# THE DERIVE - NEWSLETTER #17

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THE BULLETIN OF THE

DERIVE

USER GROUP

**Contents:**



**Revised Version 2008** **March 1995**

**Two letters from Halvor DEVOLD, Norway:**
(with some good reasons to be printed on the information page!!)

Dear Josef, … Thank you for the exe-file of my KEPLER-movie-picture and please forgive me for this delayed response! Time has simply run away for me since I came back from Plymouth last summer. Our new school reform, Reform94, introduced the graphic calculator as a compulsory tool, to be used on all first-year courses in mathematics. Most of our math teachers were taken by surprise, and re-acted accordingly, that is in the negative ---.
Hence, I have had - and still have - a hard time in persuading colleagues to see the advantage of such tools, - and to teach my nearest colleagues how to use graphic colleagues as well as *DERIVE*. We are just a few math-teachers in this country - it seems - who have realized the new situation - and the new opportunities - that has arisen in maths teaching, and who are also in regular contact - through a BBS. (Firda BBS)
In order to improve upon this situation I have borrowed your idea and made a printed bulletin with extracts from the same BBS-conference. (Its name is DRIVE)
You will find the first issue of this printed bulletin attached. It is of course by no means meant to compete with your bulletin of the DUG, but solely directed to a Norwegian public, and it is not - as you will see - devoted to *DERIVE* alone, but meant to be informative on both graphic calculators and *DERIVE*. In fact, with *DERIVE 3*, the gap between graphic calculators and CAS seems to become narrower, - that is in their user interface. Hence it will be easier to persuade people to use both, and to see and judge for themselves when to use *DERIVE*, and when a graphic calculator is sufficient. Both have their virtues.......... Best regards.

More information about FIRDA & DRIVE:
Halvor Devold, Maneweien 10, N-4900 Tvedestrand, Norway

Dear Josef, ........I am now working on a project that includes the use of **AcroSpin**:

I bought AcroSpin from MathWare, US recently in order to explore its potential as a means to extend the visual power of *DERIVE*. My first aim was to see if it could be used to present the five regular polyhedrons in ***stereo vision***. ....

***DNL***: I think, that some of you have wondered about the Option to save a 3D-plot in AcroSpin format. Here you can find the answer. I can assure you, Halvor produced a wonderful file. I asked him to inform us how to obtain AcroSpin. Halvor answered immediately:

You can order AcroSpin at MathWare, P.O.Box 3025, Urbana, IL, 61801, (800) 255 2468,
FAX (217)384 7043

The price is US$ 30.00. MathWare offers a second useful program called GyroGraphics (US$ 89.00). GyroGraphics supports AcroSpin.

## Denken Sie bitte an das

**1. Deutsche DERIVE User Group Treffen**
**im Rahmen der MNU 1995**
**Nürnberg, 9. - 13. April 1995**

**Das Treffen findet am Mittwoch, den 12.April 1995**
**um 14 Uhr statt.**

Die Räumlichkeit erfahren Sie im Tagungsbüro.
Fakultät der Wirtschafts- und Sozialwissenschaften,
Uni Erlangen-Nürnberg, Lange Gasse 20

Ich würde mich freuen, viele von Ihnen wieder zu sehen, bzw. kennen zu lernen.

*Liebe DUG-Mitglieder,*

*Ich begrüße Sie alle recht herzlich zu Beginn des 5. DUG-Jahres. Ganz besonders herzlich willkommen heiße ich die neuen DUG-Mitglieder. So haben wir seit kurzem auch eine „diplomatische" Vertretung in Neuseeland und in Kolumbien.*

*Ich habe einige Anfragen bezüglich KEPLER.EXE erhalten. Leider ist GRASP - mit diesem Präsentations- und Animationsprogramm wurde KEPLER erzeugt - nicht mit allen Grafikkarten problemlos kompatibel. Oft hilft es, ein wenig mit den Videomodi zu experimentieren. H.Devold, der Schöpfer von KEPLER, hat mit einem Einsatz von AcroSpin - DERIVE3 Anwender kennen AcroSpin zumindest von Transfer-Save-AcroSpin im 3D-Menu - wieder eine wahrhaft bezaubernde Show geschaffen.*

*Diese Ausgabe des DNL ist zu einem beträchtlichen Teil dem DERIVE - Einsatz in österreichischen Schulen gewidmet. Ich bedanke mich bei allen Kollegen für ihre Bereitschaft, uns einen Blick in ihren Schulalltag werfen zu lassen und auch für ihre Geduld bei meinen lästigen Nachfragen. Allen "glücklichen?" Nichtlehrern unter Ihnen verspreche ich, dass die nächsten Ausgaben nicht nach Kreide und Tafel riechen werden. Ich habe für dieses Exemplar einen Index für alle bisherigen Ausgaben des DNL vorbereitet, aber wie so oft, der Platz reichte trotz des deutlich gestiegenen Umfangs wieder nicht aus.*

*Ich hoffe, in nächster Zukunft auch über e-mail erreichbar zu sein. Die Auszüge aus dem BBS und aus der derive-news@mailbase.ac.uk. in Birmingham haben mir Appetit gemacht. Eine digitale Derive-Konversation - Lehrstunde können Sie ab Seite 7 verfolgen.*

*Im nächsten DNL will ich anstelle des Beitrags von Thomas Weth, dem ich an dieser Stelle ganz besonders für seine zuverlässigen Zusendungen danken möchte, einen Artikel von Peter Baum, Kassel, veröffentlichen, der aus seiner Beschäftigung mit Thomas Weths „Kurvenlexikon" entstanden ist. Es freut einen Herausgeber wirklich, wenn sein Bulletin nicht nur abonniert, sondern auch tatsächlich gelesen wird und wie ersichtlich, sogar Anregungen für eigene Arbeiten liefert.*

*Abschließend möchte ich Sie noch auf die MNU in Nürnberg - mit dem deutschen Usergrouptreffen - und auf die DERIVE DAYS Düsseldorf hinweisen.*

*Mit vielen Grüßen bis zum Sommer*
*Ihr*

*(PS.: Ihre Mitgliedsnummer finden Sie auf dem Adressaufkleber)*

*Dear DUG-Members,*

*My best regards to you all at the beginning of the 5th DUG year. My special welcome to the new DUG members. So we have opened a "diplomatic" representation in New Zealand and in Colombia.*

*I received some questions concerning KEPLER.EXE. Unfortunately it seems that GRASP - the presentation- and animation program used to produce KEPLER - is not compatible with some graphics adapters. It might help to experiment with the video modes of your graphics card. H.Devold, the creator of KEPLER, supported by AcroSpin - DERIVE 3 users know AcroSpin at least from Transfer-Save-AcroSpin in the 3D Menu - produced a new charming show.*

*A big part of this issue is dedicated to the Austrian DERIVE scene. I want to thank all my colleagues for their willingness to give us interesting insights in their daily life at school. And I thank them for their patience when I had many boring questions about their contributions. But I promise all the "fortunate?" non-teachers among you, that the next DNL will not smell of chalk and blackboard. For this issue I have prepared an index for all the DNLs from DNL#1 - DNL#16, but I ran out of paper, despite the now 42 pages contents.*

*I hope to be available by e-mail in the near future. The excerpts from the BBS and from the derive-news@mailbase.ac.uk have given me an appetite. You can follow a digital DERIVE conversation - lecture on page 5.*

*Instead of the next chapter of Thomas Weth´s "Lexicon of Curves" - let me thank him for his reliability in delivering his contributions - I want to publish an article by Peter Baum, Kassel, which emerged from dealings with Thomas Weth´s "Lexicon". Its really a pleasure for an editor, to see that people do not only renew their subscriptions but actually read the journal and find in it ideas for their own investigations.*

*With my best wishes until summer*

*Sincerely*

*(You can find your DUG membership number on the address label.)*

The *DERIVE-NEWSLETTER* is the Bulletin of the *DERIVE User Group*. It is published at least four times a year with a contents of 40 pages minimum. The goals of the *D-N-L* are to enable the exchange of experiences made with *DERIVE* as well as to create a group to discuss the possibilities of new methodical and didactical manners in teaching mathematics.

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**Contributions:**
Please send all contributions to the Editor. Non-English speakers are encouraged to write their contributions in English to reinforce the international touch of the *D-N-L*. It must be said, though, that non-English articles will be warmly welcomed nonetheless. Your contributions will be edited but not assessed. By submitting articles the author gives his consent for reprinting it in *D-N-L*. The more contributions you will send, the more lively and richer in contents the *DERIVE Newsletter* will be.

**Preview: Contributions for the next issues**

Stability od systems of ODEs, Kozubik, SLO
Prime Iterating Number Generators, Wild, UK
Graphic Integration, Probability Theory, Linear Programming, Böhm, AUS
Continued Fractions and the Bessel Functions, Cordobá a.o., ESP
LOGO in DERIVE, Lechner, AUS
DREIECK.MTH, Wadsack, AUS
IMP-Logo and Misguided Missiles, Sawada, HAWAII
"Reverse Discussion" of Curves, Reichel, AUS
Reichel - Klingen - Böhm - Splines, a triathlon, AUS & GER
3D-Geometry, Reichel, AUS
Parallel- and Central Projection, Böhm, AUS
Conic Sections, Fuchs, AUS
Müller´s Method to solve univariate equations, Speck, NZL
Vector and Vector Indices Sorting, Biryukow, RUS
Algebra at A-Level, Goldstein, UK
Tilgung fremderregter Schwingungen, Klingen, GER
A Utility file for complex dynamic systems, Lechner, AUS
and
Setif, FRA; Vermeylen, Belgium; Lymer, FRA; Leinbach, USA; Baum, GER; Weth, GER; Wiesenbauer, AUS; Keunecke, GER; .........

and messages from the derive-news@mailbase.ac.uk

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## R.Larham, Abbots Langley, UK

Alfonso J.Poblacion is mistaken about the curve (DNL#16, p6) given by the parametric equations:

```
[sin(0.99t) - 0.7 cos(3.01t) , cos(1.01t) + 0.1 sin(15.03t)]
```

It is clearly closed as it is continuous and both the *x* and *y*-coordinates are periodic with a common period of 200π. To demonstrate this try plotting the curve for $t \in [200\pi, 200\pi+2\pi]$ over the plot for $t \in [0,2\pi]$. Alternatively try simplifying:

```
[sin(0.99t) - 0.7cos(3.01t),cos(1.01t) + 0.1sin(15.03t)]- -
[sin(0.99(t + 200π)) - 0.7cos(t + 200π)),cos(1.01(t + 200π)) +
+ 0.1sin(15.03(t + 200π))]
```

If this curve does not appear to close when plotted for a large enough range for *t*, it is an artefact of the plotting algorithms and not of the curve itself.

In fact it is easy to see that any curve which can be represented in parametric form by:

```
[A sin(αt) - B cos(βt), C cos(γt) + D sin(δt)]
```

is closed, when α, β, γ, δ are rational multiples of one another.

## Dr. A.Lyubimov, St.Petersburg, Russia

Dear Sir, thank you very much for your kind letter, for invitation to join DUG and especially for DNL you have sent me. It is a very useful and fascinating issue.

I have met some difficulties with the sending of my membership due (this is the reason of my delay with the answer) because our young banks have the weak connections with Europeans ones yet. So I had to transfer my due on 1995 with my friend who was going abroad. He had to send my due and the application form to you last week from London. Please let me know if you did not receive them.

I gladly met your information about Sergey Biryukow from Moscow. Please send me his mail address and/or e-mail if possible so that I could contact him.

I am very interested in cooperation with you, too and I also know the strong background of European mathematics and teachers. Some ideas of my High School course I wrote are taken from the brilliant "Elementarmathematik vom Höheren Standpunkte aus, 1.Band" by Felix Klein that had been translated into Russian just ten years ago. In addition I am going to study German.

I hope our cooperation will be useful and fruitful. Best wishes for Christmas and New Year.

A.Lyubimov

## A.Kröpf, Hildesheim, Germany

**Message 3989: From AL POOR to AL RICH about ITERATES:**

Author `f(x) := x^2` and `ITERATES(f(x),x,x,3)`. Simplifying yields [ $x$, $x^2$, $x^8$, $x^{127}$ ]. Is this a known bug - or what´s the matter? Sincerely

***DNL:*** I don´t know, why you have $x^{127}$, but your result made me curious. Compare the two listings:

**DERIVE Version 3.01**

```
u:=x^2
ITERATES(u,x,x,3)
 [x,x^2,x^4,x^8]
F(x):=x^2
ITERATES(F(x),x,x,3)
 [x,x^2,x^4,x^8]
v:=2*x+3
ITERATES(v,x,x,3)
 [x,2*x+3,4*x+9,8*x+21]
G(x):=2*x+3
ITERATES(G(x),x,x,3)
 [x,2*x+3,4*x+9,8*x+21]
ITERATES(v,x,2,3)
 [2,7,17,37]
ITERATES(G(x),x,2,3)
 [2,7,17,37]
```

**DERIVE Version 3.04**

```
u:=x^2
ITERATES(u,x,x,3)
 [x,x^2,x^4,x^8]
F(x):=x^2
ITERATES(F(x),x,x,3)
 [x,x^2,x^8,x^128]
v:=2*x+3
ITERATES(v,x,x,3)
 [x,2*x+3,4*x+9,8*x+21]
G(x):=2*x+3
ITERATES(G(x),x,x,3)
 [x,2*x+3,8*x+21,128*x+381]
ITERATES(v,x,2,3)
 [2,7,17,37]
ITERATES(G(x),x,2,3)
 [2,7,17,37]
```

I don´t know the reason, why the "software smithies" have changed the ITERATES-function, but obviously there is now a difference whether you enter an expression - say $v = 2x+3$ or a function $G(x) = 2x+3$ as iteration rule.
In the first case we express $x_{n+1} = 2\,x_n + 3$ with $x0 = x$, in the second case we do as follows:

$x0 = x$

$x1 = G(x0) = 2x + 3$

$x2 = 2\,G(x1) + 3 = 2\,(2\,x1 + 3) + 3 = 2\,(2\,(2x + 3) + 3) + 3 = 8x + 21$

$x3 = 8\,G(x2) + 21 = 8\,G(2\,x2 + 3) + 21 = 8\,(2\,(8x + 21) + 3) + 21 = 128x + 381.$

I´ll try to find an answer from Al Rich for you (and for us!). Josef.

(Mr Kröpf, please give me your address, that I can send you an answer.)

*DERIVE 6 makes no difference and gives the results as version 3.01 did a couple of years ago.*

There was another problem occurring with the ITERATES-function. You know Johannes Wiesenbauer, our sophisticated specialist in number theory. In December he called me very disappointed because his beloved POWERMOD-function from DNL#14 didn´t run in DERIVE 3.00, although he had no problems with the Beta-version. I contacted SWHH and Al Rich faxed an answer:

Dear Josef, thank for reporting the problem DERIVE and DERIVE XM Version 3.00 have simplifying calls on the INV function. I am happy to report that the problem has been found and is now corrected by Version 3.01.
I will request that SWHE send a Version 3.01 diskette to Dr. Wiesenbauer so he can verify that all his functions now work correctly.
I sincerely apologize for any inconvenience this bug may have caused you and the other members of the DERIVE User Group.

As we change DERIVE to make it more powerful and efficient, new bugs are inevitably introduced despite our best efforts. Before releasing a new version, I run it through a test suite of over 7,000 math problems. Since I add to the test suit problems designed to detect the bugs you and others report (e.g. the INV function will now be added), at least the same bug should not occur again! (You might put a note to this effect in the DERIVE Newsletter.)

What I´ve done now, and many thanks, Al for giving us an insight into your work.

**M.Th.Langsch, Wittlich, Germany**

Gleichungen 3.Grades scheinen in der Luft zu liegen. Im Schulbuch für die 11.Klasse steht die Aufgabe, ein *s* zu finden, so dass für alle *x* > *s* gilt: $\left|\frac{1}{x^3-x-1}\right| < \frac{1}{1000}$, das führt auf $x^3 - x - 1001 = 0$. Das Lösungsbuch gab ohne Kommentar $s = 10$ an (!).Ich änderte den Nenner in $x^3 - x + 10$.

Nun waren wir bei $x^3 - x - 990 = (x - 10)(x^2 + 10x - 99) = 0$ und kannten die einzige reelle Lösung. Doch die Schüler wollten wissen, wie man Gleichungen 3.Grades löst. Mit den Cardanischen Formeln kamen wir zu:

$$x = \sqrt[3]{495+\sqrt{495^2-\left(\frac{1}{3}\right)^3}} + \sqrt[3]{495-\sqrt{495^2-\left(\frac{1}{3}\right)^3}}$$

, doch nun kamen wir mit Termumformungen nicht weiter. *DERIVE* macht daraus mit **approX** zwar 10, mit **Simplify** aber nur:

$$x=\sqrt[3]{495+\frac{299\sqrt{222}}{9}} + \sqrt[3]{495-\frac{299\sqrt{222}}{9}}$$

- warum?

Zur Ergänzung sei erwähnt, dass $495 \pm \frac{299\sqrt{222}}{9} = \left(5 \pm \sqrt{\frac{74}{3}}\right)^3$ ist, und damit trivialerweise

$$\left(5+\sqrt{\frac{74}{3}}\right) + \left(5-\sqrt{\frac{74}{3}}\right) = 10.$$

Mit freundlichen Grüßen

*Recent versions of DERIVE have no problems simplifying the nested roots!*

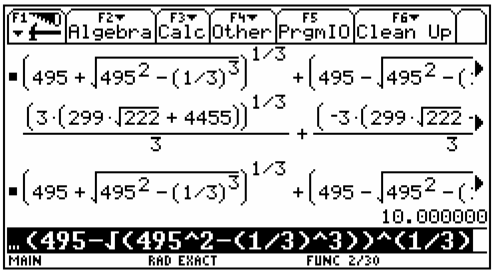
```
                3
#1:    SOLVE(x   - x - 990, x)

#2:    x = -5 - √74·i ∨ x = -5 + √74·i ∨ x = 10
```

#3: $\left(495 + \sqrt{\left(495^2 - \left(\frac{1}{3}\right)^3\right)}\right)^{1/3} + \left(495 - \sqrt{\left(495^2 - \left(\frac{1}{3}\right)^3\right)}\right)^{1/3}$

#4: 10

*The TI-Calculators don´t denest the roots and behave similar as DERIVE´s earlier versions: This the screen of the Voyage 200. TI-NspireCAS gives the same results.*




**Message 3313: From PAULONE to JERRY GLYNN about VECTOR EXPANSION**

A simple question perhaps about the result of using a vector expression and then simplifying the expression. The expression (as authored): `VECTOR(n^x,n,-2,2)`
When this expression is simplified the result is:

```
[2^x cos(pi x)+2^x i sin(pi x), cos(pi x) + i sin(pi x), ?, 1,2^x]
```

I have somewhat of any idea as to what is happening but could you give a simple explanation of what Derive is doing with this expression? Positive values of n are okay, of course, but what about the negative?

**Message 3314: From JERRY GLYNN to PAULONE about #3313 / VECTOR_EXPANSION**

Let's simplify this question further so that we can see more clearly what is happening.
New question: $(-2)^x$ If x is 1/2 we have square root of a negative number so Derive simplifies (-2)^(1/2) as sqrt(2)*i with the last symbol representing complex number i. Try Simplifying (-2)^x and see that Derive is putting this expression into a form that shows the potential complex numbers. It is a very weird use of language to say that 2^x cos(pi*x)+2^x*i*sin(pi*x) is simpler than (-2)^x.
Please try out these ideas and come back with a follow on note. Thanks for your question ... this conversation might be helpful to others if made public. May I ?

*Recent versions of DERIVE do not show the trig form, but give outputs depending on the domain of x:*

#1: $\text{VECTOR}(n^x, n, -2, 2)$

#2: $x :\in \text{Real}$

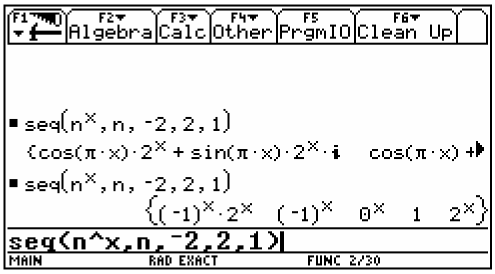
#3: $[(-2)^x, (-1)^x, ?, 1, 2^x]$

#4: $x :\in \text{Complex}$

#5: $[e^{x\cdot(\text{LN}(2) + \pi\cdot i)}, e^{\pi\cdot i\cdot x}, ?, 1, 2^x]$

#6: $x :\in \text{Real}\ (-\infty, 0)$

#7: $[(-2)^x, (-1)^x, \infty\cdot(\pm 1)^x, 1, 2^x]$




*The first calculation was done in Complex Format = Rectangular, the second one in Complex Format = Real.*

*See another difference: DERIVE gives a question mark, where the TI presents* $0^x$*.*

**Message 3333: From KEITH WILLIAMS to PUBLIC about REPLACING COLUMNS IN A MATRIX**

I am trying to come up with a user function which will delete a column in a matrix and then put another column which you specify in it's place.
For example a function such as `replace_column(m,c1,c2)`. Where `m` = matrix and `c1` = the column to be extracted and `c2` = the column to be put in the extracted columns place. If anyone can come up with such a function I would appreciate it very much.

