THE DERIVE - NEWSLETTER #34

THE BULLETIN OF THE

DERIVE

USER GROUP

+ TI 92

Contents:

revised version 2015 **June 1999**

[1] **Mathematische Bildung und neue Technologien**, W. Peschek u.a (Hrsg), Leipzig: Teubner 1999

[2] **Wiskunde voor hoger beroepsonderwijs door middel van DERIVE**, P. van der Velden & J. Verhoosel, Academic Service 1999

[3] **DERIVE**, Matija Lokar, DMFA Ljubljana 1999 (in Slowenian)

[4] **Experimente zur Mechanik mit dem CBR**, H-D. Hinkelmann, bk teachware 1999 (SR-10)

[5] **Arbeiten mit dem TI-92 SchülerInnenarbeitsheft**, E. Schneider, bk teachware 1999 (SR-11)

[6] **Einführung des Integralbegriffs mit dem TI-92**, J. Böhm & W. Pröpper, bk teachware 1999 (SR-13)

[7] **Optimization – Graphically, Numerically and Analytically with the TI-92**, J. Böhm, bk teachware 1999 (SL-09)

[9] **Scripting Guide for the TI-92 and TI-92 PLUS**, J D Child, Texas Instruments 1998

[9] **Aufgaben mit Graphikrechner, für die Klassen 8-10,** M. Ebenhöh & G. Steinberg, Schroedel 1999

[10] **Terme im M-Unterricht, unter Verwendung von Computergraphik und Computeralgebra,** E Lehmann, Schroedel 1999

[11] **Neue Ideen für den M-Unterricht**, R Fulge & A Roettger, Schroedel 1999

[12] **Analysis Alive – Ein interaktiver Mathematik-Kurs**, M Wolff & O Gloor & C Richard, Birkhäuser 1998
Das Buch wird mit CD ausgeliefert, die eine Fülle von Grafiken und Animationen zur Veranschaulichung der abstrakten mathematischen Begriffe enthält. Man benötigt aber Maple im Hintergrund. Viele Ideen des Buchs können aber in andere CAS übertragen werden.

A Collection of interesting web sites:
(The red URLs have become invalid since 1999.)

http://come.to/VisualMath/ — Online symbolic and computer algebra system

http://www.didmath.ewf.uni-erlangen.de/staatsexamensaufgaben.shtml

http://learn-line.nrw.de/Faecher/Mathematik/CAS/ — Informations about CAS and teaching units (German)

https://sites.google.com/site/edwardlaughbaum/home/ — T^3 – Information (USA)

http://www.qmark.com/questionbank/mathletics.html — Math tests (Martin Greenhow)

http://www.bham.ac.uk/ctimath — Birmingham, Cti

http://www.ti.com.calc/oesterreich — TI- Austria´s Homepage

http://www.mupad.de/EDU/interact.shtml — An interactive book
http://de.mathworks.com/products/symbolic/

http://www.clarasil.de/san/berichte/thinkquest/html — International Competition

http://mathforum.org/dr.math/ — "Ask Dr Math"

http:// http://mathforum.org/ — Informations and Problems

http://www.hh.schule.de/tak — The Transatlantic Classroom

http://www.zum.de — Center for Teaching Materials (German)

http://www.c3.lanl.gov/mega-math/index.html — Los Alamos National Labaratory

http://www.stauff.de/bewmath/dateien/bewmath.html — Moved Mathematics – Java applets

Liebe DERIVE- und TI-Freunde,

Ich wollte Ihnen gerne das vielseitige Programm der ACDCA Summer Academy vom 25.-28.8.1999 in Gösing, Nö vorstellen. Deshalb muss leider der Letter of the Editor sehr kurz ausfallen.
Beachten Sie bitte das ausführliche User Forum. Die Dateien können Sie wie letztlich von unserer Home page herunterladen.

Mit den besten Grüßen und Wünschen für einen schönen Sommer, Ihr
Josef Böhm

Dear *DERIVE*- and TI-friends,

I wanted to present the wide-ranging program of our ACDCA Summer Academy from 25 – 28 August. So you find only a very short "letter" of the editor.
Take note of the ample User Forum. You can download all files belonging to this issue from our home page.

