 1953
Oettinger, E. Die Zissoide oder „Efeuartige“, Mathe-Plus (Feb. 1985) Nr. 3, p. 4-7
Weth, Th. Ein abbildungsgeometrischer Zugang zu algebraishen Kurven 3. und höherer Ordnung, Didaiktik der Mathematik 19, 1991, H.2. S. 145-164

Die Würfelverdoppelung mit der Kissoide

Gegeben: Würfel mit Kantenlänge b und Volumen b^3.

Gesucht: Konstruktion einer Kantenlänge X, so dass $V = X^3 = 2b^3$.

Das Problem lässt sich zurückführen auf die Auflösung der doppelten Proportion:

$$b : X = X : Y = Y : 2b,$$

welche beim Auflösen $X^3 = 2b^3$ ergibt. Der Kissoidenkonstruktion (s. oben) entnimmt man:

Da ΔPSD_1 und ΔSBC rechtwinkelig sind, gilt (mit $BF = D_1E = z$):

Given: Cube with edge length b and volume b^3.

Searched: Finde edge length X such that $V = X^3 = 2b^3$.

The problem can be reduced to solving the double proportion:

$$b : X = X : Y = Y : 2b,$$

which leads to $X^3 = 2b^3$. We can derive from the construction of the cissoide (see above):

As ΔPSD_1 and ΔSBC are both rigth triangles (with $BF = D_1E = z$):

$$y : x = x : z \text{ in } \Delta PSD_1$$
$$x : z = z : (a - x) \text{ in } \Delta SBC$$

$$y : x = x : z = z : (a - x)$$

Vergleicht man mit obiger doppelter Proportion, so sind $PE = y = b$ und $CE = SF = a - x = 2b$ als gegeben anzusehen. Zu konstruieren ist also die erste der beiden mittleren Proportionalen $SE = x = X$.

Konstruktion:

- Zeichne eine beliebige Kossoide (a beliebig),
- Zeichne P' mit SP' = a/2,
- Verbinde P' mit C und schneide mit der Kissoide – ergibt Punkt P.
- Vom Schnittpunkt P fälle das Lot PE auf SC.

Nach dem Strahlensatz gilt dann:

$$PE : EC = P'S : SC = 1 : 2.$$

PE sollte aber gleich b sein (und damit $EC = 2b$). Deshalb muss noch eine Streckung durchgeführt werden, die $PE = y$ auf eine Strecke der Länge b abbildet und gleichzeitig x im selben Maß auf die Länge X vergrößert (vgl. Zeichnung). Für dieses (mit Hilfe der Kissoide) konstruierte X gilt nun

$$\boldsymbol{X^3 = 2b^3}.$$

Comparing with the double proportion from above we see that $PE = y = b$ and $CE = SF =$

$= a - x = 2b$ can be taken as given. We have ti find x with $SE = x = X$.

How to do:

- Draw any cissoid (take any a),
- Draw P' with SP' = a/2,
- Connect P' with C and intersect with the curve giving point P.
- Draw the perpendicular line PE.

According to the theorem of proportional segments we finally have:

$$PE : EC = P'S : SC = 1 : 2.$$

PE should equal b (and $EC = 2b$). We need to find a stretching, mapping $PE = y$ into b and simultaneously increasing x to X. For this X – found by means of the cissoid) now is valid:

$$\boldsymbol{X^3 = 2b^3}.$$

I tried to translate the important parts into English and I hope that the nongerman readers are able to follow Mr. Weth´s interesting contribution. Josef 1993

In 1992 Thomas Weth added a sketch of the "Delian Problem". Now we can use DERIVE 6 and its slider bars to demonstrate the procedure more general together with confirming it algebraically. Josef 2006

#1: $\text{kissoide} := \left[\frac{a\cdot t^2}{1+t^2}, \frac{a\cdot t^3}{1+t^2}\right]$

#2: VECTOR(kissoide, t, −3, 3, 0.01)

#3: $\left(x - \frac{a}{2}\right)^2 + y^2 = \frac{a^2}{4}$

#4: $\left[S := [0,\ 0],\ C := [a,\ 0],\ P\_ := \left[0, \frac{a}{2}\right]\right]$

#5: [P_, C]

#6: SOLVE(kissoide = C + t_·(P_ − C), [t, t_], Real)

#7: $t = \frac{2^{2/3}}{2} \wedge t\_ = \frac{2^{2/3}}{5} - \frac{2\cdot 2^{1/3}}{5} + \frac{4}{5}$

Question for the students: Why is t = 2^(-1/3)?