**Message 3335: From HADUD to KEITH WILLIAMS about REPLACING COLUMNS IN A MATRIX**

The following function will replace column no. "`nc`" with the elements of the vector "`c`". Is this what you want?

```
#1:   replace_column(m, nc, c) := VECTOR(VECTOR(IF(jJ = nc, ELEMENT(c, iI),
          ELEMENT(m, iI, jJ)), jJ, DIMENSION(ELEMENT(m, 1))), iI, DIMENSION(m))
```

#2: $m := \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \end{bmatrix}$

```
#3:   replace_column(m, 3, [p, q, r])
```

#4: $\begin{bmatrix} a & b & p & d \\ e & f & q & h \\ i & j & r & l \end{bmatrix}$

*Now we can write the function in a more compact form:*

#5: $\text{replace\_column}(m, nc, c) := \text{VECTOR}(\text{VECTOR}(\text{IF}(j\_ = nc, c_{i\_}, m_{i\_,j\_}), j\_, \text{DIM}(m_1)), i\_, \text{DIM}(m))$

#6: $\text{replace\_column}(m, 3, [p, q, r]) = \begin{bmatrix} a & b & p & d \\ e & f & q & h \\ i & j & r & l \end{bmatrix}$

**Message 3338: From KEITH WILLIAMS to HADUD about REPLACING COLUMNS IN A MATRIX**

Dear Hadud I am sorry to cause you so much trouble. I had previously left you a message that I could not get your solution to my problem to work. It is working now I changed it around a little. This is how I changed it.

```
REPLACE_COLUMN(M, NC, C) := VECTOR(VECTOR(IF(j = NC, ELEMENT(APPEND(C), I),
        ELEMENT(M, i, j)), j, DIMENSION(ELEMENT(M, 1))), i, DIMENSION(M))
```

I really appreciate your help in solving this problem. I tried to figure this out for quite sometime. You must be a very smart person. I wish that I could learn to come up with functions as good as you do. Do you know of any good books which would teach someone how to better learn to program Derive. Again thank you very much for helping me. keith williams

**Message 3340: From JERRY GLYNN to HADUD about REPLACING COLUMNS IN A MATRIX**

I agree with Keith's statement that you must be a smart person and I am interested in his question to you about how does one learn to write good functions in Derive? Thanks for your help.

**Message 3342: From KEITH WILLIAMS to HADUD about REPLACING COLUMNS IN A MATRIX**

Dear Hadud: Your function does work correctly. I just do not understand how the function goes about obtaining the results. Could you possibly list the steps that this function goes through and explain how the result is obtained. If you could maybe I could better understand how to create user functions. I am curious as to how you learned to write them so well. Are you a teacher of math or did you learn what you know about writing functions on your own?
If you could answer these questions I would appreciate it very much. Also any information that you could give me to better help me to understand how to write user functions would be greatly appreciated. keith williams

**Message 3346: From HADUD to KEITH WILLIAMS about REPLACING COLUMNS IN A MATRIX**

I'm sorry not to have reacted sooner. I normally log on once a week but I should have realized that after leaving a message some questions might come up!

Looking at the changes you made to make my function work, I suspect the problem was that you were in the OPTION-INPUT-(CHARACTER, INSENSITIVE) mode. I assumed the (WORD, SENSITIVE) option (my default setting, more about that below) and I should have mentioned that.

I do not understand why you put in the `APPEND` function. In the present context it has no effect at all. With just one vector-argument APPEND simply returns the argument. Thus `APPEND([a,b,c,d])` will simplify to `[a,b,c,d]`. Also, in your first message (12/4), the argument following `APPEND` should be "I", not 1, and a final parenthesis is missing. In your second message (12/7) both errors were corrected.

Your definition of the function is a good example to demonstrate a sneaky DERIVE quirk as follows: Apply your function to the matrix

$$\begin{pmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \end{pmatrix}$$

with the 3rd column to be replaced by $[p, q, r]$.

This is your result:

$$\text{REPLACE\_COLUMN}\left(\begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \end{bmatrix}, 3, [p, q, r]\right)$$

$$\begin{bmatrix} a & b & p & d \\ e & f & q & h \\ 3 & 2 & r & l \end{bmatrix}$$

The column replacement works as expected but the letters i and j in the matrix are replaced by 3 and 2. The reason is, of course, that i and j are used as indices within the function. So, when i or j are encountered anywhere they are replaced by the current value of the index. Note that this is not an error. DERIVE is behaving exactly as advertised. It can be very annoying, however, since the user of a function normally does not know which names are used inside the function and therefore should be avoided.

To get around this type of name conflict I always operate in the (WORD, CASE SENSITIVE) mode. For function-internal indices and auxiliary functions I use names of at least 2 characters starting with a LC character followed by an UC character. This runs contrary to common usage for variable names. Thus it is unlikely that a user encounters a name conflict. This explains the use of iI and jJ as indices in my definition of the function.

Incidentally, there is no name-conflict problem with argument names used in a function definition, since they are treated as dummies.

As far as gaining some facility with DERIVE programming is concerned, you really don't have to be very smart! You might find any introductory text to LISP programming helpful, but the basic principles are very simple. The key to LISP (and therefore DERIVE) is the nested function concept. Rather than writing sequential statements such as

```
y = SIN(x) + A
z = SQRT(1 + y^2)
```

DERIVE wants to see the combined (nested) function `SQRT(1 + (SIN(x)+A)^2)`

To evaluate this expression DERIVE starts with the content of the innermost parentheses (x in this case). If a numerical value for x is found it evaluates SIN(x) and then proceeds to the next set of parentheses etc.

Consider now the function REPLACE_COLUMN. The end-product of this function is a matrix. Since in DERIVE a matrix is a vector of rows, the outermost (top level) task is to form a vector of rows. So we author the expression (let's say it's #1)

```
1:   VECTOR(row,I,DIMENSION(M))
```

We haven't given any thought yet to what "row" might be, but we know the number of rows is `DIMENSION(M)`, so we let the row index "I" range from 1 to `DIMENSION(M)` (the 1 may be omitted, see manual). Now we author an expression for "row". We know each row is a vector of elements:

```
2:   VECTOR(elem,J,DIMENSION(ELEMENT(M,1)))
```

Again we have left open the form of "*elem*" but we prescribe the range of the column index "*J*". Note that `ELEMENT(M,1)` is the 1st row of "*M*". Its dimension is therefore the number of columns.
To insert expression #2 into #1 highlight #1 and do a MANAGE-SUBSTITUTE. When the name "row" comes up replace it with "#2". This results in the nested expression #3.
The next step is to define "*elem*". If we substituted `ELEMENT(M,I,J)` for "*elem*", the result would be the original matrix M, which is OK, except for column no. "*NC*" which should be replaced. So we author

```
4:   IF(J=NC,ELEMENT(C,I),ELEMENT(M,I,J))
```

This will simplify to the I-th element of C if we are in the NC-th column. Otherwise we will get the original matrix element.
Now all that's left is to substitute #4 for "*elem*" in #3 (giving expression #5) and then name the function by authoring

```
6:   REPLACE_COLUMN(M,NC,C) := #5
```

This methodology is called "top-down" (or "outside-in") programming. We could equally well have used the "bottom-up" ("inside-out") approach by recognizing that the innermost task is is the definition of an element. Thus we could start with expression #4, embed it in #2 and finally embed the result in #1.
I hope you will find this helpful. After a little practice you will get to the point where you can do all these substitutions in your head and write out the entire function in-line (as long as it isn't too involved!).

*Final comment from 2008:*

Use the built in function REPLACE(u,v,n) (see Online Help!)

```
#7:   replace_c(m, nc, c) := REPLACE(c, m`, nc)`
```

$$\#8: \quad replace\_c(m, 3, [p, q, r]) = \begin{bmatrix} a & b & p & d \\ e & f & q & h \\ i & j & r & l \end{bmatrix}$$

$$\#9: \quad replace\_c\left(\begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \end{bmatrix}, 3, [p, q, r]\right) = \begin{bmatrix} a & b & p & d \\ e & f & q & h \\ i & j & r & l \end{bmatrix}$$

**Message 3344: From KEITH WILLIAMS to PUBLIC about RECTANGULAR TO POLAR**

Can anyone come up with a user function that will take a matrix such as a 2x1,3x1 4x1 etc. which contains numbers in rectangular form and change them to polar form in degrees and reassemble them back as a matrix of whatever size 2x1 etc. that you are working with.

A function such as `Polar(x):=` Take like `[[4+3i],[1-8i],[7+3i]]` and change it to polar form which would be `[[5,36.870],[8.062,-82.875],[7.616,23.199]]`. keith williams

**Message 3350: From GREG SMITH to KEITH WILLIAMS about POLAR TO RECTANGULAR**

Here are some ideas in response to your messages #3333 and #3344. I have been interested in programming in Derive for some time, especially in connection with implementing procedures in my book "Mathematics for Operations Research" which has just come back in print as a Dover Publications paperback. I started programming in 1951 when we inherited a Navy computer at George Washington University that performed arithmetic using telephone relays, had a magnetic drum, and was programmed in octal. It was harder for me to get started in serious programming in Derive than in anything I had tried earlier but I must say it has turned out to be supremely rewarding. As a final personal note I must note that I am tied up right now in finishing our semester and it will be January before I can get back to doing much on this BBS.

In my experience, the first principle is to start with simple programs and gradually work up to the one to be written. The second principle (at least for me because I have concentrated on vectors, matrices, and some tensors, as used in my book) is to realize that Derive uses row vectors and not column vectors (as in my book); this means that even I, who for decades have forced students to use one-rowed matrices to represent, for example, Frechet derivatives, must pay due respect to row vectors. Maybe the next principle is to check every individual command (or program) to see that it works correctly.

With regard to #3333, consider starting as follows to do the job in three steps, each a small step towards the goal.

```
REPLACE_VECTOR_ELEMENT(v,j0,new):=VECTOR(IF(n_=j0,new,ELEMENT(v,n_)),n_,
                                    DIMENSION(v))

REPLACE_MATRIX_ROW(a,i0,v):=REPLACE_VECTOR_ELEMENT(a,i0,v)

REPLACE_MATRIX_COLUMN_BY_VECTOR(a,i0,v):=REPLACE_MATRIX_ROW(a',i0,v)'
```

You may wish to use abbreviated names but note that "BY_VECTOR" in the preceding name is intended to stress that we are using a (Derive) vector and not a one-row or one-column matrix. Try these to see that they work.

With regard to #3344, the following should indicate how you could move toward this goal. First, consider how you could replace one element in a matrix as the counterpart of the first command used above. In order to keep it simple, it might help to use the following two commands which are certainly not necessary but they can aid the eye in working with matrices; I would not use these for (Derive) vectors that were not actually matrices.

```
WIDTH(a):=DIMENSION(ELEMENT(a,1))
```

```
DEPTH(a):=DIMENSION(a)
```

```
REPLACE_MATRIX_ELEMENT(a,i0,j0,new):=VECTOR(IF(m_=i0,VECTOR(IF(n_=j0,new,
                   ELEMENT(a,m_,n_)),n_,WIDTH(a)),ELEMENT(a,m_)),m_,DEPTH(a))
```

Now, finally, to address your #3344, try to adapt the following to your needs. The simplest command to replace one matrix by another whose elements are functions of those for a given matrix might be the next one.

```
COPY_MATRIX(a):=VECTOR(VECTOR(ELEMENT(a,m_,n_),n_,WIDTH(a)),m_,DEPTH(a))
```

Check this to see that it works. Now, simply Replace 'ELEMENT(a, m_, n_)' in the former command by '√(ELEMENT(a, m_, n_))' and assign the name 'SQRT_ELEMENT(a)'.

```
SQRT_ELEMENT(a):=VECTOR(VECTOR(SQRT(ELEMENT(a,m_,n_)),n_,WIDTH(a)),m_,DEPTH(a))
```

Check this to see that it works on matrices with nonnegative elements and then move on to your case of interest. Good luck. w.h.marlow

**Message 3354: From KEITH WILLIAMS to W.H.MARLOW about RECTANGULAR TO POLAR**

Dear MR. MARLOW I was looking over your functions which you left in a file for me. I came up with a function ALL(M,R1,R2,R3):= where M= MATRIX,3x1. R1 to R3 are the rows of the 3x1 matrix which will be changed into polar form. I uploaded a file named POLAR.ZIP in the file area so that you could download it. This function will just work for a 3x1 matrix. I haven´t figured out how to come up with a function which will take a matrix of 2x1, 3x1, etc.... and change the rows from rectangular to polar form. Maybe you can look at the file that I uploaded come up with a function which will do this. Without your help and the help of HADUD I could never have come up with this function. So I guess I am getting better at figuring out functions. Look at the file which I uploaded and maybe you can come up with the function POLAR(M):= which will take any size matrix 2x1 and up into polar coordinates. Thank you very much for your interest in my problem. KEITH WILLIAMS

**Message 3355: From HADUD to K WILLIAMS about RECTANGULAR TO POLAR CONVERSION**

Allright, let's try it once more! This time we'll use the bottom-up approach.
What you want to do is to take the first (and only) element of the I-th row of your (n x 1)-matrix and convert it to polar form. Then you want to build up a result matrix from this.

OK, you get the element by authoring

```
1:  ELEMENT(M,I,1)
```

To convert any complex number Z you have to compute 2 numbers: amplitude and phase. To do that in one swoop, you make a vector out of the results by authoring

```
2:  [ABS(Z),180/pi PHASE(Z)]
```

Now highlight #2, do a MANAGE-SUBSTITUTE and when Z comes up replace it with #1. The result is

```
3:  [ABS(ELEMENT(M,I,1)),180/pi PHASE(ELEMENT(M,I,1))]
```

Now you have an expression for the I-th row of your desired output matrix. To put all the rows in a matrix and create the desired function you author

```
4:  REC2POL(M):=VECTOR(#3,I,DIMENSION(M)) ......
```
and that's it!

Note that I used single-character names throughout, so this will work in the OPTION-INPUT-(CHARACTER,INSENSITIVE) mode. To bring the result closer to customary notation, you might want to replace the comma in expression 2 with a < , which can be read as an angle sign:

```
2:  [ABS(Z)<180/pi PHASE(Z)]
```

DERIVE now interprets what's inside the brackets as a relational expression, but that's OK as long as don't do any further processing of the result matrix.

**Message 3356: From KEITH WILLIAMS to HADUD about RECTANGULAR TO POLAR**

Dear Hadud You have come to my rescue one more time. What can I say to you except that I really do admire your understanding of mathematics. You must be a genius. I sure wish I could figure out problems like you do. I bet that there are not many problems in math that you cannot figure out. If I need anymore help with any problems I know who to consult. The reason that I was interested in manipulating matrices is because I am taking electrical courses that involve using matrices to solve electrical circuits. In ac circuit analysis you have to deal with rectangular and polar coordinates and phase angles. I really appreciate your help on my problems. Maybe I can help you sometime. Thank you very much for your help Hadud. keith williams

**Message 3357: From KEITH WILLIAMS to HADUD about POLAR TO RECTANGULAR CONVERSION**

DEAR HADUD, The function that you gave me to change from rectangular to polar helped me out a lot. I took your function and also made it where it would change from polar to rectangular.

```
pol2rec(m):=vector([element(m,i,1)*cos(element(m,i,2)*(pi/180))+
             element(m,i,1)*(sin(element(m,i,2)*(pi/180))*#i)],i,dimension(m)).
```

This will change from polar to rectangular.   keith williams

**Message 3358:From HADUD to KEITH WILLIAMS about POLAR TO RECTANGULAR CONVERSION**

Thanks for communicating your neat POL2REC function. I get the impression that you are now well on your way to becoming an expert in DERIVE functionality!

If you prefer to work in the character input mode I have one suggestion: For the reasons outlined in my message #3346, use letters followed by underscore(s) for indices inside your function (I_ or I__ in place of I in the function POL2REC). DERIVE allows this type of variable names even when you are in the character mode (see version 2 manual , p.38)

**Message 3359: From KEITH WILLIAMS to HADUD about #3358 / POLAR TO RECT.ANGULAR CONVERSION**

Dear Hadud I know that you have heard enough about matrices and polar to rectangular conversion, but I am going to ask you one more problem. You came up with the function rec2pol which would change a 2x1,3x1 etc ... matrix from rectangular to polar form. I got to thinking that I would try to come up with a function which would take a 2x2,3x3,4x4 etc... matrices which contained elements in rectangular form to polar form. Well, I am still working on it. I tried to use your reasoning on how to figure it out using the methods which you used in figuring out the other problems. So far I haven't had much luck. I know that the end product of this function is a matrix. So think that you would need `vector(row,i, dimension(m))`. I got this from one of the problems that you solved. Then `vector(elem,j,dimension(element(m,1)))`. And then replace row with the last expression, which would yield `vector(vector(elem,j, dimension(element(m,1))),i,dimension(m))`. I know that something needs to be put in place of *elem*, but I have not figured it out yet. I tried putting `[abs(element(m,i,1)),180/pi phase(element(m,i,1))]` but this replacement doesn't seem to work. can you help me with this problem. If you will I will try my best to figure out these problems concerning matrices myself.
Thank you keith williams

**Message 3360: From HADUD to KEITH WILLIAMS about RECTANGULAR TO POLAR CONVERSION**

Everything you did is perfectly OK. The only problem is that, when replacing "*elem*", you forgot that you are no longer dealing with a (n x 1) but with a (n x m) matrix. Your element references therefore should be "`element(m,i,j)`", not "`element(m,i,1)`". If you make that change you will get a working function but the display will not be what you want. The reason is as follows:

Remember that DERIVE works inside-out. Thus when it encounters the innermost brackets [] it interprets their contents as a row. The next `VECTOR` function (which has the index J) then forms a (m x 2) matrix. The outermost `VECTOR` function (index I) then forms a vector of matrices, i.e. it will display n (m x 2)-matrices side-by-side.

The net result is something that looks like the transposed of the desired matrix. This is another DERIVE quirk (not an error) that can be frustrating.

There are two ways around the problem:

1) Put another set of brackets around the replacement for "*elem*":

```
[[ABS(ELEMENT(M,I,J)),180/pi PHASE(ELEMENT(M,I,J))]]
```

Now "*elem*" is a (1 x 2) matrix and the final result will be a matrix of matrices with the desired appearance.

2) Omit the brackets and replace the comma with a relational symbol:

```
ABS(ELEMENT(M,I,J)) < 180/pi PHASE(ELEMENT(M,I,J))
```

Now "*elem*" is just an expression, not a vector, and the result will be a (n x m) matrix of expressions.

**Message 3361: From KEITH WILLIAMS to HADUD about RECT.TO POLAR CONVERSION**

Hello Hadud  Your function worked perfect again. At one time I had the function down just like you had it except for the extra brackets. While we are really into matrix manipulation I have one more problem for you since you have not failed yet to come up with a solution to every problem which I have burdened you with. May I ask you one more question. I would say that this would be the last one but I would not want to lie to you.
You really amaze me how you can figure out all of these matrix problems.  What I would like to ask you is that your last function would convert a 2x2,3x3 etc.. matrix from rectangular to polar coordinates but can you develop a function which would do the opposite: convert a 2x2,3x3,ect... from polar form to rectangular form? If you can I don't think that I have any more questions about matrix manipulation at the present time. Thank you very much for all the help which you have given me on this subject.      keith williams

**Message 3363: From KEITH WILLIAMS to HADUD about MATRIX REDIMENSION**

Dear Hadud I have a hp48 calculator and I was looking through the manual which came with it on the functions which would manipulate matrices. In the manual there was a function called redimension which would take a 2x2,3x3 ect.. and redimension it. For example in the book there was an example which showed a 3x3 matrix with the element [[1,2,3],[4,5,6],[7,8,9]] and was redimensioned to a 9_row by 1_col matrix with the elements redimensioned as [[1],[2],[3],[4],[5],[6],[7],[8],[9]]. Would a function such as this be very hard to develop for derive? A function such as `rdm(m,row,col):=` where `m` = matrix row = number of rows and `col` = number of columns.
I was just curious wheather or not a function like this could be developed for DERIVE. Thank you for taking an intrest in all these problems which I have asked you concerning matrix manipulation. keith williams

**Message 3364: From HADUD to KEITH WILLIAMS about RECT.TO POLAR CONVERSION**

I'm a little disappointed that you couldn't do the POL2REC function yourself after all my explanations! Here it is:

```
POL2REC(M):=VECTOR(VECTOR(ELEMENT(ELEMENT(M,I,J),1,1)*(COS(ELEMENT
       (ELEMENT(M,I,J),1,2)*pi/180)+#i*SIN(ELEMENT(ELEMENT(M,I,J),1,2)*pi/180)),
       J,DIMENSION(ELEMENT(M,1))),I,1,DIMENSION(M))
```

This assumes the double-bracket version of REC2POL, i.e.

```
REC2POL(M):=VECTOR(VECTOR([[ABS(ELEMENT(M,I,J)),180/pi*PHASE(ELEMENT(M,I,J))]],
        J,DIMENSION(ELEMENT(M,1))),I,DIMENSION(M))
```

The redimension function is best done recursively by introducing the auxiliary function APPV_:

```
APPV_(n,M):=IF(n=1,ELEMENT(M,1),APPEND(APPV_(n-1,M),ELEMENT(M,n)))

REDIM(M):=ELEMENT([VAPP_:=APPV_(DIMENSION(M),M),VECTOR([ELEMENT(VAPP_,I)],
                     I,DIMENSION(M)*DIMENSION(ELEMENT(M,1)))],2)
```

All of the above functions will work for any matrix M, square or rectangular.