With my best wishes for a wonderful summer
Josef Böhm

ACDCA Summer Academy – Preliminary List of Talks

Baumann, GER	Informationssicherheit durch kryptologische Verfahren
Biryukow/Fydorova, RUS	Tzunami in DERIVE and TI-92
Böhm, AUT	Basic Skills versus Technology
Boieri, IT	The LABCLASS Project in Italy
Ersoy/Adrahan, TUR	Initiating a Project TI-92/DERIVE supported
Franzova, USA	Using TI-92 in a Traditional Calculus and DE Course
Graubner, GER	A Special Triangle
Gremillion, USA	TI Interactive!
Heugl, AUT	Important Elements of the Algebraic Competence
Himmelbauer, AUT	TI-92 Programs in Maths Lessons
Keunecke, GER	Differential Equations
Knechtel, GER	Changes in Didactics/Methods using CAS-Calculators
Kokolj-Volic, SLO	Exam Questions when using TI-92 or DERIVE
Kutzler, AUT	Computers and Calculators as Pedagogical Instruments
Lechner, AUT	HIV and the Immune System – A Mathematical Model
Leinbach, USA	Using Computer Algebra to Extract Meaning from Parameters
Lokar, SLO	Exponential Growth
Maaß, AUT	Black Boxes and Teaching Mathematics
Magiera, POL	Analytical Mechanics Problems with DERIVE
Mahmudi, INDONESIA	Using MatLab/Maple in Solving Intervall Hull ....
Mann, ISR	Are Circular Functions Trigonometric or Real?
Marlewski, POL	Lin. Discrete Least-Square Fitting Assisted by CAS
Postel, GER	A Tutorial System for High Schools
Raggett, AUT	Using the TI-92 and CBR in Physics and Applied Maths Teaching
Roanes, ESP	Truth-Tables in Propositional Multi-Valued Logics with DERIVE
Romanovskis, LAT	Kepler´s Ellipses – From Compasses to CAS
Schüller, AUT	Curricular Interfaces
Schwarz, GER	Independent Classwork
Shelby/Stoutemyer, USA	DERIVE for Windows Version 5 – Rumor becomes Reality
Sjöstrand, SWE	Computers and Mathematics at Lindälvs Gymnasium
Sugeng, INDONESIA	Maple and the Abstraction Process
Tonisson, EST	Step-By-Step Solution Possibilities in Different CAS
Torres-Skoumal, AUT	CAS in an International School Setting
Weiskirch, GER	Didaktische und Methodische Veränderungen im Leistungskurs
Weller, GER	Squaring the Circle and Leonardo
Wurnig, AUT	Using the TI-92 in the 9^{th} Grade of Austrian Grammar Schools

The *DERIVE-NEWSLETTER* is the Bulletin of the *DERIVE & TI-92 User Group*. It is published at least four times a year with a contents of 44 pages minimum. The goals of the *DNL* are to enable the exchange of experiences made with *DERIVE* and the *TI-92/89* as well as to create a group to discuss the possibilities of new methodical and didactical manners in teaching mathematics.

As many of the *DERIVE* Users are also using the *TI-92/89* the *DNL* tries to combine the applications of these modern technologies.

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Contributions:

Please send all contributions to the Editor. Non-English speakers are encouraged to write their contributions in English to reinforce the international touch of the *DNL*. It must be said, though, that non-English articles will be warmly welcomed nonetheless. Your contributions will be edited but not assessed. By submitting articles the author gives his consent for reprinting it in the *DNL*. The more contributions you will send, the more lively and richer in contents the *DERIVE & TI-92 Newsletter* will be.

Next issue: September 1999
Deadline 15 August 1999

Preview: Contributions for the next issues

3D-Geometry, Reichel, AUT
Graphic Integration, Linear Programming, Various Projections a.o., Böhm, AUT
A Utility file for complex dynamic systems, Lechner, AUT
Examples for Statistics, Roeloffs, NL
Quaternion Algebra, Sirota, RUS
Various Training Programs for the TI
A critical comment on the "Delayed Assignation" :==, Kümmel, GER
Sand Dunes, River Meander and Elastica, The lighter Side ....., Halprin, AUS
Type checking, Finite continued fractions, ..., Welke, GER
Share Holder´s Considerations using a CAS, Böhm, AUT
Kaprekar´s "Self numbers", GER
Linear Programming - Graphic Solution on the *TI*, Kirmse, GER
Implicit Multivalue Bivariate Function 3D Plots, Biryukov, RUS
LU-Decomposition, Morales, COL
ODEs with Constant Coefficients, Fernandez, ARG
Three Bodies Problem (DERIVE & Cyclone98), Fritsch, AUT