Wie would like to find the solution of $X^3 = 2b^3$ with generalized a and generalized b.

Let´s take $b = 4$ as an example.

#8: $P := \text{SUBST}\left(\left[\frac{a\cdot t^2}{1+t^2}, \frac{a\cdot t^3}{1+t^2}\right], t, 2^{-1/3}\right)$

#9: $\left[E := \left[P_1,\ 0\right],\ Q := \left[P_1,\ b\right]\right]$

#10: [E, Q]

#11: [S, P]

#12: aux := Q + t·(S − P)

#13: $\left(\text{SOLUTIONS}\left(aux_2 = 0,\ t\right)\right)_1$

#14: $\frac{2977074\cdot b}{913235\cdot a}$

#15: $E\_ := \text{SUBST}\left(aux,\ t,\ \frac{2977074\cdot b}{913235\cdot a}\right)$

#16: $E\_ := \left[\frac{575302\cdot(913235\cdot a - 2977074\cdot b)}{1359384087195},\ 0\right]$

#17: VECTOR([E + t·(E_ − E), 0], t, 0, 1, 0.01)

#18: |E_ − E|

#19: $5.502798679\cdot 10^{-14}\cdot\left|7\cdot a - 2.28960048\cdot 10^{13}\cdot b\right|$

#20: $5.502798679\cdot 10^{-14}\cdot\left|7\cdot 6.4 - 2.28960048\cdot 10^{13}\cdot 4\right|$

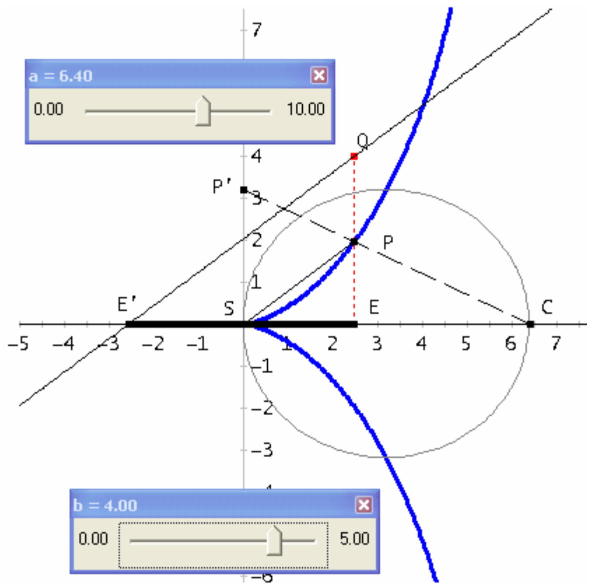
#21: 5.039684198

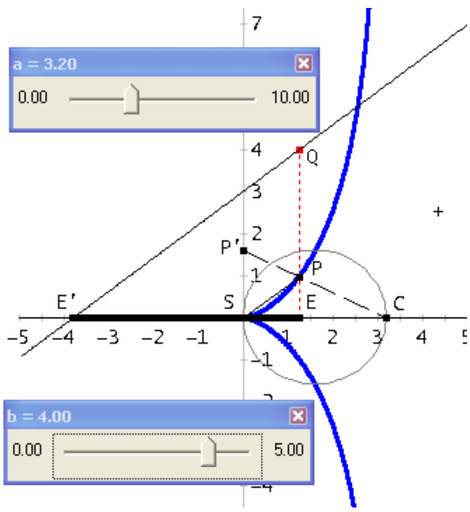
#22: $5.502798679\cdot 10^{-14}\cdot\left|7\cdot 1 - 2.28960048\cdot 10^{13}\cdot 4\right|$

#23: 5.039684198

#24: $\text{SOLVE}(x^3 = 2\cdot 4^3,\ x,\ \text{Real})$

#25: x = 5.039684199





The DERIVE Screen on the TV-Screen

H. Scheuermann & W. Eichenauer, Hofheim, Germany

When I was in Wolffenbüttel last year, Mr Scheuermann gave a DERIVE demonstration on a TV-screen using a selfmade interface. I was very much impressed and asked Mr Scheuermann to publish details about this interesting possibility to demonstrate PC-screens instead of using panels. Here is Mr Scheuermann´s report. I didn´t try to translate because I am no technician, so I don´t know the special technical expressions, but I am sure that you will find a way to translate the description if you are interested in producing this interface. Mr Scheuermann wrote about the cost: 60 German Mark. J.B.