**Message 3366: From KEITH WILLIAMS to HADUD about RECT.TO POLAR CONVERSION**

Dear Hadud I did come up with a function the way I could understand it. Mine is I think like yours except that it is longer. It is as follows:

```
pol2rec(m):=vector(vector(element(element(element(m,i,j)*1,1)*1,1)*
   cos(element(element(m,i,j),1,2)*(pi/180))+element(element(element(m,i,j)*1,1)
   *1,1)*sin(element(element(m,i,j),1,2)*(pi/180))*#i,j,dimension
   (element(m,1))),Zi,dimension(m))
```

keith williams

**Message 3367: From KEITH WILLIAMS to HADUD about MATRIX REDIMENSION**

Hadud the functions which you gave to me the other day worked. I guess I didn't make myself clear as to the kind of function I wanted. The example in the book took for example a 3x3 matrix and redimensioned into like a 3x1, 1x3, 1x9, 9x1, 2x4, 4x2, etc.. just whatever dimension that you could assemble out of what ever size matrix which you were working with. The example took a [[1,2,3],[4,5,6],[7,8,9]] matrix and changed it into different sizes. a 9_row 1_col would be [[1],[2],[3],[4],[5],[6],[7],[8],[9]] A 3x2 would be [[1,2],[3,4],[5,6]] A 2x3 would be [[1,2,3],[4,5,6]]. I have been trying to come up with a function which would do this.

For example `redim_matrix(m,col,row):=` where `m` = matrix ,`col` = number of columns wanted, and `row`=number of rows wanted. I have put to together some of it but it will just take the first element of the matrix and make whatever size matrix which you specify with all the elements in the new matrix of element 1.What I have is

```
Redim_matrix(m,col,row):=vector(vector(element(m,1,1),j,1,col),j,1,row).
```

This will recreate any size matrix which you can redimension out of the original matrix but every element win the matrix will be the same as element number one. I just thought that I would show you what I have come up with so far. Look at what I have come up with and tell me if I a proceeding in the right direction. If you come up with a function which will do this let me know what it is. Mean which I will keep trying to figure this one out. Ill be waiting to hear from you. Maybe I can figure this one out and show you that I have at least learned something. Thank you keith williams

**Message 3368: From KEITH WILLIAMS to HADUD about MATRIX REDIMENSION**

Dear Hadud I just wanted to show you this function which I came up with. It is not anything like what you can do but at least its a beginning. This function called

```
Matrix_reduce(m,dw,dd):=vector(vector(element(m,i,j),j,dimension(element(m,1))-
dw),i,dimension(m)-dd).
```

This function will take any size matrix reduce the number of rows or columns by the amount which you specify. `dw` = deduct width, `dd` = deduct depth, `m` = matrix. I just wanted to show you this. keith williams

**Message 3369:From KEITH WILLIAMS to HADUD about #3367 / MATRIX REDIMENSION**

Dear Hadud I am still trying to come up with a function which will redimension a matrix into different dimensions. I made a little progress today I came up with

```
Matrix_redim(m,col,row):=vector(vector(elem,j,col), I,row).
```

This will create any size matrix which you specify by col and row with the word *elem* in place of the elements of the matrix. I haven't figured out what to put in place of *elem* to make it redimension the elements of the matrix that you would be working with. I'll keep trying I know without a doubt that you can figure it out. If you do let me know. keith williams

**Message 3371:From KEITH WILLIAMS to HADUD about RECT.TO POLAR CONVERSION**

Dear Hadud The function which you gave me to redimension a matrix into a 1_col by x_rows works. I figured how to make the function work faster.
The one that you gave me took a 20x20 matrix and changed it into a 1_col by 400_row matrix in 309_sec. The way which I came up with did in in .3_sec. The function is:

```
redim2(m):=extract_1_row(adjoin_element(append(m),m),1).
```

I made this function from the following:

```
extract_1_elements(m,e):=[element(m,e)]
Extract_1_row(m,c1):=extract_1_elements(m,c1)'
Adjoin_element(m1,m2):=append([m1],m2)
```

keith williams

**Message 3372: From HADUD to KEITH WILLIAMS about REDIMENSIONING A MATRIX**

Bravo! Your REDIM2 function is indeed much more elegant than my monstrosity! I guess I can retire as your teacher! I learned from your function that you can use `APPEND(M)` to convert a general matrix M to a single vector, which is something I never realized. Given that, I can still do you one better, though! Try

```
REDIM1(M):=[APPEND(M)]'
```

This also holds the key to your general REDIM function:

```
REDIM(M,r,c):=VECTOR(VECTOR(ELEMENT(APPEND(M),(I-1)c+J),J,c),I,r)
```

This will generate an (r x c) matrix from the elements of the arbitrary matrix M. I hope I've understood the problem correctly now!

**Message 3373: From KEITH WILLIAMS to HADUD about REDIMENSIONING A MATRIX**

Dear Hadud I'm the one who should say bravo! to you. The `redim(m,r,c)` function was just what I wanted. I don't understand it but it works exactly as it should. Your redim1 function was faster than my redim2 function which just goes to show you that you are still a lot better than I am at figuring out functions. I can't think of any more questions concerning matrices, but don't retire yet as my teacher. I want to say to you that I appreciate all your help on all these problems which I as sure were not any challenges to you. I have enjoyed communicating with you over this bulletin board during the past few weeks. Again I want to thank you for taking an interest in my problems in math. Your friend and admirer keith williams

*Solving the problems with recent version of DERIVE:*

```
      redim(m_, r_, c_) := VECTOR(VECTOR((APPEND(m_))                , k, c_), i, r_)
#1:                                                   (i - 1)·c_ + k

      redim2(m_, r_, c_) :=
        If r_·c_ ≠ DIM(APPEND(m_))
#2:        "wrong dimension"
           VECTOR(VECTOR((APPEND(m_))↓((i - 1)·c_ + k), k, 1, c_), i, 1, r_)

#3:   m := VECTOR([RANDOM(100), RANDOM(100), RANDOM(100)], k, 4)
```

#4: $m := \begin{bmatrix} 33 & 58 & 23 \\ 51 & 93 & 96 \\ 57 & 91 & 90 \\ 54 & 83 & 44 \end{bmatrix}$

#5: $redim(m, 2, 6) = \begin{bmatrix} 33 & 58 & 23 & 51 & 93 & 96 \\ 57 & 91 & 90 & 54 & 83 & 44 \end{bmatrix}$

#6: $redim(m, 6, 2) = \begin{bmatrix} 33 & 58 \\ 23 & 51 \\ 93 & 96 \\ 57 & 91 \\ 90 & 54 \\ 83 & 44 \end{bmatrix}$

#7: $redim(m, 3, 3) = \begin{bmatrix} 33 & 58 & 23 \\ 51 & 93 & 96 \\ 57 & 91 & 90 \end{bmatrix}$

#8: redim2(m, 3, 3) = wrong dimension

#1: $m1 := \begin{bmatrix} 4 + 3\cdot i \\ 1 - 8\cdot i \\ 7 + 3\cdot i \end{bmatrix}$

#2: $REC2POL(M) := VECTOR\left(VECTOR\left(\left[\,|ELEMENT(M, i, j)|, \frac{180}{\pi}\cdot PHASE(ELEMENT(M, i, j))\right], j, DIMENSION(ELEMENT(M, 1))\right), i, DIMENSION(M)\right)$

#3: REC2POL(m1)

simplified and then approximated:

#5: $\begin{bmatrix} [5, 36.86989764] \\ [8.062257748, -82.87498365] \\ [7.615773105, 23.19859051] \end{bmatrix}$

#6: $$m2 := \begin{bmatrix} [5, 36.86989764] \\ [8.062257748, -82.87498365] \\ [7.615773105, 23.19859051] \end{bmatrix}$$

#7: $$\text{POL2REC(M)} := \text{VECTOR}\left(\text{VECTOR}\left(\text{ELEMENT(ELEMENT(M, i, j), 1)}\left(\text{COS}\left(\frac{\text{ELEMENT(ELEMENT(M, i, j), 2)}\cdot\pi}{180}\right) + i\cdot\text{SIN}\left(\frac{\text{ELEMENT(ELEMENT(M, i, j), 2)}\cdot\pi}{180}\right)\right), \text{j, DIMENSION(ELEMENT(M, 1))}\right), \text{i, 1, DIMENSION(M)}\right)$$

#8: POL2REC(m2)

simplified first and then approximated:

#10: $$\begin{bmatrix} 4 + 3\cdot i \\ 1 - 8\cdot i \\ 7 + 3\cdot i \end{bmatrix}$$

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#11: $$\text{REC2POL\_(m\_)} := \text{VECTOR}\left(\left[|v|, \frac{180}{\pi}\cdot(\text{PHASE}(v))_1\right], v, m\_\right)$$

#12: REC2POL_(m1)

Expression #12 approximated

#13: $$\begin{bmatrix} 5 & 36.86989764 \\ 8.062257748 & -82.87498365 \\ 7.615773105 & 23.19859051 \end{bmatrix}$$

#14: $$m2\_ := \begin{bmatrix} 5 & 36.86989764 \\ 8.062257748 & -82.87498365 \\ 7.615773105 & 23.19859051 \end{bmatrix}$$

#15: $$\text{POL2REC\_(m\_)} := \text{VECTOR}(v\_{}_1\cdot(\text{COS}(v\_{}_2\cdot 1°) + i\cdot\text{SIN}(v\_{}_2\cdot 1°)), v\_, m\_)$$

#16: POL2REC_(m2_)

#16 approximated:

#18: $[4 + 3\cdot i, 1 - 8\cdot i, 7 + 3\cdot i]$

More Forum contributions on page 50

# Ebene Algebraische und Transzendente Kurven (6)

Thomas Weth, Würzburg, Germany

## Cassinische Kurven - Die Lemniskate von Bernoulli

Im Jahre 1680 beschrieb Johann Dominicus Cassini eine Kurvenschar, die bis heute mit dem Namen ihres Erfinders verbunden blieb. Über den Anlass ihrer Entdeckung schreibt sein Sohn Jakob Cassini im Jahre 1749 sinngemäß: „Durch genaue Beobachtung der Sonne fand mein Vater eine von der Ellipse verschiedene Kurve, die dazu dient, die exakte Bewegung der Sonne und ihre verschiedenen Entfernungen von der Erde zu beschreiben.“ Obwohl sich herausstellte, dass die Cassinischen Kurven in keinerlei Beziehung zum eigentlichen Ausgangsproblem stehen, dienten sie doch immer wieder als Gegenstand mathematischer Untersuchungen - insbesondere um die Tragfähigkeit algebraischer Methoden und der Infinitesimalrechung auszuloten. Aufmerksamkeit schenkte man einem Spezialfall, der Bernoullischen Lemniskate, mit deren Rektifikation (Längenmessung) sich Graf Fagnano eingehend beschäftigte. In geeigneter Weise entwickelt und verallgemeinert, bilden seine Überlegungen den Kern der Theorie der elliptischen Funktionen.

1680 J.D.Cassini decribed a family of curves, connected with his name until today. His son, Jakob Cassini, wrote in 1749: “...by exact observation of the sun may father found a curve, different from an ellipse, which can serve to describe the exact motion of the sun and her different distances from earth.“ Although it was found out that the curves of Cassini don´t have any connection with the starting problem, they have been used as an object of mathematic investigation, especially to check the validity of algebraic methods of calculus. Special attention was paid to a special case: the Lemniskate of Bernoulli. Count Fagnano dealt with its rectification and his ideas can be seen as the foundation of the theory of elliptic functions.

## Definition und Konstruktion der Kurve

Gegeben sind zwei Punkte $B_1$ und $B_2$ (Brennpunkte) mit der Entfernung 2c. Die Punkte P, bei denen das Produkt ihrer Entfernungen $\rho_1$ und $\rho_2$ zu den Brennpunkten einen festen Wert $a^2$ besitzt, bilden die Cassinischen Kurven: $\rho_1 \cdot \rho_2 = PB_1 \cdot PB_2 = a^2$.

Given are two points $B_1$ and $B_2$ (focal points) with $B_1B_2 = 2c$. The curves of Cassini are the sets of points P, whose product of the distances to the focal points $\rho_1 \cdot \rho_2 = PB_1 \cdot PB_2 = a^2$ (const.). See fig.3.

Legt man die Brennpunkte symmetrisch zum Ursprung eines kartesischen Koordinatensystems auf die $x$-Achse (vgl. Abb.3), so erhält man in kartesischen Koordinaten:

$\sqrt{y^2 + (x+c)^2}\sqrt{y^2 + (x-c)^2} = a^2$ und durch Quadrieren: $(x^2 + y^2)^2 - 2c^2(x^2 - y^2) = a^4 - c^4$.

Zur Gleichung in Polarkoordinaten kommt man durch Einsetzen von $x = r\cos\varphi$ und $y = r\sin\varphi$ (mit DERIVE): $r^4 - 2c^2 r^2 \cos 2\varphi = a^4 - c^4$ . Löst man diese Gleichung nach $r$ auf und plottet in Polarkoordinaten, so liefert DERIVE: (soLve for r and plot in polar co-ordinates:)

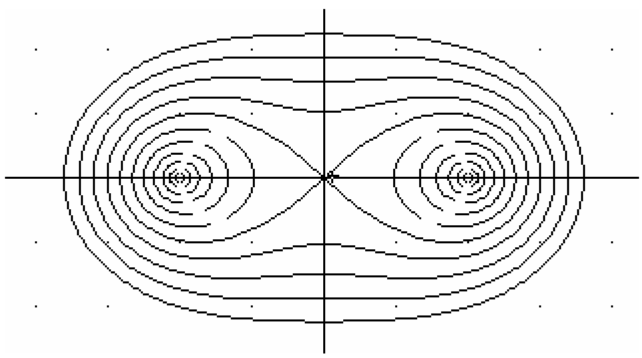


Abb1: Cassinische Kurven mit DERIVE

Mit dem Schieberegler lässt sich nun auch die implizite Darstellung treffen. Daneben ist das Bild einer Schar von Cassinischen Kurven in impliziter Form gegeben.

Applying slider bars for *a* and *c* and the implicit form it is easy to investigate the curves. On the right hand side you can see a family of "Cassinis".

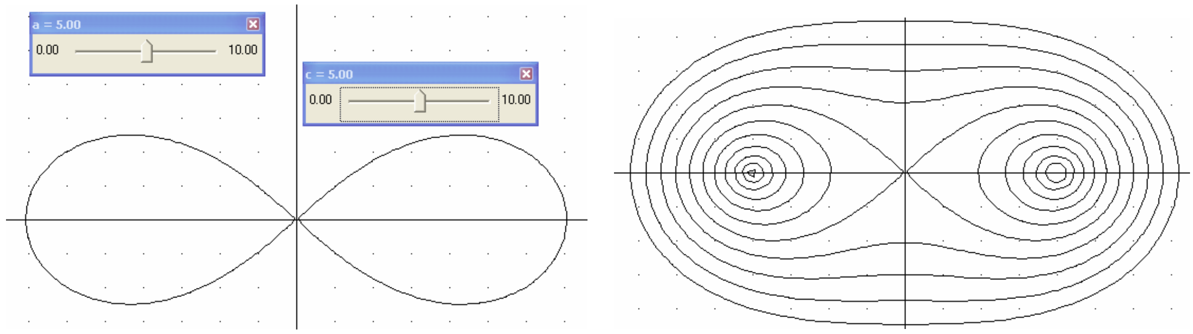



Abb2: Cassinische Kurven mit Schieberegler und impliziter Plot (Kurvenschar)
Cassini Curves with slider bars for *a* and *c* and a family of implicit plots

Zur Zirkel- und Linealkonstruktion der Kurve benötigt man zu gegebenem $a^2$ zwei Strecken $\rho_1$ und $\rho_2$ , für deren Produkt gilt: $\rho_1 \cdot \rho_2 = a^2$. Um die gegebenen Brennpunkte $B_1$ und $B_2$ werden dann Kreise mit diesen Radien gezeichnet. Die Schnittpunkte dieser Kreise sind dann Punkte der Cassinischen Kurven.
Die Struktur der Gleichung $\rho_1 \cdot \rho_2 = a^2$ legt zur Konstruktion von $\rho_1$ und $\rho_2$ die Verwendung des Kathetensatzes (oder des strukturgleichen Höhensatzes) im rechtwinkligen Dreieck nahe. Gibt man auf der $x$-Achse den Punkt A($a$,0) vor, und verbindet A mit einem beliebigen Punkt R der $y$-Achse, so ist das Dreieck AOR rechtwinklig. Die Höhe von O auf die Hypotenuse AR liefert den Höhenfußpunkt T und den zu Kathete OA zugehörigen Hypotenusenabschnitt AT der Länge $\rho_1$. Mit der Hypotenusenlänge $\rho_2$ gilt nach dem Kathetensatz: $\rho_1 \cdot \rho_2 = a^2$.
Zeichnet man nun die Kreise k($B_1$, $\rho_1$) und k($B_2$, $\rho_2$), so bilden deren Schnittpunkte P - soweit sie existieren - Punkte der Cassinischen Kurven (siehe Abb. 3).

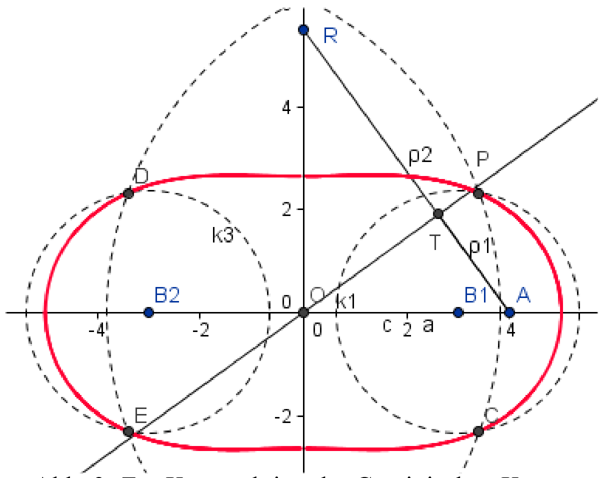



Abb. 3: Zur Konstruktion der Cassinischen Kurve

To construct the curve using ruler and compass you need to a given $a^2$ two distances $\rho_1$ and $\rho_2$ with $\rho_1 \cdot \rho_2 = a^2$. Then you have to draw circles with centres $B_{1,2}$ and $\rho_{1,2}$ as radius. The intersections of these circles are points of the curves. The structure of $\rho_1 \cdot \rho_2 = a^2$ leads to use the Pythagorean Theorem. Take any arbitrary point A($a$,0) on the $x$-axis and connect A with an arbitrary point R on the $y$-axis, then the triangle AOR is a right one. The height OT with respect to AR gives T, which divides the hypotenuse AR = $\rho_2$. With AT = $\rho_1$ you find finally RA · TA = $OA^2$ = $\rho_1 \cdot \rho_2 = a^2$. So draw two circles k($B_1$, $\rho_1$) and k($B_2$, $\rho_2$), and their intersection points are - if they do exist - two points of the curves.
See figure 3.

**Die Bernoullische Lemniskate**

Der Sonderfall $c = a$ liefert einen Spezialfall der Cassinischen Kurven, die Lemniskate von Bernoulli, deren Namen sich ableitet vom griechischen λεμνισκοζ (Schleife, Band).