and
Setif, FRA; Vermeylen, BEL; Leinbach, USA; Speck, NZL; Biryukow, RUS
Wiesenbauer, AUT; Aue, GER; Koller, AUT, Ibrahim & Córdoba, ESP, ...

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David Millward, USA davidmillward@lineone.net

Hi,

I am a registered user of Derive and bought all of the Derive texts but am having a problem solving difference equations using the RECUREQUN.MTH utility. (Now RecurrenceEquations.mth)

In particular, the problem is with non-homogeneous equations such as

`u(n+1)-8u(n)+16u(n-1)=4n^2`

and

`u(n)-6u(n-1)+9u(n-2)=3^n`

Figures in brackets are subscript to denote the order of the term.
CAN Derive be used to solve such problems ? If so, HOW !????

I do hope you can help me - it's driving me around the bend !

David Millward

From Josef to Johann Wiesenbauer (my man for all cases!)

As I am - unfortunately enough - not a specialist in difference equations I forward this request and hope that you can find an answer. Many thanks in advance, Josef

Johann´s answer:

To be honest, it doesn't exactly take "a specialist in difference equations" to solve the difference equations mentioned above. I could serve as an example myself. Nevertheless, I had no problems whatsoever when computing the following lines in DfW 4.11 (using RECUREQN.MTH for the very first time!)

$$\text{LIN2\_CCF}(-8,\ 16,\ 4\cdot(n+1)^2,\ n) = 2^{2\cdot n}\cdot(c1 + c2\cdot n) + \frac{4\cdot(3\cdot n^2 + 10\cdot n + 11)}{27}$$

$$u1(n) := 2^{2\cdot n}\cdot(c1 + c2\cdot n) + \frac{4\cdot(3\cdot n^2 + 10\cdot n + 11)}{27}$$

$$u1(n+1) - 8\cdot u1(n) + 16\cdot u1(n-1) = 4\cdot n^2$$

$$\text{LIN2\_CCF}(-6,\ 9,\ 3^{n+2},\ n) = 3^n\cdot\left(c1 + c2\cdot n + \frac{n^2}{2} - \frac{n}{2}\right)$$

$$u2(n) := 3^n\cdot\left(c1 + c2\cdot n + \frac{n^2}{2} - \frac{n}{2}\right)$$

$$u2(n) - 6\cdot u2(n-1) + 9\cdot u2(n-2) = 3^n$$

You see? It's simply a matter of evoking a library function. The online help for `LIN2_CCF()` says "`LIN2_CCF(p, q, r, x, c1, c2)` simplifies to a general solution of the constant coefficient second order linear difference equation $y(x+2) + p \cdot y(x+1) + q \cdot y(x) = r(x)$, in terms of two symbolic constants c1 and c2."

i.e. just what the doctor prescribed! The only thing you have to take care of is the leading term of the difference equation which is $u(n+2)$ and not $u(n)$!! Maybe it was this that drove David around the bend? Well, whatever...

Cheers, Johann

On my request there was also a very similar answer from David Stoutemyer, SWHH. Many thanks.

See more 2015 comments on difference equations on page 57, Josef.

Alfonso Población, Valladolid, Spain alfonso@gauss.mat.eup.uva.es

Dear Josef,

Two comments about the last newsletter, #33.

