THE DERIVE - NEWSLETTER #28

ISSN 1990-7079

THE BULLETIN OF THE

DERIVE®

USER GROUP

+ TI 92

Contents:

revised version 2012 **December 1997**

[1] **Lineare Algebra und Geometrie mit *DERIVE*,** Hans-Jürgen Kayser, 160 Seiten, Dümmlerbuch 4526, Dümmler, Bonn,1997, ISBN 3-427-45261-1
Ein Lehr- und Arbeitsbuch, das die Themen 2D- und 3D-Grafik (basierend auf der Matrizenrechung), lineare Gleichungssysteme, 3D-Geometrie (Vektorrechnung) und Vektorräume behandelt. Abgeschlossen wird das gelungene Buch mit einigen Anwendungen und Projekten. Diskette ist erhältlich.

[2] **Tolle TI-92 Programme - Bd 1**, B. Kutzler & D.R.Stoutemyer, bk-teachware, ISBN 3-901769-00-5
Deutsche Fassung der im vorigen DNL vorgestellten *Great TI-Programs, Vol 1*.

[3] **Lineare Gleichungen lösen mit dem *TI*-92**, B. Kutzler, bk-teachware, ISBN 3-901769-04-8

[4] **Solving Linear Equations with the *TI*-92**, B. Kutzler, bk-teachware, ISBN-3-901769-03-X

[5] **Explorations, Engineering Math on the *TI*-92: Electronic and Electrical Applications**, Adams & Hergert, Texas Instruments, ISBN 1-886-309-10-8

[6] **Analysis mit dem TI-92**, Berichte über Mathematik und Unterricht, Hsg.: U. Kirchgraber, ETH-Zürich, Mai 1997
Ein ausgezeichnetes Skriptum zur Evaluation eines schweizerischen Schulversuchs, dessen Schwerpunkte Anwendungsbezogenheit und die Modellbildung sind.

[7], [8] **Proceedings of the 8th and 9th ICTCM**, Addison-Wesley-Longman, ISBN 0-201-69558-8 and ISBN 0-201-34312-6

Interesting WEB sites:

http://math.la.asu.edu/~kawski	Visit Matthias Kawski's homepage.
http://www.upv.es/derive/	There you can find all the news about DERIVE and the TI-92 (in Spanish) that the DELTA-Group receives. (A DERIVE and TI-User Group which communicates in Spanish)
http://www.rhombus.de	Have a look at Jan Vermeylen´s web site

These websites are valid in 2012,Josef

Don´t forget to renew your DUG Membership = DNL Subscription for 1998. We are looking forward to meeting you again in 1998.

Vergessen Sie bitte nicht, Ihre DUG-Mitgliedschaft = DNL Abonnement für 1998 zu erneuern. Wir freuen uns darauf, Sie 1998 wieder zu treffen.

Dear DERIVE and TI-friends,

This DNL is a bit special. All its articles are - with one exception: the Titbits - talks held at the DERIVE and TI-92 Symposium "Fun in Teaching Mathematics" in Saeroe, Sweden, this summer. I have to thank all lecturers who submitted their presentation in written form. Additional explications to Sergey´s Implicit Plots and to the various projections used in the Window of VIVIANI will follow next year. But I have the pleasure to announce that there are some contributions left for the 1998 issues of the DNL which reached me after the deadline, among them our friend Terence Etchells´ exciting "high end" talk "To Booleanly go where no Math has gone before!". Johann Wiesenbauer announced his Titbits (12) as the "best Titbits ever". Make up your own mind. His search for the coefficient of x^{3000} *from Problem 4 (DNL#27, page 25) is really an adventure - although I must admit that this is somewhere beyond my knowledge.*

We will continue the running themes (Hidden Lines, Linear Programming, a lot of ACDCs,.....) in the next DNLs. I can promise Bert Waits´ presentation of the new TI-92 Plus module and a more than complete program package to deal with dynamic systems from Josef Lechner. As you can see on page 2 our members are as productive as ever. Each new DNL shows new articles and new authors. I have the intention letting DERIVE and the TI-92 converge together more and more. In my experience, and I am sure that many of you will agree, most of my former DERIVE projects can be transferred onto that little miracle, the TI, and on the other hand, projects carried out on the TI make people aware how to realize projects on the PC using DERIVE. It is obvious that there are some restrictions, but in most cases the important issue is the inspiring idea.

A Merry Christmas and a Happy New Year 1998 from your DUG-team Noor and Josef

Another highlight of fall was the 2nd US DUG Meeting in Chicago. It was a pleasure and an honour for me to meet a lot of American friends. I have to thank for your attentiveness and your valuable ideas for further development of the DUG. My special thanks to Karen and David Stoutemyer for their presence and for organizing the meeting. I must not forget Lisa Townsley-Kulich who found time to support our meeting within the framework of the ICTCM. At that meeting I gave a "Brief History of the DUG". Do you know how many pages of DNL have been produced until now? How many authors have contributed? Guess!! I´ll give the answers next time.

We got many suggestions about how to spread the DNL and how to make it even more widely known.

Having arrived back home I received an e-mail message from my friend Ed Laughbaum, Ohio State University, who gave a summary of good ideas. You can find them with some others below. What do you think?

Develop a "Beginners Corner". To do this, try to get a current member to do a regular column for beginners.

Perhaps also include a column on "Enhancing the Teaching and Learning of Mathematics with DERIVE and/or the TI-92". Again maybe try to find someone to author the column for each issue.

We will include a link to your web page from the College Short Course Web page if this is OK with you.

If you would make a 2-page flyer promoting DUG, I would include it with the materials that every College Short Course participant gets when they attend one of our courses. The flyer should include a membership form.

Send a few newsletter samples (in Word) that we can put it on Bert Waits´ Web page. They should be useful to educators.

Additional ideas were: Ask people how they attracted other people to use technologies for doing maths! Ask your members to install a link in their web page - if they have one - to your web site! (You can find my web-address on the information page.) I can see the DNL on the web in the future, then it would be easy for you to download contributions or parts of them. Karen Stoutemyer gave a new impulse for American Users. Andreas Krause from TI discussed some ways to let people know that there is a DERIVE (and TI) Newsletter. Many wonderful ideas.

Thank you all for your cooperation. You see, there is a lot to do in 1998. Let´s begin and hope for a good and properous future.

I hope to meet many of you in Gettysburg. Don´t forget to register!! Or even better: submit a paper.

The *DERIVE-NEWSLETTER* is the Bulletin of the *DERIVE User Group*. It is published at least four times a year with a content of 40 pages minimum. The goals of the *DNL* are to enable the exchange of experiences made with *DERIVE* as well as to create a group to discuss the possibilities of new methodical and didactical manners in teaching mathematics.
We have established a section dealing with the use of the *TI*-92 and we try to combine these modern technologies.

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Contributions:
Please send all contributions to the Editor. Non-English speakers are encouraged to write their contributions in English to reinforce the international touch of the *DNL*. It must be said, though, that non-English articles will be warmly welcomed nonetheless. Your contributions will be edited but not assessed. By submitting articles the author gives his consent for reprinting it in *D-N-L*. The more contributions you will send, the more lively and richer in contents the *DERIVE Newsletter* will be.

Next issue: March 1998
Deadline: 15 February 1998

Preview: Contributions for the next issues

More talks from the FUN-Conference, Saeroe 1998
3D-Geometry, Reichel, AUT
The new TI-92 PLUS Module, Waits, USA
Linear Programming, Various Projections, Word Problems, Böhm, AUT
A Utility file for complex dynamic systems, Lechner, AUT
Examples for Statistics, Roeloffs, NL
About the "Cesaro Glove-Osculant", Halprin, AUS
Hidden lines, Weller, GER
Fractals and other Graphics, Koth, AUT
Implicit Multivalue Bivariate Function 3D Plots, Biryukov, RUS
Riemann, a package for the TI-92, Böhm & Pröpper, AUT/GER
Parallel Curves, Wunderling, GER
Quaternion Algebra, Sirota, RUS
150 Years of π´s 250 decimal places, Romanovskis, LAT
Concentric Curve Shading, Speck, NZL
Drawing in the Plane with *DERIVE*, Mata & Torres, ESP
Information Technologies in Geometry, Rakov & Gorokh, UKR

and
Setif, FRA; Vermeylen, BEL; Leinbach, USA; Halprin, AUS; Weth, GER; Aue, GER; Koller, AUT; Mitic, UK; Tortosa, ESP; Schorn, GER; Santonja, ESP; Welke, GER; Dorner, USA and and and ....

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Opening Address
for the Symposium in Saeroe, July 1997

Chairman Helmut Heugl, Austria

Despite major progress in the algorithmic basics of mathematical software systems, the latter have spent the last thirty years in a deep fairytale-like sleep or they have only been well known to specialist users. Yet, the picture has changed dramatically within just a few years. Today systems like DERIVE are familiar to many students at schools and universities and are used in the fields of informatics or the different engineering disciplines, as well as in business, psychology and medicine. I see different reasons for this explosive proliferation and for the growing awareness of mathematical software systems:

1. The main reason can be seen in the pragmatic circumstance that some few scientists, such as David Stoutemyer, took the entrepreneurial risk of creating a professional software product out of the know-how existing in the many experimental systems and of marketing it. Success was by no means certain, but the professional start paid off. Since such professional systems as DERIVE have penetrated the market, not only has the problem-solving capacity of these systems attracted considerable attention, also but the fundamental problem-solving capacity of mathematics in a way that is unique in the history of mathematics.

2. Hand in hand with the commercialization of these systems there arose the necessity and the possibility to perfect the systems' user interfaces. In school teaching especially it is no longer necessary for the pupils to learn the language of the computer, but the computer now understands the mathematical language of the students. Only by this means have such systems access to schools. But, as I will explain later, this process is far from over.

3. Of course, hardware is making a significant contribution to this development. When I first attended an international congress on computer algebra in 1985 and listened with fascination to the lectures given by David Stoutemyer or Bruno Buchberger I was amazed by the potential of computer algebra systems on mainframes. Ten years later the pocket calculator TI-92 can solve much more complicated problems than the hardware of 1985, which had greater volume, but not greater capacity.

4. In spite of this magnificent progress in hard and software, such a broad effect would not be conceivable if politicians responsible for education in some countries were not willing to exploit the potential of information technology in school as well. For especially if pupils already grow up with this technology and if they witness this new type of mathematics, they will utilize meaningfully and effectively these technological opportunities in their later working lives.

These new applications of computer algebra systems (CAS) caused an intensive and partly heated discussion about the role and the goals of mathematics and especially of mathematics education. On the one hand, there are justified hopes that mathematics education might become more practical, more meaningful and hence more interesting for pupils, on the other, articles appear in Austrian newspapers entitled „A pupil's dream come true - the abolition of mathematics in school is only a question of time„. Journalists reason that such systems will render mathematics teaching superfluous. So opinions on the importance of CAS for mathematics education fluctuate between it being a curse and a blessing.

For this very reason international congresses like this one are so significant because teachers cannot only be given diskettes or pocket calculators like the TI-92, we also need educational concepts and an open discussion about the goals of teaching mathematics. At the beginning of the 90s this was the reason for us to found our organization ACDCA, supported by the Austrian Ministry of Education, with the aim of researching these issues. But in the age of communications technology it would be absurd to want to work on these questions in isolation in a small country like Austria. Vital ideas, without which our work could not have been successful, came and still come from our contacts abroad. Our cooperation with David Stoutemyer's team deserves special mention. If we call David and Karen the father and th mother of DERIVE, this is not only a friendly gesture, but also expresses their importance for our work.

The second major impulse is and was the series of conferences we launched in Krems in 1992 with our first DERIVE conference. We are very proud that a worldwide movement has evolved from it. Every two years a small symposium like this one takes place with invited participants, and every other two years there are major conferences like in Plymouth or Düsseldorf. The next two conferences are sure to take place - in Gettysburg next year, and in 1999 we will host a small meeting in Austria.

At this point I would like to express our gratitude to our Swedish colleagues, above all David Sjöstrand, for continuing the conference series, in such magnificent surroundings and with such an excellent programme.

Let me now give you a brief survey of the activities of ACDCA in Austria. Our first research project, the DERIVE project, was completed in 1994. Our book probably represents the best report about our results. In our project we dealt not only with mathematical subjects, but also with didactical concepts. Keywords like the White Box/Black Box Principle, the Module Principle and the Creativity Spiral, which describes the learner's way into mathematics with the aid of the computer, have fortunately become commonplace in the didactical discussion.

One of our findings was that a greater acceptance of CAS can only be expected once there is an algebra pocket calculator in every school bag and we thus become independent of using the computer labs. I would not have expected that such tools would come onto the market so fast. The TI-92 fulfills all these conditions, again thanks to David Stoutemyer. So we have immediately launched a new project to investigate the effects of TI-92 in mathematics education.

This project was initiated in January of this year thanks to the support of the Austrian Ministry of Education and Texas Instruments. Originally we had planned to equip five experimental classes in each of the nine Austrian provinces with the TI-92. Meanwhile the interest of students and teachers in this project has surged. Now we have more than 60 experimental classes with more than 1300 pupils from the 7th to the 11th grades. In the province of Lower Austria, which I am responsible for, there are more than 30 classes.

The goal of the project is to analyze the impact of permanently available algebra pocket calculators
* on the contents and objectives,
* on the organization and methodology of mathematics education,
* on the motivation of pupils and teachers,
* on the curriculum,
* on the learning media,
* on the development of in-service training for teachers,
* on, finally, the development of teaching materials.

The school year 1997/98 will be the year under investigation, then the evaluation phase will follow. The next international congress after this present meeting, projected for spring 1999, will also present our results.

Whereas our first project might be termed a wide-angle project, that is investigating global aspects of mathematics teaching, in our second project we also want to include telephoto-like studies in the form of „observation windows„. All the teachers in every grade will have to test a sequence of teaching units, laid down by the central planning group. Subsequently the effects on teaching and learning will be assessed in evaluation tests. One example might be the approach to the chapter „Exponential Function„. One section of the experimental classes will adopt the traditional approach by using the terms of the functions, whereas the other section will attempt an approach via „recursive models„.Other goals of our project will be so-called „Rahmenthemen„, in which we will give the teachers of the experimental classes specific guiding questions on certain chapters such as „Trigonometry„ and „Analysis„. An evaluation will be carried out by means of tests and questionnaires given to pupils and teachers. A third area of research will investigate the practical realization of the didactical principles we dealt with in our book. Again the findings of the project will be evaluated externally by the Austrian Center for School Development.

Above and beyond this venture we have an abundance of visions for the future. We are already thinking about how CAS will influence the learning media of the pupils in the future. Because of the results from our first project we are convinced that CAS not only have an amplifying effect, i.e. what has been done up to now, can be done more effectively, but CAS will also become part of cognition, thus fundamentally changing the learning media themselves. So the necessity and the opportunity arise to perfect the user interfaces of such systems by exploiting the manifold possibilities of graphics, animation, sound, and hypertext and even by producing interactive and electronic 'textbooks'. Moreover, we learnt from our first project that the teachers of the experimental classes had to do so much extra work as there were no suitable learning media at their disposal, and this work load cannot be expected of teachers in everyday school routine.

In the future pupil-oriented learning concepts will require instruments to help teachers and students construct appropriate learning media. Like a composer, the teacher will write the score, using the instruments given by the computer, and like a conductor he will guide the learning process. Bruno Buchberger of the RISC Institute of the University of Linz has made great progress in this research area. We can only tell our requirements to these scientists or we can investigate the practicability of some ideas by conducting experiments in everday school life. We are glad that the Austrian Ministry of Education is also very interested in this research work and so we are confident that visionary results can be achieved by networking research in the fields of hardware and software, educational experts, practising teachers and those responsible for educational policy. Austria is too small a country to be able to compete in the areas of hardware and software development. Our strength might be dealing with the question of what we can do with the software. But in this field, too, we cannot work in isolation from worldwide trends. For this reason I am very grateful for conferences like this one, because they enable the exchange of opinions between experts from all over the world.

I wish all of you an interesting and fruitful symposium over the next few days and would again like to express my thanks to David Sjöstrand and his team for organizing this conference.

The participants of the FUN Symposium in front of Tjolöholm Castle

Teaching Geometry of Curves with DERIVE

Vladimir Y. Rovenskii, Krasnoyarsk, Russia

Computer Algebra Systems ***DERIVE, MAPLE, REDUCE, MATHEMATICA, MUPAD,*** etc. are perspective instruments not only in scientific investigations, but also in classroom studies of mathematics, physics, etc. We use ***DERIVE*** in teaching "*Geometry of Curves*", where operations of algebra and calculus with vector- functions are doing in symbolic form, the images of curves are constructed, see [1]. Learning mathematics with the computer was based on writing programs in ***BASIC*** (see [2]), ***PASCAL***, etc. ***DERIVE*** allows to develop traditions of basic and entertaining (see [3]) mathematics and to do next steps in studying the remarkable world of mathematics using personal computers.

The students acquaint with ***DERIVE*** in the first lessons and then prepare the following practical works:

(1) Plane curves - graphs
(2) Plane parameterized curves
(3) Curves in polar coordinates
(4) Implicit given plane algebraic curves
(5) Level curves and extremum
(6) Asymptotes of plane curves
(7) Tangent line to a curve
(8) Enveloping curve ("mathematical embroider")
(9) Singular points of a plane curve
(10) Length of a curve
(11) Curvilinear coordinates
(12) Curvature of a curve and osculating circle
(13) Constructions with plane curves
(14) Space Curves, etc. (1)

The initial lessons (1) - (4) are devoted to visual modeling (with change of parameters of curves) and very simple calculations. Later, starting from "extremum" and "asymptotes", the tasks include 3 parts:

(a) Ploting curves given by their equations or conditions
(b) Analytic deducing of characteristics, equations, constructions, etc.
(c) Visualization of the results of part (b)

The "right" final figure is one of the indications of a successful work. The students solve problems, write essays, course work using ***DERIVE***, take a part in student conference. Below we give some fragments "without long calculations" from lessons *(1) - (14).*

(2) ***Cycloidal curves*** are given in parameter form. We plot tables of curves with change of parameters ***a, b*** in modulus ***m = a/b***.

---

(1) Vladimir provided an additional file for "Computations for Space Curves". (You will find ROVADD.MTH on the diskette).

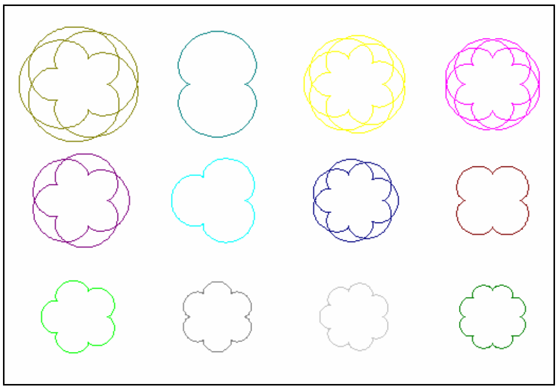
Epicycloids:

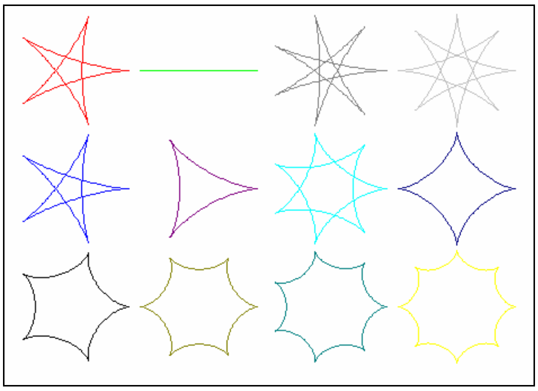
```
VECTOR(VECTOR([(1+a/b)*COS(a/b*t)-a/b*COS(t+a/b*t)+5*b-32.5,
       (1+a/b)*SIN(a/b*t)-a/b*SIN(t+a/b*t)+4.4*a-9.2],a,1,3),b,5,8)
```

Hypocycloids

```
VECTOR(VECTOR([(1+a/b)*COS(a/b*t)-a/b*COS(t+a/b*t)+2.2*b-23,
       (1+a/b)*SIN(a/b*t)-a/b*SIN(t+a/b*t)-2.1*a-4.3],a,-3,-1),b,5,8)
```

$(0 \le t \le 16\pi)$





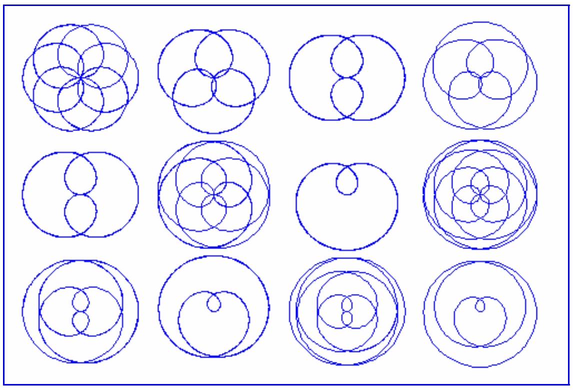
(3) The task includes ***Spirals***, ***Roses*** and other beautiful curves in polar coordinates. The following formula fills the table with ***Roses*** $\rho = \cos((a/b)*\varphi)$.

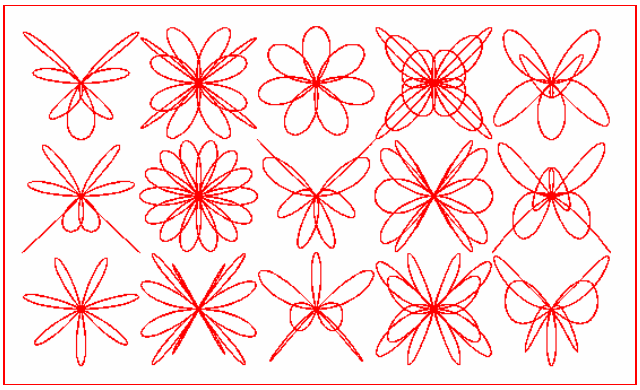
```
VECTOR(VECTOR([SIN(a/b*t)*COS(t)+2.3*b-12, SIN(a/b*t)*SIN(t)+2.1*a-4.2],
       a,1,3),b,4,7)
```

$(0 \le t \le 14\pi)$

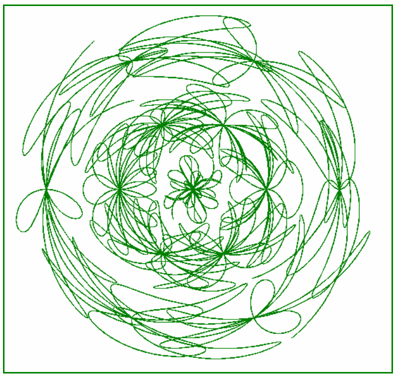
Experiment (see [2]) with polar coordinates and parameter equations:

```
VECTOR(VECTOR([SIN(7*t)*COS(a*t)+2*b-6,SIN(7*t)*SIN(b*t)+2*a-4],a,1,3),
       b,1,5)
```



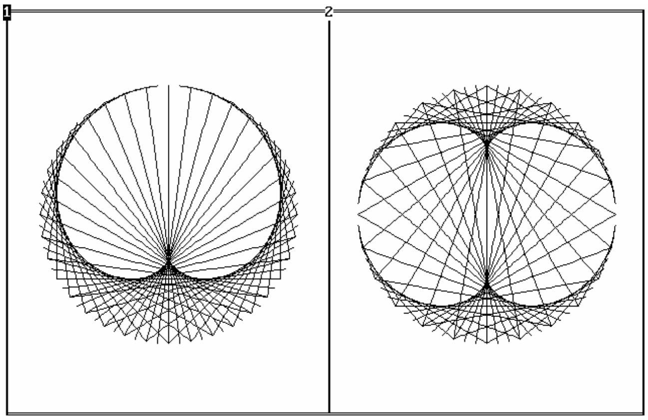


This is the sequence of the red flowers from above but plotted in polar coordinates! Josef



(8) *Mathematical Embroider* is the method of constructiong curves using the families of line segments or circles [3]. *DERIVE* allows to realize all procedures of Mathematical Embroider, which are given in [3], to explain them with deducing the equations of enveloping curves, and to find new similar interesting examples. Consider the embroider of a Cardioid (***m = 2***) and a Nephroid (***m = 3***) by line segments:

```
VECTOR([t*COS((90+5*n*m)*°)+(1-t)*COS((5*n-90)*°),
      t*SIN((90+5*n*m)*°)+(1-t)*SIN((5*n-90)*°)],n,-36,36)
```



Consider the embroider of a *Cardioid* (analogously *Limacon of Pascal*) and a *Nephroid* by circles:

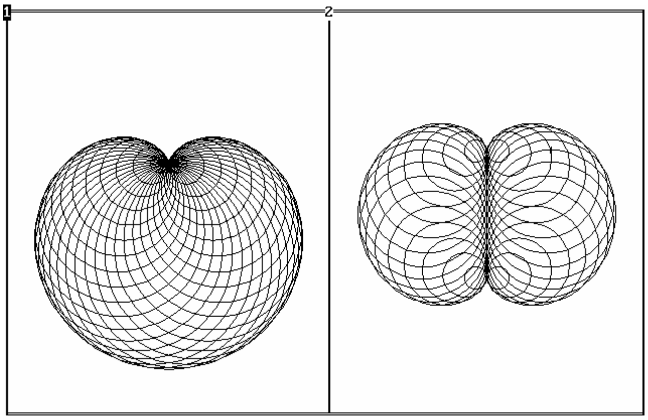
```
VECTOR([COS(10*n*deg)+c*COS(t),SIN(10*n*deg)+c*SIN(t)], n,-18,18)
```

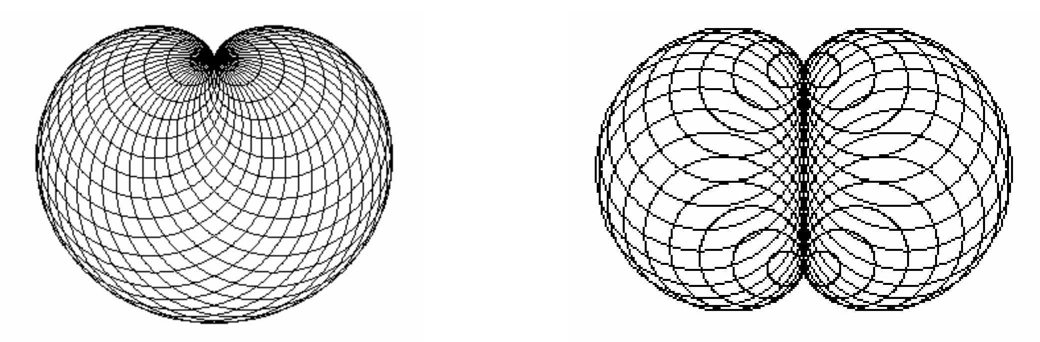
with

```
c:= ABS(COS(10*n*deg))                              →
c:= SQRT(COS(10*n*deg)^2+(1-SIN(10*n*deg))^2) →
```

Nephroid

Cardiod





The graphs in *DERIVE 6*

(10) *The Length of the right inscribed polygon* is compared with the *Length of a Curve* using integral of velocity. We use a remarkable chance to recall regular and *star-shaped polygons* (from elementary geometry) and their relation with ruled surface (*hyperboloid of one sheet* from Analytical Geometry). The consecutive vertices of star-shaped polygons have coordinates

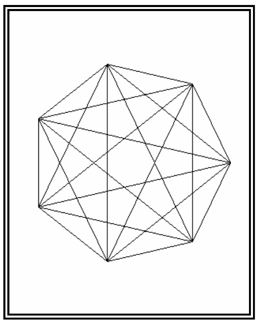
$x(i) = \cos(2\pi\, i\, m/n,\ y(i) = \sin(2\pi\, i\, m/n)$.