The special case c = a gives the Lemniscate of Bernoulli. (λεμνισκοζ = loop, cord)

In bipolaren Koordinaten ergibt sich $\rho_1 \cdot \rho_2 = c^2$. (bipolar co-ordinates).
Als algebraische Kurvengleichung erhält man daraus: $(x^2 + y^2)^2 = 2c^2\,(x^2 - y^2)$
und in Polarkoordinaten: $r^2 = 2\,c^2 \cos 2\varphi$.

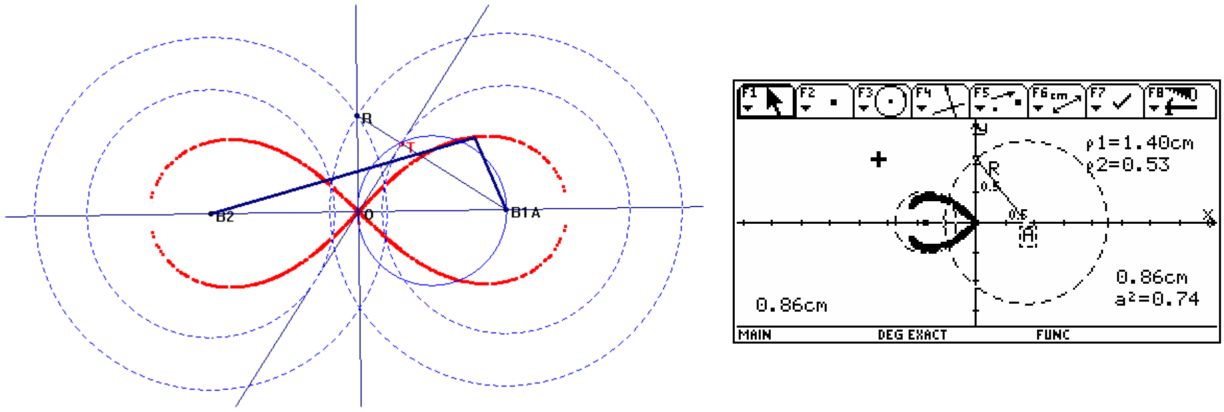



Abb. 4. Lemniskate von Bernoulli (with Cabri on the PC and on the TI-Handheld)

Eine weitere Konstruktion, die sich als „wörtliche Übersetzung" der algebraischen Kurvengleichung verstehen lässt findet man in „Kurven in geometrischer und algebraischer Sicht" (Didaktik der Mathematik 2, 1992 (112-138)).

Einige interessante Eigenschaften der Lemniskate, die sich mit DERIVE überprüfen lassen:

- Die Kurve schneidet sich im Ursprung selbst in einem rechten Winkel.
- Die Lemniskate ist eine anallagmatische Kurve, dh sie wird unter einer (geeigneten) Inversion am Kreis auf sich selbst abgebildet.
- Den von der Lemniskate eingeschlossenen Flächeninhalt erhält man (DERIVE !):

$$A = 4\left(\frac{1}{2}\int_{\varphi=0}^{\frac{\pi}{4}} r^2\, d\varphi\right) = 2c^2$$

Der eingeschlossene Flächeninhalt ist also halb so groß wie ein Quadrat, dessen Seitenlänge gleich der Entfernung der beiden Brennpunkte ist.

The lemniscate shows some interesting properties, which easily can be proofed using DERIVE:

- The curve is intersecting itself orthogonally.
- The lemniscate is anallagmatic, i.e it can be mapped by an appropriate inversion at a circle onto itself.
- The area enclosed by the lemniscate is $c^2$, is half the area of a square with the distance between the foci as side length.

**Details from the DERIVE session:**

#16: $r^4 - 2\cdot c^2\cdot r^2\cdot \mathrm{COS}(2\cdot\phi) = 0$

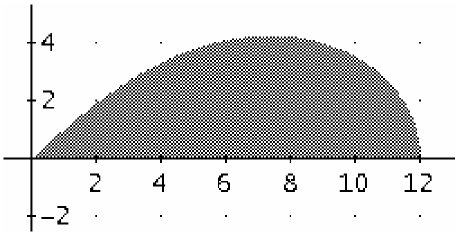
#17: $\mathrm{SOLVE}(r^4 - 2\cdot c^2\cdot r^2\cdot \mathrm{COS}(2\cdot\phi) = 0,\ r)$

#18: $r = -\sqrt{2}\cdot|c|\cdot\sqrt{(\mathrm{COS}(2\cdot\phi))} \vee r = \sqrt{2}\cdot|c|\cdot\sqrt{(\mathrm{COS}(2\cdot\phi))} \vee r = 0$

#19: $r = \sqrt{2}\cdot|c|\cdot\sqrt{(\mathrm{COS}(2\cdot\phi))}$

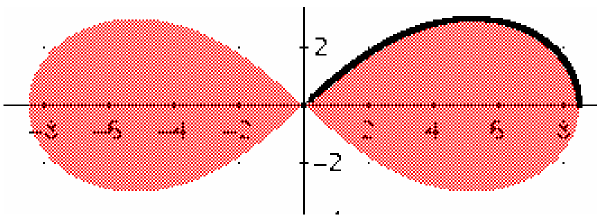
Area: $0 \le \phi \le \pi/4$ gives on 1/4 of the curve

#20: $r < \sqrt{2}\cdot|c|\cdot\sqrt{(\mathrm{COS}(2\cdot\phi))} \wedge 0 \le \phi \le \frac{\pi}{4}$

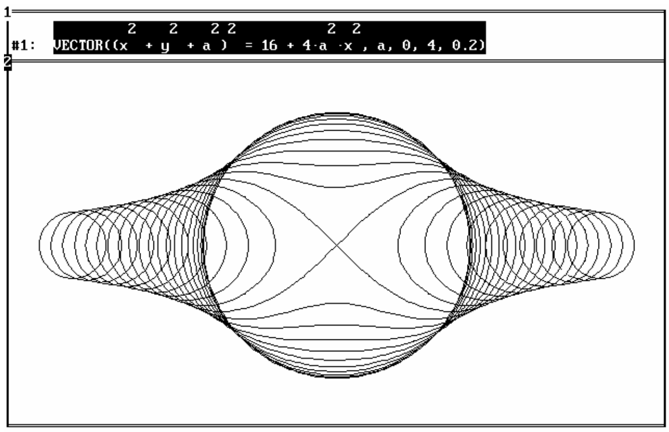



#22: $r < \sqrt{2}\cdot|c|\cdot\sqrt{(\mathrm{COS}(2\cdot\phi))}$

#23: $\mathrm{VECTOR}\left([\sqrt{2}\cdot|c|\cdot\sqrt{(\mathrm{COS}(2\cdot\phi))},\ \phi],\ \phi,\ 0,\ \frac{\pi}{4},\ \frac{\pi}{5000}\right)$




#21: $4\cdot\left(\frac{1}{2}\cdot\int_0^{\pi/4} (\sqrt{2}\cdot|c|\cdot\sqrt{(\mathrm{COS}(2\cdot\phi))})^2\ d\phi\right) = 2\cdot c^2$




Family Cassini (Memories on DOS-Times)

# A DERIVE–Toolbox

Eugenio Roanes-Lozano , Eugenio Roanes-Macías & Josef Böhm

*This contribution has a nice history. Eugenio jr sent a FAX with his "A Trick about 2D Plots in DERIVE" and I gave the answer, that I had written a little utility to draw broken lines and shaded areas to complete Sergey's labelling utility from DNL#16. So Eugenio suggested publishing a Spanish - Austrian co production. This is the result:*

## A Trick About 2D Plots in DERIVE

**Abstract**

There are some commands in DERIVE that have to be selected from the **Command line** and do not have a correspondent function in the **Author line**. This is the case with colours. It is not possible to choose directly the desired colour for the plot of a function from the **Author line**.

This is sometimes an inconvenience. For instance, when preparing a .MTH - file for a Teaching Unit, if we want to identify a certain curve by its colour.

We shall show how it is possible to choose the colour of a plot from the **Author line** (and to keep it unaltered even if it is plotted more than one time). It works both in DERIVE 2 and the new DERIVE 3.

### 1.1 Plotting a function

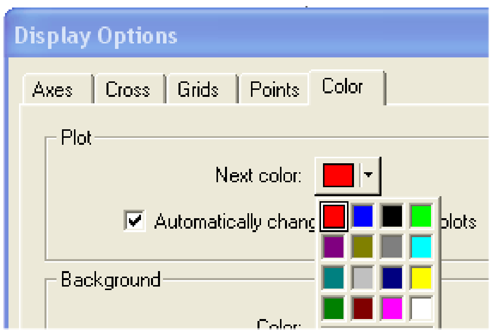
We shall assume that our hardware uses a 16-colour VGA graphics-card. If that is not the case, what follows also stands, altering it accordingly to the number of colours of the corresponding graphics-card.

In DERIVE, by default, if the last plot was drawn in colour *i*, the next plot will be drawn in colour *i-1*. The other possibilities given in the **Command line** are:

- From now onwards, plot all the functions in the same colour (**Plot/Options/Color/Auto/No**)
- Choose manually the colour of the next plot (**Plot/Options/Color/Plot/Next Plot/ ...**)

If we leave it as it is by default, after 15 plots (the background colour is not used), the colour of the plot will be the same. Moreover, a vector can be plotted. Therefore we can merge these two ideas to obtain any plot in the desired colour. We have to build a function **DRAW(fu,co)** that constructs a 15 items vector, with the function **fu** in position 15 - co+1. The other 14 functions must plot nothing (in the not required colours). We have chosen **y = î**, but this can obviously be changed.

Comment on this for the recent version of DERIVE: VGA-graphics adapters are the past. The order of the colours did also change. We can choose from 16 colours as ever. The standard colours are presented in a box and can be numbered from 1 to 16 (1 = red, 2 = blue, …, 16 = white):



Let´s define the function **DRAW(fu,co)** that draws the function **fu** in colour **co** this way:

```
#1:   draw(f, co) := APPEND(VECTOR(ı, k, co - 1), [f], VECTOR(ı, k, 15 - co))
```

Instead of "plotting" the imaginary unit you can plot "nothing" expressed by the questions mark!

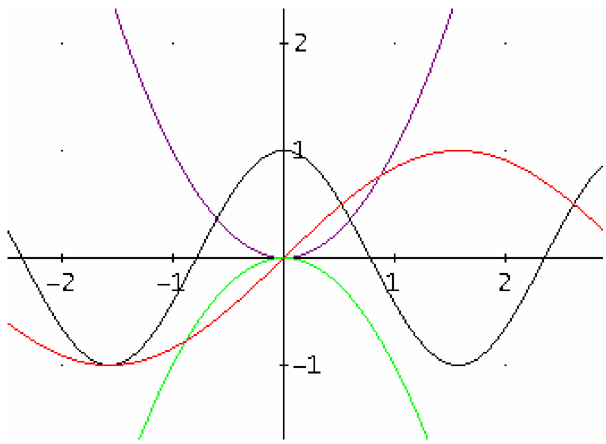
```
#2:   draw(f, co) := APPEND(VECTOR(?, k, co - 1), [f], VECTOR(?, k, 15 - co))
```

In order to have the colours in a correct ordering, before using DRAW for the first time, it is convenient to start a new DERIVE session to confirm in the Display Options Color Box, that red is the colour for the next plot and that "*Automatically change color of new plots*" ticked.

Let´s create the first "programmed" graphs of functions:

```
            2                2
#3:   [draw(x , 5), draw(- x , 4), draw(SIN(x), 1), draw(COS(2·x), 3)]
```

Expression #3 should show the first parabola in purple (colour #5), the second one in light green (colour #4), the sine-wave in bright red (colour #1) and the cosine in black (colour #3). After each plot the colour table is set with red to start.

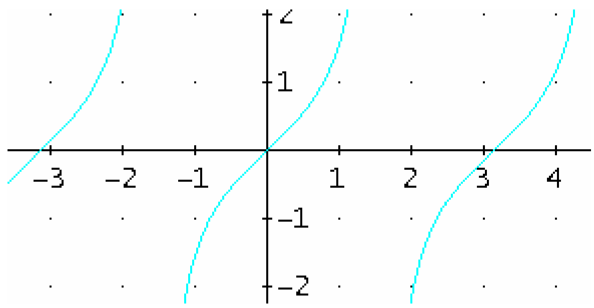



You see that this is working properly.

## 1.2 Creating a Colour Table

Assuming that your colour palette looks like the palette shown on page 24 we can assign:

```
#4:   [red := 1, blue := 2, black := 3, l_green := 4, purple := 5, m_green := 6,
       d_gray := 7, l_blue := 8]

#5:=  [grayblue := 9, l_gray := 10, d_blue := 11, yellow := 12, green := 13,
       d_brown := 14, pink := 15, white := 16]

#6:   draw(TAN(x), l_blue)
```



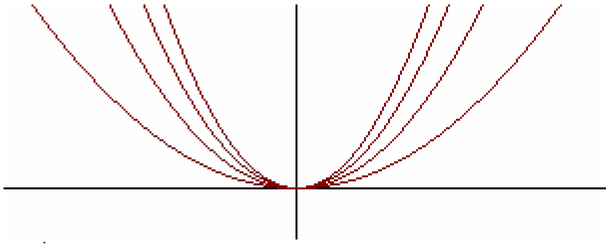
### 1.3 Plotting several functions at the same time

Now, to plot several functions with just one line of code, we shall do a **Plot** of a VECTOR of the vectors created by DRAW (i.e., a **Plot** of a matrix).

```
                2
#7:   VECTOR(a·x , a, 4)
```

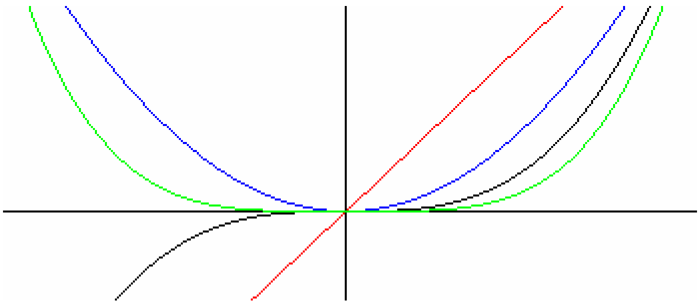
This does not address the colours. Assume that you want to have all parabolas plotted dark brown, then enter and plot:

```
                     2
#8:   VECTOR(draw(a·x , d_brown), a, 4)
```



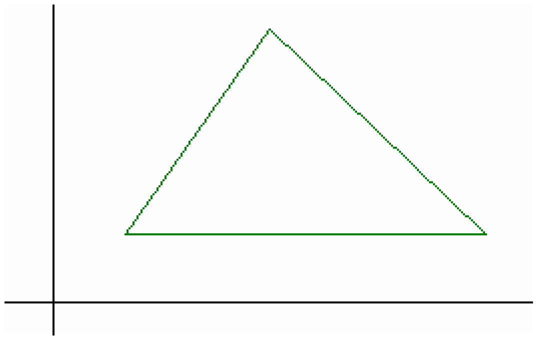
Plot in different colours $y = x^j$ with $j$ ranging from 1 to 4.

```
                    j
#9:   VECTOR(DRAW (x , j),j,1,4)
```



Final example: draw a green triangle

```
#10:  [a := [1, 1], b := [6, 1], c := [3, 4]]

#11:  draw([a, b, c, a], green)
```



## References:

[1] **J.L.Llorens:** *Introducción al uso de DERIVE*, Universidad Politécnica de Valencia, 1993

[2], [3] **A.Rich, J.Rich, D.Stoutemyer:** *DERIVE User´s Manual 2 & 3,* Soft Warehouse

*In the last DNL you could find Sergey´s Labelling Tool. Often it is useful to draw broken or dotted lines and to shade areas in a plot to demonstrate or to emphasize a certain fact. Using the next little functions you can easily do this and draw pure DERIVE - sketches without using any other drawing program.*

## Broken lines and shaded areas

See first four applications of my functions:

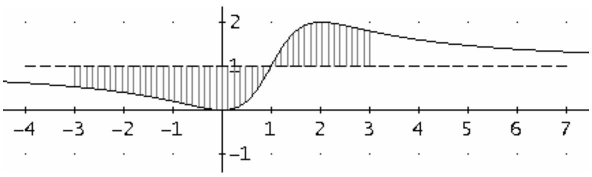



Figure 1

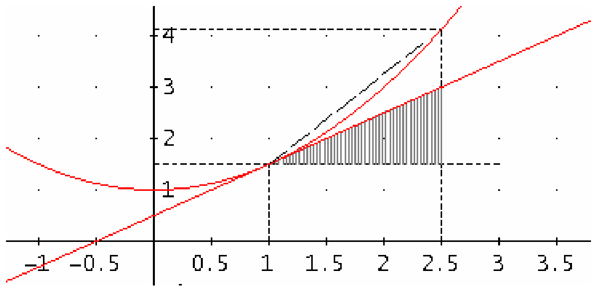



Figure 3

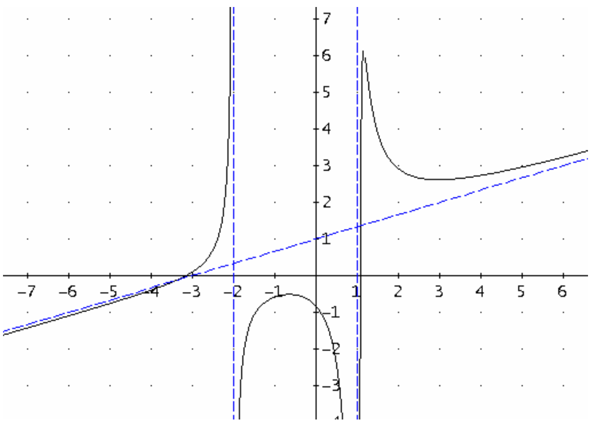



Figure 2

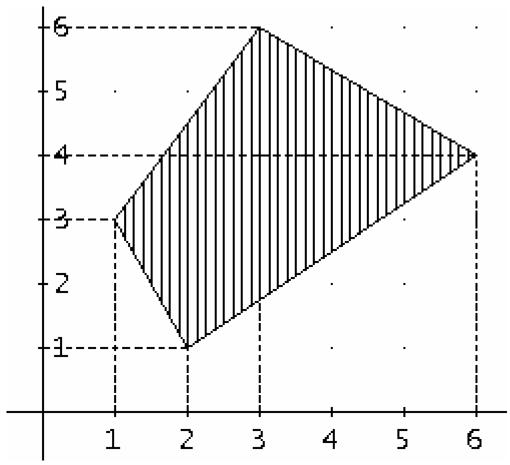



Figure 4

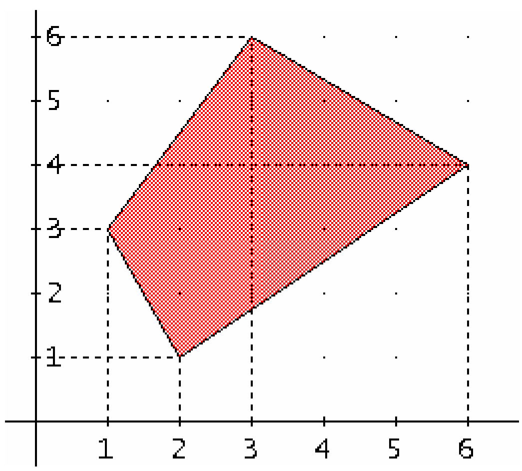



Figure 4a (DERIVE 6 – shading using inequalities, plotting u ≤ y ≤ v)

I used the following DERIVE Utility file BROKEN.MTH:

#1: [Notation := Decimal, NotationDigits := 3]

#2: $$\text{NORM}(x1, y1, x2, y2) := \frac{1}{\sqrt{(x2 - x1)^2 + (y2 - y1)^2}} \cdot [x2 - x1, y2 - y1]$$

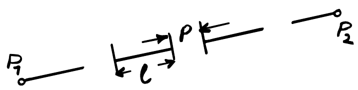
#3: $$\text{STR}(x1, y1, x2, y2, l, p) := \text{APPEND}\left(\text{VECTOR}\left([[x1, y1] + (i - 1)\cdot(l + p)\cdot \text{NORM}(x1, y1, x2, y2), [x1, y1] + (i\cdot l + i\cdot p - p)\cdot \text{NORM}(x1, y1, x2, y2)], i, 1, \text{FLOOR}\left(\frac{\sqrt{(y2 - y1)^2 + (x2 - x1)^2}}{p + l}\right)\right), \left[\left[[x1, y1] + \text{FLOOR}\left(\frac{\sqrt{x1^2 - 2\cdot x1\cdot x2 + x2^2 + (y1 - y2)^2}}{l + p}\right)\cdot(l + p)\cdot \text{NORM}(x1, y1, x2, y2), [x2, y2]\right]\right]\right)$$

#4: ORDNER(x, y, l, p) := [STR(0, y, x, y, l, p), STR(x, 0, x, y, l, p)]

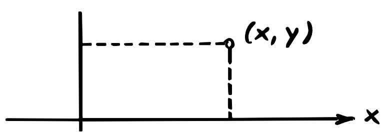
#5: $$\text{SHADE}(u, v, x, a, b, n) := \text{VECTOR}\left(\begin{bmatrix} a + \frac{i\cdot(b - a)}{n} & \lim_{x\to a + i\cdot(b - a)/n} v \\ a + \frac{i\cdot(b - a)}{n} & \lim_{x\to a + i\cdot(b - a)/n} u \end{bmatrix}, i, 0, n\right)$$

#6: SHADE_C(u, v, x, a, b, n, ca, ce) := APPEND([[IF(ca ≤ x ≤ ce, u, ?)]], [[IF(ca ≤ v ≤ ce, v, ?)]], SHADE(u, v, x, a, b, n))

STR(x1,y1,x2,y2,l,p) creates a broken line connecting points $P_1(x_1,y_1)$ and $P_2(x_2,y_2)$:




ORDNER(x,y,l,p) creates a pair of broken lines parallel to the axes containing $P(x,y)$:



`SHADE(u,v,x,a,b,n)` shades the area between the graphs of $u$ and $v$ with $x$ as variable, dividing the area between the graphs into $n$ stripes ranging from $x = a$ to $x = b$.