The following iterative algorithm

```
"PIPOB.MTH"
Y(n):=ITERATES((1-(1-k_^4)^(1/4))/(1+(1-k_^4)^(1/4)),k_,SQRT(2)-1,n)
A(n):=IF(n = 1,(1+(Y(n))↓(n+1))^4*(6-4·√2)-2^(2n+1)*(Y(n))↓(n+1)*
               (1+(Y(n))↓(n+1)+(Y(n))↓(n+1)^2),
               (1+(Y(n))↓(n+1))^4*A(n-1)-2^(2n+1)*(Y(n))↓(n+1)*
               (1+(Y(n))↓(n+1)+(Y(n))↓(n+1)^2))
PIAPPROX(n_):=1/A(n_)
```

gives approximations of PI: approXimating PIAPPROX(1), you have 8 exact decimals; PIAPPROX(2), 41; PIAPPROX(3), 171;PIAPPROX(4), 694; etc, and only in a few seconds (I used DFW). As I don´t work with TI92, I don´t know if these good results are also obtained with it, so any interested user could check it and tell us. I think that it should be faster than the seven hours of the Spigot algorithm. With this iterative method, you don´t need to store previous digits either. (I include this algorithm in a chapter of a book for secondary levels that it is in press by now).
There are other algorithms based on modular equations.

On the other hand, you comment some implementations of the Koch curve in page 34. There is another one, extensive to any Iterated Function System: the one I explained in Plymouth, do you remember? It can be seen in the last chapter ("Construction of Fractal Sets") of the book "Mathematical Activities with DERIVE", edited in 1997 by Chartwell-Bratt. It also contains files to represent the fractal structure of some types of infinite series.

Cheers, Alfonso

```
PrecisionDigits := 8
PIAPPROX(1)
3.1415926
PrecisionDigits := 41
PIAPPROX(2)
3.1415926535897932384626433832795028841971
PrecisionDigits := 171
PIAPPROX(3)
3.14159265358979323846264338327950288419716939937510582097494459230781640628620899862803482534211706798214808651328230664709384460955058223172535940812848111745
028410270193
PrecisionDigits := 694
PIAPPROX(4)
3.14159265358979323846264338327950288419716939937510582097494459230781640628620899862803482534211706798214808651328230664709384460955058223172535940812848111745
02841027019385211055596446229489549303819644288109756659334461284756482337867831
65271201909145648566923460348610454326648213393607260249141273724587006606315588
17488152092096282925409171536436789259036001133053054882046652138414695194151160
94330572703657595919530921861173819326117931051185480744623799627495673518857527
24891227938183011949129833673362440656643086021394946395224737190702179860943702
77053921717629317675238467481846766940513200056812714526356082778577134275778960
9173637178721468440901224953430146549585371050792279689
"DERIVE's built in π"
π
3.14159265358979323846264338327950288419716939937510582097494459230781640628620899862803482534211706798214808651328230664709384460955058223172535940812848111745
02841027019385211055596446229489549303819644288109756659334461284756482337867831
65271201909145648566923460348610454326648213393607260249141273724587006606315588
17488152092096282925409171536436789259036001133053054882046652138414695194151160
94330572703657595919530921861173819326117931051185480744623799627495673518857527
24891227938183011949129833673362440656643086021394946395224737190702179860943702
77053921717629317675238467481846766940513200056812714526356082778577134275778960
9173637178721468440901224953430146549585371050792279689
```

(Compare with Peter Witthinrich´s results in DNL#33, Josef)

<u>G P Speck, Wanganui, New Zealand</u>

You might find the following comments, in whole or in part, suitable for inclusion in the Forum section of the *DERIVE* Newsletter.

I found the questions and discussion from Hugo, Jakobcic, Wiesenbauer and Stegenga concerning piecewise defined functions on pages 5 & 6 of *DNL*#32 in the *DERIVE* User Forum quite interesting. All of the issues and shortcomings raised in this discussion can be resolved easily using the characteristic functions defined in my article **Concentric Curve Shading** in *DNL*#31.

(The functions can be found in the file SHAD_UT.MTH on the Diskette of the Year 1998. Or you can download the file from SWHH´s home page – MTH31. Look for the DUG-section. Josef)

Now it seems that there are three issues raised concerning piecewise defined functions expressed via characteristic functions in the *DNL*#32:

1) Johann Wiesenbauer rightfully points out that `CHI(a,x,b)` as defined is not a function since SIMPLIFYING, e.g., `CHI(0,0,1)` or `CHI(0,1,1)` gives ± 0.5 + 0.5. This deficiency is corrected in our `CHOPEN(x,a,b)` characteristic function; it has value 1 at each x in the open interval (a,b) and has value 0 elsewhere. It is essentially the real-valued counterpart of `CHI(a,x,b)` with restrictions, a < b, a ≠ ∞, and b ≠ ∞, which pose no major problems with characteristic functions such as `CHLESS(x,a)` and `CHGRT(x,a)` defined.