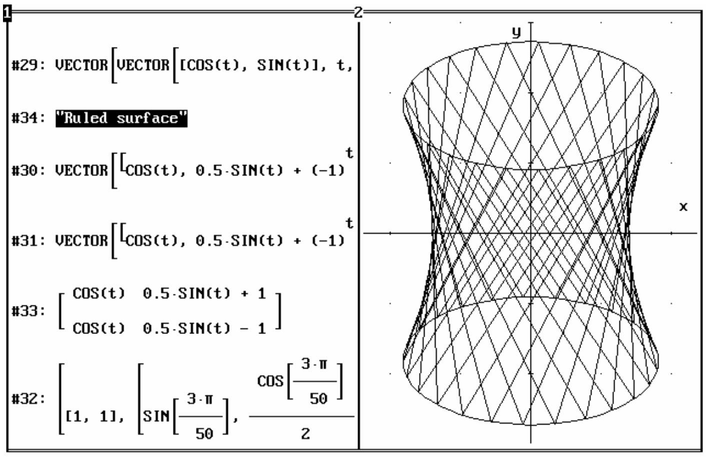
```
VECTOR([COS(t),SIN(t)],t,0,2*m*pi,2*m*pi/n)
```

This polygon has one component when natural ***n, m*** are mutually simple. For ***n* = 7** we obtain three cases ***m* = 1, 2, 3** (see Fig. below). Using these data we find a simple method to project the part of a hyperboloid of one sheet with two families of rulings. We inscribe star-shaped polygons in an ellipse (a parallel projection of a circle) and then move it up and down along ***OY*** on the distance c to obtain two star-shaped polygons. Then we connect the i-th vertex of the down polygon with the **(*i*+1)**-th vertex of the up polygon and so on, see the figure.

```
[[COS(t),0.5*SIN(t)+1],[COS(t),0.5*SIN(t)-1]]
```
→ *the two ellipses*

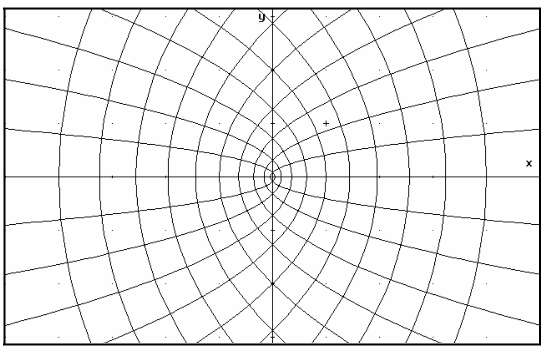
```
VECTOR([COS(t),0.5*SIN(t)+(-1)^(t*n*m/(2*pi))],t,0,2*m*pi,2*m*pi/n)
```



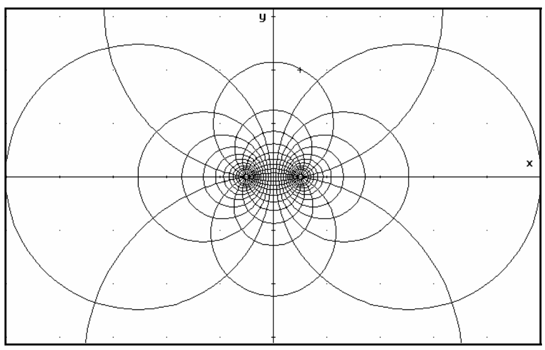


Ruled Surface (*m = 11* and *n = 50*)

(11) For parabolic coordinates ***u, v*** on a plane coordinate lines form two families of mutually orthogonal parabolas with a common focus ***O*** (singular point) and opposite axes ***OX*** and ***–OX***. For bi-polar coordinates ***u, v*** ($u \in R$, $0 \le v \le \pi$) the coordinate lines are (1) circles through the points (0,a), (0,-a) and (2) circles orthogonal to family (1).



Parabolic Coordinates ($-\pi \le u,v \le \pi$)



Bipolar Coordinates

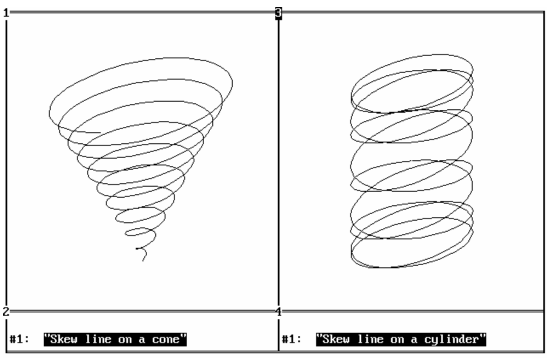
```
VECTOR([u^2-v^2,2*u*v],u,-2,2,0.2)

VECTOR([u^2-v^2,2*u*v],v,-2,2,0.2)

VECTOR([SINH(u)/(COSH(u)-COS(v)),SIN(v)/(COSH(u)-COS(v))],u,-2,2,0.2)

VECTOR([SINH(u)/(COSH(u)-COS(v)),SIN(v)/(COSH(u)-COS(v))],v,-pi,pi,3/10)
```

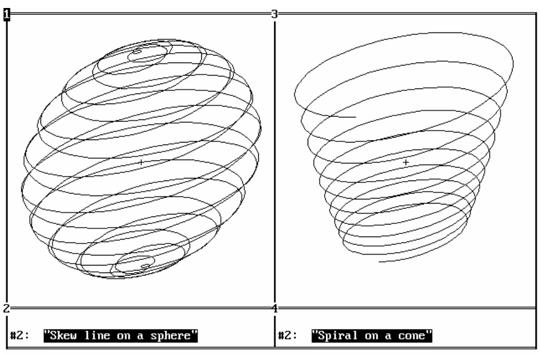
(14) We plot the space curve ***r(t) = [x(t), y(t), z(t)]*** using parallel projection. Let ***{O; i, j, k}*** be the space rectangular coordinate system with axes ***OX, OY, OZ***, and ***{O1; e1, e2}*** the "2D-window" rectangular coordinate system with axes O_1X_1, O_1Y_1. Consider two projections ***p : R3 → R2*** close to well-known *isometric* and *dimetric* projections. Assume that axes ***OX, OY*** project under angles ***±(π/2+ATAN(1/2)) ≈ ± 117°*** to axis O_1Y_1 and the projection of axis ***OZ*** coincides with O_1X_1, mainly $p(O) = O$, $p(i) = - e_1 - e_2 /2$, $p(j) = e_1 - e_2 /2$, $p(k) = e_2$. In view of linearity $p(x\, i + y\, j + z\, k) = (y - x)\, e_1 + (z - (x+y) /2)\, e_2$. Hence $x' = y - x$ and $y' = z - (x+y)/2$. This projection is called **ISOMETRIC** (see the file **GRAPHICS.MTH**). We consider an anologous projection **DIMETRIC** when the axes ***OY*** and ***OZ*** project onto O_1X_1 and O_1Z_1 and the projection of ***OX*** is a bisectrix of the 3D-coordinate quadrant, mainly: $x' = y - x/2$, $y' = z - x/2$. A *surface of revolution* in space is given by the vector function ***r(u,v)*** = [***f(u) cos(v), f(u) sin(v), g(u)***] of two real parameters ***u, v***. Here the curve ***l(u): x = f(u), y = 0, z = g(u)*** in the plane ***XOZ*** rotates around ***OZ*** and v is the angle of rotation.

A transparent surface of revolution would be visible if we round it by a thread. In case of a torus such "thread" can be closed and we obtain knot ***K(m,n)***. This idea of "knitting" on a surface of revolution can be realized using ***DERIVE***. We only need to substitute ***a*u*** instead of the parameter ***v*** (with ***a*** sufficiently large, for instance ***a = 30***) and to use concrete functions ***f*** and ***g*** in the equation of a surface. For ***K(2,39)*** we set ***u:=39*t, v:=2*t***.

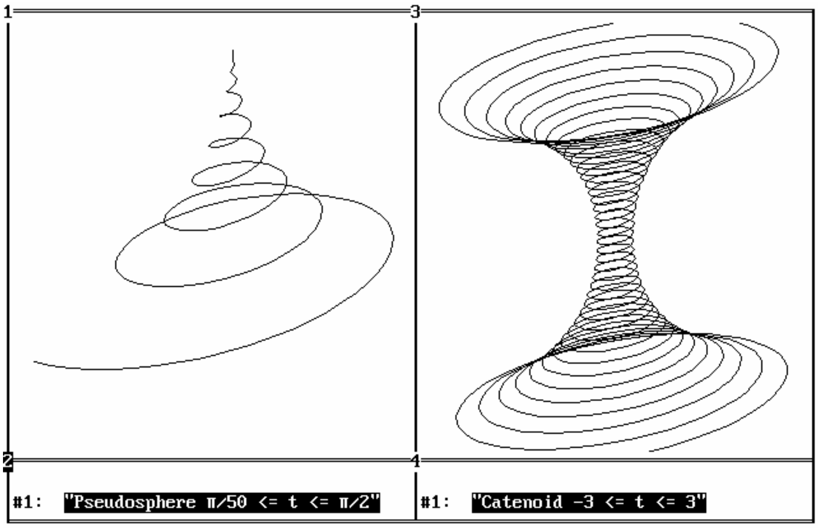
We can start with a set of formulae, analogous to **AXES** and **ISOMETRIC** from **GRAPHICS.MTH**.

```
diaxes:=[[-t_/2,-t_/2],[t_,0],[0,t_]]
DIMETRIC(v):=[v SUB 2-v SUB 1/2,v SUB 3-v SUB 1/2]
DIMETRIC([f*COS(a*t),f*SIN(a*t),g])                              (*)
```



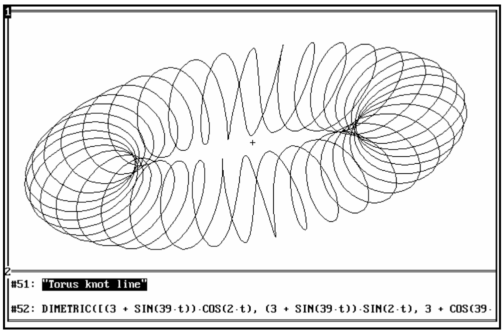


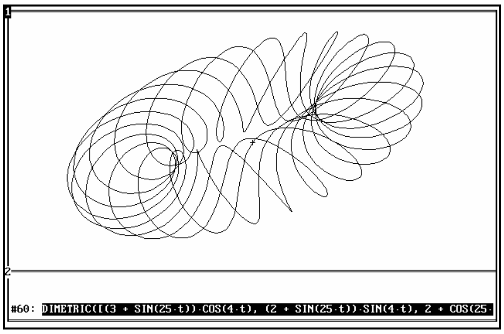
Substitute into (*):	***F = 2, G = 3*SIN(t)***	to obtain a skew line on a cylinder
	F = 0.1*t, G = 0.2*t	a skew line on a cone
	F = 0.3*#e^(0.1*t), G = 0.25*#e^(0.1*t)	a spiral on a cone
	F = COS(t), G = SIN(t)	a skew line on a sphere



Knitting on a Catenoid

F = COSH(t), G = 4*t, (a=40)	Catenoid
F = COS(t)+LOG(TAN(t/2)), G = 3*SIN(t)	Pseudosphere
F = 3+SIN(b*t), G = COS(b*t)	Knots on a torus (various ***a,b***)





Knot ***K(2,39)*** on a torus

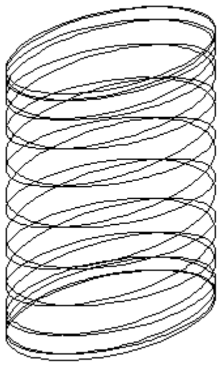
The above organization of the teaching process (1) helps active students mastering a subject and training of geometric imagination, (2) gives effective practice in computer symbolic transformations. This work in is at a point of development and search.

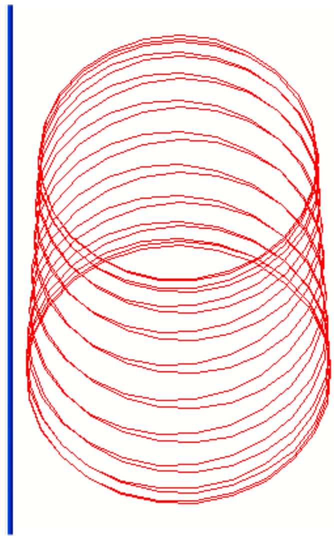
[1] V.Rovenskii *Theory of Curves (Lectures on Differential Geometry) with Appendix: Curves on a display with* ***DERIVE,*** Krasnoyarsk, **1996,** 196pp.

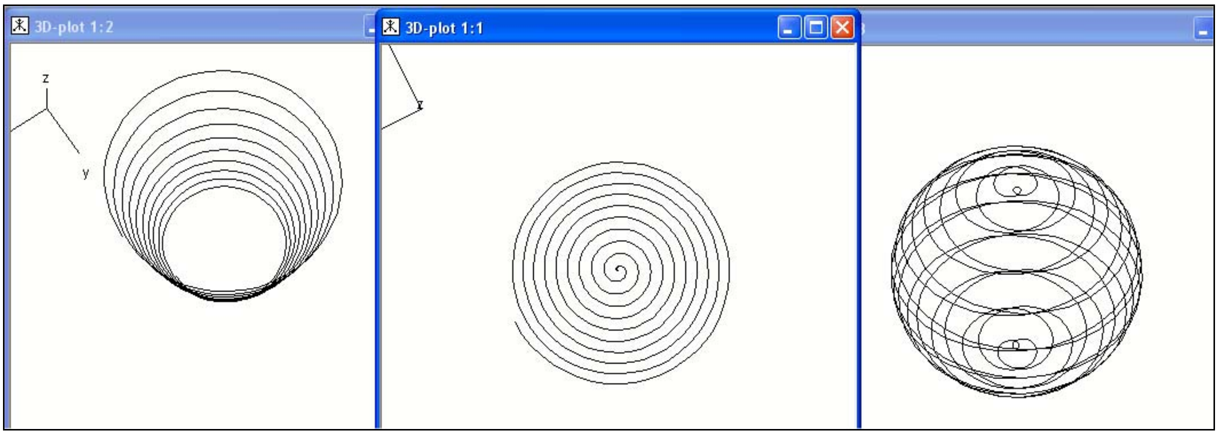
[2] C.Kosniowski *Fun Mathematics on your Microcomputer,* Cambridge University Press, **1984**, 192pp.

[3] D.Pedoe *Geometry and the Liberal Arts,* Pengiun Books Ltd., **1976**, 332 pp.

The next page shows some of Vladimir's graphics produced with DERIVE 6. We don't need projections for displaying 3D objects. Nevertheless studying the various projections is still an interesting and valuable issue in mathematics, Josef.






3D-plot 1:2
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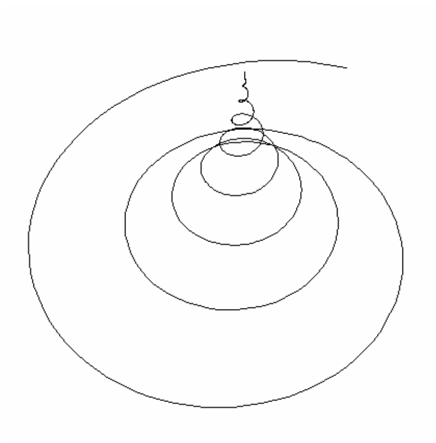
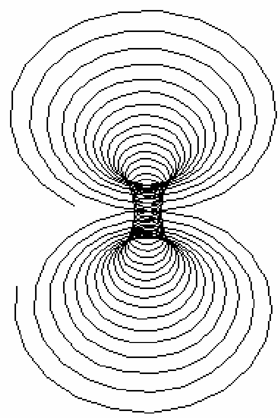
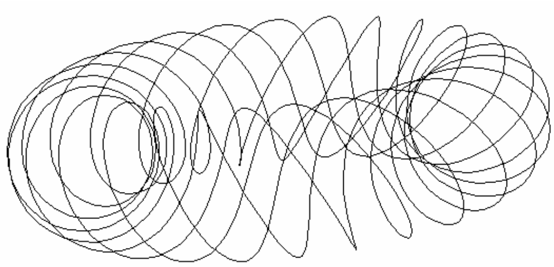
z
y
z







Why Do We Save the "Good Stuff" for Last?

Presented by
L. Carl Leinbach
Gettysburg College
Gettysburg, PA 17325

Introduction

My basic thesis in this paper is that the way we teach mathematics is inverted. We present the theoretical and technical aspects of the subject first, leaving the part that our students find as the interesting and "fun" aspects of the subject until the end. Most of our students came to our classes because they were told that they needed to study mathematics as a prerequisite for some other course that they want to take or as a requirement imposed by a professional or graduate school. Many of the students have no idea about why the requirement is necessary, and the way we teach our subject does little to enlighten them. They see mathematics as a "hurdle" that must be cleared before they can do what really interests them. The subject becomes nothing more than a collection of meaningless manipulations and a collection of facts for which they have little use.

When we teach our classes, most of us begin our presentations with "motivating examples." Unfortunately, the examples are either grossly simplified from their genesis in the real world or presented so quickly that students fail to see the connection between the example and the mathematics that they will be studying. It is rare in the presentation of the motivating example that students have time to work in a meaningful way with the example, to "play" with it, and to ask and analyze the "what if?" questions that make up the excitement and fun of studying mathematics. They are, instead, quickly thrown into the technical aspects of the mathematics with the promise that they will return to the motivating example at a later time. Our argument is that our students need to develop a maturity before doing a detailed analysis of the example. The unfortunate truth is that because they have found nothing in mathematics to hold their interest, they take the minimum number of courses to complete their requirement and leave before they are allowed to return to the example that may have sparked their interest and curiosity.

My proposal is inspired by an article by Paul J. Campbell ("Finite Mathematics as Environmental Modeling", The UMAP Journal, volume 17, number 4, pp 415-430). In this article he describes a course that he taught at Beloit College to an audience of Biology Majors. Campbell's basic premise is that a course taught for students taking an introductory mathematics course, much more interest can be developed using computer models based on the mathematical concept of growth and decay than on a course that is primarily a standard presentation of the mathematical theory underlying these concepts. Indeed many of the issues of modern life can be cast in terms of the concept of growth and decay. Typical examples are population growth, the use of non renewable resources, the spread of diseases, and the theory of investments. The fact is that for the most part they are discussed in a qualitative way with little attention or understanding of the quantitative aspects of the phenomenon.

The main effort for the students goes into the front end of the process, not in the solution of the model using a variety of mathematical techniques to derive a solution to a problem arising from a model that is presented in its full blown form to the student as a passive observer. The fact is that with Computer Algebra Systems the part that is usually left for the student to do, can be easily solved by machine. In fact, a survey of recent graduates who had majored in mathematics and were employed by industry as mathematicians found that the part of their major that they used the least was the part on which most of their educational effort had been spent, the analytic solution of equations of various types. The graduates' experience is that in industry those processes are done using modern computational tools. (Math Horizons; February, 1995 pp18-24 & April, 1995, pp 26-31). It is also the case that students doing graduate work in mathematics, use the modern computational tools to do most of their technical work. This frees them to examine the underlying structure of the object of their study and explore the relationships that define them or hold clues to their hypotheses.

Our form of pedagogy begins early in the educational process. We teach children to memorize addition and multiplication tables. We build on that to have them doing large calculations by pencil and paper. While one can not deny that children need to learn certain "number facts," one can certainly question the amount of time that is spent on drill with its only reward being that it has been accomplished. Why can't we use the "number facts" to have some fun and make discoveries that are more important to their understanding of mathematics than endless work sheets of sums, differences, and products.

In this presentation, I will look at two examples of how we can make the teaching and learning of mathematics more fun. In the first example, I will consider the use of some elementary number facts that can be used to solve puzzles and lead to deeper investigations. At a crucial stage in each investigation, technology can be used to assist the solver in their exploration. The second example is an interactive modeling tool that implements many of the ideas expressed by Campbell. In both examples, I will use the TI-92 as the technological tool to assist in the investigation. Its portability, affordability, and versatility make it an ideal tool. CAS's on larger computers can, of course, operate faster, but the combination of the CAS and programming capability make the TI-92 an ideal tool for my endeavor.

Fun with Numbers

When Marvin Brubaker of Messiah College and I teach workshops together we always try to give participants something to think about over their lunch hour or over night. We offer "prize problems." This means that Marvin and I each put up a dollar which is awarded to the first person to solve the problem. This in itself adds to the fun component of the exercise. Marvin presents the problems to the participants. They have been listening to me going over the syllabus material long enough and need a break. Here are some typical examples of problems. They all involve playing with numbers and can lead to deeper explorations.

1. A social security number in the United States consists of nine digits. As a warm up exercise, does anyone have a prime social security number? (Use the factor command on the TI-92.) Now for the prize part: Construct a social security number that has the following properties. Each of the digits 1 through 9 is used exactly once in the number. No zeros appear and the dashes between parts of the number are omitted. Here is the hard part.

   The first digit of the number has one as a factor. The first two digits taken together have two as a factor. The first three taken together have 3 as a factor. The first four have four as a factor, and so it goes up to nine. How many such numbers are there?