`SHADE_C(u,v,x,a,b,n,ca,ce)` has the same effect as `SHADE()`, but adds the graphs of $u$ and $v$ for $x \in [ca, ce]$.

I show the commands producing figures1 to 4 to illustrate the functions:

```
LOAD(BROKEN.MTH)
```

Figure 1

$$f1 := \frac{x^2}{x^2 - 2\cdot x + 2}$$

```
STR(-4, 1, 7, 1, 0.2, 0.1)

SHADE(f1, 1, x, -3, 3, 50)
```

Figure 2

$$f2 := \frac{x^4 + 3\cdot x^3 - 5}{3\cdot(x - 1)^2\cdot(x + 2)}$$

```
STR(-12, -3, 12, 5, 0.25, 0.1)

STR(1, -6, 1, 6, 0.25, 0.1)

STR(1, -10, 1, 10, 0.25, 0.1)

STR(-2, -10, -2, 10, 0.25, 0.1)
```

Figure 3

$$f3(x) := \frac{x^2}{2} + 1$$

$$x + \frac{1}{2}$$

```
STR(1, f3(1), 2.5, f3(2.5), 0.3, 0.1)

[STR(0, f3(2.5), 2.5, f3(2.5), 0.04, 0.04), STR(0, f3(1), 3, f3(1), 0.04, 0.04),
STR(2.5, 0, 2.5, f3(2.5), 0.08, 0.08), STR(1, 0, 1, f3(1), 0.08, 0.08)]
SHADE(x + 0.5, f3(1), x, 1, 2.5, 50)
```

Figure 4, you can use piecewise defined functions for shading polygons!

$$\begin{bmatrix} 1 & 3 \\ 2 & 1 \\ 6 & 4 \\ 3 & 6 \end{bmatrix}$$

$$u := IF\left(1 \le x \le 3, \frac{3\cdot(x + 1)}{2}, IF\left(3 \le x \le 6, \frac{2\cdot(12 - x)}{3}, ?\right)\right)$$

$$v := IF\left(1 \le x \le 2, 5 - 2\cdot x, IF\left(2 \le x \le 6, \frac{3\cdot x - 2}{4}, ?\right)\right)$$

```
SHADE(u, v, x, 1, 6, 40)

[ORDNER(1, 3, 0.08, 0.08), ORDNER(2, 1, 0.08, 0.08), ORDNER(3, 6, 0.08, 0.08),
ORDNER(6, 4, 0.08, 0.08)]
```

In my last figure I used additionally LABEL.MTH from DNL#16:

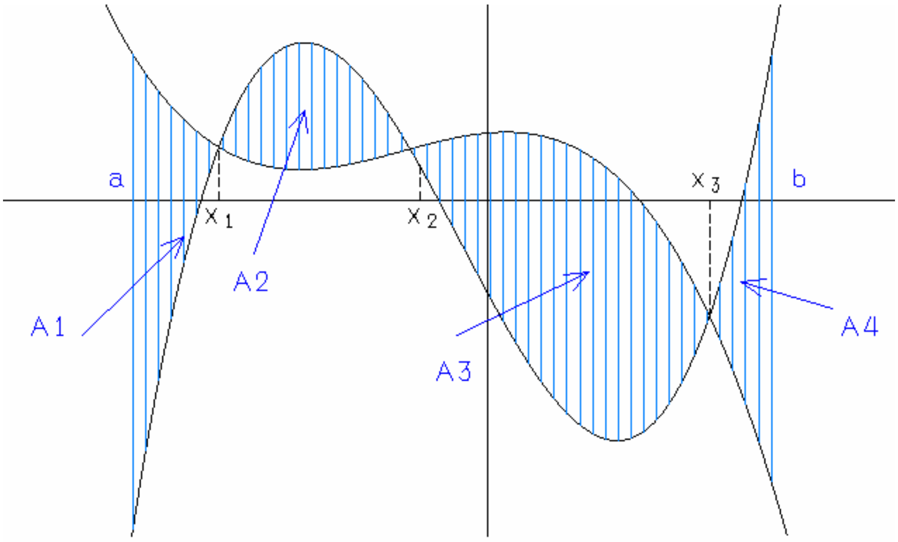
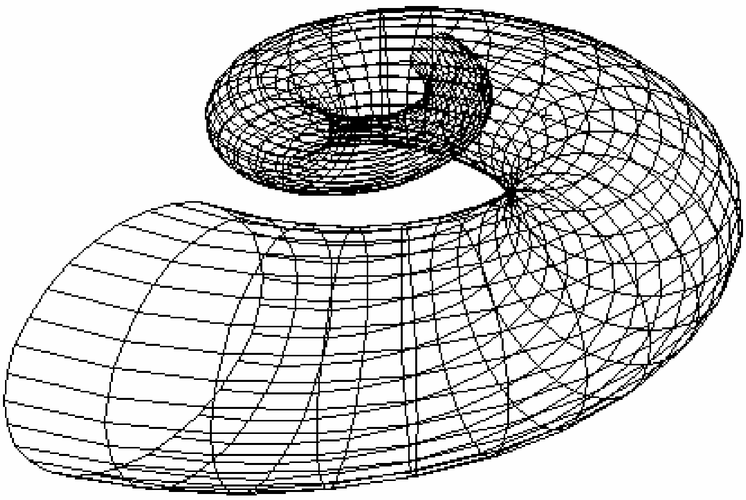



Figure 5

```
f4(x) := 0.3·(x + 2.5)·(x + 1)·(x - 2.7) + 1

                           2
f5(x) := 0.05·(3 - 2·x)·(x  + 4·x + 5)

SHADE(f5(x), f4(x), x, -3.5, 2.8, 50)

SOLVE(f4(x) = f5(x), x)

x = -0.763 ∨ x = -2.65 ∨ x = 2.19

[STR(-2.65, 0, -2.65, f4(-2.65), 0.1, 0.05), STR(-0.673, 0, -0.673, f4(-0.673),
0.1, 0.05), STR(2.19, 0, 2.19, f4(2.19), 0.1, 0.05)]
```

It is not very difficult to extend the given functions: so you could add scale factors to consider different scalings for the two axes in ORDNER().



I found this nice snail-shell in a Mathcad book and I tried to produce it with DERIVE.

# Visiting DERIVE in Austrian Schools

Some Austrian teachers

## 1 DERIVE in a 3rd form (age 13), Alfred Eisler, Tulln

### Some impressions and results of using DERIVE in a 3rd form of a gymnasium.

Use of the program
We don´t use DERIVE continuously, but we use the PC if its usage is "fitting well" to the teaching situation.

Knowledge of the pupils
Round 30% of the pupils are using a PC at home, but none of them has DERIVE available. Until now we have used DERIVE in connection with two items:

**1. Solving equations**
**2. Working with co-ordinates.**

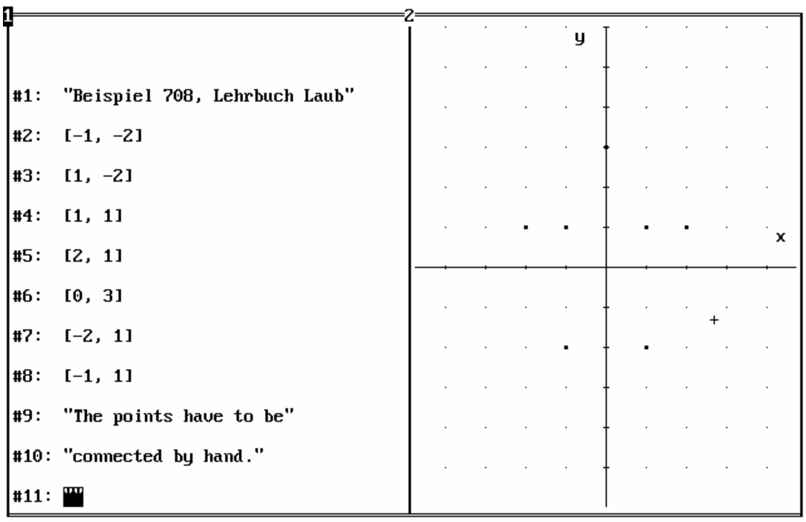
to 1) Solving equations was used to introduce the program and to discuss the most important commands (Author, Simplify, Solve and Remove) and to show some of DERIVE´s capabilities. We talked about the difference between working with DERIVE and working with a pocket calculator.

to 2) The pupils had to find the co-ordinates of points, Author them and plot the resulting figure. Between 1) and 2) there was an interval of more than a week.

I made the following experiences:

- The pupils like to go into the PC-lab.
- The soLve-command creates a very strong "Aha - effect".
- The biggest problems occured in editing the expressions. Pupils make many mistakes - they don´t find the characters on the keyboard and that causes different expressions and then different results.
- We need too much time for editing and then correcting the input.
- The pupils are forgetting the most things within one week. I had to start explaining the first steps from the beginning in the next lesson. If we don´t use the PC continuously then it would be better to work in 2 hours´ lessons.

The last three points seem to be the problem. Until now I cannot give an answer to the question "What is the PC in this age good for (in maths, of course)?", which is satisfying me. I shall report again in the future.



## 2 With DERIVE on the floor (age 12), Bernhard Wadsack, Vienna

### DERIVE and large prime numbers

In connection with the chapter prime numbers in teaching mathematics in a 2nd form the pupils mostly are very astonished that there is an infinite number of prime numbers. The fact that there exists no largest prime number and that you are able to proof this thesis is nearly incredible for them.

At the other hand for every moment in maths´ history exists a largest known - and published - prime number. A special part in hunting larger and larger prime numbers were playing the so called Mersenne prime numbers of the form $2^p - 1$ with $p \in \{2,3,5,13,17,19,31,61,89,107,127,521,607,1279, 2203,2281, ....\}$. Using computers additional prime numbers could be found among the Mersenne numbers. In 1986 it could be shown, that $2^p - 1$ with p = 216091 is a prime number. This number has 65050 digits.

Now I tried whether DERIVE 2.56XM is able to calculate this huge number. First I quickly reduced the decimal places to 2 to harass DERIVE not too much and then I was curious whether DERIVE would **Simplify** $2^{216091} - 1$. I waited and waited, and ........ after 43.5 sec it was done. DERIVE had mastered the calculation. Now it took some time until an endless line appeared on the screen. There it was at last, the - until 1986 - largest known prime number. 31 pages - thirty one in letters - full of digits, that were unfolded 9m of printer output. In that very moment I had the idea to expand this huge prime number in the classroom of my 2nd form.

The pupils made big eyes when I told them about this enormous prime number, but their astonishment became even greater, when I started to unfold the printout on the floor. To provoke a longer activity with this number I set them the task to look for chains of same digits or sequences of digits like 123.... All the pupils dispersed at both sides of this "number carpet" and doggedly searched for columns of same digits: soon we heard messages of success about groups of three digits the same, then sporadically about groups containing 4 digits and it lasted a while until one pupil had discovered five fours.

You easily can imagine that this lesson ended too soon for many pupils. Many of them had liked to search a longer time when I had to pick up "my number" in the break. The lesson had been very interesting, the pupils were enthusiastic. Whenever had it been, that they could glide on the floor during a maths lesson.

I am sure that they will not forget this lesson for a long while. In the next lessons many questions followed, e.g. whether they themselves would be allowed to work with this program which is able to handle with such large numbers. I will comply their wish with pleasure because I´ve intended to use DERIVE in maths teaching since long.

Comments: You can **Simplify** $2^{216091} - 1$ only with DERIVE XM. (I worked on a 386, 8 MB RAM). I couldn´t calculate $2^{500000}$ in **Exact Mode**.

-Sincerely Bernhard W.

*(Enclosed were 31 pages ≈ 9m ≈ 120g:* `746093 ................ 28447`*, Josef)*

(With DERIVE 6.10 $2^{216091}$-1 takes 0.64 sec and $2^{500000}$ takes 3.41 sec = 40 pages

*On the next two pages you can find two working sheets which I use to get the pupils accustomed with DERIVE and to develop some maths skills. On the first paper they hopefully achieve a feeling for the structure of the expression and the different levels of the subexpressions. In the second paper they can see the expressions on the screen - they see and they learn to see the mistakes they have made in setting wrong parentheses. When they later will work with their pocket calculator they will not see the expressions, they often fail in calculating similar numerical expressions. Additionally they have to make a difference between decimal places and significant places.*

## 3 Rediscovering structures with DERIVE (age 16), Josef Böhm, Würmla

**DERIVE/III/1.2**

$$\frac{a^2+2b^2}{(a+3b)^3}\cdot\frac{3a+4b}{5b+c^2}+\frac{(2a+3b^2)^3}{3a^3+(a-4b)^4}$$

Edit this terrible rational expression! Compare carefully the result with the expression on the paper.

Then highlight using the arrow keys the designated subexpressions. Switch off the **Lin**e Edit Mode in the message line (using the F6 - key). Start always with ←. ↓ will bring you one logic level deeper into the expression; with → and ← you can move within this level. Using ↑ you will climb up one level. (You can help yourself with ↑, ↑, ..... at any position.)

Note how many keystrokes you needed:

$$\frac{a^2+2b^2}{(a+3b)^3}\cdot\boxed{\frac{3a+4b}{5b+c^2}}+\frac{(2a+3b^2)^3}{3a^3+(a-4b)^4}$$

$$\frac{a^2+2b^2}{(a+3b)^3}\cdot\frac{3a+4b}{5b+c^2}+\frac{(2a+3b^2)^3}{\boxed{3a^3+(a-4b)^4}}$$

$$\frac{\boxed{a^2+2b^2}}{(a+3b)^3}\cdot\frac{3a+4b}{5b+c^2}+\frac{(2a+3b^2)^3}{3a^3+(a-4b)^4}$$

$$\frac{a^2+2b^2}{(a+3b)^3}\cdot\frac{3a+4b}{5b+c^2}+\frac{(2a+3b^2)^3}{\boxed{3a^3}+(a-4b)^4}$$

$$\frac{a^2+2b^2}{(\boxed{a+3b})^3}\cdot\frac{3a+4b}{5b+c^2}+\frac{(2a+3b^2)^3}{3a^3+(a-4b)^4}$$

$$\frac{\boxed{a^2}+2b^2}{(a+3b)^3}\cdot\frac{3a+4b}{5b+c^2}+\frac{(2a+3b^2)^3}{3a^3+(a-4b)^4}$$

$$\frac{a^2+2b^2}{(a+3b)^3}\cdot\frac{3a+4b}{5b+c^2}+\frac{(2a+\boxed{3b^2})^3}{3a^3+(a-4b)^4}$$

$$\frac{a^2+2b^2}{(a+3b)^3}\cdot\frac{3a+4b}{5b+c^2}+\frac{(2a+3b^{\boxed{2}})^3}{3a^3+(a-4b)^4}$$

$$\frac{a^2+2b^2}{(a+3b)^3}\cdot\frac{3a+4b}{5b+c^2}+\frac{(2a+3b^2)^3}{3a^3+(a-\boxed{4}b)^4}$$

**DERIVE/III/1.3**

Calculate the numerical expressions using your pocket calculator. Note the precision wanted: (n **d** = n decimal places; n **s** = n significant places). Check your results using DERIVE. You have to use the **approX** command. Pay attention to **Option Precision** and **Notation** and the **Digits.**

| Task: | Pocket Calculator | PC: |
|---|---|---|
| $18{,}45\ .(-678{,}17)\ .\ 0{,}0854$ | **(3 d)** | **(3 d)** |
| $\frac{345{,}3\ .\ 3{,}8927\ .\ 0{,}1525}{8{,}52\ .\ 2{,}593\ .\ 39\ .\ 45\ .\ 0{,}72}$ | **(5 d)** | **(5 d)** |
| $\frac{-29{,}573\ .\ 8{,}26\ .\ 10^4}{2857{,}3\ .\ 147{,}12}$ | **(4 s)** | **(4 d)** |
| $\left(\frac{14{,}21\ .\ 0{,}852}{3{,}71}\right)^3$ | **(2 d)** | **(5 s)** |
| $\sqrt{\left(\frac{1}{7{,}42\ .\ 2{,}483}\right)^5}$ | **(4 s)** | **(4 d)** |
| $\sqrt{6-\sqrt{7}}$ | **(2 d)** | **(3 s)** |
| $\frac{13{,}72^2+16{,}95^2-8{,}42^2}{5{,}32\ .\ 2{,}35}$ | **(3 s)** | **(1 d)** |
| $\sqrt{12-\sqrt[3]{12-\sqrt{11}}}$ | **(4 d)** | **(5 s)** |
| $6{,}23412\ .\left(\sqrt[6]{\frac{732{,}4}{0{,}8721}}-1{,}2146^3\right)^3$ | **(0 d)** | **(4 d)** |
| $28{,}1^2\ .\left(\sqrt[4]{0{,}75}+\frac{305{,}9}{417}\right)$ | **(4 s)** | **(4 d)** |
| $\frac{2{,}31+0{,}451}{0{,}76^2}:\frac{3{,}76^2-2{,}45^3}{10{,}564-14{,}1022}$ | **(6 d)** | **(6 s)** |
| $\frac{\frac{1002{,}17\ +\ 2431{,}6}{1{,}28\ .\ 0{,}46}}{\frac{308{,}96\ -\ 81{,}748}{\sqrt{128{,}48}}}$ | **(4 s)** | **(4 d)** |

Note: The n-th root $\sqrt[n]{expression}$ must be entered in DERIVE as $(expression)$^$(1/n)$.

## 4 Dynamic Systems (age 18), J. Lechner & H. Achleitner, Amstetten

*Josef Lechner and Helmut Achleitner are teachers at the gymnasium in Amstetten, Lower Austria. This is one of Austria´s "DERIVE show - schools". They both are working since a long time using DERIVE. Because this school is an "Informatics Gymnasium" they spend more time for mathematics tuition and they also can give more attention to the programming aspect. Supported by their teachers the students develop among others tools with DERIVE, which they later on can use as a "Black Box". See on this page one of these tools concerning dynamic systems.*
*Josef & Josef have sent several examples of examination papers. I tried to translate them. So, if there are any mistakes, then they are my fault, don´t blame Josef & Helmut. (But blame Josef_2, that´s me.)*
*I didn´t translate the toolbox for the Dynamic Systems, „Wachstumsprozesse Vernetzte Systeme" (Growth processes, Cross-linked (= dynamic) systems)*

# Wachstumsprozesse
# Vernetzte Systeme

### Wertetabelle für explizite Funktionen

- `F(x) :=` *Funktionsterm*
- `TAB(x0, sw, n) := VECTOR([k, F(k)], k, x0, x0 + (n – 1)sw, sw)`

### Wertetabelle für rekursive Funktionen

- `F(x) :=` *Rekursionsgleichung*
- `TAB_R(x0, n) := VECTOR([k, ITERATE(F(v), v, x0, k)], k, n)`

### Vernetzte Systeme (Phasendiagramm)

- `F(sys_par1, sys_par2) := [`*System 1, System 2*`]`
- ```
  TAB_PH(st_1, st_2, n) := ITERATES(F(ELEMENT(v, 1), ELEMENT(v, 2)),
                       v, [st_1, st_2], n)
  ```

### Wertetabelle für die Systemvariable 1

- ```
  TAB_PARSYS1(st_1, st_2, n) := [VECTOR(k, k, 0, n),
                        ELEMENT(TAB_PH(st_1, st_2, n)`, 1)]`
  ```

### Wertetabelle für die Systemvariable 2

- ```
  TAB_PARSYS2(st_1, st_2, n) := [VECTOR(k, k, 0, n),
                        ELEMENT(TAB_PH(st_1, st_2, n)`, 2)]`
  ```

### Darstellung im $x_n$ - $x_{n+1}$ - Diagramm

- ```
  TABS(x0, n) := ITERATES([ELEMENT(v, 2), F(ELEMENT(v, 1))], v,
                        [x0, x0], n)
  ```

**The examples:**

1. Cats and Mice

A region is populated by cats and mice.

a) The "Standard Model"

The number of mice is determined by its natural increase of 30% annually and its decrease caused by their enemies, the cats. This decrease is proportional to both the number of cats and mice with a factor of proportionality b = 0.005.