2) Johann Wiesenbauer suggests defining a function, say CHI2(a,x,b,c,d) which is 1 on the open interval (a,b), has value c at x = a, has value d at x = b, and is 0 elsewhere. This is, of course, trivial using our characteristic functions:

```
CHI2(a,x,b,c,d):=CHOPEN(x,a,b)+c*CHPOINT(x,a)+d*CHPOINT(x,b)
```

3) To display Allan Hugo´s piecewise defined function using characteristic functions, first consider:

```
G(x):=x*CHOPENRT(x,0,1)+x^2*CHOPENRT(x,1,3)+(2*x+3)*CHGRTEQ(x,3)
[G(-1),G(0),G(1),G(3)]=[0,0,1,9]
```

G(x) is a characteristic function representation of Allan Hugo´s function except that G is defined as 0 for x < 0 while Hugo´s function is undefined for x < 0. Johann, again rightfully, points out that this is not satisfactory and feels that resort to the use of one IF() seems to be necessary. However, even here, we can escape using an IF(). Consider:

```
F(x):=FLOOR(SQRT(x-10))*CHLESS(x,0)+G(x)
```

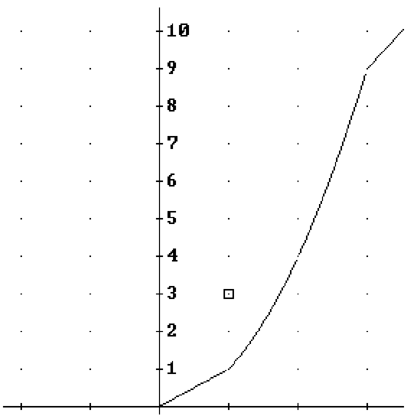
Note that the FLOOR() function is undefined at any non-real complex number, so that FLOOR(SQRT(x-10)) is undefined at any x < 0. The following values of F illustrate that it is undefined at x < 0:

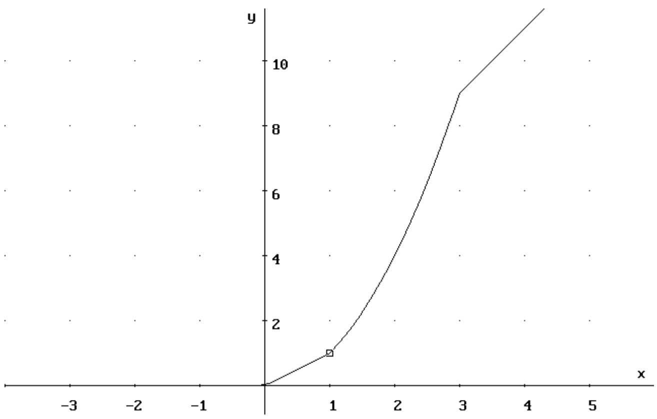
```
[F(-100),F(-1),F(0),F(1),F(3)]=
          [FLOOR(SQRT(110)*î),FLOOR(SQRT(11)*î),0,1,9]
```

This does require, of course, that when we get FLOOR(c), where c is non-real complex, returned when we SIMPLIFY F(r), we interpret this as meaning that F is undefined at x = r.

Note 1: The FLOOR(SQRT(x-10)) function can easily be modified to show that a function is undefined on other subsets of the real numbers.

Note 2: All of the above functions PLOT satisfactorily in my DFW Version 4.