2. A new school has a thousand lockers lining a hallway. A thousand students are lined up and given consecutive numbers from one to a thousand. The first student goes through and opens all of the locker doors. The second student closes all the even numbered locker doors. The third student changes the state of the door (open to closed or closed to open) of all lockers having numbers that are divisible by three. The fourth does the same for all lockers divisible by four and so on until the last student changes the state of the door of locker number 1000. At the end of the exercise, what doors are remaining open? Which door was touched the most?

3. Consider the following equation:

$$\left(x^2 - x - 1\right)^{\left(x^2 - 7x + 12\right)} = 1.$$

Find all of its solutions. How do you know that you have them all? What is the the maximum number of solutions that an equation of the form

$$\left(ax^2 + bx + c\right)^{\left(dx^2 + ex + f\right)} = 1$$

can have. Construct different examples that will have 0, 1, … up to the maximum number of solutions.

4. Chicken nuggets are bite-sized pieces of chicken meat which are breaded and fried. Fast food restaurants sell them in boxes of 6 nuggets, 9 nuggets, or 20 nuggets. Can you make up an order that will have exactly 55 nuggets? What about 37? What is the largest order for nuggets that can not be filled exactly using the marketing scheme of the restaurants?

Each of these problems requires the solver to think deeply about the natural numbers and number facts. In the first example the size of the search can be impossible unless the solver knows and uses some basic facts about divisibility by 2, 3, 4, 5, 6, 8, and 9. The solver also needs to either explicitly or implicitly construct a search tree, a valuable problem solving skill. One of Brubaker's students at Messiah College extended this problem into a search for numbers with analogous properties in other number bases and gave a student paper presentation of her results.

The second problem has two parts. Both involve a knowledge of factorization and finding perfect squares. The second part is best handled by using a calculator to construct the number based on some insight about the characteristics of doors whose last state will be open. Try the problem for a smaller line of lockers, say 30.

For the third problem, you, of course, have to know about solving quadratic equations, but you also have to know about the possible values for u and v for which $u^v = 1$. This is not hard task, but it is amazing how many people do not see through the specifics to the basic number facts.

For the fourth problem, I wrote a small TI-92 program for the following problem: Given a natural number, n, are there non-negative integers, i, j, and k such that

$$n = 6i + 9j + 20k\ ?$$

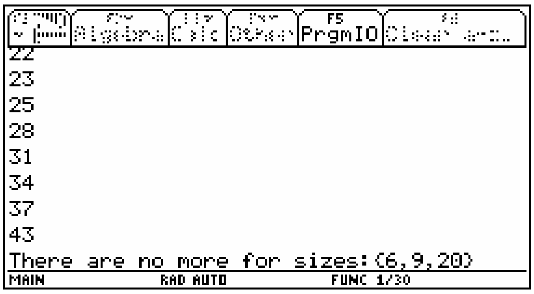
The program is quite straight forward and involves little computer algebra, but it does involve a feeling for numbers. It is environmentally sound in that it searches for the solution that uses the fewest containers. By the way a solution for 55 is one container of 6 nuggets, one container of 9 nuggets, and 2 containers of 20. There is no combination that yields exactly 37 nuggets.

In order to answer the second question, the solver needs to realize that once there is a run of six numbers in a row for which the order can be filled, all orders any larger than the start of this run can be filled. The TI-92 program that finds the largest number for which an order can not be filled generates a sequence of numbers of the form

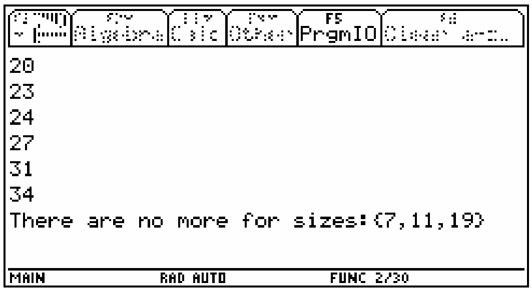
$$6i + 9j + 20k$$

sorts it and finds a run of 6 consecutive numbers. The program prints out all orders that can not be filled and stops when it finds a run of 6. The following is the output.

Note that the program only begins its search following the size of the largest individual box of nuggets.



The interesting thing is that the solution of this problem raises more questions than it answers. For example, is there any relation to the 43 and the box sizes of 6, 9, and 20? What about if we try other numbers? Here is the result for 7, 11, and 19.



Even though the size of the required run is longer, the maximum is smaller, and there are fewer numbers for which an order can not be filled. What is the relationship of box sizes to orders that can be filled? How is the maximum order that can not be filled related to the box sizes? These are challenging questions that I believe do not have answers. They are, however, accessible at an early level, and they are fun to explore and think about.

In each of the above examples the problem solvers are thinking about mathematics, not just doing rote exercises that have no context or apparent goal beyond that of drill. Mere drill does not inspire one to notice the number facts that are contained in their computations. It has the same effect on the student as playing scales has on the music student who has to play endless scales. Certainly the computations need to be mastered just as a musician must master scales. Years ago Bach realized the tedium of practicing scales. He incorporated the scales into musical compositions for his students. Instead of tedium, the students were making music. We need to supply the same experience for our students. We want our students to make mathematical music.

Front End Loading via Model Building

As mentioned earlier, Paul Campbell wanted his biology students to have a deep understanding of the mathematics underlying their discipline. He knew that if he insisted on the learning of all of the mathematical problem solving techniques required to give analytical solutions he would not have time to reach his objective. Instead he relied on a computer modeling program, known as STELLA which stands for Systems Thinking Experimental Laboratory Learning with Animation. Clearly a name concocted to make an acronym. However, the tool is very useful for encouraging users to think deeply about constructing a model of a growth/decay situation.

While I am not skilled in constructing graphical interfaces, I do think that the ideas behind STELLA and some of its predecessors, such as DYNAMO invented by Jay Forrester to study industrial dynamics and later to analyze complex growth situations, have value in moving the student from a qualitative analysis to a quantitative one. This is a necessary move if one is going to analyze the effects of different policy decisions.

In my approach that uses the TI-92, I go back to the earlier ideas of DYNAMO in that the basic model is constructed using paper and pencil techniques, but I use the modeling format of STELLA. After the conceptual model template has been constructed, the students then use the TI-92 in an interactive way to enter their model using the language of their model. The program translates their input to equations and generates a numerical solution that can be easily translated into a graphical output. They can rerun the program changing selected portions of their model and compare the results.

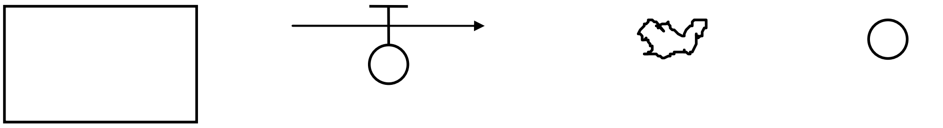
Note that there are three points in the modeling process where students need to do deep analysis. The first is when the model template is constructed. What are the variables that are pertinent to this model? What are the relationships between the variables? Does the model reflect the influence of the variables and their relationships? The second stage is the translation of the relationships and influences into the language of the model. What is the mathematical nature of the relationship? How does one weight the influence of a variable within a relationship? What are appropriate initial conditions? Finally, the analysis of the results of the simulation requires an understanding of the model. What part of the output is due to the basic structure of the model? The initial conditions? The weighting of the variables?

After the model is completed, come a series of important what if questions. It is through these questions that students come to understand the model and the role of the variables within the model. As a result the students are getting out of their mathematics course what they came for. They can get wrapped up in the analysis and, indeed, have fun in their course while they are indeed working very hard doing mathematical analysis.

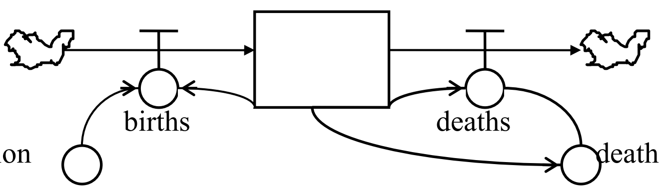
Building the Model

The STELLA model uses only four items to describe a dynamic growth/decay model. The first is called a *stock*, which is an accumulation of a model variable. The variable is modified by a *flow*, which is described as a function of time, i.e. the variable changes with each unit of time. The flow is controlled by a *valve*, which can be either one way or two way. If a flow is influenced by a factor outside of the system, this factor is represented by a *cloud*. The effect of the valve that describes the magnitude of the flow has elements that are called *converters*. These are, in effect, weights that are assigned to the elements of a relationship.

Schematically the modeling items look like this:



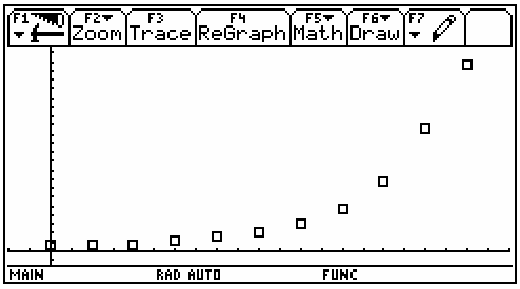
The use of these items for model building can be illustrated by this simple model of population growth that appears in a review of STELLA by J. Warner (*The UMAP Journal*, volume 17, number 4, pp. 373-396) Consider a simple model of a deer population that is influenced only by the births and deaths within the population. The model for this situation can be described via the following diagram.



Note that in this model, the birth fraction is treated as a constant while the death fraction is dependent upon the population size. This represents a belief that as the population increases, the death rate is influenced. Warner gives the following data to support this relationship:

Population:	0	10	20	30	40	50	60	70	80	90	100
Death fraction:	0.025	0.03	0.035	0.055	0.075	0.095	0.135	0.22	0.36	0.625	0.95

A plot of this data shows a typical exponential shape.



An exponential regression on the above data yields the equation:

$$\text{death fraction} = .0187 * 1.04^{pop(t)}$$

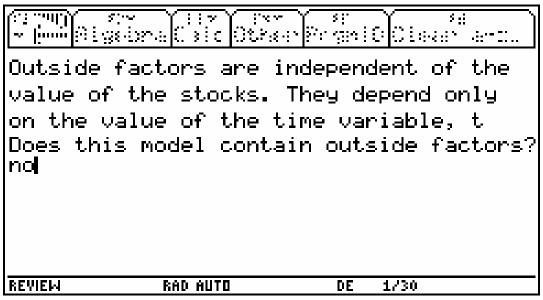
where pop(t) is the size of the population at time, t, given in terms of the percentage of the carrying capacity. If we assume a constant birth rate of .5 (an annual increase of one half of a percent of the carrying capacity), we then have the following for the growth/decay, or in this case, birth/death, equation

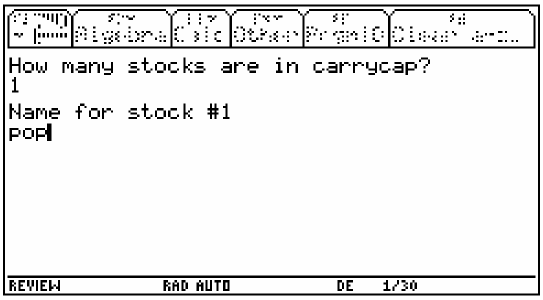
$$\text{change in population} = .5*pop(t) - .0187*1.04^{pop(t)}$$

If we assume an initial population size of 2% of the carrying capacity, we are ready to run the simulation.

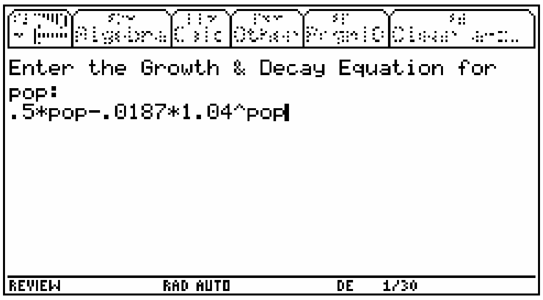
Using the Simulation Program

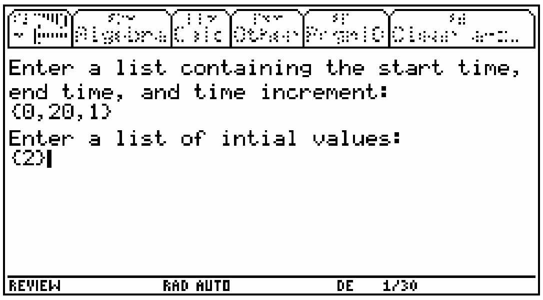
The program that is written for the TI-92 is an interactive program that allows the user to enter the information about the model in a way that is consistent with the STELLA-type model that was created above. Several Screens appear that ask information about the model. After giving a name to the model, the first scree asks about any outside factors (*clouds*). Then the user tells how many stocks are involved in the model.



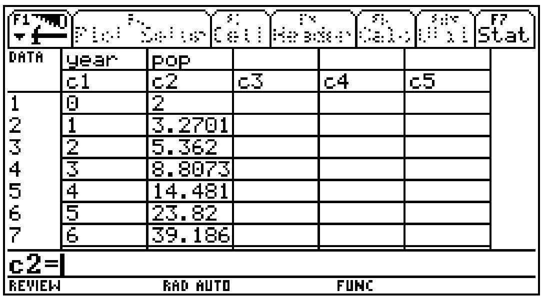


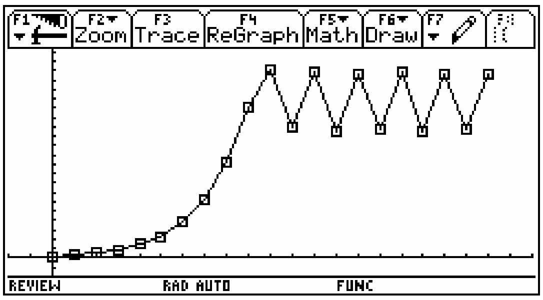
The next step is to enter the information about the flows and the information necessary to generate numerical data



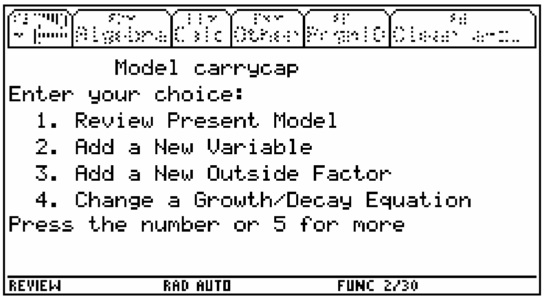


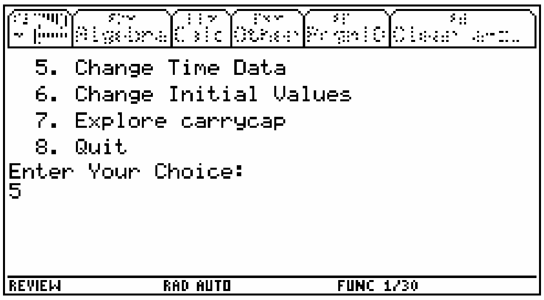
There is now enough information for the program to run the simulation. The result is a table of numerical data that is shown below. The graph can be plotted using the plot setup interface provided by the TI-92.

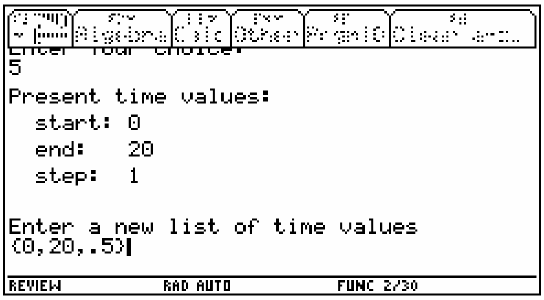


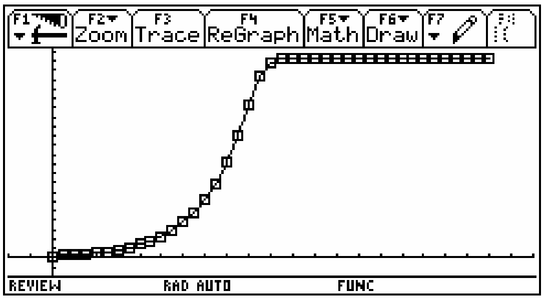


The result shows that the population increases in an exponential manner until it reaches a certain level. It then oscillates about an equilibrium level. Is the oscillation due to the time increment or the nature of the model? Let's find out. The tool includes a program called "rerun()" . Here are a few screens form that model. Note that we change the time data without having to start from the beginning.









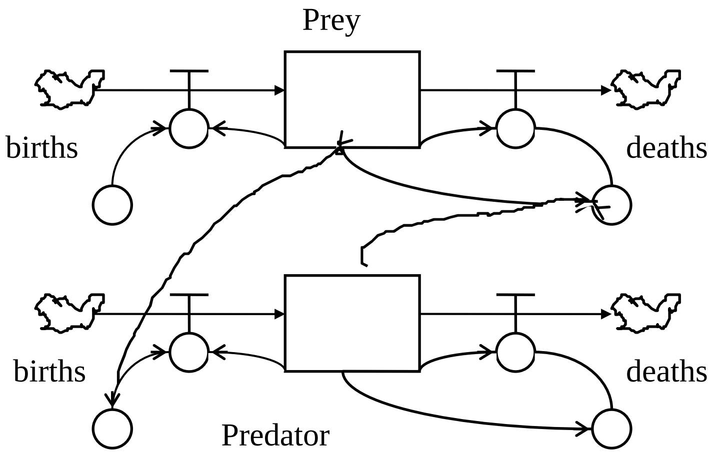
Now the question is which is the more realistic representation of the situation? For a deer population, the first may be more realistic because of the birth patterns of the population. Other areas of exploration may include checking the model for sensitivity to the birth fraction. Outside influences such as the imposition of a hunting season can also be explored.

The modeling tool uses Fourth Order Runge-Kutta to generate the numerical data. This fact need not concern the students at this point. Their job is to explore and analyze the model. To do this they need to be able to give a mathematical interpretation to their qualitative assumption. They need to understand the quantitative effect of their assumptions about the growth and decay equations. Finally, they need to take the graphical results and make meaningful qualitative statements about them. The ability to do this is what we hope for as an end result for students who are preparing for disciplines where they will be consumers of mathematics.

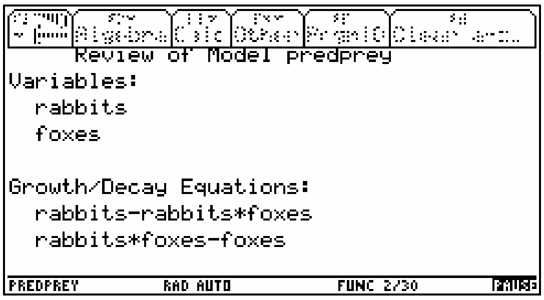
Obviously, we do not spend the entire course ignoring the fundamentals. We have sparked their curiosity for questions such as, how does mathematics handle rates of change and their influence on the dependent variable? This is a tremendous lead in to derivatives, linear approximation, and even integration. The difference is that now the students have a meaningful context for the concepts and the resulting technicalities.

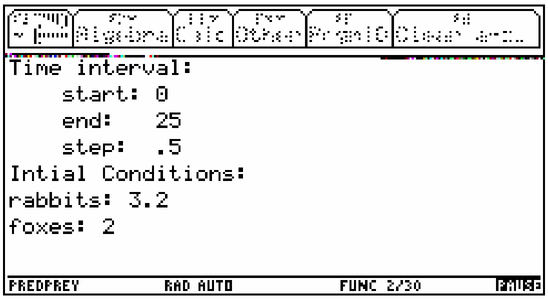
Some Other Models

The model given above is rather simplified for presentation purposes. Let's look at some other possibilities. The first is the classical predator-prey model. This is of interest because it has two interacting populations.

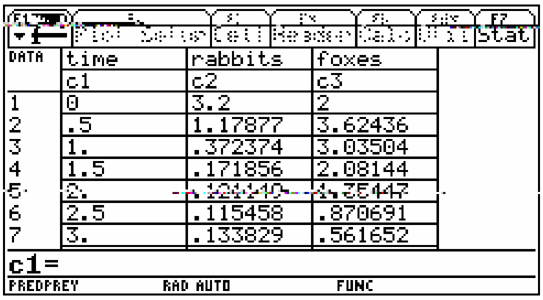


In this example, we called the Prey rabbits and the Predator foxes. The following are screens from the Model Review option of the program Rerun( ).



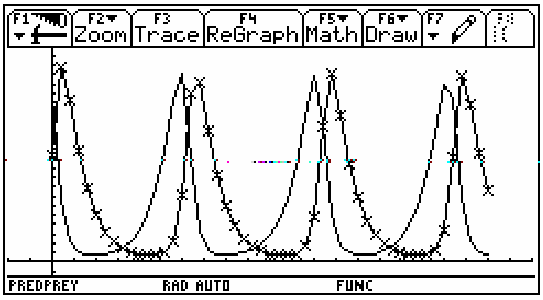


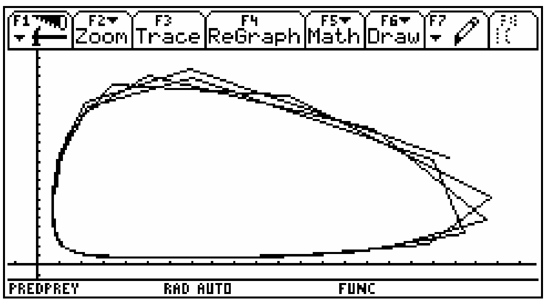
The following are the table and graphs of the populations versus time.



DATA	time	rabbits	foxes
	c1	c2	c3
1	0	3.2	2
2	.5	1.17877	3.62436
3	1.	.372374	3.03504
4	1.5	.171856	2.08144
5	2.	.121140	1.35447
6	2.5	.115458	.870691
7	3.	.133829	.561652

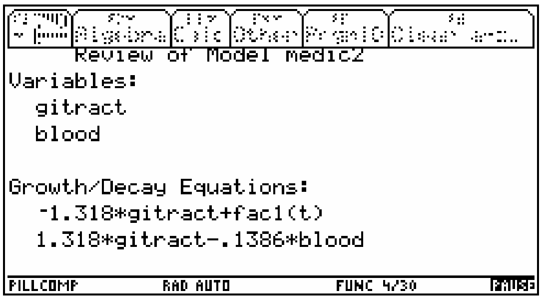
c1=
PREDPREY RAD AUTO FUNC

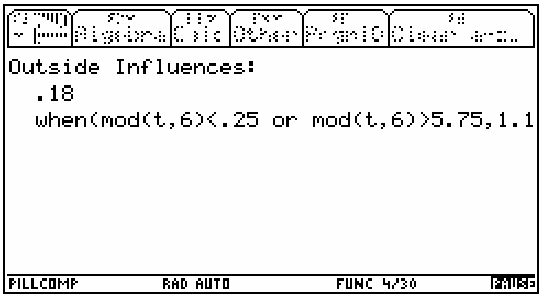




This shows the usual pattern of the peaks and valleys in the Predator population (marked with crosses) following those in the Prey population. Of course, the more instructive graph is that of the phase plane representation of Predator vs. Prey shown on the next page.

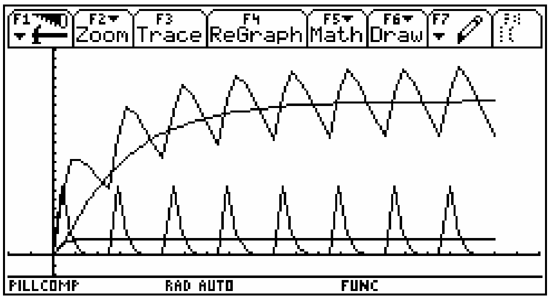
Another example is a model that has outside influences. It is a model of diffusion of medicine to the blood stream and the gastrointestinal tract. The students can examine two different strategies for administering the medicine: One is with a continuous acting pill and the other is with a pill that is administered every six hours.





Notice that the outside influence is added on as a function of the time variable. We have two equations from which to choose. The first is a constant representing the continuous acting capsule. The second is the pill that is taken every six hours. The time interval and the condition is written to encompass any round off error that may occur when adding time increments. The else condition which is not showing on the screen above is that the dosage is 0.

On left hand side is the result of two runs of the simulation. The upper graphs are the amount of medication in the blood stream. The lower graphs show the amount in the gastrointestinal tract. It is fairly obvious which graphs correspond to the continuous acting capsule and which correspond to the pill taken every six hours.