Because each of the cats claims a certain district only 150 cats can survive in the whole region. Hence the increase of the cats is proportional to the number of mice and to the existing growth potential (proportionality factor d = 0.0001). The cats´ natural annual decrease is 15%. When observation starts there are living 2000 mice and 40 cats in this region.

State the difference equations and investigate the long-time evolution of the mice´ and cats´ population. Will there appear any position of balance? If yes, then calculate it and explain the reason for the existence of equilibrium.

(Plot the graph and save as CAT1!)

b) "Immigration of Cats"

In the region observed there are a lot of cat lovers, so annually z animals are added. At the other hand it can be assumed, that the cats are loosing partially their natural instinct with the result, that the increase of the "Standard Model" is only proportional to the number of the mice and cats (and does not longer depend on the growth potential).

Form again the difference equations and investigate the behaviour of the system (take z = 3 and don´t change the other parameters). What can be observed? What is the effect of an increasing z?

(Save as CAT2.)

2. A recursive sequence

Show: The sequence $\langle x_n \rangle$ defined by the recursive equation (difference equation):

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right), \qquad a > 0$$

with $x_0$ as starting value converges to $\overline{x} = \sqrt{a}$.

Represent the sequence in a $x_n$ - $x_{n+1}$ - diagram. ($x_0 = 10$, $a = 2$).

(Plot and save as HERON1.)

Explain why this difference equation is not linear.

3. Life in the Soil

Life in the soil causes an annual increase of the concentration of nutritive substances of 32% (i.e. forming humus). Grain cultivation consumes these substances so without fertilizing the ground less grain is growing and the extraction of the nutrients - starting with a 400 units quantity - will show a decrease rate of 5% annually.

a) How will the quantity of nutrients $x_n$ develop within the next years, if there are now 1070 units of quantity in the ground? Could it be possible to exploit totally the soil, i.e. the nutrients would decrease to 0? (if so, then find the time.)

b) How many units of quantity could be consumed annually by cultivating grain that the quality of the soil could remain the same? Give reasons!

c) What is the type of the difference equation presented and calculate the first four elements of the sequence.

d) Annual increase of nutrients is not constant 32% but will reduce annually by 1% (31%, 30%, ....). Set up the corresponding difference equations.

(Save as GROUND1)

*The three examples from above are accomplished with a fourth one from probability theory and form a 2 h – written examination (100 min) in a 7th form (age 17). Here are two more examples, one of them dealing with financial mathematics:*

## 4. Fish and Fishermen

A population of 950 fish has an annual growth rate of 50%. The annual catch numbers are increasing by 10%, starting with 400 fish in the first year (then 440, 484, .......).

a) Describe the growing process using a difference equation.
($x_n$ is the number of fish at the end of year $n$; $x_0 = 950$).

b) How many fish will represent the population in the 3rd, 4th and 5th year?

c) Using DERIVE simulate the process for the first ten years.

d) When will the population die out? Reduce the catch rate, so that the fish will not die out.

## 5. Mr Akiar and his loan

Mr Akiar takes a loan of AS 250000 with an interest rate of 11% annually. Exactly one year later he starts repayment. His first payment is AS 25000. His trade union has promised an annual 8% increase of salary, so Mr Akiar has the intention to raise accordingly his repayments also by 8% annually.

a) Give a representation of the debts in respect to time (as a value table). Will the bank director accept Mr Akiar´s offer? Will Mr Akiar be able to finish repayment? Write down reasons for your answers!

b) Use DERIVE to represent the development of Mr Akiar´s debt.

c) Union´s negotiations are failing year for year and the promised rises of salary don´t become true. At how many percents has the annuity at least to be increased annually that the loan can be repaid? What is the “limit interest rate“, that repayment is yet possible (2 decimal places).

*At last I'd like to present an example from an end examination incl. solution. In Austria we call this end examination “Matura“, what is the “Abitur“ in Germany. When a student has passed the Matura, then he/she is “maturus“ (lat., = ripe) to attend university.*

## 6. Growth under self poisoning (Environment pollution)

A biotope has food for round 20000 individuals maximum. This population produces metabolic products, which are leading to a decrease of the population. The natural growth rate is proportional to the existing growth potential, the rate of decrease caused by the metabolites is proportional to the number of the present individuals and to the amount of the metabolites.

The increase rate of the metabolic products is proportional to the number of individuals. These products are decomposed by bacteria. The decrease rate is proportional to the existing amount of the products.

| | |
|---|---|
| $x_n$ | number of individuals at time $n$ ($x_0 = 5000$) |
| $s_n$ | amount of metabolites at time $n$ ($s_0 = 0$) |
| $z$ | increase rate of the individuals (proportionality factor of the growth potential) ($z = 0.3$) |
| $a$ | decrease rate caused by the influence of the toxic metabolites ($a = 0.001$) |
| $c$ | increase rate of the metabolites ($c = 0.01$) |
| $d$ | decrease rate of the metabolites caused by the bacteria ($d = 0.1$) |

Describe the system using difference equations. Investigate the long-time behaviour of the population's quantity and the amount of the metabolites. (Give the first 50 pairs of values!)

Give an interpretation of the result.

Is there any equilibrium possible? Make your conclusions from the long-time-simulation.

Express the fix points by the parameters.

Give a graphical representation using an appropriate tool.

Give a simulation for the special case, when d = 0 and interpret the outcome.

Comment: Save the simulation on a diskette. Produce a hardcopy of the simulation and the graph. Label the plot by hand.

### Solution:

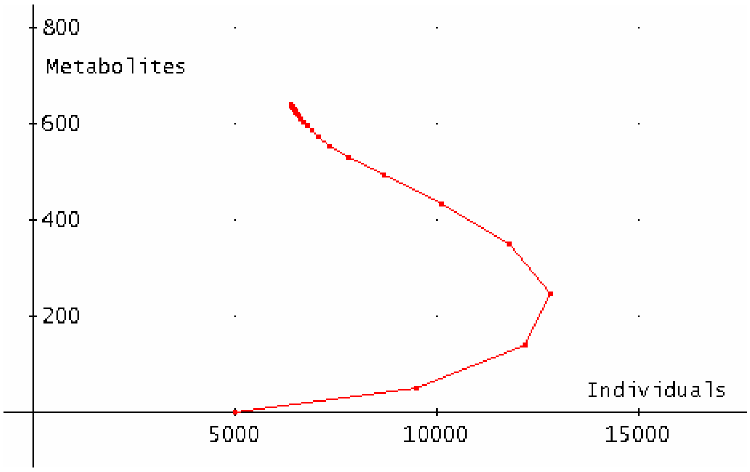
**The model**:

$$x_{n+1} = x_n + z\,(20000 - x_n) - a \cdot x_n \cdot s_n$$
$$s_{n+1} = s_n + c \cdot x_n - d \cdot s_n$$

```
#1:   LOAD(D:DYN_SYS.MTH)

#2:   F(x, y) := [x + z·(20000 - x) - a·x·y, y + c·x - d·y]

#3:   [z := 0.3, a := 0.001, c := 0.01, d := 0.1]

#4:   TAB_PH(5000, 0, 50)
```

#5:

| | |
|---|---|
| 5000 | 0 |
| 9500 | 50 |
| $1.2175\cdot10^4$ | 140 |
| ... | |
| ... | |
| .. | . |
| 6390.046754 | 638.9654646 |
| 6390.013534 | 638.9693857 |
| 6389.986451 | 638.9725824 |




#6: TAB_PARSYS1(5000, 0, 50)

#6:

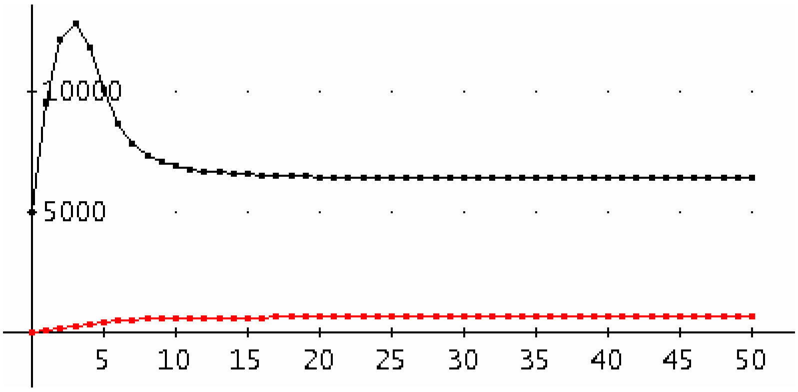
| | |
|---|---|
| 0 | 5000 |
| 1 | 9500 |
| 2 | $1.2175\cdot10^4$ |
| ... | |
| | ... |
| | ... |
| 48 | 6390.046754 |
| 49 | 6390.013534 |
| 50 | 6389.986451 |

#8: TAB_PARSYS2(5000, 0, 50)

#9: (TAB_PARSYS2(5000, 0, 50)) [48, ..., 51]

#10:

| | |
|---|---|
| 47 | 638.9606551 |
| 48 | 638.9654646 |
| 49 | 638.9693857 |
| 50 | 638.9725824 |




**Interpretation:** After a big increase of the population at the beginning (black), we can see the consequences of self poisoning and after a while appears a state of equilibrium.

**The values of equilibrium**: $x_n = x_n + z\,(20000 - x_n) - a \cdot x_n \cdot s_n$

$s_n = s_n + c \cdot x_n - d \cdot s_n$

Solve this system of equations for $x_n = x$ and $s_n = s$

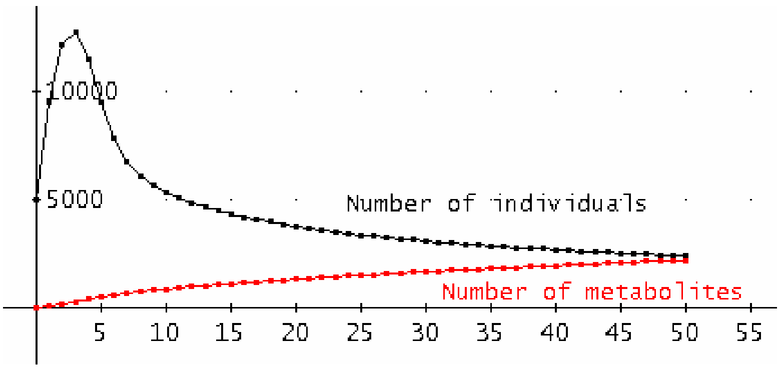
```
      SOLVE(x = (F(x, y))  ∧ y = (F(x, y)) , [x, y])
#11:                     1                2
```

```
#12:  (x = -9389.87 ∧ y = -938.99) ∨ (x = 6389.87 ∧ y = 638.99)
```

Equilibrium in x = 6340 and y = 639

Case d = 0

```
#13:  d := 0

#14:  TAB_PARSYS1(5000, 0, 50)

#15:  TAB_PARSYS2(5000, 0, 50)
```



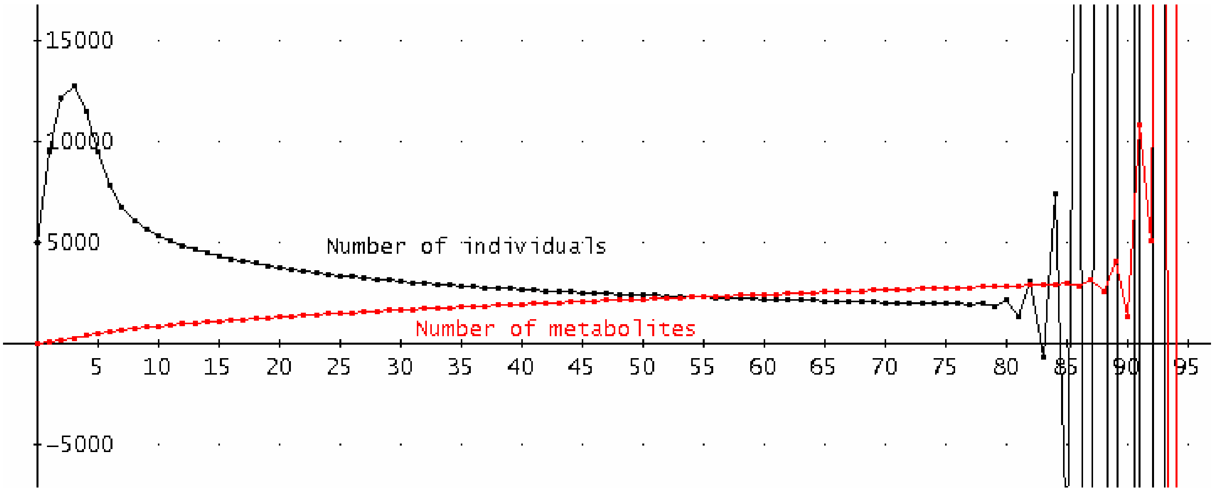

It seems to converge to an equilibrium!

```
      SOLVE(x = (F(x, y))  ∧ y = (F(x, y)) , [x, y])
#16:                     1                2

#17:  false
```

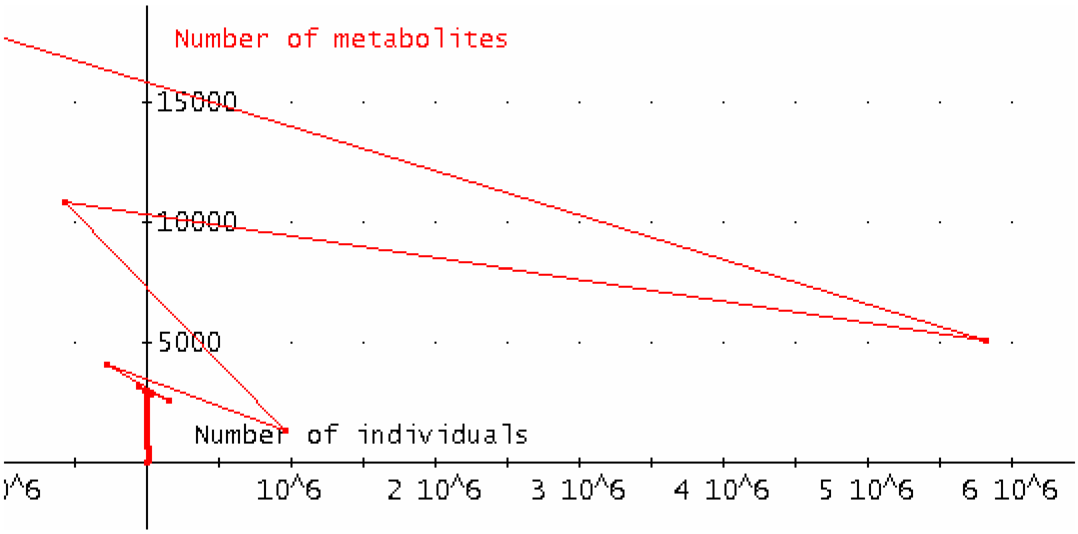
But there is no solution for the equilibrium. Let´s investigate the long time behaviour!

#18: TAB_PARSYS1(5000, 0, 100)

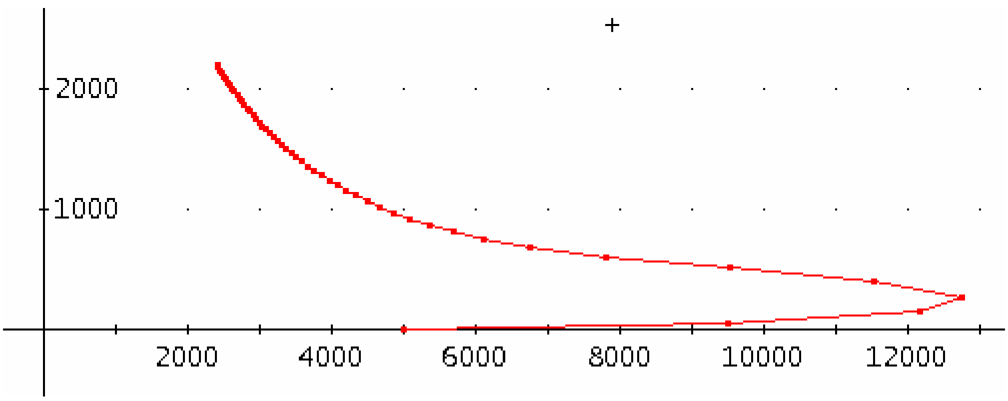
#19: TAB_PARSYS2(5000, 0, 100)



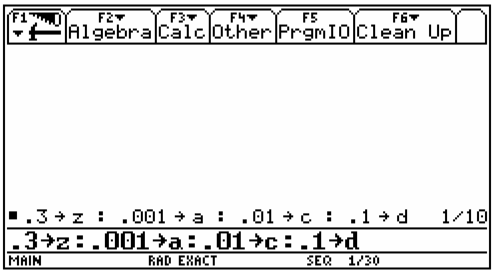

#22: TAB_PH(5000, 0, 50)

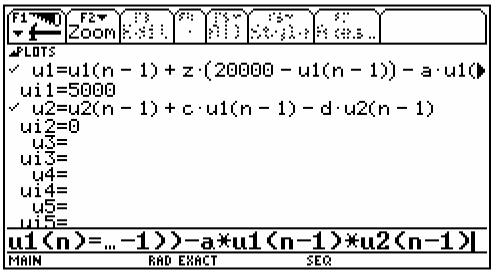


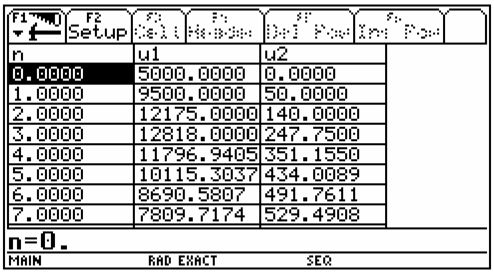

#22: TAB_PH(5000, 0, 50)

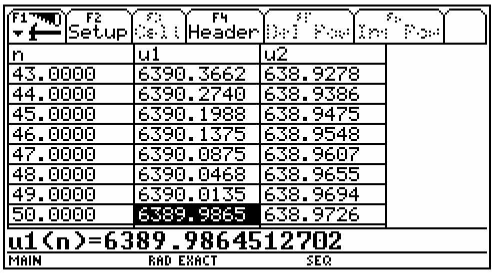


Treating the problem with the TI-92 / Voyage 200

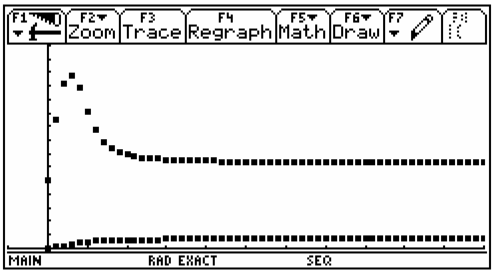


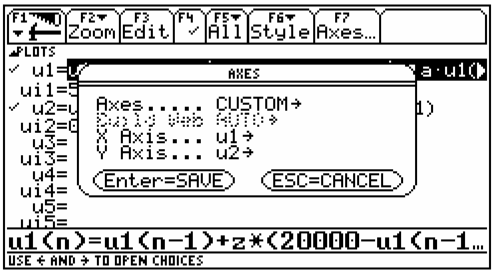



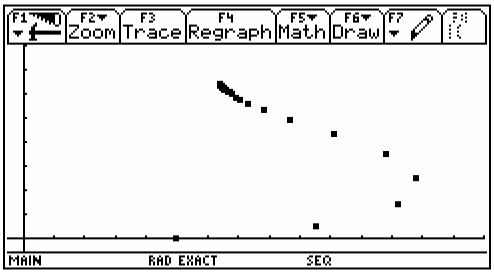



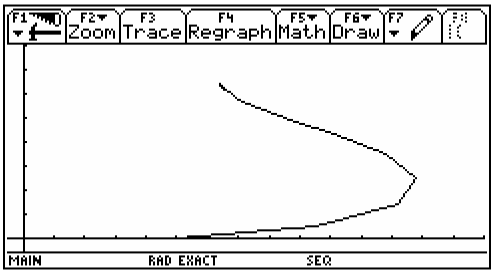

We define the two sequences in the Y=Editor (first of all turn to the Sequence Mode!). The table gives the numbers of the individuals for the following years. Compare with the DERIVE results.