Francisco Cabo Garcia, Spain pcabo@esgueva.eco.uva.es

We have a problem with DfW 4.06.
If we ask the computer to calculate SIN(ASIN(x)), the answer is x but when we ask to calculate ASIN(SIN(x)), the outcome is:

$$\mathrm{SIN(ASIN(x))} = x$$

$$\mathrm{ASIN(SIN(x))} = \mathrm{SIGN(COS(x))}\cdot\left|x - \pi\cdot\mathrm{FLOOR}\left(\frac{x}{\pi} + \frac{1}{2}\right)\right|$$

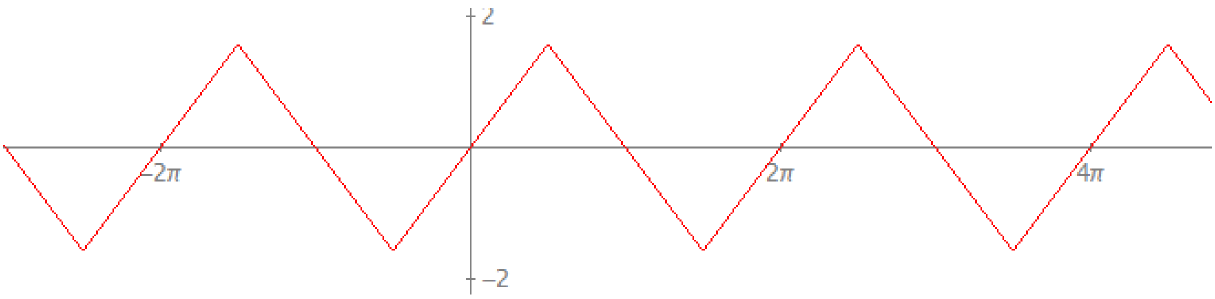
We would like to know whether it is possible to ask DERIVE to return x instead of the expression above?

Dr. Nigel Backhouse, Liverpool sx52@liverpool.ac.uk

Francisco,
The fact is that ASIN(SIN(x)) does not coincide with the function x. It is a sawtooth type of function, with discontinuities. Try to plot the output function.

(I tried, and you can see how it looks like in DERIVE 6.10:)

$$\mathrm{ASIN(SIN(x))} = \mathrm{SIGN(COS(x))}\cdot\left(\pi\cdot\mathrm{FLOOR}\left(\frac{1}{2} - \frac{x}{\pi}\right) + x\right)$$



Johan Vegter, Enschede, Netherlands cad@hartman.nl

Hello DERIVERS,
I´m a relative new user of DERIVE and want your help with a (small) problem:
I have two equations:

$$(1) \quad \sin\frac{360\,(L+x)}{4\,\pi\,R} = L\,\frac{R}{2} \qquad (2) \quad f = R - \sqrt{R^2 - \left(\frac{L}{2}\right)^2}$$

I am interested in the relation between *f* and *x*. *R* is unknown and *L* is a constant value. So, the idea is to solve this set of equations for *R,* and make one (large) equation with *f*, *x* and *L* (R is eliminated). Thanks in advance.

Terence Etchells, Liverpool T.A.Etchells@livjm.ac.uk

Hi Johan,

it is not possible to express f in terms of L and x (mathematics is the problem not Derive). However we can express x in terms of f and L, so you can find particular f's given L. I attach a MTH file that goes through steps. Hope this helps.

Cheers, Terence

This is Terence's procedure performed with DERIVE 6.10.

```
#1:  CaseMode := Sensitive
```

Equation 1 and Equation 2

$$\#2:\quad \text{SIN}\left(\frac{360\cdot(L+x)}{4\cdot\pi\cdot R}\right)=\frac{L\cdot R}{2}$$

$$\#3:\quad f=R-\sqrt{\left(R^2-\left(\frac{L}{2}\right)^2\right)}$$

Square both sides of #3 and then solve for R

$$\#4:\quad \left(f-R=-\sqrt{\left(R^2-\left(\frac{L}{2}\right)^2\right)}\right)^2=\left((R-f)^2=\frac{4\cdot R^2-L^2}{4}\right)$$

$$\#5:\quad \text{SOLVE}\left((R-f)^2=\frac{4\cdot R^2-L^2}{4},R\right)=\left(R=\frac{L^2+4\cdot f^2}{8\cdot f}\right)$$

Substitute for R in Equation 1 and then simplify

$$\#6:\quad \text{SIN}\left(\frac{360\cdot(L+x)}{4\cdot\pi\cdot R}\right)=\frac{L\cdot R}{2}$$

$$\#7:\quad \text{SIN}\left(\frac{720\cdot f\cdot x}{\pi\cdot(L^2+4\cdot f^2)}+\frac{720\cdot L\cdot f}{\pi\cdot(L^2+4\cdot f^2)}\right)=\frac{L\cdot(L^2+4\cdot f^2)}{16\cdot f}$$