Zur Demonstration von Funktionsgraphen bzw. Textzeilen aus DERIVE im Rahmen von Vorführungen und nicht zuletzt Unterricht ist das Bild eines herkömmlichen PC-Monitors mit einer Bildschirmdiagonalen von 14“ bzw. 16“ in der Regel zu klein. Großbildmonitore mit Bildschirmdiagonalen von 24“ (und mehr) geben die Bildschirmausgabe in höchster Auflösung wieder; jedoch eignen sich diese auf Grund ihres enormen Gewichts nur für einen festen Standort. Darüber hinaus ist diese Lösung – auch heute noch – recht kostspielig. Monochrome LCD-Displays erweisen sich auf Grund der schlechten Unterscheidbarkeit von den einzelnen Graphen zB bei der Darstellung von Kurvenscharen (zB bei Parametervariationen) aber auch bei der grafischen Ausgabe von Funktionen als problematisch. Farbige LC-Displays sind als Lösung in der Anschafung ebenfalls kostenintensiv. Im Allgemeinen erfordern diese Displays auch noch einen speziellen Arbeitsprojektor mit geringer Wärmeabgabe und eine gewölbte, stark reflektierende Projektionswand (häufig muss auch dann der Raum an hellen Tagen noch – leicht – verdunkelt werden.) Damit schließt auch diese Alternative einen beweglichen Einsatz weitgehend aus. Deshalb haben Kollegen an unserer Schule eine Schnittstelle entwickelt, die das Bild des Computers an den Eingang eines Farbfernsehers anpasst und überträgt. Das Gerät ist auf Grund seiner kleinen Abmessungen und seiner schnellen Einsetzbarkeit (wird nur von „außen“ an PC und TV angeschlossen) speziell für den flexiblen Einsatz in verschiedenen Klassenräumen konzipiert worden. Da sich das Gerät auch im Eigenbau zusammensetzen lässt, ist es relativ preiswert; es verfügt aber nicht über alle Möglichkeiten der vorher beschriebenen Alternativen. Für viele PC-Anwendun-gen vor einer größeren Anzahl von Zuschauern ist die Darstellung des Textmodus oder der Grafik in der groben Auflösung von 600 x 200 Punkten via Adapter und handelsüblichem Fernsehgerät eine zufrieden stellende Lösung.

Für den interessierten Leser folgt eine kurze Beschreibung der technischen Voraussetzungen für den Betrieb des Geräts und der Schaltplan der Schnittstelle *):

PC: Die im PC eingesetzte Grafikkarte muss einen RGB-Anschluss (9-polig Sub-Min D) aufweisen. Bei CGA- und EGA-Karten ist dies immer der Fall. Auch an älteren VGA-Karten befindet sich vielfach dieser Anschluss. Bei den neueren SVGA-Karten ist leider nur noch ein 15-poliger Anschluss vorhanden, sodass hier keine Anschlussmöglichkeit mehr besteht.