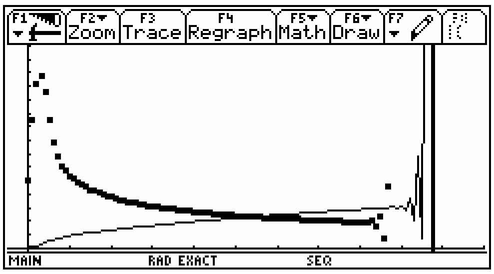



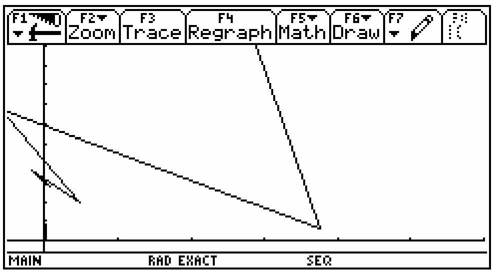

We can plot the two sequences presenting the number of individuals and metabolites.




The plot of the phase diagram can be achieved by customizing the axes – either in points or as a line.




This are the respective plots for the special case $d = 0$.

## 5 **Let´s work in pairs** (age 18), Hubert.Voigt & Karl Öhlinger, Perg

*Hubert and Karl are both teachers at a "Handelsakademie", the same type of school on which I am working as a teacher. So I am very interested how they are dealing with the phenomenon "Teaching maths with a PC". They have made the experiment to allow their pupils to work in pairs in their assessments. And they think, that it was successful. The marks didn´t change significantly, but the students reaction was enthusiastic. Hubert and Karl could observe a considerable learning process in developing teamwork and cooperation.*

*See one of their tests:*

3. Test. IVA, Group A, 20.4.1994

Pay attention: you have to write down the results on your paper (try to do it in words, like "The profit is .....“). Short comments between the calculations are welcome.

What you have to work through together:

1.a) $f(x) = a\,x^3 + b\,x^2 + c\,x$
The graph shows a maximum for $x = 0.5486$, a zero at $x = 1$ and a point of inflexion at $x = -1/3$.
Find the equation of the function. ($y = x^3 + x^2 - 2x$).

b) The graph from a) is intersected by the graph of $y = \sqrt{x+6} - 2$.
Calculate the area between the graphs.

2. A tank must hold 500 litres. Its length is 4m and its cross section is a rectangle with a semicircle at the bottom. A cover plate is not needed. Which dimensions minimize the consumption of material?

3. Given are a supply function $p_S = 0.25\,x^2 + 3$ and a demand function $p_D = 10 - e^{0.4x}$.
a) Give a sketch of both functions, according to the plot on the PC-screen.
b) What is the new position of the equilibrium if there is a 10% subsidy?
Are there any surplus goods? What is the reason for that fact?
c) Which tax rate is necessary to give a total tax yield of 5 monetary units?

This part has to be worked through separated (you may help each other).

4.a) Find the price elasticity for the demand function $p_D(x) = 10 - \ln(4x)$. (without using the PC)
Simplify the result.

b) What is the elasticity $\varepsilon_D$ for $x = 5$ quantity units. Interpret the result for a 10% increase in price. What are the new price and the new demand? Check $\varepsilon_D$ using the new data.

c) Explain the conception "elasticity". When can we speak of an "elastic" demand?

5. Find the antiderivative of $\dfrac{\sqrt[3]{x^4} - 4x^2 + 3\,\sqrt[5]{x^6}}{\sqrt{x}}$. Write down the result using root signs.

The partners were:

----------------------------------------

*I believe that example 3 could be interesting for those of you who are not accustomed with economic mathematical models. So I asked Hubert to send his solution:*

**Solution for 3.):**

The intersection point of the two curves is ($x \approx 3.46$, $p \approx 6.00$). That is the equilibrium point

**10% Subsidies:**

Subsidies might lower the supply-price, if they are transmitted to the consumers, so with

$p_S(x) = (0.25x^2 + 3) \cdot 0.9$ we obtain the new equilibrium:

$(0.25x^2 + 3) \cdot 0.9 = 10 - e^{0.4x} \quad \Rightarrow \quad x = 3.65$ QU and $p = 5.69$ MU

If subsidies would raise the producent´s income, then they lead to an increase of the quantity of supply.
The new price is 6.00 (from the equilibrium) · 1.10 = 6.60. We find the new quantity of supply:

$6.60 = 0.25x^2 + 3 \quad \Rightarrow \quad x = 3.80$ QU.

The sold quantity is 3.46 QU, so we obtain a difference of 0.34 QU. (Surplus of grain, milk, etc.)

**Total tax yield of 5MU:**

Taxes raise the supply price. $p$ is the tax rate

$p_S(x)(1 + p) = p_D(x) \quad \Rightarrow \quad p_S(x) \cdot p = p_D(x) - p_S(x)$ = the tax/QU (*)

Total tax = quantity · tax/QU $\Rightarrow$ $5 = x \cdot (p_D(x) - p_S(x))$. Solutions: $x_1 = 0.94$; ($x_2 = 2.89$).

Substituting for $x$ in (*) and solving for $p$ yields: $p_1 = 1.65$, i.e. a taxation rate of 165% and the second solution $p_2 = 0.34 = 34\%$

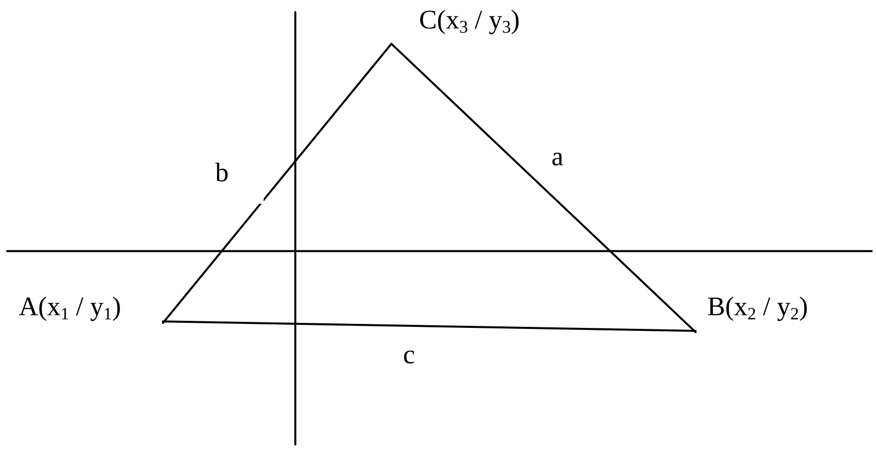
*Together with some other tests Hubert sent a paper as a result of an idea from DNL#13 (The circumscribed circle of a given triangle in CAS and Spreadsheets, D.Sjöstrand). Hubert prepared two teaching units - one for the circumscribed, the other for the inscribed circle. Here is the second one:*

## The inscribed circle of a triangle

The calculation of the incircle is more difficult. You will need some information.

*The centre of the incircle is the intersection point of the angle bisectors. You will obtain its radius if you draw the perpendicular line from the centre with respect to one side.*

Task 1: Draw the incircle into the given triangle:

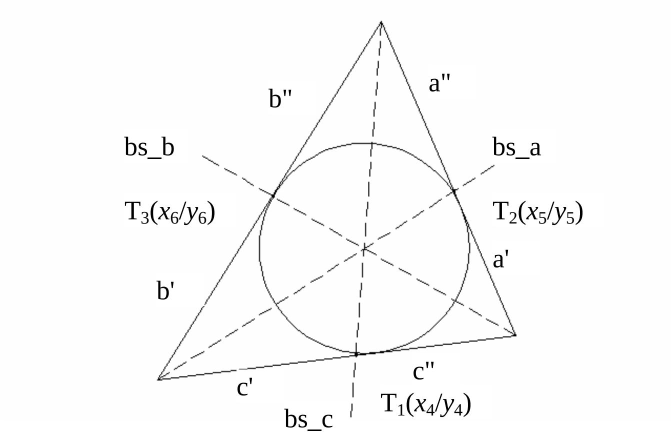


To find the equation of the angle bisectors you must know, that **each angle bisector divides the opposite side proportional to the adjacent sides.**

Task 2:

In the triangle above measure the sides a, b, c; and the sections a', a'', b', b'', c', c'' and calculate the ratios.
Example: c : b = a' : a''.

Sketch:




We obtain 6 equations for the co-ordinates of the 3 dividing points $T_{1,2,3}$. But we need only 4 of them:

$(x_4 - x_1) : (x_2 - x_4) = b : a$
$(x_5 - x_2) : (x_3 - x_5) = c : b$
$(x_6 - x_3) : (x_1 - x_6) = a : c$
$(y_4 - y_1) : (y_2 - y_4) = b : a$
$(y_5 - y_2) : (y_3 - y_5) = c : b$
$(y_6 - y_3) : (y_1 - y_6) = a : c$

The solutions of this system of equations are the co-ordinates of the dividing points $T_{1,}, T_2, T_3$.
Note that these points are **not** the osculating points!!

Task 3:

Given is the triangle A(–5/–3), B(5/–4),C(2/4). Draw this triangle on a separate sheet of paper. Calculate the point $T_3$ $(x_4/y_4)$ and add it to your sketch. Notice: the lengths of a, b and c are the distances of two points.

Calculation of the other points is not a big problem, but now we will use DERIVE. For this purpose load a so called **Utility file** by **File** > **Load** > **Utility File**: **`incircle.mth`.** You cannot see the file and the formulae.
You have to **Author** the co-ordinates of the vertices: [x1:= - 5, y1:= - 3, x2 := 5, ...... etc ].

The following functions are provided for you.

| | | | |
|---|---|---|---|
| `a` | side a | `bs_a` | bisector of $\alpha$ |
| `b` | side b | `bs_b` | bisector of $\beta$ |
| `c` | side c | `bs_c` | bisector of $\gamma$ |
| `triangle` | the triangle | `radius` | the radius of the incircle |
| `d_points` | the div. points | `centre` | the centre of the incircle |
| | | `circle` | the incircle |

**Author** the name of the function needed, then **Simplify** or **Approximate**.

Task 4:

We want to calculate the angle bisectors - remember the equation of a line which passes through two given points (2nd form).

The line through $P(x_1/y_1)$, $Q(x_2/y_2)$: $y = \frac{y2 - y1}{x2 - x1}(x - x1) + y1$

Find the equation of one angle bisector using a vertex and the opposite dividing point:

Then let DERIVE calculate all the angle bisectors and check your result(s). (Use **Expand**)

Task 5:

Take any two of the bisectors and calculate the intersection point. Compare your result with DERIVE´s answer on the provided function **CENTRE**!

At last plot the triangle, the dividing points, the bisectors, the centre and the incircle. For plotting the triangle set **Option** > **Display** > **Points** > **Connected** > **Yes**. Then set this option back to **Discrete**.

If you like you can see the "DERIVE - program", with **Transfer Load Derive: incircle**.

#1: incircle.mth

#2: InputMode := Word

#3: [x1 := -3, y1 := -2, x2 := 5, y2 := -1, x3 := 2, y3 := 6]

#4: $\text{triangle} := \begin{bmatrix} x1 & y1 \\ x2 & y2 \\ x3 & y3 \\ x1 & y1 \end{bmatrix}$

#5: $a := \sqrt{(x2 - x3)^2 + (y2 - y3)^2}$

#6: $b := \sqrt{(x1 - x3)^2 + (y1 - y3)^2}$

#7: $c := \sqrt{(x1 - x2)^2 + (y1 - y2)^2}$

#8:

#9: Radius:

#10: $s := \dfrac{a + b + c}{2}$

#11: $\text{radius} := \dfrac{\sqrt{s\cdot(s - a)\cdot(s - b)\cdot(s - c)}}{s}$

#12: $x4 := \text{IF}\left(x1 = x2,\ x1,\ \left(\text{SOLUTIONS}\left(\dfrac{x4 - x1}{x2 - x4} = \dfrac{b}{a},\ x4\right)\right)_1\right)$

#13: $y4 := \text{IF}\left(y1 = y2,\ y1,\ \left(\text{SOLUTIONS}\left(\dfrac{y4 - y1}{y2 - y4} = \dfrac{b}{a},\ y4\right)\right)_1\right)$

#14: $x5 := \text{IF}\left(x2 = x3,\ x2,\ \left(\text{SOLUTIONS}\left(\dfrac{x5 - x2}{x3 - x5} = \dfrac{c}{b},\ x5\right)\right)_1\right)$

#15: $y5 := \text{IF}\left(y2 = y3,\ y2,\ \left(\text{SOLUTIONS}\left(\frac{y5 - y2}{y3 - y5} = \frac{c}{b},\ y5\right)\right)_1\right)$

#16: $x6 := \text{IF}\left(x1 = x3,\ x1,\ \left(\text{SOLUTIONS}\left(\frac{x6 - x3}{x1 - x6} = \frac{a}{c},\ x6\right)\right)_1\right)$

#17: $y6 := \text{IF}\left(y1 = y3,\ y1,\ \left(\text{SOLUTIONS}\left(\frac{y6 - y3}{y1 - y6} = \frac{a}{c},\ y6\right)\right)_1\right)$

#18: $d\_points := \begin{bmatrix} x4 & y4 \\ x5 & y5 \\ x6 & y6 \end{bmatrix}$

#19: $bs\_a := \text{IF}\left(x1 = x5,\ x = x1,\ \frac{y5 - y1}{x5 - x1}\cdot(x - x1) + y1\right)$

#20: $bs\_b := \text{IF}\left(x2 = x6,\ x = x2,\ \frac{y6 - y2}{x6 - x2}\cdot(x - x2) + y2\right)$

#21: $bs\_c := \text{IF}\left(x4 = x3,\ x = x3,\ \frac{y4 - y3}{x4 - x3}\cdot(x - x3) + y3\right)$

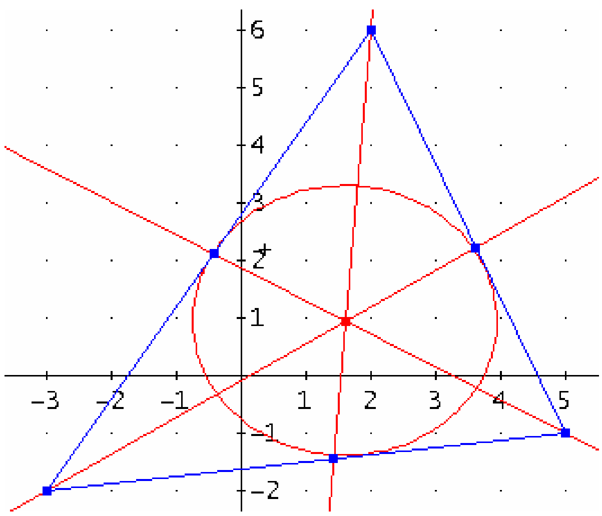
#22: $m := (\text{SOLUTIONS}(bs\_a = bs\_b,\ x))_1$

#23: $n := \lim_{x \to m} bs\_a$

#24: $centre := [m,\ n]$

#25: $circle := (x - m)^2 + (y - n)^2 = radius^2$

*Note: Vertical bisectors can be plotted easily in DERIVE 3.x because of its implicit plotting capability. In earlier versions you can help yourself by creating a special* plot_bisector *function:* pbs_a:=[[x1,y1],[x5,y5]], *etc. In this file you find the circle given in implicit form, too.*



# Titbits from Algebra and Number Theory

**by Johann Wiesenbauer, Vienna**

As this is the first article of my series after the arrival of DERIVE 3, I am going to try out some of the new features of it. My first impression is that the performance of DERIVE has been enhanced greatly by a lot of very useful functions, even though the first official versions of DERIVE 3 contained some tantalising bugs (as a matter of fact, more serious ones than the Beta version 3.0y). In particular, some of the following routines will only work properly, if you have version 3.04 of DERIVE or later.

Speaking of bugs, the DERIVE-routine for the μ-function in DNL #15 yields the wrong value 0 for n=1 and hence should be changed to

```
μ(n) ≔ IF(n = 1, 1, IF(DIM(FACTORS(n)) = ∑((FACTORS(n))↓↓2),
       (-1)^DIM(FACTORS(n)), 0))
```

The reason for this is a tiny inconsistence in the current DERIVE: factors(factor(1)) returns [1] instead of the proper value [ ].
The following simple function derived from μ is also sometimes useful:

```
SQUAREFREE(n) ≔ IF(μ(n)^2 = 1, true, 0)
```

Let us turn to more demanding tasks now, e.g. what about the DERIVE-routine for the number δ(n) of divisors of n?" Try it on your own first!

```
δ(n) ≔ DIM(SORT(VECTOR(∏(u_), u_, {[1]}·∏(VECTOR(MAP_LIST([v_↓1^k_], k_,
       {0, ..., v_↓2}), v_, FACTORS(n))))))

VECTOR(δ(i),i,20) = [1, 2, 2, 3, 2, 4, 2, 4, 3, 4, 2, 6, 2, 4, 4, 5, 2, 6, 2, 6]
```

One of the most important functions of number theory is certainly Euler's φ-function which can be programmed as follows:

```
ϕ(n) ≔ n·∏(1 - 1/v_↓1, v_, FACTORS(n))

VECTOR(ϕ(i), i, 20)

[1, 1, 2, 2, 4, 2, 6, 4, 6, 4, 10, 4, 12, 6, 8, 8, 16, 6, 18, 8]
```

(Note that because of the same DERIVE-bug as above the case n=1 needs an extra definition.)

Among others, the φ-function can be used to calculate the inverse of a mod m, if m is of moderate size:

```
INV(a, m) ≔ IF(GCD(a, m) = 1, MOD(a^(ϕ(m) - 1), m))

INV(2, 1234567891234567891234567) = 617283945617283945617284

MOD(2·617283945617283945617284, 1234567891234567891234567) = 1
```

Of course, the inv-function based on Euclid's algorithm is far superior to the preceding one (0.1 sec versus 6.3 sec for the example above!) :

```
INV_(a, m) ≔ IF(GCD(a, m) = 1, MOD((ITERATE(IF(MOD(a_, b_) = 0,
             [a_, b_, c_, d_], [b_, MOD(a_, b_), d_, c_ - FLOOR(a_, b_)·d_]),
             [a_, b_, c_, d_], [a, m, 1, 0]))↓4, m))
```

Even so, in view of the huge power that had to be calculated, the first implementation worked surprisingly fast. The reason for this: My prayers as to the DERIVE-implementation of the powermod-function (cf. #14, p.29) have been heard at last! This must have happened very recently , judging by two facts: Firstly that ridiculous powermod-function in the DERIVE-library is still there, and secondly, what is more, they forgot to change the mods-function, too! But don´t look a gift horse in the mouth, as the saying goes, I am sure that this will be remedied before long. In the meantime take this version of mods:

```
MYMODS(a, n, m) ≔ MODS(MOD(a^n, m), m)
```

Here is still another function, that makes use of the improved version of mod:

```
LEGENDRE(a, p) ≔ IF(¬ PRIME(p), "Input error!", MODS(MOD(a^((p − 1)/2), p), p))
```

Of course, `Legendre(a,p)` is the well known Legendre symbol (a/p) from the theory of quadratic residues, which is defined for all integers a and primes p. There is an important generalisation called after Jacobi which cannot so easily be implemented. Here is my solution:

```
JACOBI(a, b) ≔ IF(MOD(b, 2) = 0 ∨ b ≤ 0, "Input error!", IF(GCD(a, b) > 1, 0,
               IF(a = 1, 1, IF(ABS(a) ≥ b, JACOBI(MODS(a, b), b),
               IF(a < 0, (−1)^((b − 1)/2)·JACOBI(−a, b),
               IF(MOD(a, 2) = 0, (−1)^((b^2 − 1)/8)·JACOBI(a/2, b),
               IF(MOD(a, 4) = 1 ∨ MOD(b, 4) = 1, JACOBI(b, a),
                − JACOBI(b, a))))))))
```

As to the performance of these two functions again an example:

```
LEGENDRE(10^60 + 1, 10^68 + 99) = 1          (0.015 sec)
JACOBI(10^60 + 1, 10^68 + 99)= 1             (0.000 sec)
```

Both results show that the congruence $x^2 \equiv 10^{60}+1 \quad mod\ (10^{68}+99)$ is solvable. One of its solutions, which differ by the sign only, is given by

```
c:=MOD((10^60+1)^((10^68+100)/4),10^68+99)
```

as can be seen by

```
MOD(c^2,10^68+99)=
1000000000000000000000000000000000000000000000000000000000001
```

The general proof for this is so simple that I cannot resist the temptation to insert it here: If $a \equiv b^2 \bmod p$ for integers *a*,*b* and a prime *p*, which is of the form 4*k*+3 and does not divide *a*, then *a*^((*p*+1)/4) (in DERIVE-notation) is always a solution of square root of *a* because of

$$\left(a^{\frac{p+1}{4}}\right)^2 = b^{p+1} = b^2 \cdot b^{p-1} \equiv a \ mod\ p$$

where we made only use of Fermat´s Theorem. The case *p* = 4*k*+1 is far more difficult and maybe I will talk about it another time.
As a final example for an application of the new (power)modfunction in DERIVE, I am going to deal with Pépin´s primality test for Fermat numbers $F_m = 2^{2^m}+1\ (m \geq 0)$ which states that $F_m$ is prime for *m*>0 iff (if and only if!)

$$3^{(F_m-1)/2} \equiv -1 \ mod\ F_m .$$

And here the DERIVE-implementation of it:

```
MYMODS(a,n,m):=MODS(MOD(a^n,m),m)

PEPINTEST(m):=IF(m=0,true,IF(MYMODS(3,2^(2^m−1),2^(2^m)+1) = −1, true,false))

VECTOR(PEPINTEST(i),i,0,10)=[true,true,true,true,true,false,false,false,false,
                             false,false]
```

The calculation above for the first 11 Fermat numbers took only 2.8 (!) seconds, even though the last of the tested Fermat numbers has already 309 digits! Let me conclude with this most impressive example for the performance of the new DERIVE. If you have any suggestions as regards possible improvements of the routines in this article or interesting topics I would be very pleased to hear from you! (jwiesenb@email.tuwien.ac.at)

**G P Speck, Wanganui, New Zealand**

I am using DERIVE Version 2.6 an a IBM 386 compatible. Having just recently joined the DUG, I am not conversant with the material in the first 14 Newsletters. However, I offer the following two Notes which I feel may be of interest to the User Group if the comments in these notes have not already been covered in previous Newsletters.