Conclusion

We need to design strategies that attract students into mathematics courses, not just as a way of recruiting majors, but also as a way of developing more informed clients. Our present pedagogical strategy is not doing this. If students take mathematics, it is generally because of a requirement that must be endured. What I propose in the above amounts to a reordering of our structure, not a radical change in the curriculum topics. It calls for a horizontal structure that is commonly found in many of the sciences and social sciences. We have the computational tools to accomplish this organization. Our vertical structure made sense when our only tool was the analysis provided by finding closed form solutions. We are living in a different age. We can go beyond the limitations of a closed form analysis and encourage our students to ask and answer interesting questions. The fact that we use a machine to assist us does not debase the value of the analysis.

From Fun to Joy

Sergey V. Biryukov,
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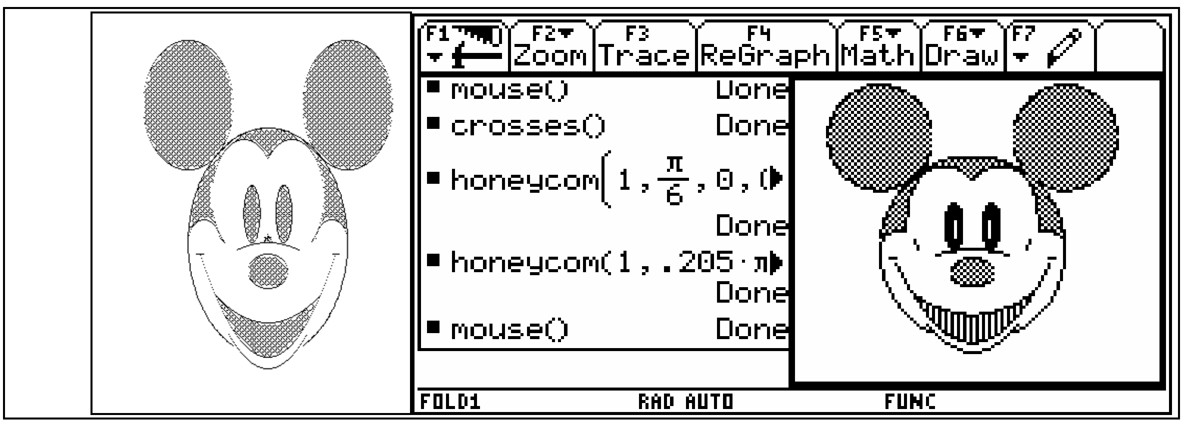
	DERIVE	*TI*-92
1.Graphing Equations & Inequalities		
Funny Mouse	+	+
Crosses in Polar Coordinates	+	+
2. The Joy of 3D Plots		
Fast Acrospin Call	*DfD/DfW*	–
3D Box, Scaling, Numbering & Labeling	*DfD/DfW*	–
Implicit 3D Plots	*DfD/DfW*	–
3. Simulations		
Honey Comb Shape	+	+
"Bottle" Submarine (Cartesian Diver)	+	+

Software Support

	DERIVE for DOS	*DERIVE for WINDOWS*	*TI*-92
1.	MOUSE.MTH, FILL.MTH		MOUSE.92P
	CROSSES.MTH, THICURVE.MTH		CROSSES.92P
2.	FASTACRO.ARJ	FAST_3d.REC	
	BOX_INST.ARJ	BOX_INST.ARJ	
	IMP_SURF.ARJ	IMP_SURF.ARJ	
3.	SOTI.MTH		HONEYCOMB.92P
	BOTTLE.MTH		BOTTLE.92P



Different ↑ Colors Blinking ↑ Eyes

1.1 Graphing Equations & Inequalities

(Funny Mouse with *DERIVE* & *TI*-92)

(M. S.Carlisle's Equations from "Fun with Graphing", US-Russia JointConf on Education, Moscow, Oct, 1994) (x:-20 20, y:-15 20)

1. $y = -1$
   $8 \le |2-x| \le 9$

2. $x^2 + y^2 + 24y = 4(x-1)$
   $3y + 2 \ge 0$

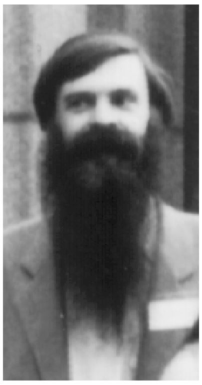
3. $x^2 + y^2 = 4x + 96$
   $y + 9 \ge 0$

4. $8y + 76 = x^2 - 4x$
   $y \le -1$

5. $8y + 84 = x^2 - 4x$
   $y \le -8$

6. $y + 10 > \frac{1}{4}(x-2)^2$
   $y + 7 > \frac{1}{18}(x-2)^2$

7. $x^2 + \frac{(y-2.5)^2}{6.25} \le 1$



8. $x^2 + y^2 + 16x - 24y + 176 \le 0$

9. $x^2 + y^2 + 256 = 24(x+y)$

10. $(x-4)^2 + \frac{(y-2.5)^2}{6.25} \le 1$

11. $\frac{(x-2)^2}{6.25} + \frac{(y+2.5)^2}{2.25} \le 1$

12. $(x-2)^2 + y^2 \le 100$
    $(x+6)^2 + y^2 \le 4$
    $(x-10)^2 + y^2 \le 4$
    $\frac{(x+1)^2}{16} + \frac{(y-4)^2}{25} \le 1$
    $\frac{(x-5)^2}{16} + \frac{(y-4)^2}{25} \le 1$
    $y \le 0$

See also DNL#27.

FILL.MTH Utility: (See DNL 26, p.43 by Sebastiano Capuccio & Nurit Zehavi)

`PLOT_WITH(expr,relations)`	–	emulates IF() for 2D Plots;
`FILL_2FX(f1,f2,x,x1,x2)`	–	Shade between f1(x) & f2(x) at [x1,x2]
`FILL_IMP(f,x,y,xs,ys)`	–	Shade f(x,y)=0 starting from [x,y]=[xs,ys]
`INTERSECT(a,b); UNION_(a,b)`	–	Shade intersection/union of a&b
`COLOR([[f1,c1],[f2,c2],...])`	–	Plot each Function in a Defined Color

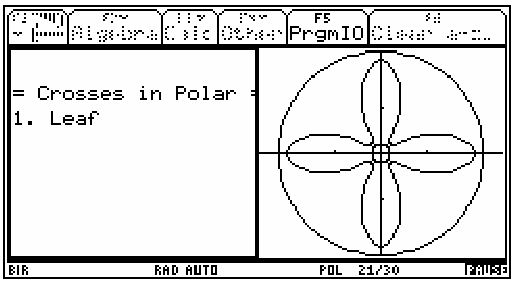
1.2 Crosses in Polar Coordinates

1. Leaf Cross

$$\max(2\left|\cos^2(2\theta)\right|, 0.3)$$

For DERIVE:

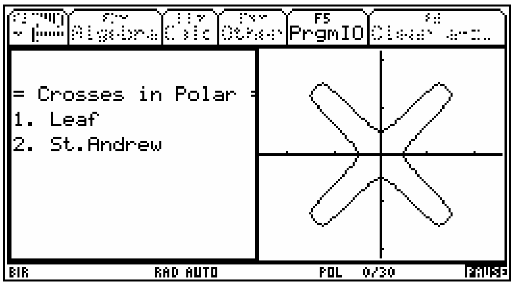
```
[MAX(2·ABS(COS(2·x)^2), 0.3), 0.3,
2.1]
```



2. St. Andrew Cross

$$\min\left(\left|\frac{-1}{2\cos(2\theta)}\right|, 2\right)$$

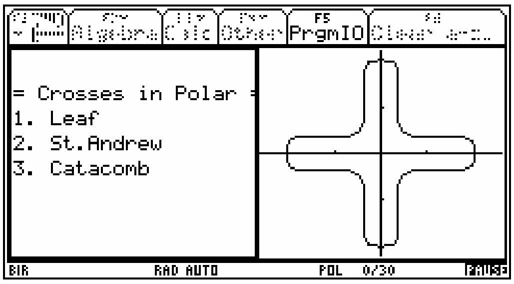
```
MIN(ABS((-1)/(2·COS(2·x))),2)
```



3. Catacomb Cross

$$\min\left(\left|\frac{2}{3\sin(2\theta)}\right|, 2\right)$$

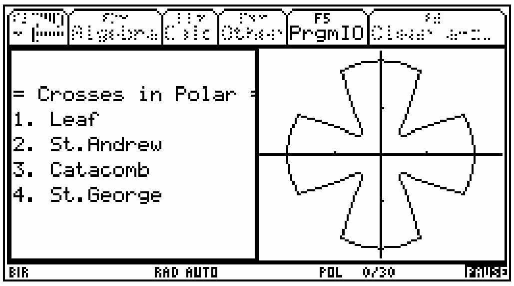
```
MIN(ABS(2/(3·SIN(2·x))),2)
```



4. St. George Cross

$$\min\left(\left|\frac{9}{10\sin(2\theta)}\right|^5, 2\right)$$

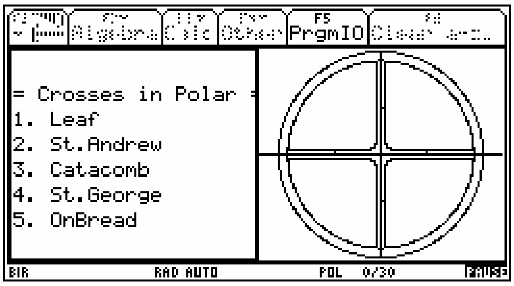
```
MIN(ABS(9/(10·SIN(2·x)))^5,2)
```



5. On - Bread Cross

$$\min\left(\left|\frac{2}{10\sin(2\theta)}\right|, 2\right)$$

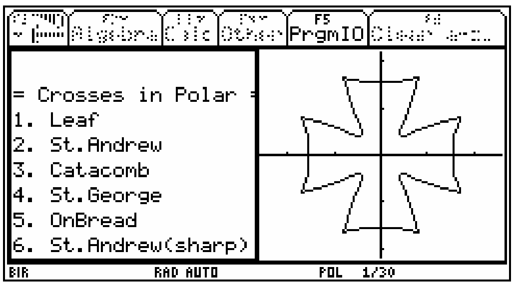
```
[MIN(ABS(1/(10·SIN(2·x))),1),1,1.1]
```



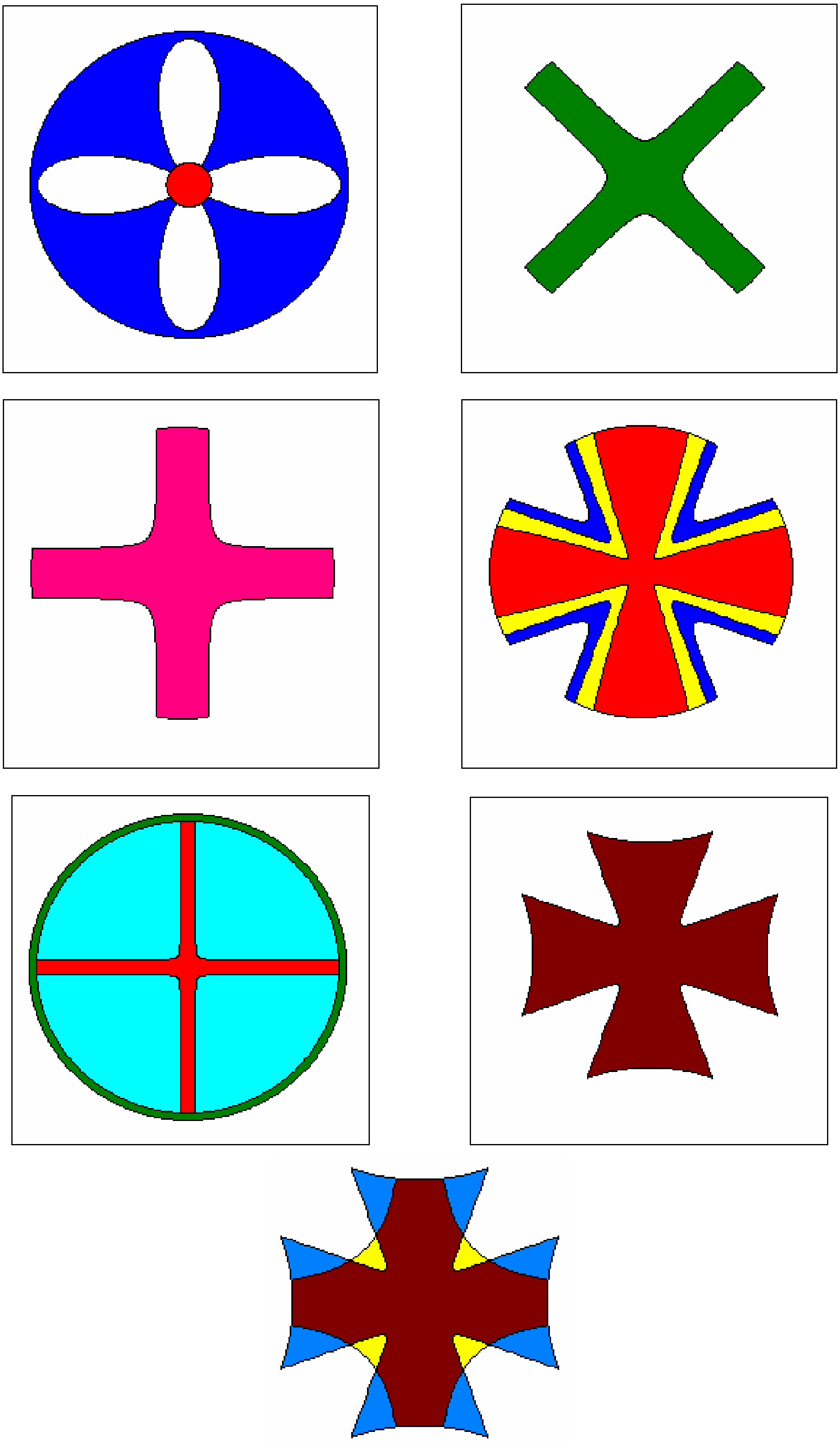
6. St. George Cross (Sharp)

Equation ?

```
MIN(ABS(-3/COS(2·x))^0.4,
ABS(0.9/SIN(2·x))^5)
```



The Crosses coloured produced with DERIVE 6 (Josef)



2. The Joy of 3D Plots

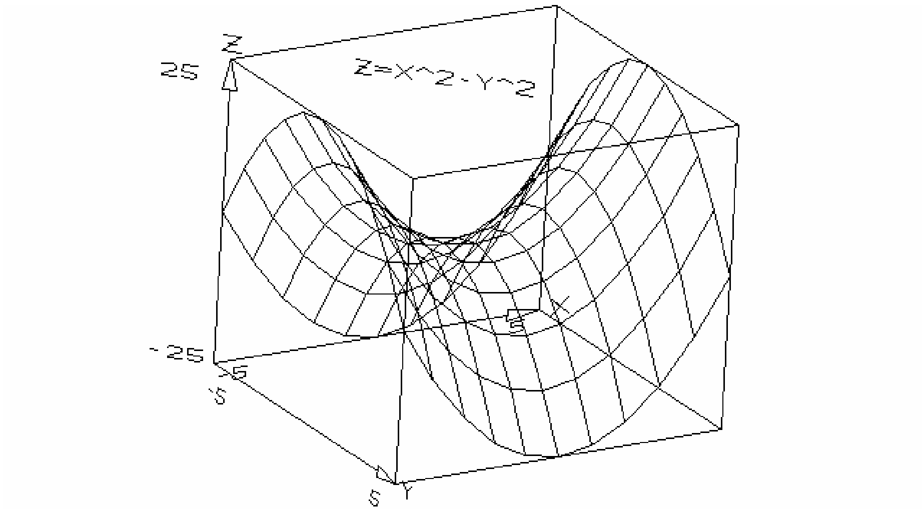
2.1 Fast Acrospin Call

DfD: FASTACRO.BAT , Ctrl-P - in Algebra/3D for ACROSPIN

DfW: RECORDER FAST_3D, Alt-P (Plot); Alt-Shift-P (ACROSPIN)

2.2 3D Scaling, Box, Numbering & Labeling (DOS/Win)

(Run BOX_INST.BAT in ACROSPIN Directory)

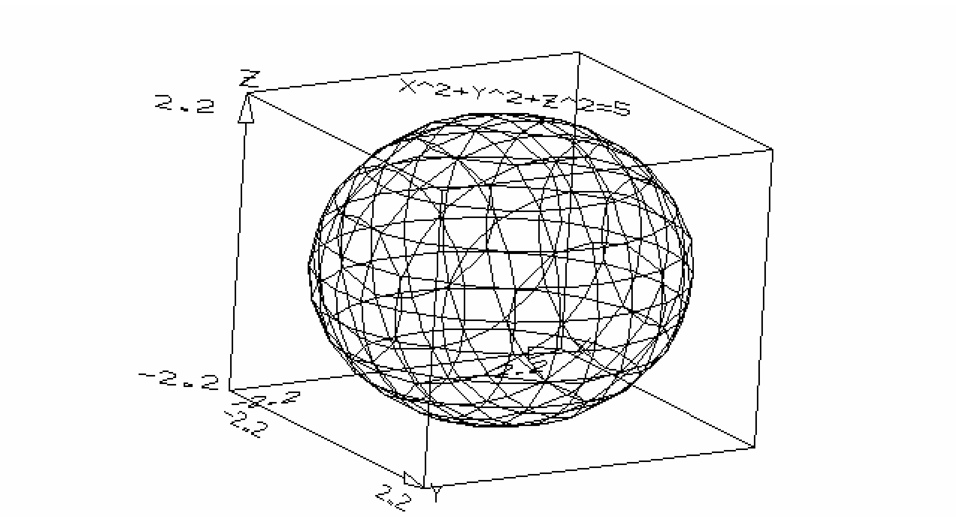


3D Axes Labels & Heading Control via DOS Shell

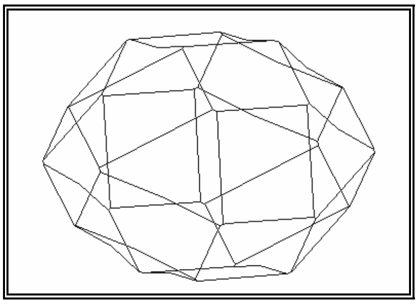
Format	Example
LET ScaleBase Base	LET ScaleBase 100
LET AxesLabels FontSize XLab YLab ZLab	LET AxesLabels .05 X Y Z1

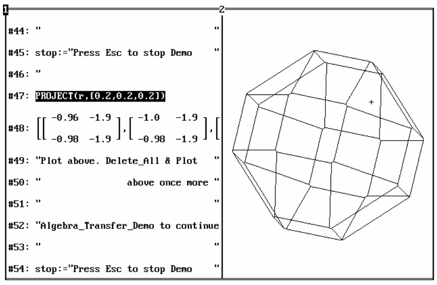
LET Heading FontSize Text X0 Y0 Z0 dX dY dZ Slope

LET Heading .05 Z=X^2-Y^2 .2 .2 1.1 .04 .04 0 0



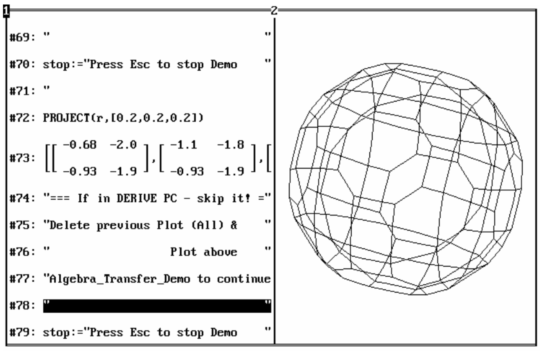
2.3 Implicit 3D Plots

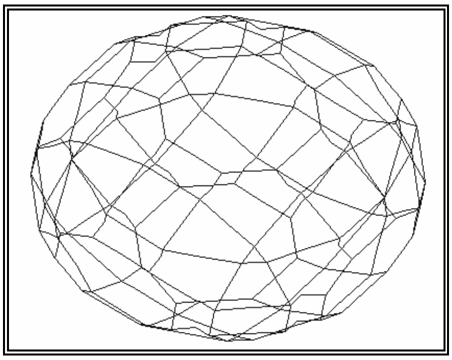




You can find all files compressed in **IMP_SURF.ARJ** except ACROSPIN and DERIVE.

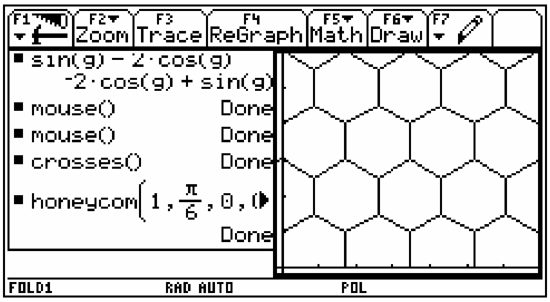
(In one of the next DNLs we will deal with Sergey´s Implicit Plots. Josef)



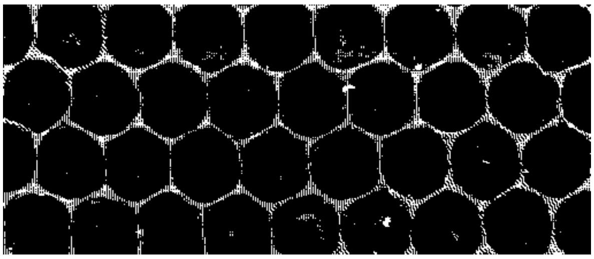


3. Simulations

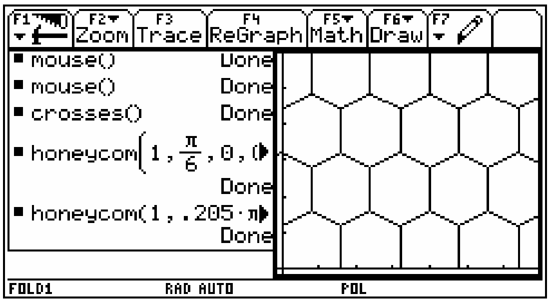
3.1 Honey Comb Shape



Minimum Wax Criterion (α=π/6)→



Equal period in x&y directions →
(α=0.205 π ≈ π/5)



3.2. "Bottle" Submarine (Cartesian Diver)

- Demonstration. Funny Play with a Bottle. Air Colas Observation
- Physics Model. Heavy Mass & a Collapsing Air__
- Math Model. Linear System
- Analysis Conditions for 2 stable points
- Physics Experiment Fine Adjustment & State of the Art Play

The Merging of Calculators and Computers: A Look to the Future of Technology Enhanced Teaching and Learning of Mathematics

Bert K. Waits and Franklin Demana, Ohio State University, USA

Introduction

Ten years ago desk top computers and calculators were viewed as quite different. Computers were powerful, expensive, and ran sophisticated software. Calculators were inexpensive and did only elementary numerical computations. Electronic calculators are now over 25 years old while desk top computers are only about 20 years old. The first electronic calculators were simple "four function" devices that did only basic arithmetic such as the Texas Instruments "DataMath" costing $120 US in 1972. They were soon followed by the so called "scientific calculator" or "electronic slide rule" that did sophisticated transcendental computations with 8 to 12 digit accuracy. The first scientific calculator was the HP-35 introduced in 1972 (it cost $395 US). The first desktop computers were slow and had little memory 32K(!) but were powerful and hinted at things to come. In 1979 the first spreadsheet VisiCalc was introduced for the Apple II PC and suddenly the world saw a reason to buy a desktop computer!

Scientific calculators are now very inexpensive ($10 to 20 US) and have significantly changed some of the mathematics curriculum taught in most countries. For example, we no longer spend valuable lecture time teaching paper and pencil methods to compute values of transcendental functions. More time could be spent on applications and conceptual understanding of transcendental functions as scientific calculator use became widespread. For many years desktop computers have remained expensive and thus still are not used nearly as widely as they should be in the teaching and learning of mathematics in colleges and universities.

Ten years ago calculators took a giant evolutionary step and added new software functionality in ROM found only desktop PC computers. These were the so-called graphing calculators, first invented by Casio in 1985. Graphing calculators started a revolution in the teaching and learning of mathematics in the United States and in many other countries as well. Inexpensive graphing calculators were really hand-held computers with built-in graphing software. Graphing calculators could be viewed as computers available to all students because of their low cost, ease of use, and portability [Demana and Waits, 1992].

Before graphing calculators, professors had to rely exclusively on expensive computers (usually housed in a separate computer laboratory) to deliver computer enhanced visualization in mathematics teaching and learning. Only a few elite colleges and universities could provide such an experience to all mathematics students on a regular basis. A CAS (computer algebra system), available usually only on expensive PC's, generally consists of three main software packages -symbol manipulating software, numerical solvers, and computer graphers. Graphing calculators provided two of these (all but the symbol manipulating software) at far less cost and often in a more user friendly environment. The pedagogical significance to the mathematics community of the small, inexpensive, hand-held graphing calculators should not be under estimated. Problem solving is now a mainstay of curriculum incorporating graphing calculators.