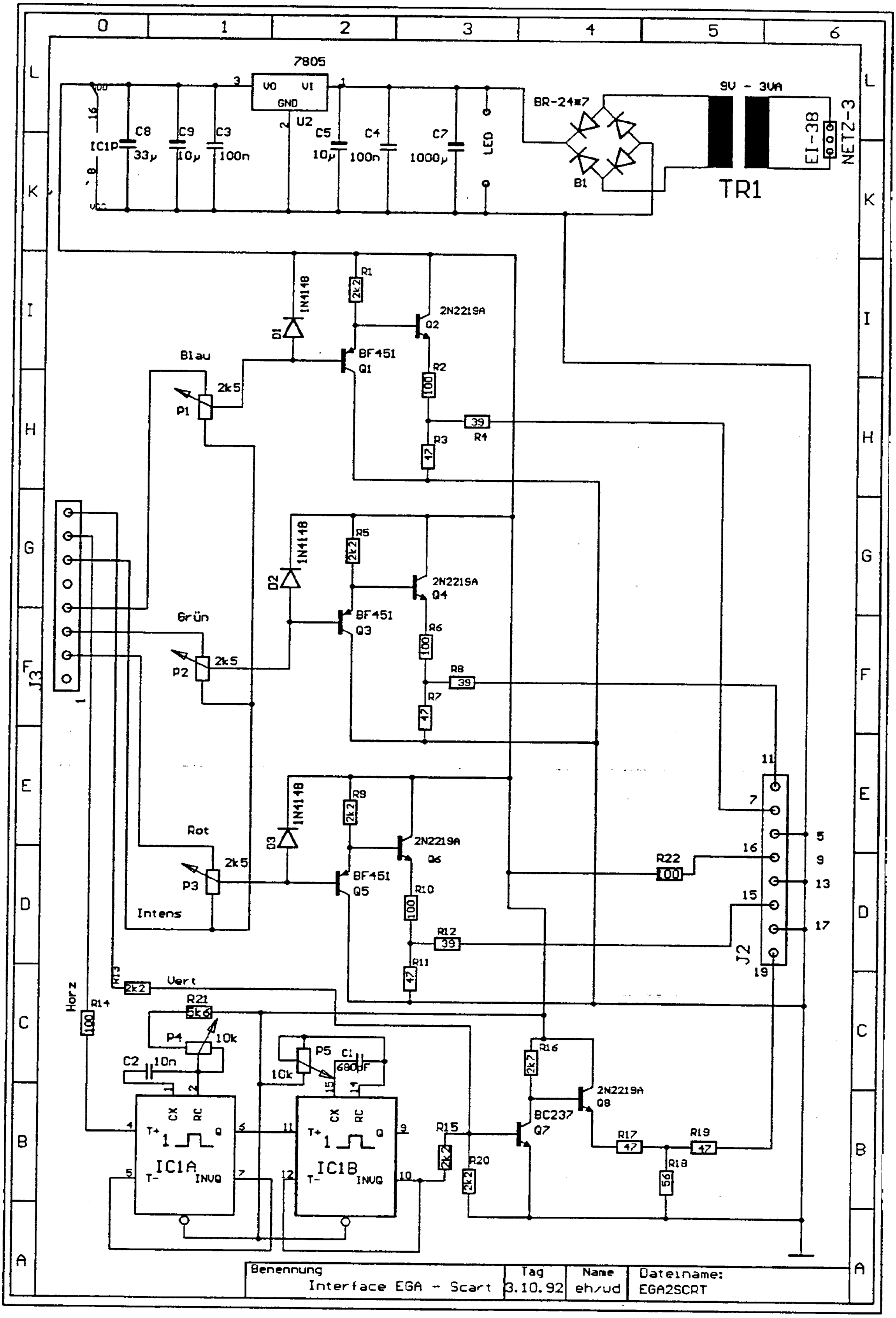
TV-Gerät: Bei den meisten Geräten ist mittlerweile eine Euro-Scart-Buchse (21-polig) vorhanden.

Software: Es können nur solche Programme verwendet werden, die eine Auswahl des Grafikmodus durch den Anwender zulassen.

Installation: Ein TV-Gerät stellt ein Bild mit einer Zeilenfrequenz von 15625 Hz dar. VGA-Grafikkarten haben je nach Auflösung Zeilenfrequenzen bis über 40000 Hz. Um nun das Bild des PC auf einem TV-Gerät darzustellen, muss es softwaremäßig gelingen, die Grafikkarte auf eine niedrige Zeilenfrequenz zu schalten. Da dies von der Grafikkarte und der verwendeten Software abhängig ist, können keine allgemein gültigen Verfahren angegeben werden. Nachfolgend zwei Installationsbeispiele:

DERIVE 2.5: Mit Option Display EGA Color erhält man eine Auflösung von 640x200 Punkten bei 16 Farben. Damit ist allerdings nur DERIVE auf dem TV-Gerät darstellbar.

Treiberdisk.: Bei den Grafikkarten liegt meist eine Diskette des Herstellers mit verschiedenen Hilfsprogrammen bei. Mit einem Programm (zB CGA.COM) kann u.U. der Bildschirmmodus für alle Programme umgestellt werden.



*) An dieser Stelle möchten wir dem Entwickler der Schnittstelle, Herrn Wolfram Eichenauer, für die Erlaubnis zur Veröffentlichung des Schaltplans danken.