Note 1: A fault in the DERIVE Precision Option?

When using Options Precision Approximate Digits: 5 I find that on returning to a view of the Digits entry it has changed from 5 to 6. The same is true if I enter Digits: 11; it changes from 11 to 12; similarly, if
Digits: 17 is entered the 17 changes to 1; in general, if 5 + 6n is entered for the Digits, then the 5n + 6 changes to 6 + 6n. Either 5 + 6n or 6 + 6n significant figures may in fact be displayed for each of the entries 5 + 6n and 6 + 6n depending on the intermediate entries and the order of the entries 5 + 6n and 6 + 6n. If this is indeed a fault, it is a VERY SERIOUS FAULT! Have any other readers encountered this difficulty?

***DNL:*** *This fault does not occur in versions 3.x any longer. But thanks for your useful comment. I am sure that there are many of our members who are still using earlier DERIVE versions. And many thanks for your contribution: Müller´s Method. Your second note might be a nice addendum to a DNL#16´s article.*

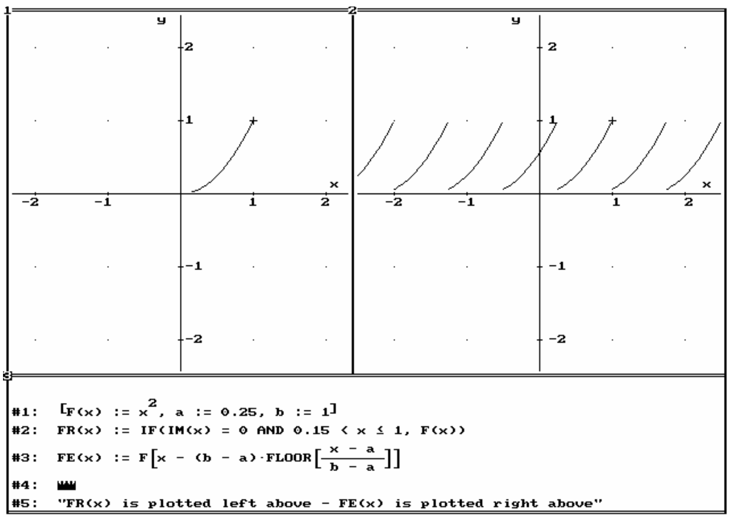
Note 2: A note on plotting Periodic Extensions

Students are usually intrigued by the following elementary result when it is first encountered.

If function F is defined for all real x then the periodic extension on the whole real line of F restricted to the half-open interval [a,b) is given by

```
F(x - (b-a) FLOOR((x - a) / (b - a))).
```

A DERIVE example of this result is shown below.



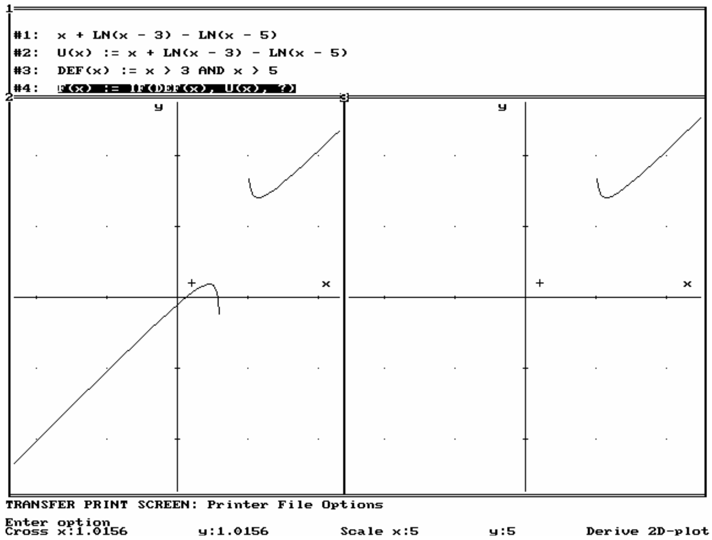
**H.Voigt, Perg, Austria**

1. The cubic $30x^4 - 91x^3 + 15x^2 - 91x + 30$ could be solved using DERIVE 2.60 in both modes Exact and Mixed as well. But now with DERIVE 3.x it does not work in Mixed Mode. The system will hang.

2. Students tried to plot y(x) = x + ln(x–3) – ln(x–5), and they could see a plot in a region, where the function should not exist, because it is defined for real x > 5. Do you have an advice?

***DNL:*** *Yes. I have. Obviously DERIVE simplifies the logarithms internally to* $ln\frac{x-3}{x-5}$ *So it is very informative for the students to define the domain - as they have to do, when sketching the graph by hand.*

```
U(x):=x+LN(x-3)-LN(x-5)
DEF(x):=x>3 AND x>5
F(x):=IF(DEF(x),U(x),?)
F(x)
```




**Dr.H.J.Kayser, Düsseldorf, Germany**

## Eine Alternative zu INVNORM.MTH aus DNL#15

\- - - - - - - Function for the inverse normal distribution - - - - - - -

\- - - - - - - INVNORM in DNL#15 - - - - - - - -

$$\#1:\ \text{INVNORM}(p,\mu,\sigma) := \text{ITERATE}\left( a - \frac{\dfrac{\text{SIGN}(\sigma)\cdot\text{ERF}\left(\dfrac{\sqrt{2}\cdot a}{2\cdot\sigma} - \dfrac{\sqrt{2}\cdot\mu}{2\cdot\sigma}\right)}{2} + \dfrac{1}{2} - p}{\dfrac{\sqrt{2}\cdot e^{-a^2/(2\cdot\sigma^2) + a\cdot\mu/\sigma^2 - \mu^2/(2\cdot\sigma^2)}}{2\cdot\sqrt{\pi}\cdot|\sigma|}},\ a,\ \mu,\ 5 \right)$$

\- - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - -
\- - - - - - - NORMAL-INV, the Challenger - - - - - - -

\- - - - - - - Definition of the function Φ-inverse (ΦINV) - - - - - - -
\- - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - -

```
#2:   [a0 := 2.51551, a1 := 0.802853, a2 := 0.010328, b1 := 1.43278, b2 := 0.189269,
      b3 := 0.001308]

#3:   W(y) := √(- 2·LN(y))
```

$$\#4:\quad V(y) := \frac{(a2\cdot W(y) + a1)\cdot W(y) + a0}{((b3\cdot W(y) + b2)\cdot W(y) + b1)\cdot W(y) + 1} - W(y)$$

```
      ΦINV(y) :=
        If y > 0 ∧ y < 1
#5:        If y > 0 ∧ y ≤ 0.5
              V(y)
              - V(1 - y)
```

\- - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - -
\- - - - - - - error-estimating: |δ| ≤ 4.5·10^-4

\- - - - - - - Definition of function NORMAL_INV
\- - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - -

```
#6:   NORMAL_INV(y, μ, σ) := μ + σ·ΦINV(y)
```

\- - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - -
\- - - - - - - Test 1 (accuracy) - - - - - - -
\- - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - -

```
#7:   VECTOR([y, INVNORM(NORMAL(y, 105, 7.5), 105, 7.5), NORMAL_INV(NORMAL(y,

        105, 7.5), 105, 7.5)], y, 120, 127.5, 1.5)
```

#8:

| | | |
|---|---|---|
| 120 | 119.999 | 120.003 |
| 121.5 | 121.495 | 121.503 |
| 123 | 122.964 | 123.003 |
| 124.5 | 124.329 | 124.503 |
| 126 | 125.472 | 126.002 |
| 127.5 | 126.309 | 127.502 |

\- - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - -
\- - - - - - - Test 2 (velocity)
\- - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - -

```
#9:   VECTOR([y, INVNORM(NORMAL(y, 105, 7.5), 105, 7.5)], y, 120, 127.5, 1.5)
```

#10:

| | |
|---|---|
| 120 | 119.999 |
| 121.5 | 121.495 |
| 123 | 122.964 |
| 124.5 | 124.329 |
| 126 | 125.472 |
| 127.5 | 126.309 |

0.031 seconds (2.3 seconds 1995)

```
#11:  VECTOR([y, NORMAL_INV(NORMAL(y, 105, 7.5), 105, 7.5)], y, 120, 127.5,
        1.5)
```

#12:

| | |
|---|---|
| 120 | 120.003 |
| 121.5 | 121.503 |
| 123 | 123.003 |
| 124.5 | 124.503 |
| 126 | 126.002 |
| 127.5 | 127.502 |

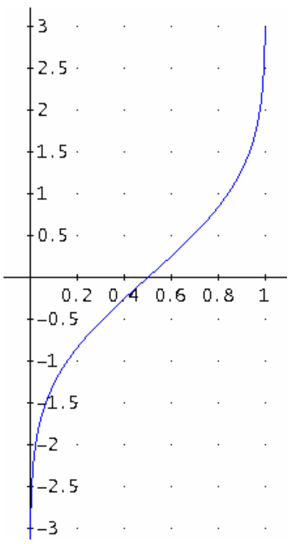
0.000 seconds (0.7 seconds 1995)

```
#13:  INVNORM(x, 0, 1)
```

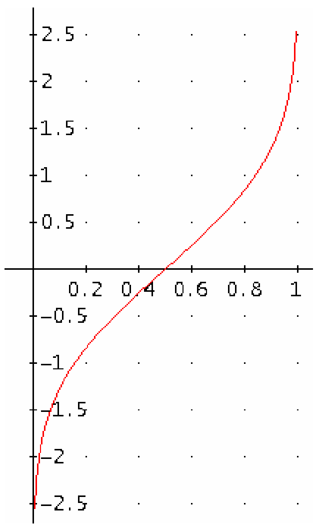
Takes about 30 seconds

```
#14:  NORMAL_INV(x, 0, 1)
```

Plots in an instant.



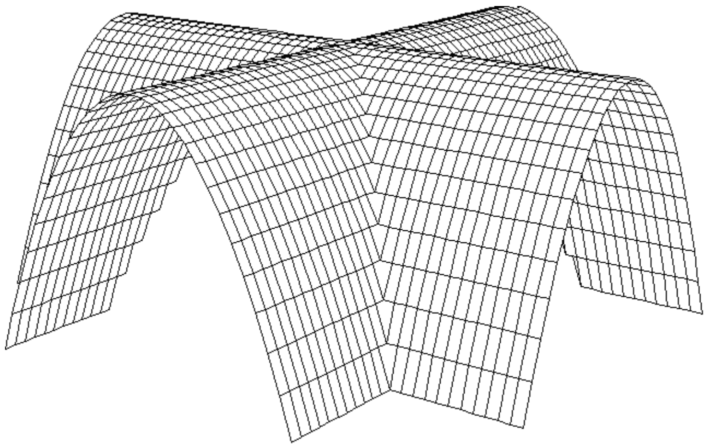

Plot of #13



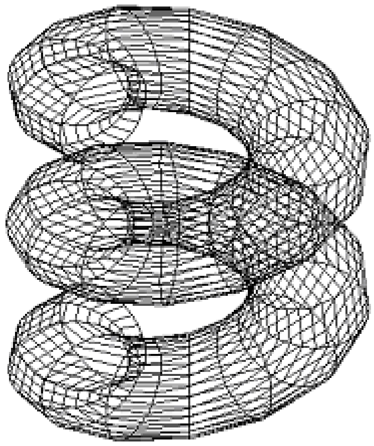

Plot of #14

Final Comment:
Accuracy of function INVNORM could be increased by increasing the number of iteration steps but time of computation - in any case higher than applying NORMAL_INV - would be increased simultaneously.

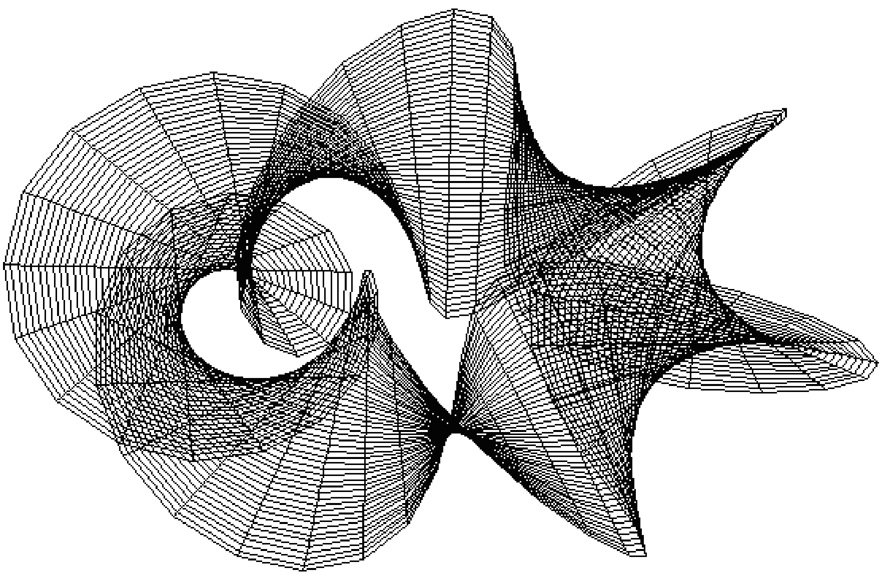
# Three Pictures from my 3D-Gallery



Intersection of two parabolic cylinders



Heliconical tube



Helix of a Rotoide

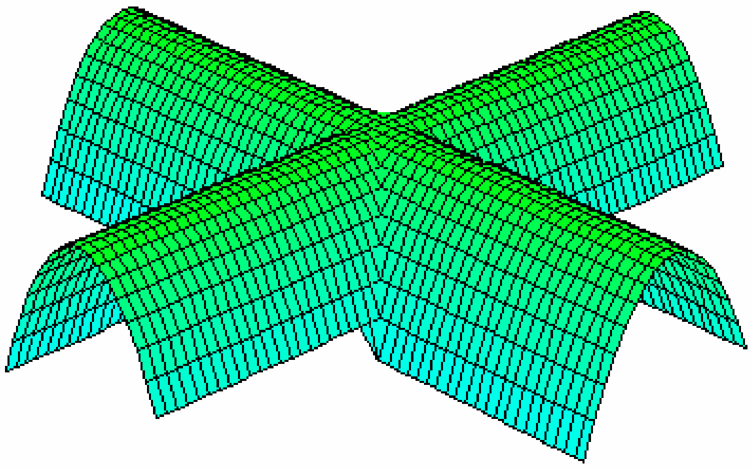
This is the DERIVE session:

```
#1:   Precision := Approximate

#2:   Notation := Decimal

#3:   NotationDigits := 3

#4:   The two parabolic cylinders

#5:   length: 5,5,2; grids: 50,50

#6:   IF(|x| < 1.5 ∨ |y| < 1.5, IF(|y| ≥ |x| ∧ |x| < 1.5, - x·x,
               IF(|x| > |y| ∧ |y| < 1.5, - y·y, 0)))

#7:   A heliconical tube

#8:   r := 2 + COS(p)

#9:   HE(p, f) := [r·COS(f), r·SIN(f), SIN(p) + 0.8·f]

#10:  ISOMETRICS(HE(p, f), f, π/2, 6·π, 30, p, 0, 2·π, 30)

#11:  COPROJECTION(ISOMETRICS(HE(p, f), f, π/2, 6·π, 30, p, 0, 2·π, 30))

#12:  A helix of a rotoide – Rotoidenwendelfläche

#13:  r_ := 3 - p·COS(3·f)

#14:  RW(p, f) := [r_·COS(f), r_·SIN(f), p·SIN(3·f)]

#15:  ISOMETRICS(RW(p, f), f, 0, 2·π, 60, p, -2, 2, 50)

#16:  COPROJECTION(ISOMETRICS(RW(p, f), f, 0, 2·π, 60, p, -2, 2, 50))
```

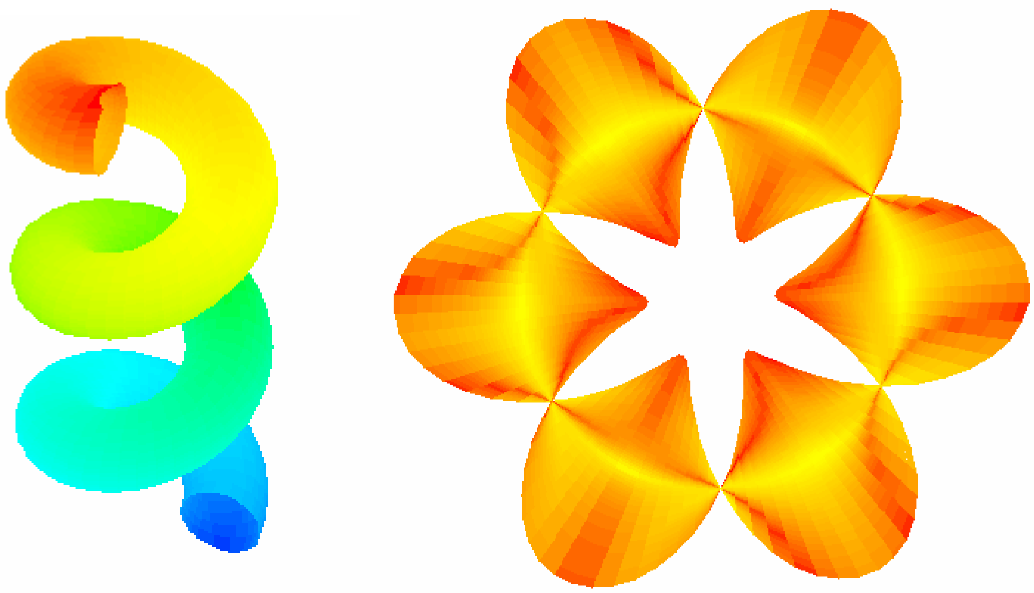
`ISOMETRICS` gives one family of parameter curves – plot with small connected points – and `COPROJECTION` gives the other family.

The isometric projections of the surfaces in the 2D-Plot window had and still have their specific charm – and investigating the various mappings have their mathematical value.

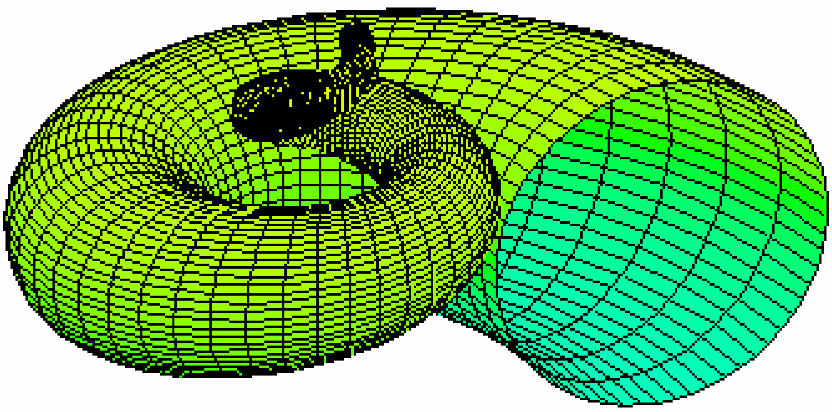
At the other hand recent DERIVE version offers 3D plots of functions $z = z(x,y)$ and of surfaces given in parameter form – like the surfaces from above.



The two cylinders



The tube and the helical rotoide (top view)



This is the snail shell from page 30 as a 3D-plot