Graphing calculators now provide millions of students useful and exciting experiences enhancing their mathematics learning with computer visualizations. Teachers are now able to present mathematical ideas, concepts, and applications in both traditional symbolic as well as computer generated numerical and graphical representations. Powerful new approaches to learning mathematics have been made possible by graphing calculators. It is now well established in many countries that a richer mathematics curriculum is possible when all students have access to graphing calculators.

Graphing calculators do have powerful built-in numerical and graphical software. However, they lack three very significant software applications for enhancing mathematics teaching and learning that are commonly available on expensive desktop computers: computer symbolic manipulators available on most popular CAS systems such as MapleTM and DERIVETM, computer interactive geometry like CabriTM, and spreadsheets. Of particular significance for mathematics curriculum reform is student use of computer symbol manipulators and computer interactive geometry .A practical and inexpensive device for delivering powerful software for mathematics teaching and learning was needed. The next great leap in the evolution of hand-held calculators and computers was provided by Texas Instruments in 1995.

A Look to the Future: Calculators and Computers Merge

In late 1995 Texas Instruments introduced the TI-92, a relatively inexpensive hand-held computer with built-in computer symbolic algebra system (using powerful Derive™ algorithms) and computer interactive geometry (an almost complete version of Cabri II™). It was about 2 times the cost of a graphing calculator but probably 25 times more powerful! It was the first of a no doubt new generation of powerful hand-held computers for mathematics education. Other calculator manufacturers will surely follow with similar products. These new inexpensive and easy to use hand-held *personal* student computer tools contain very powerful and versatile computer software and now really represent "a computer for *all* mathematics students." These new tools and their successors will very significantly change the mathematics curriculum of the future. We will move from an almost exclusive focus on paper and pencil manipulative skills to a focus on some paper and pencil manipulative skills, mental skills, reasoning and understanding, concepts, problem solving, and applications.

Implementing a Technology Enhanced Approach to Instruction

Many professors have simply opted to *avoid* using computer symbolic algebra and computer interactive geometry in their mathematics classes because it was simply not practical or possible to do so. One reason was the high cost of computer laboratories, their upkeep, and related training issues. Dependence only on desk top computers and expensive software housed in computer laboratories is still a major barrier to implementing serious curriculum reform in mathematics. In the future hand-held computers like the TI-92 and its successors will provide an inexpensive computer lab in a suitcase so any college classroom can become a computer laboratory on demand.

The Twilight of Traditional Paper and Pencil Algebraic Manipulative Skills

In the past, teaching of traditional (paper and pencil) algebra symbolic manipulative skills in college mathematics was necessary because they were the *only* procedures available for algebraic manipulation necessary to "solve" problems. Today this simply is not the case. Computer symbolic manipulator algorithms now do algebraic and arithmetic calculations much faster and with far better accuracy than possible with the "traditional" paper and pencil methods.

We believe what is needed in the future is a university mathematics curriculum that takes advantage of computer technology to assist students in gaining mathematical understanding, in becoming powerful and thoughtful "thinkers", communicators, and problem solvers. We seek a **balanced approach** to the use of technology in the future. In the past there was none. In the future there will still be a need for mental mathematics skills (perhaps even a greater need than in the past), some paper and pencil manipulative skills, and certainly computer assisted symbolic algebra and calculus manipulative skills.

A New Challenge for Mathematics Professors

Our community can *no /onger ignore* how student use of computer symbolic algebra and computer interactive geometry can impact the mathematics curriculum. This new generation of hand-held student computer tools will no doubt become as popular as graphing calculators are today. We *must* deal with the fact that computer symbolic algebra and computer interactive geometry are better -far better - tools than paper and pencil for doing many of the "manipulations" associated with mathematics. What we must come to agreement on is what paper and pencil skills are necessary.

These new tools can also be used to *better* illustrate important concepts and applications of mathematics. We must redefine "basic skills" to include those paper and pencil manipulative tacks necessary to understand algebra as a language of representation. Some traditional paper and pencil skill will continue to be necessary for mathematical activities as will traditional mental skills. However, we *must* also agree to stop spending large portions of our time teaching obsolete paper and pencil algebra and calculus manipulations. These obsolete paper and pencil skills *must* be identified. We must determine *what are* the essential paper and pencil computation skills. That is our challenge for the future!

Examples

Consider every one's favorite algebra topic offactoring algebraic expressions. Factoring *is still* very important. After all, the *Fundamenta/ Theorem of A/gebra* is a *factoring* theorem. This theorem is central to good mathematical understanding, as are the very important connections among factors, x-intercepts of the graph of functions, zeros of functions, and the behavior of functions. Figure 1 shows the result of applying the **factor(** CAS command

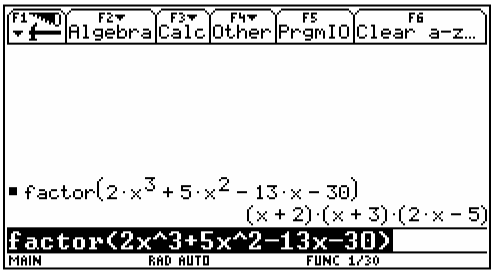


Figure 1. A CAS factoring example.

to the given cubic polynomial. The real mathematical significance of this result is not the actual factoring procedure. In some sense, we really don't care *how* the procedure is done as long as it is done correctly! What is important is the factors (by inspection) give us the solutions to the equation "expression" = O *and* reveal more information about the behavior of the polynomial function than in non-factored form. The non-factored form gives us some **global** understanding (the cubic polynomial behaves like $y = 2x^3$ for x large) and the y- intercept. However the factored form gives us a great deal of local behavior (x-intercepts and some idea of existence and location of extrema). We also like to explore the connections among the factors, zeros, and graph of the polynomial as shown in Figure 2.

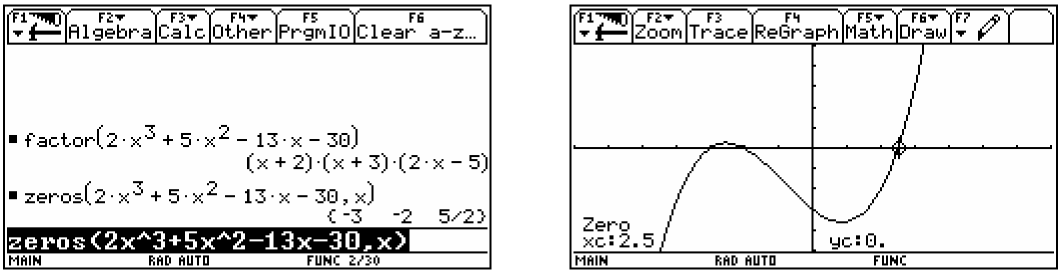


Figure 2. Exploring connections.

It is clear to us that inexpensive CAS technology will change the nature of the current style of "computing" in the teaching and learning of mathematics from an almost exclusive paper and pencil symbol manipulation approach to a more balanced approach as mentioned earlier. Clearly in the future we will still need do appropriate mental, some paper and pencil symbolic manipulations, *and;* we believe, a great deal of computer assisted symbolic manipulations. For example in the past it was common practise in algebra to require **all** computations like “perform the indicated operation and simplify: $\frac{2x-3}{x-5}-\frac{2-x}{x-7}$ “ to be done with paper and pencil methods. In the future it will become commonly recognized that a CAS tool (like the TI-92) or CAS software like MAPLE™. on a PC will provide the appropriate method of computation. Such was the case in the past with computing sin 20.125° using paper and pencil table interpolation which was replaced by a scientific calculator “tool”. For the mathematics community to deny the inevitable change is to bury our heads in the sand.

There will be some interesting surprises along the way as we move to a more balanced technology enhanced approach to teaching and learning mathematics. For example, using the problem from the previous paragraph, the result of inputting the problem in the edit line and then pressing the ENTER key on the TI-92 gives an unexpected result (to most students) as shown in Figure 3.

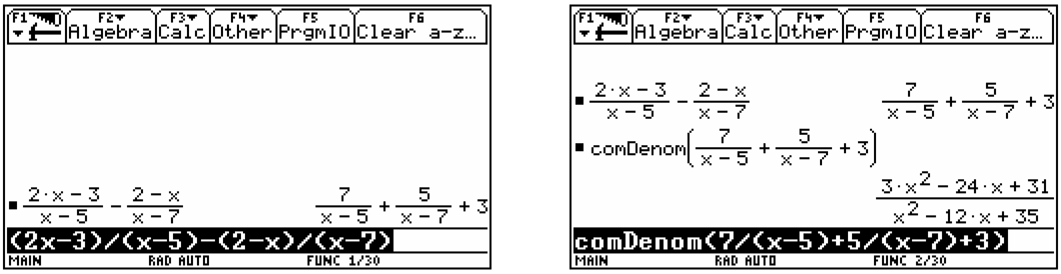


Figure 3. A surprise!

The TI -92 automatically did a partial fraction decomposition "looking" ahead to calculus. What students would usually expect to see is what is shown in the second panel of figure 3 using the comDenom(CAS command. Some may have even expected the result shown in the first panel of Figure 4 using the TI-92 **factor(** computer algebra command. Notice the

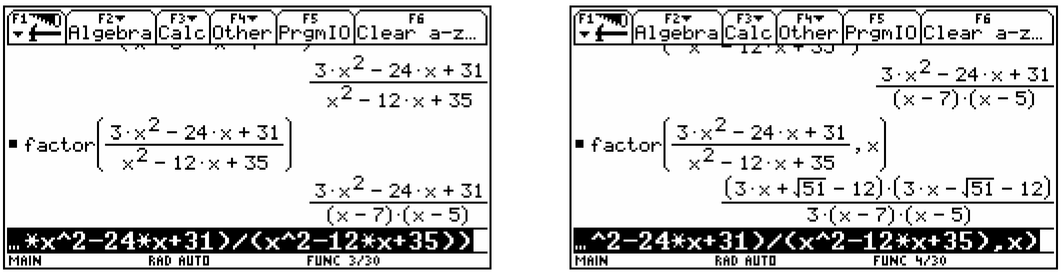


Figure 4. Some CAS commands in action.

factorization displayed is trivial (rational). No student would ever expect the result shown in the second panel of Figure 4 of using the **factor(expression, x)** syntax on the *same expression*. This syntax asks the TI-92 for *real* factors. The moral of this short excursion is that there is a learning and understanding issue from both the student and teacher of using any CAS system.

Another "partial fraction" example from calculus involves using the TI-92 computer algebra command **Expand(** as a black box procedure. Figure 5 shows the result of applying the **Expand(** CAS command to a simple rational function which one could hardly expect anyone to mentally integrate. Notice it is now very easy to integrate the result (a partial fraction decomposition) by finding anti derivatives of each part mentally from basic integration facts!

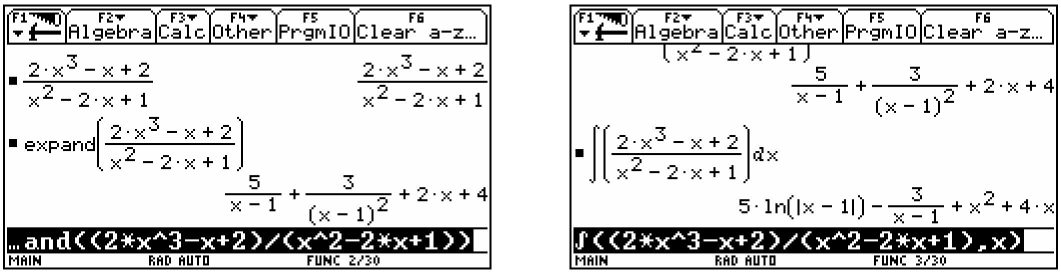


Figure 5. A CAS partial fraction decomposition and CAS integration.

The result of applying the "Integrate" CAS command *before* the CAS partial fraction decomposition is provided in the second panel of Figure 5 just to check our "mental integration" skills! .Here we use the integrate procedure as a white box procedure. That is, we allow use of some algebraic, non-calculus, black box procedures while not allowing any black box integration procedures (until the skill or concept is learned). The white-boxlblack box principle was first introduced by Professor Bruno Buchberger from the Research Institute for Symbolic Computation in Linz, Austria. This wonderful principle is outlined in detail along with other excellent examples in the book by Heugl, Klinger, and Lechner (Addison- Wesley Publishing Company, 1996).

We believe professors should definitely continue to teach techniques of factoring, partial fraction decomposition, integration, and so on, and what they mean and why they are useful. However, the *tools* they will use "to factor", "to find a partial fraction decomposition", and "to integrate" will include CAS tools as well as paper and pencil and mental tools. This means professors *should* teach the old standard and comfortable topics but should spend far *less time* with paper and pencil methods (manipulations) and far *more time* with both CAS tools and new mathematics topics possible with these tools. Our thrust should be not to delete traditional topics but to reduce the time spent and change the tools used for these topics and add some of the new mathematics content made possible.

A Mathematics Image Problem

The mathematics community must do a better job of addressing an international "mathematics image problem." The public often associates "doing mathematics" with only the mental and paper and pencil arithmetic and algebraic computations and manipulations they learned in school and college. We need to convincingly communicate to the general public that "doing mathematics" in the 21st. century means much more than "doing" the mathematics of the past. Collegiate level mathematics in the future will be far more technology enhanced, richer, interesting, and applicable than in the past.

Summary

Calculators with built-in *graphing* software for enhancing mathematics teaching and learning are now over ten years old. Casio invented and marketed the first graphing calculator in 1985 and started a revolution in delivering powerful and useful *computer graphing* to millions of mathematics students. Inexpensive graphing calculators have certainly fulfilled our dream of making it possible for *all* mathematics students to use computer visualization on a regular basis for both in-class and out-of-class activities -a dream that never could have been realized with desk-top PC's in computer labs. Texas Instruments has produced the first inexpensive, hand-held computer symbolic algebra and computer interactive geometry tool designed for mathematics and science students. There will surely be similar products from other calculator companies in the future.

We now need to be more specific and explicit about a controversial issue. We can no longer spend out time in the mathematics classroom doing everything we did in the past paper and pencil era *and* adding on the many topics and methods our students need for the technological intensive future they face. We have much to learn about our future mathematics curriculum and the details of how we will get there. However, in a short twenty years *we know* both what we are teaching and the tools we and our students are using will be dramatically different from today. It will be fun to watch this evolution.

References

Demana, Franklin and Bert K. Waits."A Computer for *All* Students." *Mathematics Teacher* 85 (February 1992): 94-95.

Waits, Bert K. and Franklin Demana. "A Computer for *All* Students -Revisited." *Mathematics Teacher* 89 (December 1996): 712-714.

Helmut Heugl, Walter Klinger, and Josef Lechner. *Mathematikunterricht mit Computeralgebra-Sytemen: Ein Didaktisches Lehrerbuch mit Erfahrungen aus dem österreichischen DERlVE-Projekt.* Addison-Wesley: Bonn, Germany, 1996.

Another picture from Sweden:

The two Davids, David Sjöstrand from Sweden and David Stoutemyer from Hawaii together with Josef from Wuermla.

The pictures are from Wolfgang Pröpper. As you can see, he is with his camera as handy as with his TI-92.

Many thanks, Wolfgang for your photographs. But unfortunately enough we don´t have your picture.

The TI-92 as a Medium in Math Classes

by Wolfgang Pröpper, Nürnberg, Germany,

held at the

Fun in Learning Mathematics Symposium at Kungsbacka, Sweden

1. Introduction

- Situation of school mathematics in the state of Bavaria, Germany

  I am working as a teacher at a "Gymnasium", a type of school that prepares pupils normally aged 10 to 19 for attending university. We have a "Zentralabitur" which means, the topics for the final examinations are the same all over Bavaria. There are positive aspects in this regulation, especially as I know quite a lot of my colleagues!! But on the other hand a Zentralabitur hinders innovations, because a chain is just as strong as it´s weakest link.

  Furthermore we have an order from 1978 (which is still in force!!) on the use of calculators in math lessons which says: calculators may do basic arithmetical operations and elementary functions, should have a memory for constants and an at least 8-digit display. But they must not be programmable.
  In the year 1988 this order got a supplement containing the words "technical innovations" and said programmability for instance also means the possibility to solve quadratic equations and the display of a calculator at schools must not be able to show graphs of functions.

  This does not require any comment.

- The situation at my school

  My school is a special "Gymnasium" for young adults (age ranging from 20 to 40, average 25) who could not get access to university at a normal Gymnasium. They must have trained for an occupation and must have been working for a while. So they are extremely achievement oriented. They want to get the best final examination at the least expense. (This does not mean our pupils were lazy, but in most cases they have to earn their living besides school and in parts have even to take care for a family or children.)

  This means that they are rather reserved with experiments, especially in mathematics which is a compulsory subject. The offer to use a symbolic calculator during normal lessons without the chance to use it in tests or even the final examination was rejected by a majority.

  (Quotation: "My conventional calculator already has more keys than I can tell apart. With the TI-92 I would get hopelessly lost.")

- Consequences for using the TI-92

  We are 7 teachers. 2 of them are using the TI-92 together with the view-screen as a medium in math-lessons. It is used to compare the results of (home-)work or for quick decisions in questions that arise from an unprepared situation. A third scope is to illustrate mathematical subjects and concepts with more or less elaborate programs.

  Three of such programs will be presented in this talk.

2. Demonstration of the examples

- To reduce too high expectations I must emphasize that the examples shown are nothing new at all. They were in various ways already presented with chalk and sponge, with transparencies and pens or with a PC and a projectable screen. The new thing is the elegant and simple posibility to introduce the TI-92 with a view-screen for making math lessons more instructive. But nevertheless I force my pupils (or at least recomend them to) do all these examples with paper and pencil.

- The three programs deal with
  - the introduction of the idea of the derivative and the derivative function,
  - Galton´s Board as a toy for introducing binomial distributions and
  - to illustrate the transition from binomial to normal distribution using standardisation.
  - All the text shown in the programs is in German but it can easily be transcribed into English or any other language. It is just a problem of time.

• Ableit()

The Program ableit() persues two aims:

1. The idea of the derivative at a point is being defined as the limit of the gradients of straight lines through this point on the graph of a function. We also regard the case that the function cannot be derived at that point.
2. Calculating these limits at different points shows that a definite, that means functional connection exists between the arguments and the derivatives of the function. This leads to the idea of the derivative function.

The realisation:

1. The program is started by ableit()
2. With F2 a (safe) input dialog is opened; but it is more comfortable to look for selected examples by F6. After an ok the graph is being plotted and the toolbar offers: slope of secants F3, gradient of tangents F4 or derivative function F5. (*Fig. A1*)
3. Preparing this lesson the pupils had to make a homework for the first example $f(x) = \frac{1}{3}x^2 - x^2 + \frac{5}{3}$. The program is used to check the homework.
4. The same function will be investigated at $x_0 = 1$. Even for $x_1 = 1$ the TI-92 yields a result: the slope of the tangent. The table of the results can be shown. The results are written down. (*Fig. A2* and *Fig. A3*)
5. The same method with the function $f(x) = |x^2 - 1| - 1$ at point 1 shows different gradients depending on the approach. That means the limit of gradients does not exist. (*Fig. A4*) The function can not be derived at x = 1.
6. Again as a homework calculating the slopes of tangents is given. The results are being controlled (*Fig. A5*) and guessings on the connection between the argument x and the value of the drivative are made more precisely.
7. With the F5 key the prior course is being automatisied and shows the derivative function in a very intuitive way. Especialy beautiful examples are f(x) = sin x (*Fig. A6*) and $f(x) = e^x$.
8. The program is closed by pressing F1 and F2. This way all garbage is collected and the program does not interfere with other applications.

• Galton()

galton() simulates a Galton´s Board. Compared with the genuine experiment the simulation has the advantage

1. to work with a device without the normal handling problems,
2. to have a hardly available Galton´s Board, with an ajustable probability of deflection and
3. to have an extremly economical device. (Probably the TI-92 with view-screen is much cheaper than a Galton´s Board produced with perfect craftsmanship.)

The realisation:

1. The program is started with galton()
2. The entering dialog first asks for the probability of deflection and then for the mode of simulation.

3. For the first experiment we choose p = 0.5 and mode 1. One can adjust the speed and trace the falling of the marble. (*Fig. G1*)
4. In mode 2 the experiment goes off automatically. One has only to indicate how many marbles are in the container. The falling down can be stopped (*Fig. G2*) with the [S]top key. If all marbles are through, their distribution can be shown in a histogram.
5. For the evaluation in lessons we use mode 2; see exercise sheet.
6. In mode 3 we make an abstraction from Galton´s Board: Only the collector for the marbles is visible (*Fig. G3*). For that the marbles are falling as quickly as the TI-92 can do. The falling down can be stopped and one can switch to a histogram (*Fig. G4*) and then return to the falling marbles again.

- BinVert()

Binomial distributions with variing parameters have different appearances. The program binvert() will show that by means of standardisation the common characteristics of different binomial distributions become visible. For standardised distributions an approximation can be given. The retransformation yields the normal distribution.

The realisation:

1. The program is started with binvert()
2. The entering dialog asks for length and probability of the related Bernoulli-chain. We start with n = 20 and p = $\frac{1}{2}$. The characteristic binomial distribution B(20; $\frac{1}{2}$) (*Fig. B1*) appears.
3. The binomial distributions B(20; 0.8) (*Fig. B2*) and B(20; 0.3) (*Fig. B3*) are calculated and plotted. Result: similar shape; maximum near the expectation.
4. From B(20; 0.3) we switch to U(20; 0.3). Then U(20; 0.8) and U(20; $\frac{1}{2}$) (*Fig. B4*). In each case the expectation is near 0 and the x-units have changed. They are 1/σ. (Plotting is more quickly, because the values don´t have to be calculated again.)
5. Attempts to find an approximation: for instance 0.4/(1+x^2) is good (*Fig. B5*) but not sufficient. The Gaussian error function suits (*Fig. B6*); perhaps show equation.
6. Returning to binomial distribution and graph of normal distribution (*Fig. B7*).
7. In both screens the affiliated cumulative distribution can be plotted (*Fig. B8* and *Fig. B9*).
7. Calculations: P(| X – μ | ≤ 3) (*Fig. B10*)

3. Conclusion

- In the meantime further programs are in progress or the ideas are still in the back of my brain. For me it is very important to exchange experiences, ideas and stimulations and criticts for instance via
  - DNL/TI-92 Corner or
  - eMail: w.proepper@wpro.franken.de or w.proepper@nk.n.by.schule.de

- But I would much more appreciate it if in my home state of Bavaria Holy Aloysius came down from heaven. He then could give a clue to those people in the department of education who are responsible for the order of 1988 which implies that solving quadratic equations or calculating tables of functions would be mathematical key-qualifications in the year of 1997. But there is still hope: The first order on electronic calculators came out in 1978. The second followed in 1988. Soon we are in the year 1998.

- The programs which were shown can be downloaded together with a little instruction as zipped *.exe files from the web. URL is http://www.kolleg.nuernberg.de/ti92.htm.
- I thank you for your attentiveness and your patience.