Solve for x

$$\#8:\quad \left[\text{SOLUTIONS}\left(\text{SIN}\left(\frac{720\cdot f\cdot x}{\pi\cdot(L^2+4\cdot f^2)}+\frac{720\cdot L\cdot f}{\pi\cdot(L^2+4\cdot f^2)}\right)=\frac{L\cdot(L^2+4\cdot f^2)}{16\cdot f},x\right)\right]'$$

$$\#9:\quad \left[\begin{array}{c}\dfrac{\pi\cdot(L^2+4\cdot f^2)\cdot\text{ASIN}\left(\dfrac{L\cdot(L^2+4\cdot f^2)}{16\cdot|f|}\right)}{720\cdot|f|}-L\\[2ex] -\dfrac{\pi\cdot(L^2+4\cdot f^2)\cdot\text{ASIN}\left(\dfrac{L\cdot(L^2+4\cdot f^2)}{16\cdot|f|}\right)}{720\cdot|f|}-\dfrac{\pi^2\cdot L^2+720\cdot L\cdot f+4\cdot\pi^2\cdot f^2}{720\cdot f}\\[2ex] \dfrac{\pi^2\cdot L^2-720\cdot L\cdot f+4\cdot\pi^2\cdot f^2}{720\cdot f}-\dfrac{\pi\cdot(L^2+4\cdot f^2)\cdot\text{ASIN}\left(\dfrac{L\cdot(L^2+4\cdot f^2)}{16\cdot|f|}\right)}{720\cdot|f|}\end{array}\right]$$

Hugh Porteous, England H.L.Porteous@shu.ac.uk

If you integrate SIGN(SIN(x))*SIN(x) from 0 to 2π then you correctly get the answer 4. However, if you declare n to be an integer and integrate SIGN(SIN(x))*SIN(nx) from 0 to 2π then you get the answer 0. Does this mean that *DERIVE* does not consider 1 to be an integer?

Ray Girvan, England ray.girvan@zetnet.co.uk
http://www.users.zetnet.co.uk/rgirvan/

I´m probably missing something, but why would this imply *DERIVE* not considering 1 an integer? The answer does surprise me: with n otherwise undefined, I can´t see why *DERIVE* returns 0, rather than a function of n.

$$\int_0^{2\cdot\pi} \text{SIGN}(\text{SIN}(x))\cdot\text{SIN}(x)\ dx = 4$$

n :ε Integer

$$\int_0^{2\cdot\pi} \text{SIGN}(\text{SIN}(x))\cdot\text{SIN}(n\cdot x)\ dx = 0$$

If you try the integral with n real, the answer comes out as $f(n)$ ....

n :ε Real

$$\int_0^{2\pi} \text{SIGN}(\text{SIN}(x))\cdot\text{SIN}(n\cdot x)\ dx = \frac{\text{COS}(2\cdot\pi\cdot n)}{n} - \frac{2\cdot\text{COS}(\pi\cdot n)}{n} + \frac{1}{n}$$

$$\frac{\text{COS}(2\cdot\pi\cdot n)}{n} - \frac{2\cdot\text{COS}(\pi\cdot n)}{n} + \frac{1}{n}$$

4

.... which gives the required answer for integer values of n. Particularly, $f(n=1) = 4$.

This is the DfW 4 result. See below the DERIVE 6 result.

n :ε Real

$$\int_0^{2\cdot\pi} \text{SIGN}(\text{SIN}(x))\cdot\text{SIN}(n\cdot x)\ dx = \frac{\text{COS}(\pi\cdot n)\cdot(\text{COS}(2\cdot\pi\cdot n) - 1)}{n\cdot(\text{COS}(\pi\cdot n) + 1)}$$

$$\text{SUBST}\left(\frac{\text{COS}(\pi\cdot n)\cdot(\text{COS}(2\cdot\pi\cdot n) - 1)}{n\cdot(\text{COS}(\pi\cdot n) + 1)},\ n,\ 1\right) = ?$$

$$\lim_{n\to 1} \frac{\text{COS}(\pi\cdot n)\cdot(\text{COS}(2\cdot\pi\cdot n) - 1)}{n\cdot(\text{COS}(\pi\cdot n) + 1)} = 4$$