The programs are among the files contained in MTH28.zip which can bedownloaded, Josef

Some screen shots on Ableit(), Galton() and BinVert()

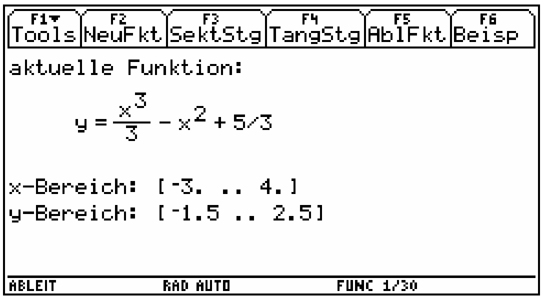


Fig. A1 The Ableit()-Screen with actual funtion

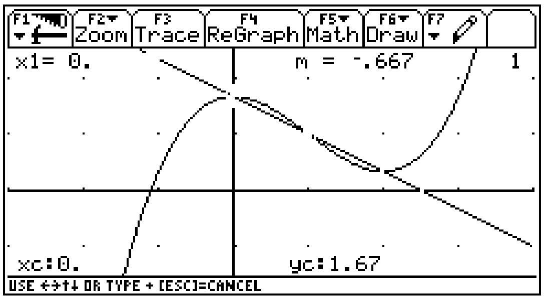


Fig. A2 Secant through $P_0(1/1)$ and $P_1(0/1.67)$

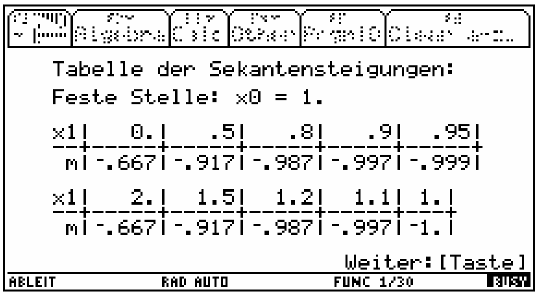


Fig. A3 Table of all slopes investigated

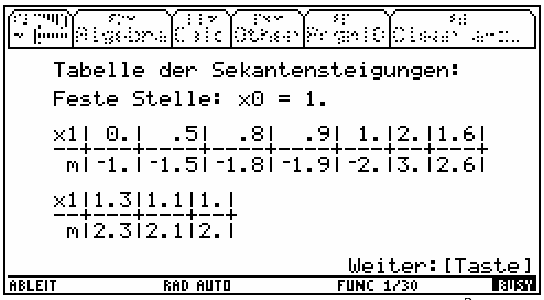


Fig. A4 Like before, but $f(x) = |x^2-1|-1$

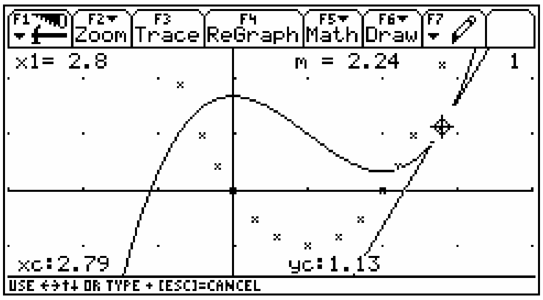


Fig. A5 Slope of tangents marked by asterisks

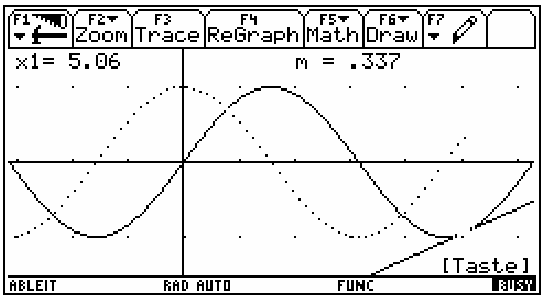


Fig. A6 Derivative function of $f(x) = \sin(x)$

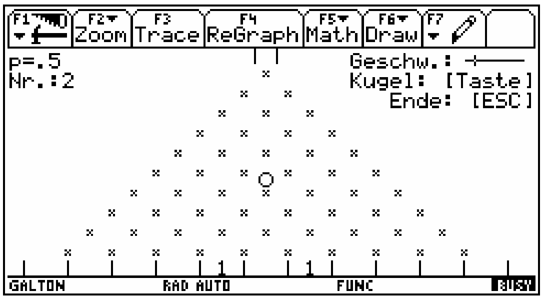


Fig. G1 Galton´s Board in mode 1

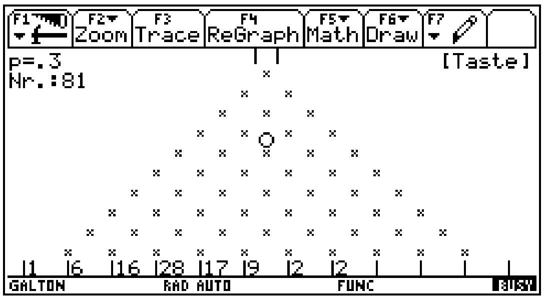


Fig. G2 GB in mode 2, stopped at marble 81

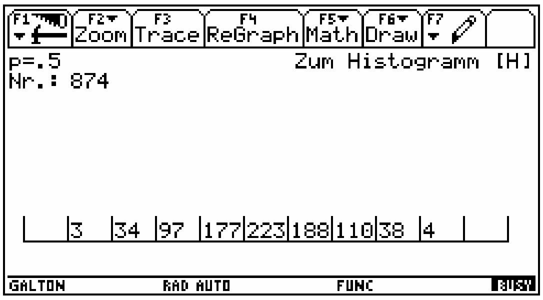


Fig. G3 GB in mode 3 after 874 marbles

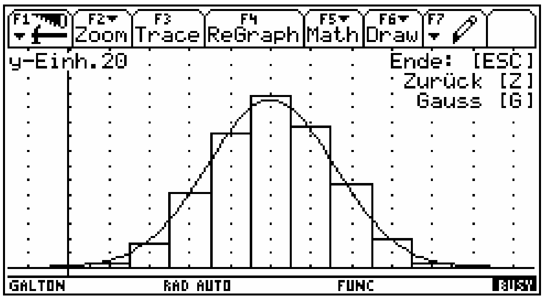


Fig. G4 Histogram of *G3* with Gaussian Curve

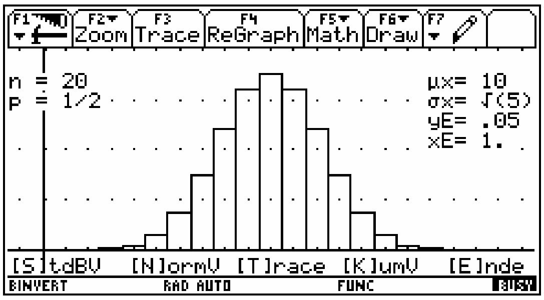


Fig. B1 Histogram of B(20; $\frac{1}{2}$)

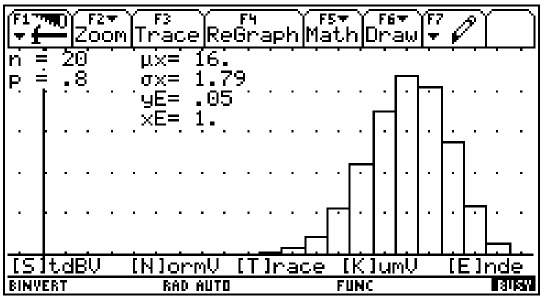


Fig. B2 Histogram of B(20; 0.8)

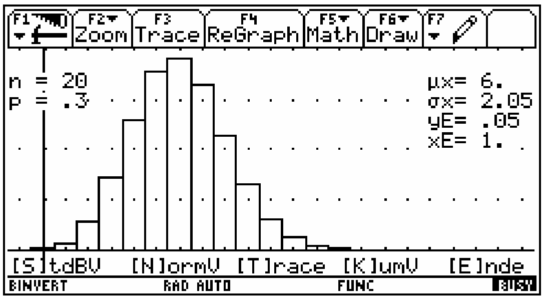


Fig. B3 Histogram of B(20; 0.3)

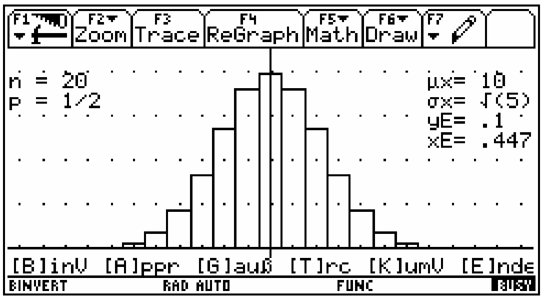


Fig. B4 Standardised histogram U(20; $\frac{1}{2}$)

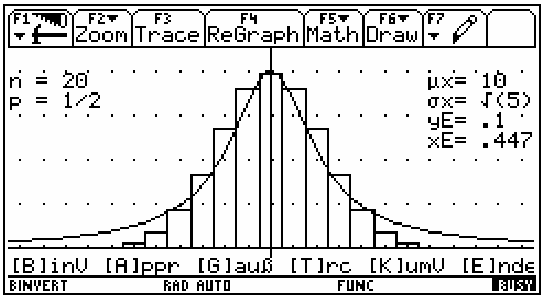


Fig. B5 Approximation by $f(x) = \frac{0.4}{1+x^2}$

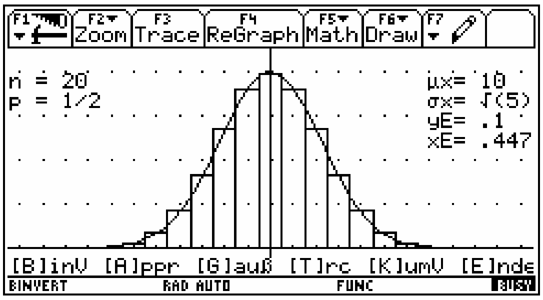


Fig. B6 The Gaussian Error Function

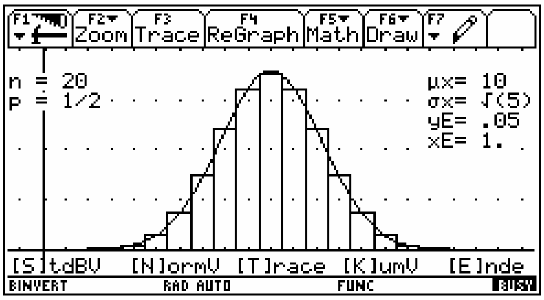


Fig. B7 The Normal Distribution

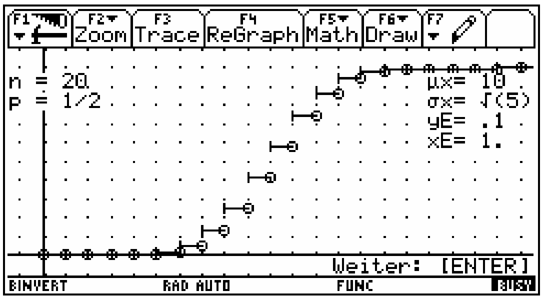


Fig. B8 Cumulative distr. belongig to *B7*

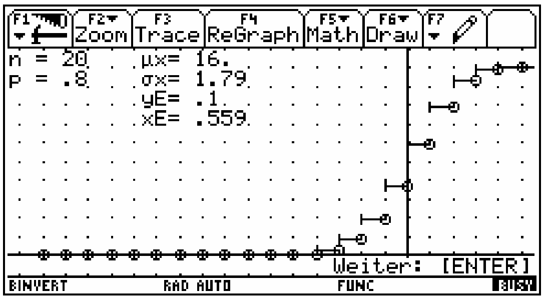


Fig. B9 Cum. distr. of U(20; 0.8); compare *B2*

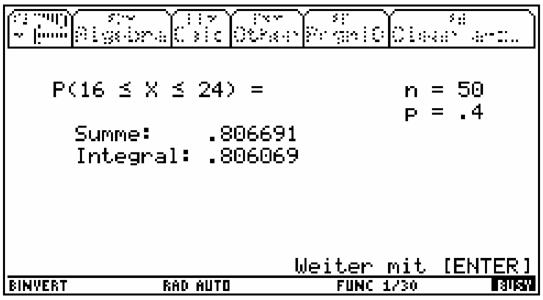
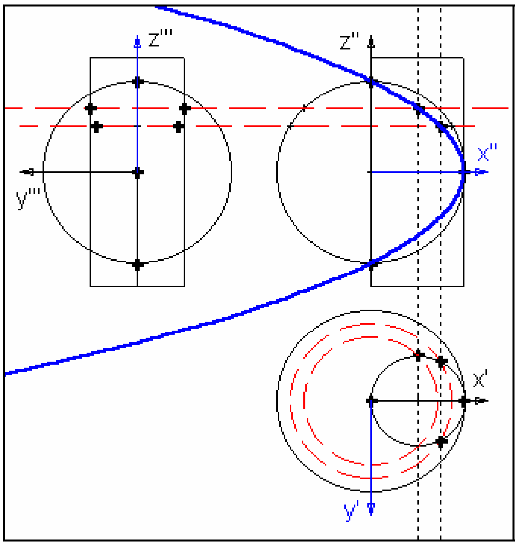


Fig. B10 An exact and an approx. calculation

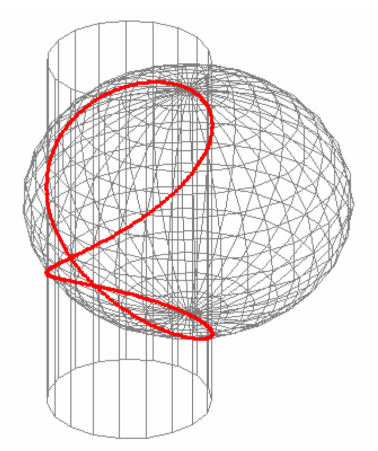
A View through the WINDOW of VIVIANI

Josef Böhm, Würmla, AUT

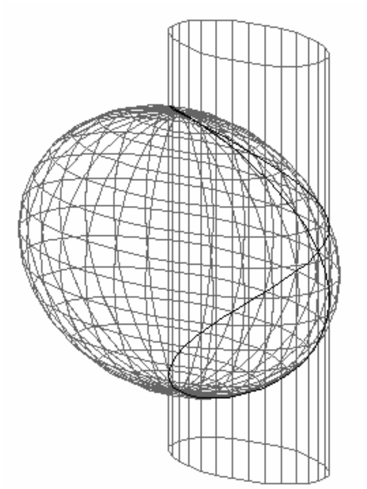
The *Hippopede* is a space curve which appears as intersection curve between a sphere and a cylinder. A special form is the *Window of Viviani*. This is the case when the sphere (radius r) is interected by an osculating cylinder with radius r/2. The figure shows the construction of this space curve. In side view the curve appears as an "Eight" - that is the "Window" -, in front view as a parabola.



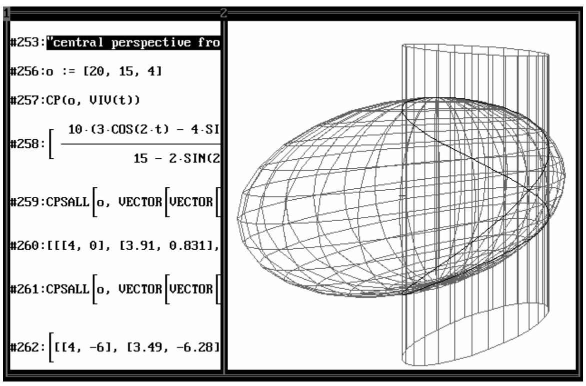
The Hippopede is a remarkable curve. We will use *DERIVE*'s algebraic capabilities to find the equation of the curve and further to find several differential geometric objects which are connected with that curve. The utility file GRAPHICS.MTH from SWHH provides a lot of tools for calculating and representing these objects as well. We only have to select them and pick up the appropriate tools. But using the provided tools we are restricted on isometric projection. An additional package of functions could help to produce other projections like oblique view, axonometric projection and central perspective projection.



isometric projection



oblique view (k=2/3, α=30°)



central perspective

Many of us have ACROSPIN at their disposal, a wonderful cheap piece of software, which produces animated presentations in different colours and layers - and shows some other nice features, too - in a very comfortable way. Unfortunately *DERIVE* is not able to save arbitrary 3D-objects in ACROSPIN-format. A self written program makes that possible now (see *DNL*#24). We will use ACD to present the following objects (in a very tight connection to *DERIVE*).

In times of DERIVE 6 we don't need ACROSPIN for presenting 3D objects, Josef.

I intend to use *DERIVE* to calculate some objects which are connected with the "Window" and then produce some projections of these objects (isometric, axonimetric, central perspective projection). At last I´ll try to animate the objects using ACROSPIN and ACD as well.

List of objects:

Sphere, Cylinder
Intersection curve (Window of Viviani)
Tangents in the double point (singularity)
Surface of the tangents (torse)
Surface of the normals
Surface of the binormals
some "accompanying Three-Pods" (tangent, normal and binormal in one point)
Osculating circles
Tube

The problem divides into two main parts:

- Calculate the object and represent the object by a list (or by several lists) of points
- Find a projection of these lists of points to achieve a 2D representation of the 3D object

or

- Find a way to animate the object using ACROSPIN.

I want to describe the calculations and use various projections (You can find the whole procedure in the file VIVPROJ.dfw). Let me begin with the calculation. The file is headed by some projection tools. The functions will not be displayed here.

```
#31:  A sphere with r = 4 osculates a straight cylinder Zylinder (r = 2) from the inside

#32:  Cylinder with r = 2

#33:  cyl := [2 + 2·COS(t), 2·SIN(t), k]

#34:  Sphere with r=4, C(0/0/0)

#35:  sph := [4·COS(u)·COS(v), 4·COS(u)·SIN(v), 4·SIN(u)]

                 2     2     2
#36:  sph2 := x  + y  + z  = 16

                                                          2
#37:  SUBST(sph2, [x, y], [cyl , cyl ]) = (8·COS(t) + z  = 8)
                              1     2

                                 2
#38:  SOLUTIONS(8·COS(t) + z  = 8, z)

#39:  [2·√2·√(1 - COS(t)), - 2·√2·√(1 - COS(t))]

#40:  Substitution t --> 2t

#41:  SUBST([2·√2·√(1 - COS(t)), - 2·√2·√(1 - COS(t))], t, 2·t)

#42:  [4·|SIN(t)|, - 4·|SIN(t)|]

#43:  Parameter form of the Hippopede - the Window of Viviani

#44:  VIV(t) := [2 + 2·COS(2·t), 2·SIN(2·t), 4·SIN(t)]
```

```
#46:  1. Isometric: plot the next expressions

#47:  ISOMETRICS([2 + 2·COS(t), 2·SIN(t), k], t, 0, 2·π, 19, k, -6, 6, 1)

#48:  COPROJECTION(ISOMETRICS([2 + 2·COS(t), 2·SIN(t), k], t, 0, 2·π, 19, k, -6, 6, 1))

#49:  ISOMETRICS([4·COS(u)·COS(v), 4·COS(u)·SIN(v), 4·SIN(u)], u, 0, 2·π, 19, v, 0, 2·π, 19)

#50:  COPROJECTION(ISOMETRICS([4·COS(u)·COS(v), 4·COS(u)·SIN(v), 4·SIN(u)], u, 0, 2·π, 19, v, 0, 2·π, 19))

#51:  ISOMETRIC(VIV(t))
```

See page 41!

In the next step we create an oblique view of the curve with a kind of shading for the "Eight-Curve" together with the tangents in the double point.

```
#53:  Shading the intersection figure with π/2 <= t <= 3π/2

      ┌ 2 + 2·COS(2·t)    2·SIN(2·t)    4·SIN(t) ┐
#54:  │                                           │
      └ 2 + 2·COS(2·t)  - 2·SIN(2·t)    4·SIN(t) ┘

#55:  Tangents in the double point (t = 0, t = π)

#56:  VIV(0) = [4, 0, 0]

             d
#57:  lim  ── VIV(t) = [0, 4, 4]
      t→0  dt

#58:  [4, 0, 0] + k·[0, 4, 4]

#59:  normalizing the direction vector

#60:  √([0, 4, 4]·[0, 4, 4]) = 4·√2

                     k·[0, 4, 4]
#61:  [4, 0, 0] + ─────────────
                        4·√2

#62:  [4, 0.5·√2·k, 0.5·√2·k]

#63:  end points of the tangent segment

#64:  VECTOR([4, 0.5·√2·k, 0.5·√2·k], k, -3, 3, 6)

      ┌ 4   -2.12   -2.12 ┐
#65:  │                    │
      └ 4   2.12    2.12  ┘

#66:  the 2nd tangent - much quicker now

#67:  [2 + 2·COS(2·π), 2·SIN(2·π), 4·SIN(π)]

#67:  [2 + 2·COS(2·π), 2·SIN(2·π), 4·SIN(π)] = [4, 0, 0]

#68:  [- 4·SIN(2·π), 4·COS(2·π), 4·COS(π)] = [0, 4, -4]

                     k·[0, 4, -4]
#69:  [4, 0, 0] + ──────────────
                         4·√2

#70:  [4, 0.5·√2·k, - 0.5·√2·k]
```

#71: or:

#72: $\mathrm{VIV}(\pi) + k\cdot \mathrm{SIGN}\left(\lim_{t\to\pi} \frac{d}{dt}\mathrm{VIV}(t)\right) = \left[4, \frac{\sqrt{2}\cdot k}{2}, -\frac{\sqrt{2}\cdot k}{2}\right]$

#73: $\left[4, \frac{\sqrt{2}\cdot k}{2}, -\frac{\sqrt{2}\cdot k}{2}\right]$

#74: VECTOR([4, 0.5·√2·k, − 0.5·√2·k], k, -3, 3, 6)

#75: $\begin{bmatrix} 4 & -1.5\cdot\sqrt{2} & 1.5\cdot\sqrt{2} \\ 4 & 1.5\cdot\sqrt{2} & -1.5\cdot\sqrt{2} \end{bmatrix}$

#76: $\begin{bmatrix} 4 & -2.12 & 2.12 \\ 4 & 2.12 & -2.12 \end{bmatrix}$

#77: Put #65 and # 76 together into one matrix

#78: $\left[\begin{bmatrix} 4 & -2.12 & -2.12 \\ 4 & 2.12 & 2.12 \end{bmatrix}, \begin{bmatrix} 4 & -2.12 & 2.12 \\ 4 & 2.12 & -2.12 \end{bmatrix}\right]$

#79: The oblique view with k=2/3 and α=30°

#80: $\mathrm{OBLS}\left(\left[\begin{bmatrix} 4 & -2.12 & -2.12 \\ 4 & 2.12 & 2.12 \end{bmatrix}, \begin{bmatrix} 4 & -2.12 & 2.12 \\ 4 & 2.12 & -2.12 \end{bmatrix}\right], \frac{2}{3}, 30\cdot 1°\right)$

#81: $\mathrm{OBL}\left(\mathrm{VIV}(t), \frac{2}{3}, 30\cdot 1°\right)$

#82: $\mathrm{OBLS}\left(\mathrm{VECTOR}\left(\begin{bmatrix} 2 + 2\cdot\mathrm{COS}(2\cdot t) & 2\cdot\mathrm{SIN}(2\cdot t) & 4\cdot\mathrm{SIN}(t) \\ 2 + 2\cdot\mathrm{COS}(2\cdot t) & -2\cdot\mathrm{SIN}(2\cdot t) & 4\cdot\mathrm{SIN}(t) \end{bmatrix}, t, \frac{\pi}{2}, \frac{3\cdot\pi}{2}, \frac{\pi}{50}\right), \frac{2}{3}, 30\cdot 1°\right)$

...
...

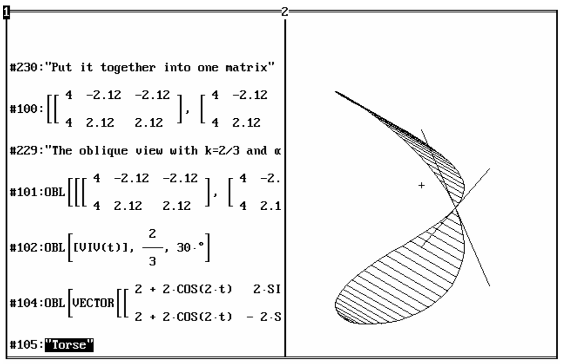
#90: We'd like to add the surface of the tangents – in German: die Torse

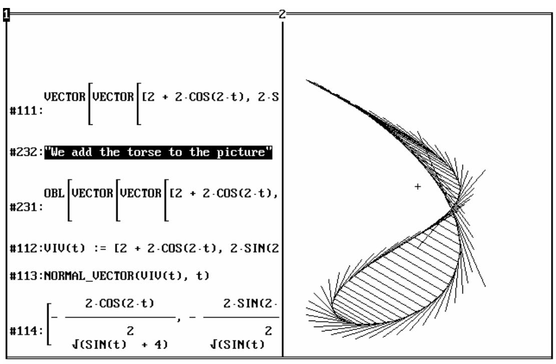
#91: $\mathrm{VIV}(t) + k\cdot\mathrm{SIGN}\left(\frac{d}{dt}\mathrm{VIV}(t)\right)$

#92: $\left[2\cdot\mathrm{COS}(2\cdot t) - \frac{k\cdot\mathrm{SIN}(2\cdot t)}{\sqrt{(\mathrm{COS}(t)^2 + 1)}} + 2, \frac{k\cdot\mathrm{COS}(2\cdot t)}{\sqrt{(\mathrm{COS}(t)^2 + 1)}} + 2\cdot\mathrm{SIN}(2\cdot t), \frac{k\cdot\mathrm{COS}(t)}{\sqrt{(\mathrm{COS}(t)^2 + 1)}} + 4\cdot\mathrm{SIN}(t)\right]$

#93: We include this surface

#94: $\mathrm{OBLS}\left(\mathrm{VECTOR}\left(\mathrm{VECTOR}\left(\mathrm{VIV}(t) + k\cdot\mathrm{SIGN}\left(\frac{d}{dt}\mathrm{VIV}(t)\right), k, 0, 2, 2\right), t, 0, 2\cdot\pi, \frac{\pi}{40}\right), \frac{2}{3}, 30\cdot 1°\right)$



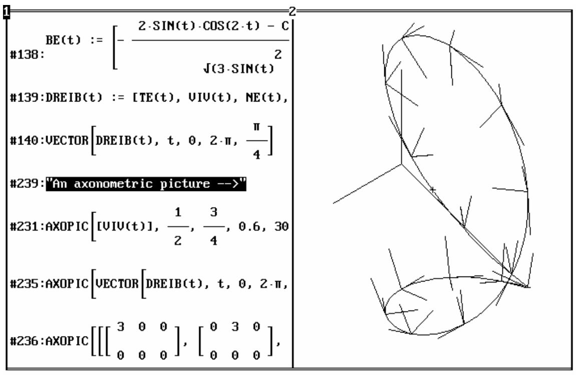


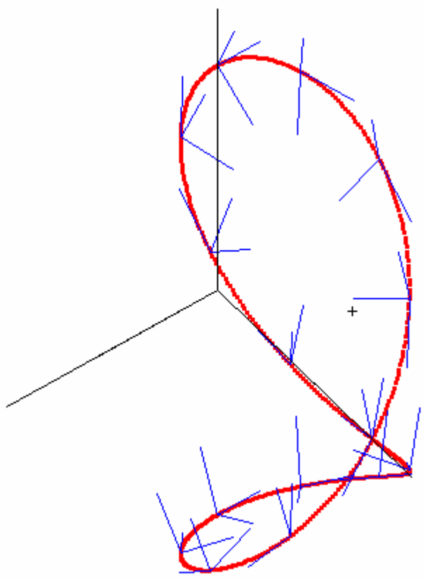
In a similar way we would find the surfaces of the normals and binormals. In each point of the curve there is a "Threepod" (*Frenet frame*) built up by the tangent unit vector, the normal normal unit vector and the binormal unit vector. We will represent the curve with some of these "Accompanying Threepods" in an axonometric view. I show how to find the endpoint of the normal unit vector:

```
#115: NE(t) := VIV(t) + 1·NORMAL_VECTOR(VIV(t), t)
```

TE(t), NE(t) and BE(t) are the endpoints of the unit vectors of the tangent, the normal and the binormal, respectively. The point on the curve VIV(t), TE(t), NE(t) and BE(t) build up a "space polygon" and are describing the "Threepod" THREEPOD(t):

```
#121: the Frenet frame - das begleitende Dreibein

#122: THREEPOD(t) := [TE(t), VIV(t), NE(t), VIV(t), BE(t)]

#123: VECTOR(THREEPOD(t), t, 0, 2·π, π/4)

#124: An axonometric picture -->

#125: AXOPIC(VIV(t), 2/3, 3/4, 4/5, 120·1°, 135·1°)

#126: AXOPICS(VECTOR(THREEPOD(t), t, 0, 2·π, π/8), 2/3, 3/4, 0.8, 120·1°, 135·1°)

#127: AXOPICS([[5 0 0; 0 0 0], [0 5 0; 0 0 0], [0 0 5; 0 0 0]], 2/3, 3/4, 4/5, 120·1°, 135·1°)
```





The Curve from 1997 and from 2012 (right).

In the file you can find additional calculations and representations of osculating circles in some points. But I will close with the most challenging figure and most challenging form of representation as well.

We will have a tube around our space curve and produce a central persepctive view of it. SPACE_TUBE() from GRAPHICS.MTH will support us: The eye position is placed in (10,8,6). CP(eye,curve) and CPS(eye,families) give pictures of a single curve or of a family of parameter lines. CPSALL(eye,families) shows both families of parameter lines.

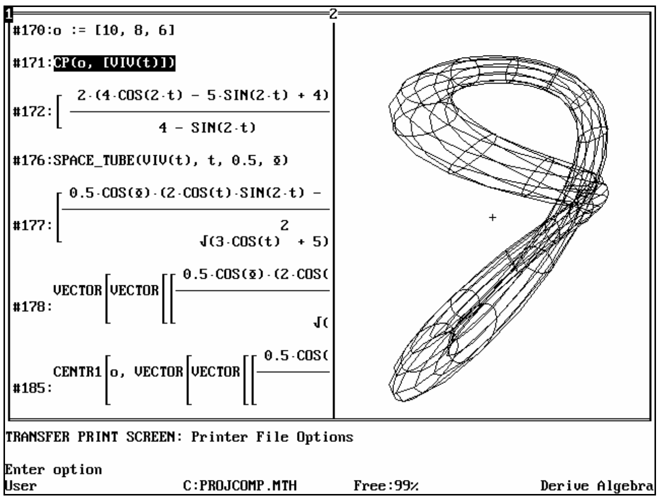
#216: o := [10, 8, 6]

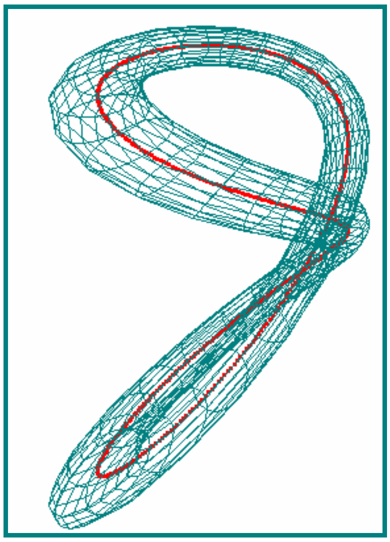
#217: CP(o, VIV(t))

#218: SPACE_TUBE(VIV(t), t, 0.5, φ)

#219: $$\left[\frac{\cos(\phi)\cdot(2\cdot\cos(t)\cdot\sin(2\cdot t)-\sin(t)\cdot\cos(2\cdot t))}{2\cdot\sqrt{3\cdot\cos(t)^2+5}}-\frac{\sin(\phi)\cdot\cos(2\cdot t)}{\sqrt{\sin(t)^2+4}}+2\cdot\cos(2\cdot t)+2,\ -\frac{\cos(\phi)\cdot(2\cdot\cos(t)\cdot\cos(2\cdot t)+\sin(t)\cdot\sin(2\cdot t))}{2\cdot\sqrt{3\cdot\cos(t)^2+5}}-\frac{\sin(\phi)\cdot\sin(2\cdot t)}{\sqrt{\sin(t)^2+4}}+2\cdot\sin(2\cdot t),\ \frac{\cos(\phi)}{\sqrt{3\cdot\cos(t)^2+5}}-\frac{\sin(\phi)\cdot\sin(t)}{2\cdot\sqrt{\sin(t)^2+4}}+4\cdot\sin(t)\right]$$

#220: $$\text{CPSALL}\left(o,\ \text{VECTOR}\left(\text{VECTOR}\left(\left[\left[\frac{\cos(\phi)\cdot(2\cdot\cos(t)\cdot\sin(2\cdot t)-\sin(t)\cdot\cos(2\cdot t))}{2\cdot\sqrt{3\cdot\cos(t)^2+5}}-\frac{\sin(\phi)\cdot\cos(2\cdot t)}{\sqrt{\sin(t)^2+4}}+\right.\right.\right.\right.\right.$$

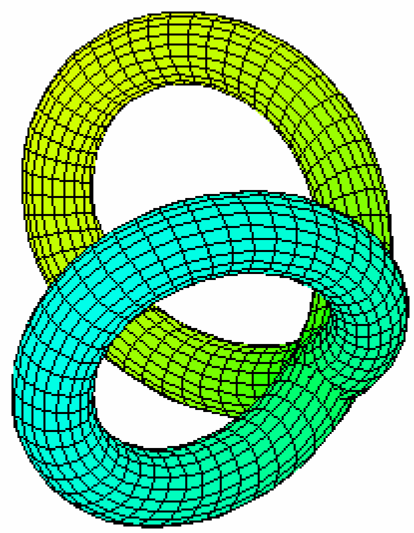


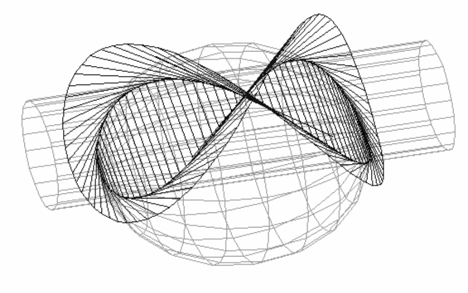


The graph from DOS-times (left) compared with the DERIVE 6 2D-plot (right).

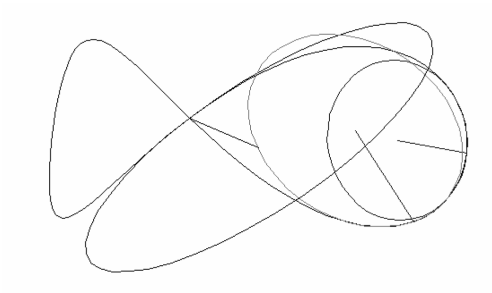
In DERIVE 6 you can immediately plot #218 in the 3D Plot Window in order to obtain a nice representation of the "Viviani Tube".

The next page shows more pictures produced as projections (from 1997) and as 3D plots (from 2012).

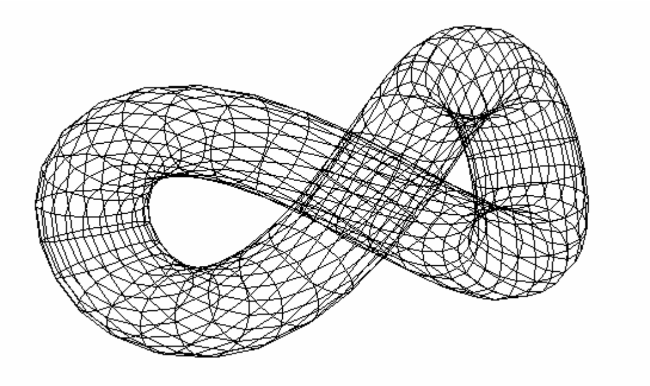




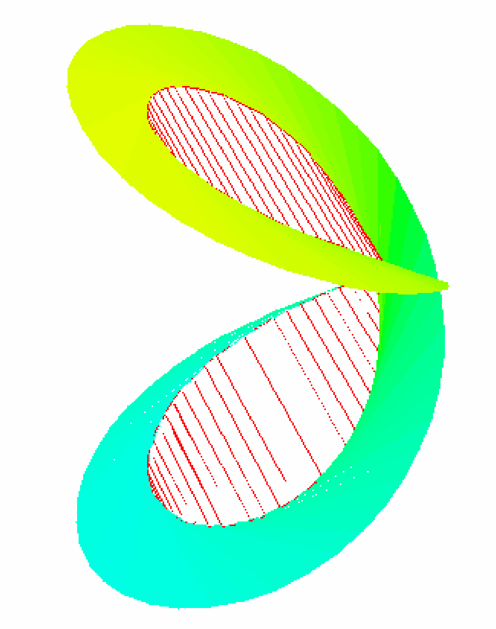
Sphere, Cylinder, Surface of tangents

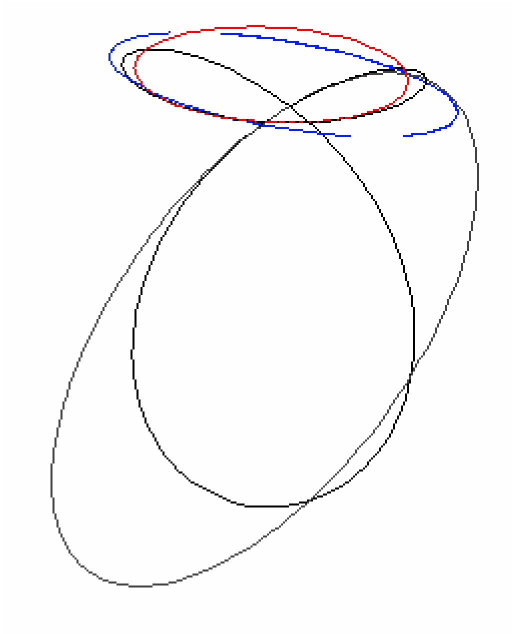


Space curve & some osculating circles & normals



The tube





Unfortunately the graphic capabilities of *DERIVE 6* are not as excellent as its algebraic ones. It seems to be that the "hand made" 2D projections are looking quite nicer in some cases. *DERIVE 7* should have performed better!!

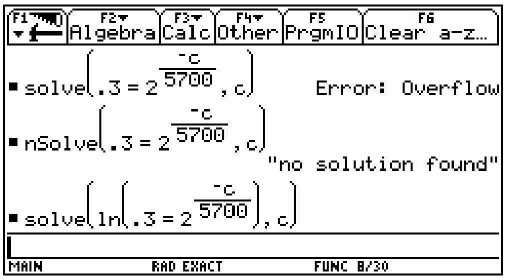
Hubert Voigt, Perg, AUT

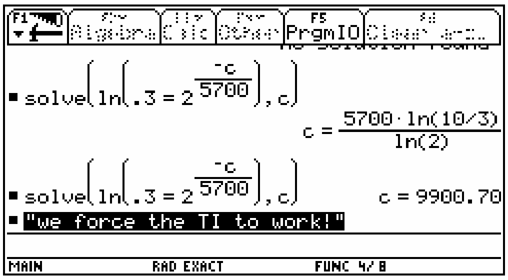
I came across a strange behaviour of the *TI*-92 solving ordinary half life problems. The exponential equation

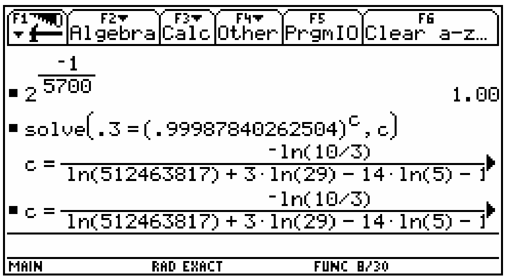
$0.3 = 2^{-\frac{c}{5700}}$ seems to be unsolvable by the *TI*-92.

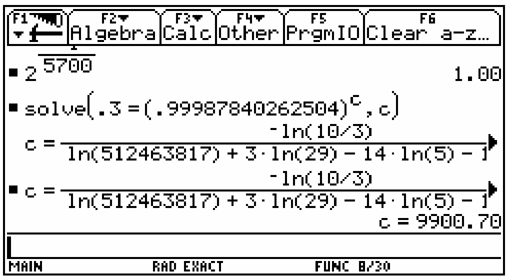
DNL: Hubert, you are right. it needs some tricks to overcome the *TI*'s boundaries.
I reported this David Stoutemyer, he confirmed the fact and answered, that the bug has been fixed. Josef

```
C:solve(.3=2^(ªc/5700),c)
C:nSolve(.3=2^(ªc/5700),c)
C:solve(ln(.3=2^(ªc/5700)),c)
C:solve(ln(.3=2^(ªc/5700)),c)
C:"we force the TI to work!"
C:2^(ª1/5700)
C:solve(.3=(.99987840262504)^c,c)
C:c=ªln(10/3)/(ln(512463817)+3*ln(29)-14*ln(5)-11*ln(2))
```









Albert White, New York **AWHITE@sbu.edu,BBG**

How do I get Derive to display the TAYLOR(1/x,x,1,5), Taylor polynomial approximation for 1/x about 1 as

$$1 - (x - 1) + (x - 1)^2 - (x - 1)^3 + ..$$

instead of

$$- x^5 + 6*x^4 - 15*x^3 + 20*x^2 - 15*x + 6?$$

Thank you for the assistance. Al

Terence Etchells, Liverpool T.A.ETCHELLS@livjm.ac.uk

Hi,
Get the DERIVE form $-x$^5+6x^4+ etc. then substitute (u+1) into this expression for x, simplify it and then substitute (x–1) for u but do not simplify. Voila!

$$\#1: \quad \text{TAYLOR}\left(\frac{1}{x}, x, 1, 5\right)$$

$$\#2: \quad -x^5 + 6\cdot x^4 - 15\cdot x^3 + 20\cdot x^2 - 15\cdot x + 6$$

$$\#3: \quad -(u+1)^5 + 6\cdot(u+1)^4 - 15\cdot(u+1)^3 + 20\cdot(u+1)^2 - 15\cdot(u+1) + 6$$

$$\#4: \quad -u^5 + u^4 - u^3 + u^2 - u + 1$$

$$\#5: \quad -(x-1)^5 + (x-1)^4 - (x-1)^3 + (x-1)^2 - (x-1) + 1$$

Leonardo, Firenze 425631_VOLPI_LEONARDO.FI@mailbox.enel.it

Dear Al,
If your problem is to make the Taylor polynomial form similar to the book one, I suggest another drastic method. Do not generate the polynomial!

I explain better: For my students I developed this simpler function:

```
TAYLOR_TERMS(y,x,x0,n):=FACTOR(VECTOR(1/i!*LIM(DIF(y,x,i),x,x0,0)*
                              (x-x0)^i,i,0,n),x)
```

This produces a vector containing each term of polynomial expansion, in ascending order from left to right.
For example:

```
TAYLOR_TERMS(1/x,x,1,6)
```

Looks like to:

$$[1,\ 1-x,\ (x-1)^2,\ (1-x)^3,\ (x-1)^4,\ (1-x)^5,\ (x-1)^6]$$

and

```
TAYLOR_TERMS(ln(x),x,1,5)
```

Looks like to:

$$0,\ x-1,\ -\frac{(x-1)^2}{2},\ \frac{(x-1)^3}{3},\ -\frac{(x-1)^4}{4},\ \frac{(x-1)^5}{5}$$

Well, they are not true polynomials, but my students seem not too disturb for that and they can easily check them with their exercises made by hand. In the same way, I developed a function for Taylor 2D expansion.
If you have same problems with this function, please tell me. I will send you my file.
Best regards

Leonardo

Rüdiger Baumann baumann-celle@t-online.de

Lieber Herr Böhm, >> Dear Josef <<,

many thanks for DNL#27 with a lot of interesting contributions. The discussion between R.Schaeck, T.Etchells and W. de Jong about the sum of the digits (pages 8-9) shows that the concept of recursion is not sufficiently spread even within the *DERIVE* - community. See the **recursive solution** of the problem:

(DIGREC.MTH)

```
       QS(num) :=
         If num = 0
#1:         0
            MOD(num, 10) + QS(FLOOR(num, 10))

#2:    QS(1234) = 10

              3
            9
#3:    QS(9  ) = 3249

       QZ(num) :=
         If num ≤ 9
#4:         num
            QZ(QS(num))

              3
            9
#5:    QZ(9  ) = 9
```

You can find another solution of this problem in Johann Wiesenbauer´s Titbits (page??). Josef

Christof earl@wohnheim.wad.org

Could anyone provide a formula for Derive to calculate the amount of litres contained in an oil-tank (a lying regular cylinder) ? The variables are the length of the cylinder, the radius and the height of oil.

front view

```
      ----
    -      -
  -_________-     ← = heigth of oil (given in centimetres from the bottom
    -      -
      ----
```

P.S.: I hope this isn't off topic. I just like to calculate things from everyday life in *DERIVE*.

Johann Wiesenbauer, Vienna J.Wiesenbauer@tuwien.ac.at

Dear Christof,

Defining r as a positive variable and simplifying

2·l·INT(SQRT(r^2 - x^2), x, r - h, r) (l length, r radius, h height)

should do the trick. Regards, Johann

Allan Holmes, Lincoln, USA alholmes@ucollege.edu

The *TI*-92 , and several other graphics calculators, have list based statistics available. Lists are especially useful in statistical analysis, but also graphing families of curves, doing similar calculations on multiple values, and collecting values for further calculation.

Does *DERIVE* have the capability of dealing with lists? Has anyone written a utlity file for this purpose? Is one being developed?

Al Rich, SWHH, Hawaii **swh@aloha.com**

In *DERIVE* lists are called vectors. Lists of lists are called matrices. *DERIVE* has extensive built-in and library functions for manipulating vectors and matrices. Statistical functions include functions for least-squares curve fitting, standard deviation, root mean square, chi-square distribution function, etc.

DERIVE also supports sets (i.e. unordered lists) with set operators (union, intersection, etc) and Boolean algebra simplification. Aloha, Albert D. Rich, Applied Logician

Jan Vermeylen, Kapellen, BEL **math@rhombus,be & http://www.rhombus.be**

Please find attached a simple utility file for basic univariate stat plots with *DERIVE,* just like the ones you can obtain on a calculator.

`a:=` is a vector (list) with given raw statistical data (which can be imported from a spreadsheet!!)

Simplify **`FREQTAB(a)`** to obtain a table of all the data in numerical order with their respective frequency.

Simplify and Plot **`HISTO(a)`** to obtain the plot of `FREQTAB`.

Simplify **`GROUP_FREQTAB(a,n)`** to obtain a table of all data grouped in n classes with their resp. frequency.

Simplify and Plot **`BLOCKS(a,n)`** to obtain a plot of `GROUP_FREQTAB(a,n)`.

I wonder if there is a more simple way to get the data of a vector in a numerical order.
Greetings from Belgium, Jan Vermeylen.

José Luis Llorens, Valencia, ESP **llorens@mat.upv.es**

DfW is working clearly worse than DfD in this example:

```
#1: int(sqrt(1+(4x^2)/(9x^2-81)),x,3,10)
#2: (Dubious accuracy, 0)
```

The expression #2 is approX(#1). I have tried to change the Precision, choose Branch-Any, etc. and I always obtain the same result.

What's the meaning of 0?

This result is useless to conjecture if the integral is convergent.

In DfD, we obtain the same message (Dubious accuracy) but, also WE OBTAIN a result depending on the precision: 9.11, 9.63721, 9.64305033, with 3, 6 and 9 digits respectively, which leads us to "suppose" that the integral must be convergent, and that its value is, aproximately, 9.64...

Therefore, with DfW none of this is possible. It is possible to correct this problem?

DNL: *DFW 4.07 works correctly*:

$$\int_3^{10} \sqrt{\left(1 + \frac{4\cdot x^2}{9\cdot x^2 - 81}\right)}\, dx = 9.63721$$

But there were two other interesting answers:

G.F.Feissner, USA **feissnerg@snycorva.cortland.edu**

The integral int(sqrt(1+(4x^2)/(9x^2–81)),x,3,10)

is a convergent integral, by comparison to int(1/sqrt(x^2–9), x, 3, 10)

(it seems simpler to get that factor of 9 from the denominator out of the way).

Interestingly enough, DfD gives a value (with the warning of dubious accuracy) which seems to be approximated by expressions of the form

int(1/sqrt(x^2–9), x, 3.001, 10) (I used 3.1, 3.01, 3.001, etc)

The improper integral, when attempted with DfW does give the value 0 (with the dubious accuracy warning).

Interestingly enough, when the lower limit is replaced by 3.00006, a value in agreement with DfD is returned, but when a lower limit of 3.00005 is used, the value of 0 is returned.

This certainly isn't an explanation, but does, I hope, provide a little more basis from which to seek one.

George F. Feissner

Leonardo, Firenze **425631_VOLPI_LEONARDO.FI@mailbox.enel.it**

Hi everyone,
As George F. Feissner and Jose Luis Llorens explained in their last letters about generalized integrals, DERIVE for Windows is less powerful (only for this subject, of course) than the older version for DOS . It happens when we should approximate an irresolvable generalized integral.
Surprisingly, the problems proposed can be solved (approx) by DFW, with substitution technique that trasforms the closed finite interval in a semi-infinite interval. That is:
INT(SQRT(1 + 4x^2/(9x^2 - 81)), x, 3, 10) (= 9.643883487...)
substitution the variable x = 1/(3t)+3, and rearranging, we get:
INT(SQRT((324t^2 + 234t + 13)/(18t + 1))/(9t^2), t, 1/21, +inf)
Assume to work with a Penthium 133 machine and Win/95 in In DERIVE 4.03 forWindows we get, with 6 precision-digits, in about 3 sec : 9,64106 (error about 10^-3) in In DERIVE 2.03 for DOS we get, with 6 precision-digits, in about 1 sec :
9,64100 (error about 10^-3)

But many times the same thing does not so well.
For example take the following semi-infinite integral:
INT(1/SQRT(x^3 + 1), x, 0, +inf) , (= 2.804364210....)
In DERIVE 2.03 for DOS
with 6 precision-digits , after 0.6 sec , we get: 2.80128 (error about 10^-3)
with 10 precision-digits , after 3.2 sec , we get: 2.804227.. (error about 10^-4)
with 15 precision-digits , after 26 sec , we get: 2.804358... (error about 10^-6)
(Note the characteristic sharply growth of the time with precision).
DERIVE gives also the warning of dubious accuracy because, properly, not all the digits displayed are correct.
In DERIVE 4.03 of Windows we get only the message above and 0 value and this is quite useless.

In other case, with functions of higher infinitesimal order, DFW can compute a good approximate value, but it takes much more time the older DFD.

For example take the following semi-infinite integral:
INT(1/SQRT(x^6+ 1), x, 0, +inf) . (= 1.402182105....)
In DERIVE 2.03 for Windows , (finally) with 10 precision-digits , after 30 sec , we get: 1.40218210 (error about 10^-9)
while in DERIVE 2.03 for DOS , we reach the same result in about 3 sec ,.

I do not know why the powerful algorithm of DOS version works so different in the Windows version (I sent it back to Soft Warehouse) but, at this moment, the fact is that DFW can not help very well about this kind of problems.
To overcome this, it is possible to program an algorithm to numerically approximate semi-infinite integrals, like DERIVE do it. This kind of problems can be resolved by adaptive quadrature-algorithms. As this formulas are not so diffuse as the other , I write below the formulas (they come from 4? order Cavalieri-Simpson's rule) that I have used:

dI = h/3(f(x0) + 4f(x0+h) + 2f(x0+2h) + 4f(x0+3h) + f(x0+4h))
E = h/45 (–f(x0) + 4f(x0+h) – 6f(x0+2h) +4f(x0+3h) – f(x0+4h))

For each iteration you compare the E value with a prefixed E0 value; if E is much more than E0 you reduce the step h (usually we take half step in order to save 50% of function computation) and recalculate the integral and error again; if E is much less than E0 you duplicate the step ad recalculate; finally, if the E is the same order of E0 you accumulate dI and repeat from the new starting point x0+4h. The loop stopped when dI is lower then a prefixed value.
I have tested this algorithm and have found similar results to the DFD.

For the examples above I have got:
with 6 precision-digits , after 11 sec , we get: 9.64420... (error about 10^-4)
with 7 precision-digits , after 4.5 sec , we get: 2.804405 (error about 10^-5)
with 10 precision-digits , after 3.5 sec , we get: 1.402182148 (error about 10^-8)

By the way, the values used as reference, with 9 digits exact, are obtained by the same algorithm, (but in EXCEL for better perform).

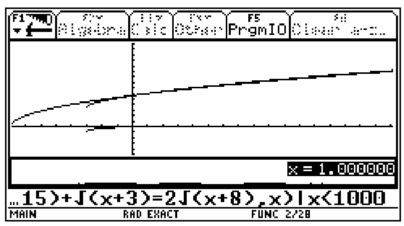
Of course, I can send my file INT_INF.mth, if anyone need it. Nice computation to everybody.
Leonardo

<u>Josef Gludovatz, Vienna, AUT</u> **JOSG%mimi@magic.itg.ti.com**

Hi all: $\sqrt{x+15} + \sqrt{x+3} = 2\sqrt{x+8}$ x = 1, but
the *TI*-92 shows the "solution" 10^26. What is going wrong? Please send me any information you have about this problem.

<u>DNL:</u> *The TI-graph shows the both sides of the equation and their difference as well. By inspection you might guess that both sides are showing the same end behaviour.*
And indeed. I use DERIVE to find the limit:

```
LIM(SQRT(x+15)+SQRT(x+3)-2*SQRT(x+8),x,inf)=0
```



So x = ∞ *should be an improper solution.*

If you work with Nsolve and use | x < 1000 *or* | x < 10000 *then you will obtain the expected solution. Bernhard Kutzler sent a similar answer.*

Titbits from Algebra and Number Theory(12)

by Johann Wiesenbauer, Vienna

Every now and then I have pangs of conscience when looking at the title of this series. For one thing I am not sure whether the rather boastful term "titbits" (or "tidbits" for my American readers) is really justified - maybe in your opinion all that stuff is simply a load of rubbish - and for another purely algebraic topics have been underrepresented so far (although, even if it's hard to believe, my roots are in abstract algebra). As for the latter unevenness I have every intention of dealing with more algebraic problems beginning in this issue, but before doing so I would like to finish off the review of the new functions in the utilty-file NUMBER.MTH.

First a remark on the routines SORT(v) and SORT2(v) in the last issue. In contrast to what I said here, their use is restricted to nonzero numbers only. (Zeros are swallowed by them, so to speak.) In most cases this will do no harm, but you should be aware of it.

Let's turn to the next block of new functions starting with

```
GEN_LUCAS(n, p, q, l0, l1) := IF(n = 0, l0, ITERATE((((p^2 - 2·q)·l0 - p·l1)·l_SUB1 +
  (2·l1 - p·l0)·l_SUB2)/(p^2 - 4·q), l_, ITERATE(IF((n AND v_SUB4) = 0, [v_SUB1^2 -
  2·q^v_SUB3,  v_SUB1·v_SUB2 - p·q^v_SUB3, 2·v_SUB3, v_SUB4/2],   [v_SUB1·v_SUB2 -
  p·q^v_SUB3,  v_SUB2^2 - 2·q^(v_SUB3 + 1), 2·v_SUB3 + 1, v_SUB4/2]), v_, [2, p, 0,
  2^FLOOR(LOG(n, 2))],    FLOOR(LOG(n, 2)) + 1), 1))
```

This routine returns the *n*-th term of the "Lucas-type" sequence (L_n) which is defined recursively by

$$L_n = PL_{n-1} - QL_{n-2},\ L_0 = l_0, L_1 = l_1 \quad (P, Q, l_0, l_1 \in \mathbf{Z})$$

assuming that $D = P^2 - 4Q \neq 0$ for the discriminant *D* of the characteristic polynomial

$$x^2 - Px + Q \in \mathbf{Z}[x]$$

(If the latter condition is not fulfilled the output will be a question mark!) Many important sequences of number theory are obtained in this way, e.g.

U_LUCAS(n, p, q) := GEN_LUCAS(n, p, q, 0, 1)

V_LUCAS(n, p, q) := GEN_LUCAS(n, p, q, 2, p)

FIBONACCI(n) := GEN_LUCAS(n, 1, -1, 0, 1)

PELL(n) := GEN_LUCAS(n, 2, -1, 1, 0)

The first two sequences are the "original" Lucas-sequences (U_n) and (V_n), as it were. Among other things they play an important role in the area of probabilistic primality-testing. In order to explain how this is done we need another important new function, namely the so-called Jacobi symbol (a/n). If p is an odd prime then the Jacobi symbol (a/p) coincides with the Legendre symbol (a/p) which is defined by

$$(a/p) = \begin{cases} 1, & \text{if } (p,a) = 1 \text{ and } x^2 \equiv a \bmod p \text{ is solvable} \\ -1, & \text{if } (p,a) = 1 \text{ and } x^2 \equiv a \bmod p \text{ is unsolvable} \\ 0, & \text{if } (p,a) \neq 1 \end{cases}$$

If now $n \in \mathbf{N}$ is an odd number and $n = p_1 p_2 \dots p_r$ is a decomposition into (not necessarily distinct) odd primes then for all $a \in \mathbf{Z}$ the Jacobi symbol (a/n) is defined by

$$(a/n) = (a/p_1)(a/p_2)\dots(a/p_r)$$

With this notion at hand one can show that an odd prime r with $(r, QD) = 1$ fulfills the conditions

$$U_{r-(D/r)} \equiv 0 \bmod r \qquad (*)$$

as well as

$$V_r \equiv P \bmod r. \qquad (**)$$

In order to take advantage of these facts we desperately need two things: Firstly, a fast modulo-arithmetic for the calculation of the sequences (U_n) and (V_n) mod m and secondly an efficient *DERIVE*-implementation of the Jacobi symbol (a/n). Here you are! (Note that U_MOD(n,p,q,m) makes use of INVERSE_MOD(a,m) !)

```
U_MOD(n, p, q, m) := IF(n = 0, 0, ITERATE(MOD((- p·l_SUB1 + 2·l_SUB2)·INVERSE_MOD(
   p^2 - 4·q, m), m), l_, ITERATE(IF((n AND v_SUB4) = 0, [MOD(v_SUB1^2 - 2·MOD(
   q^v_SUB3, m), m), MOD(v_SUB1·v_SUB2 - p·MOD(q^v_SUB3, m), m), 2·v_SUB3, v_SUB4/2],
   [MOD(v_SUB1·v_SUB2 - p·MOD(q^v_SUB3, m), m), MOD(v_SUB2^2 - 2·MOD(q^(v_SUB3 +
   1), m), m), 2·v_SUB3 + 1, v_SUB4/2]), v_, [2, p, 0, 2^FLOOR(LOG(n, 2))], FLOOR(LOG(n, 2)) +
   1), 1))

V_MOD(n, p, q, m) := IF(n = 0, MOD(2, m), (ITERATE(IF((n AND v_SUB4) = 0, [MOD(
   v_SUB1^2 - 2·MOD(q^v_SUB3, m), m), MOD(v_SUB1·v_SUB2 - p·MOD(q^v_SUB3, m), m),
   2·v_SUB3, v_SUB4/2], [MOD(v_SUB1·v_SUB2 - p·MOD(q^v_SUB3, m), m), MOD(v_SUB2^2 -
   2·MOD(q^(v_SUB3 + 1), m), m), 2·v_SUB3 + 1, v_SUB4/2]), v_, [2, p, 0, 2^FLOOR(LOG(n, 2))],
   FLOOR(LOG(n, 2)) + 1))SUB1)

JACOBI(a, b) := IF(MOD(b, 2) = 0 OR b <= 0, "Input error!", IF(GCD(a, b) > 1, 0, IF(a = 1 OR
   b = 1,1, IF(ABS(a)>= b, JACOBI(MODS(a, b), b), IF(a < 0, (-1)^((b - 1)/2)·JACOBI(-a, b),
   IF(MOD(a, 2) = 0, (-1)^((b^2 - 1)/8)·JACOBI(a/2, b), IF(MOD(a, 4) = 1 OR MOD(b, 4) = 1,
   JACOBI(b, a), - JACOBI(b, a))))))))
```

In the current version of *DERIVE* V_MOD(n,p,q,m) is actually called LUCAS_MOD(n,p,q,m) and the U-counterpart is missing. (Hopefully Al Rich, when reading this, comes to the conclusion that the arrangement above is the better choice!) Furthermore, I have corrected a tiny bug in JACOBI(a,b) that occurred whenever b had the value 1. (Sorry!)

Now we are ready for the Baillie-Wagstaff test that makes use of the first condition (*). For a given odd natural number N we carry out the following steps:

1. First we check whether N is a square in which case N fails the primality test. Otherwise we select the first D in the sequence 5,9,13,17,21,... s.t. (D/N)= -1. (If (D/N) happens to be 0 for some trial D this usually leads to a nontrivial divisor of N, too, except for some very small values of N.)
2. We take for P the smallest odd integer $> \sqrt{D}$ and set $Q := (P^2 - D)/4$.
3. If U_MOD(N+1,P,Q,N)=0 then N has passed the primality test, otherwise it is composite. Note that as usual there is a small chance of N being composite also in the first case.

And here is the corresponding *DERIVE*-routine that performs those steps automatically along with a computation of the first ten composite numbers that pass the Baillie-Wagstaff test:

```
BAILLIE_WAGSTAFF(n) := IF(NUMBER(SQRT(n)), false, (ITERATE(IF(p_ = 0, [2·FLOOR(
  - FLOOR(- SQRT(d_)), 2) + 1, ((2·FLOOR(- FLOOR(- SQRT(d_)), 2) + 1)^2 - d_)/4, d_],
  [p_, q_, IF(d_ >= n, true, IF(MOD(n, 2) = 0 OR GCD(d_, n) > 1, false, U_MOD(n + 1, p_, q_, n) =
  0))]), [p_, q_, d_], [0, 0, ITERATE(IF(JACOBI(d_, n) < 1, d_, d_ + 4, d_), d_, 5)], 2))SUB3)

SELECT(¬ PRIME(k) AND BAILLIE_WAGSTAFF(k), k, [1, 3, ..., 5000])=

  [323, 377, 1349, 2033, 2651, 3569, 3599, 3653, 3827, 4991]
```

Just in case you are disappointed about the small size of the numbers in the list above, I hasten to point out that the real power of the Baillie-Wagstaff test becomes only visible in a combination with other probabilistic primality tests, in particular with Rabin-Miller tests. Rabin-Miller tests are based on Fermat's Little Theorem and the fact that $x^2 \equiv 1 \bmod p$ has only the solutions $x \equiv \pm 1 \bmod p$, if p is a prime. A *DERIVE*-implementation could look like this (n must be an odd number !!)

```
RABIN_MILLER(n, a) := IF((ITERATE(IF(k_ = 2 OR a_ = -1, [a_, k_], [MODS(a_^2, n), k_/2]),
  [a_, k_], ITERATE([- ABS(MODS(a^o_, n)), (n - 1)/o_], o_, ITERATE(IF(MOD(n_, 2) = 1, n_,
  n_/2), n_, n - 1), 1)))SUB1 = -1, true, false)
```

Rabin-Miller tests are also used by DERIVE internally to check a number for primality and I am very proud to report that the routine above is for 100 digit-numbers N only 10-15% slower than the built-in Rabin-Miller test! Above all, Rabin-Miller tests have the nice property that for any composite $N > 9$ there are at most $\varphi(N)/4$ bases a in the range $0 < a < N$ such that N passes the Rabin-Miller test with respect to a. In particular, by choosing k bases at random the probability that a composite N passes all k tests is certainly smaller than $1/4^k$. On the other hand, for any given set of bases there is an N that passes all tests with respect to those bases! Look at the following example where the set consists of the first 10 primes:

```
n := 1195068768795265792518361315725116351898245581
VECTOR(RABIN_MILLER(n, a), a, [2, 3, 5, 7, 11, 13, 17, 19, 23, 29])
[true, true, true, true, true, true, true, true, true, true]
BAILLIE_WAGSTAFF(n) = false
"0.1 s and 1.1 s, respectively."
```

And here another example that came as a shock to David Stoutemyer when I presented it at the DERIVE-conference in Sweden this summer. (I remember a discussion we had on an earlier occasion where he claimed that I would never be able to find such an example. We both were laughing and poking fun at each other. Ever since I have been wondering why he was so sure.)

```
106219·212437 = 22564845703
FACTOR(22564845703) = 22564845703
PRIME(22564845703) = true
PRIME(22564845703, 7) = false
BAILLIE_WAGSTAFF(22564845703) = false
```

In contrast to this example no number has been found so far that passes a Rabin-Miller test for the single base a=2 along with the Baillie-Wagstaff test and it was shown that the smallest number of this kind must be $> 25\cdot10^9$.

What about the second condition (**)? Can it also be used for an efficient primality test? Let's check it again for the number N from above:

```
n := 1195068768795265792518361315725116351898245581
V_MOD(n, 1, 1, n) = 1
V_MOD(n, 2, 1, n) = 2
V_MOD(n, 3, 1, n) = 3
V_MOD(n, 2, 3, n) = 796712512530177195012306062527273384978690280
```

After three failures we finally arrived at a pair (P,Q) that shows that N must be composite. Although the situation isn't always that bad, it is true that there is no comparison between this test and the Baillie-Wagstaff test. There are even composite numbers N s.t. V_MOD(N,P,Q,N)=N for all pairs (P,Q). The smallest one is N = 443372888629441:

```
n := 443372888629441
FACTOR(n) = 17·31·41·43·89·97·167·331
V_MOD(n, 123456789, 987654321, n) = 123456789
```

For so-called Mersenne numbers, i.e. numbers of the form $\mathrm{M}_p = 2^p - 1$ where p is a prime, a test based on (*) becomes particularly simple, if we choose the discriminant D in such a way that $(D/M_p) = -1$. For p > 2 we could take D = 12, P = 4, Q = 1. It can be shown that in this case

$$V_{2^{p-1}} \equiv 0 \bmod \mathrm{M}_p$$

is a necessary and sufficient condition for the primality of M_p. If we set $s_k := V_{2^k}$ then the sequence (s_k) can be easily computed by the recursion

$$s_1 = 4, \quad s_n = s_{n-1}^2 - 2 \quad (n > 1)$$

and the condition above becomes

$$s_{p-1} \equiv 0 \bmod M_p .$$

Checking it is nothing else but the famous Lucas-Lehmer test for Mersenne numbers M_p where p is an odd prime. Believe it or not, only since DERIVE-version 4.06 this extremely simple test has become available in NUMBER.MTH (at last!). Moreover some of the functions in the following block had to be rewritten to make use of it. (Note that I have also updated the list of known Mersenne primes !)

```
LUCAS_LEHMER_TEST(p) := IF(ITERATE(MOD(s_^2 - 2, 2^p - 1), s_, 4, p - 2) = 0, true, false)

NEXT_MERSENNE_DEGREE(n) := IF(n < 2, 2, ITERATE(IF(LUCAS_LEHMER_TEST(p_),
  p_, NEXT_PRIME(p_)), p_, NEXT_PRIME(n)))

MERSENNE_LIST(n) := ITERATES(NEXT_MERSENNE_DEGREE(n_), n_, 2, n - 1)

MERSENNE(n) := 2^ITERATE(NEXT_MERSENNE_DEGREE(j), j, 1, n) - 1

MERSENNE_DEGREE(n) := ELEMENT([2, 3, 5, 7, 13, 17, 19, 31, 61, 89, 107, 127, 521, 607,
1279, 2203, 2281, 3217, 4253, 4423, 9689, 9941, 11213, 19937, 21701, 23209, 44497, 86243,
110503, 132049, 216091, 756839, 859433, 1257787, 1398269, 2976221], n)

MERSENNE(n) := 2^MERSENNE_DEGREE(n) - 1

PERFECT(n) := ITERATE((m_ + 1)·m_/2, m_, MERSENNE(n), 1)
```

As you may have noticed, I have also rewritten the code for PERFECT(n) which formerly made use of TRIANGULAR(n) (which makes use of POLYGONAL(n, p) which makes use of POLYGONAL_PYRAMID(n, p, d) ... Oh god, why must things be so complicated at times?!)

On behalf of all number theory enthusiasts I don't want to close this review without thanking Al Rich for including those new functions in NUMBER.MTH, which is definitely not to be despised in its present form. (Well, I still don't see what the functions from POLYGONAL(n, p) to RHOMBIC_DODECAHEDRAL(n) are actually good for, but this could be due to my ignorance...Don't look a gift horse in the mouth, as the saying goes!)

The remaining space I would like to use for some algebraic "titbits". (I am very confident that you agree this time!)

First of all, I would like to keep a promise which I gave Carl Leinbach in Sweden this summer concerning a function that decomposes a given permutation into its disjoint cycles. Sorry Carl for that long delay but this really belongs to the problems DERIVE is definetely not suited for. As a matter of fact, I am still not convinced that I found the best solution when it comes to the brevity of the program. Therefore all programming wizards are called upon to get their teeth into it. My first approach looked like this:

```
TOCYCLES(p) := ITERATE((ITERATE(IF(DIMENSION(r_) = 0, [IF(DIMENSION(q_) > 0,
p_·[q_], p_), [ ], r_], IF(DIMENSION(q_) = 0, [p_, [r_SUB1], DELETE_ELEMENT(r_)],
IF(o_SUB(q_SUBDIMENSION(q_)) = q_SUB1, [p_·[q_], [ ], r_], [p_, APPEND(q_,
[o_SUB(q_SUBDIMENSION(q_))]), SELECT(s_ / = o_SUB(q_SUBDIMENSION(q_)), s_, r_)]))),
[p_, q_, r_], ITERATE([IF(DIMENSION(r_) = 0, [[1]], 1), [ ], r_], r_, SELECT(o_SUBk_ / = k_, k_,
[1, ..., DIMENSION(o_)]), 1)))SUB1, o_, IF(pSUB2SUB1, p, pSUB2, p), 1)
```

To check it I used a function RPERM(n) that generates random permutations of $[1,2,..,n]$:

```
RPERM(n) := [[1, ..., n], APPEND(ITERATE(ITERATE([APPEND(p_, [q_]), SELECT(s_ / = q_,
  s_, r_)], q_, r_SUB(RANDOM(DIMENSION(r_)) + 1), 1), [p_, r_], [[ ], [1, ..., n]], n - 1))]
```

Everything looked fine at first:

$$\text{RPERM(6)} = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 3 & 4 & 2 & 1 \end{bmatrix}$$

$$\text{TOCYCLES} \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 3 & 4 & 2 & 1 \end{bmatrix} = [\,1 \quad 6\,] \cdot [\,2 \quad 5\,]$$

But then I came across a very nasty DERIVE-bug. Look:

[[1, 2]] • [[3, 4]] • [[5, 6]] = [[1, 2]] • ([[3, 4]] • [[5, 6]])

In other words, DERIVE takes the liberty to insert brackets where it shouldn't !! (By the way, it is also impossible to replace the dot by a space as they promised in the output options!) Therefore my final version looks like this:

```
TOCYCLES(p) := ITERATE((ITERATE(IF(DIMENSION(r_) = 0, [IF(DIMENSION(q_) > 0,
APPEND(p_, [q_]), p_), [ ], r_], IF(DIMENSION(q_) = 0, [p_, [r_SUB1],DELETE_ELEMENT(r_)],
IF(o_SUB(q_SUBDIMENSION(q_)) = q_SUB1, [APPEND(p_, [q_]), [ ], r_], [p_, APPEND(q_,
[o_SUB(q_SUBDIMENSION(q_))]), SELECT(s_ / = o_SUB(q_SUBDIMENSION(q_)), s_, r_)]))),
[p_, q_, r_], ITERATE([IF(DIMENSION(r_) = 0, [[1]], [ ]), [ ], r_], r_, SELECT(o_SUBk_ / = k_, k_,
[1, ..., DIMENSION(o_)]), 1)))SUB1, o_, IF(pSUB2SUB1, p, pSUB2, p), 1)
```

For the sake of convenience it suffices to input only the second line of a permuation:

$$\text{RPERM(10)} = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 7 & 1 & 3 & 4 & 5 & 8 & 2 & 9 & 10 & 6 \end{bmatrix}$$

$$\text{TOCYCLES} \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 7 & 1 & 3 & 4 & 5 & 8 & 2 & 9 & 10 & 6 \end{bmatrix} = [[1, 7, 2], [6, 8, 9, 10]]$$

TOCYCLES([7, 1, 3, 4, 5, 8, 2, 9, 10, 6]) = [[1, 7, 2], [6, 8, 9, 10]]

Actually I have prepared a whole DERIVE-package for computations in the symmetric groups but there is not enough space left so I will postpone its introduction.

For now, I would like to present my solution of problem 4 of the CAS-competition (cf. DNL #27, p 25). My first approach (well, not the very first one!) took DERIVE 27.1 s, and here is my final solution using ideas of Flajolet-Salvy (cf. http://pauillac.inria.fr/algo/papers/html/ FlSa97/ FlSa97.html).

```
(ITERATE([DELETE_ELEMENT(INSERT_ELEMENT(- a_·b_/c_, a_), 8), b_ + [3, 5, 6, 6, 5, 3, 1],
  c_ + 1], [a_, b_, c_], [[1, 0, 0, 0, 0, 0, 0], - [3500, 10005, 16512, 19518, 20020, 14515, 6006], 1],
  3000))SUB1SUB1
```

39739422655800..........20476320 (1427 digits!) (*Find the complete calculation in the file!*)

It took DERIVE 2.9 s on my Pentium 200 PC to get this result thereby demolishing the solutions of all other CAS. (For example, the fastest approach of Mathematica took 270 s on a Pentium 200 PC, too. Well, obviously mind and machine were not in harmony ...)

By the way, there was a competition, as it were, in the last DNL about the best way of summing up the digits of a number. May I join in?

```
S(n) := SUM(MOD(k_, 10), k_, ITERATES(FLOOR(n_, 10), n_, n))
```

You can see at first glance that this is the shortest routine of all. I leave it to you to check that it is also the fastest. See you next time! (j.wiesenbauer@tuwien.ac.at)

When Johann sent his contribution he closed with the sentence:

P.S.: Du könntest unter Umständen. auch zu einer Competition um die schnellste TI-92 Implementierung von Problem 4 aufrufen!

So he invites the TI-92 specialists to compete for the fastest TI-92 implementation of problem 4´s solution. You can find the complete solution with all its 1427 digits on the diskette. Josef