The results are not identical!

$$\frac{\cos(2\cdot\pi\cdot n) - 2\cdot\cos(\pi\cdot n) + 1}{n} - \frac{\cos(\pi\cdot n)\cdot(\cos(2\cdot\pi\cdot n) - 1)}{n\cdot(\cos(\pi\cdot n) + 1)} = \frac{2}{n} - \frac{2\cdot(-1)^n}{n}$$

Francisco Marcelo Fernandez marcelo@diquite.edu.ar

I obtained the following results for the integral SIN(*x*)*SIN(*n x*) from 0 to 2π:

1) 0 when *n* is defined to be an integer
2) π when *n* = 1
3) SIN(2π *n*)/(*n*^2-1) when *n* is real. This expression yields the result π when *n* → 1

Johann Wiesenbauer, Vienna, Austria J.Wiesenbauer@tuwien.ac.at

Hi all,
Frankly, those results aren´t that unexpected in my eyes. In the first place, assuming that *n* is any real number the exact result in DERIVE-notation would be

IF(*n* = 1, π, if(*n* = -1, -π, SIN(2 π *n*)/((*n* + 1)*(*n* – 1))))

as easily can be checked. As on many other occasions (try e.g. the simple example INT(*x*^*n*,*x*)!), DERIVE gives only the general term and leaves the treatment of the special cases to you. This is a compromise I for my part can live with, because I can imagine that otherwise things might become fairly complicated if this result with its nested IF-statements is not a final one but a subterm in a much bigger term!

If *n* is declared as integer then obviously DERIVE simply replaces the numerator SIN(2 π *n*) of the general term by 0, wich leads to the value 0 for the fraction in general, but as before the cases *n* = 1 and *n* = -1 are exceptions which aren´t considered by DERIVE. (Interestingly enough, even Marcelo seems to have overlooked the latter case!)

Thus in my opinion, unlike the original example by Hugh Porteous which is still a puzzle to me, the result given by DERIVE in this case is exactly what you can expect of a CAS.
Cheers, Johann

Lorenz Jaeneke, Switzerland ljaeneke@datacomm.ch

The result of the considered integral INT(SIGN(...........,0,2π) is indeed the expression given below, which leads for some integers to

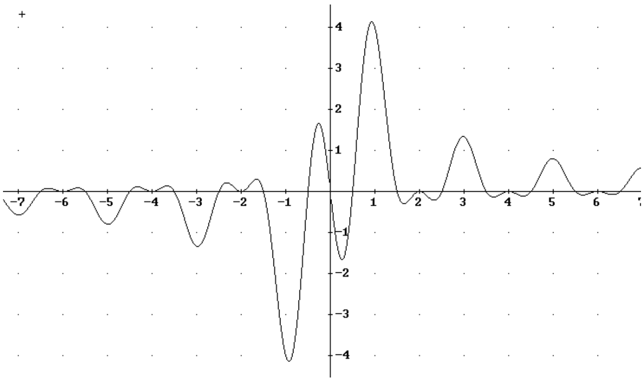
$$\text{VECTOR}\left(\frac{\cos(2\cdot\pi\cdot n)}{n} - \frac{2\cdot\cos(\pi\cdot n)}{n} + \frac{1}{n},\ n,\ 0,\ 10\right)$$

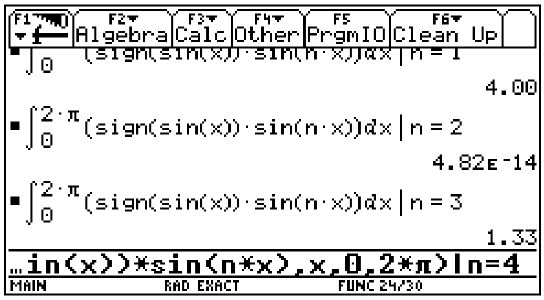
$$\left[?,\ 4,\ 0,\ \frac{4}{3},\ 0,\ \frac{4}{5},\ 0,\ \frac{4}{7},\ 0,\ \frac{4}{9},\ 0\right]$$

The quotation mark tells us about the both vanishing numerator and denominator. So, we have to calculate the limit:

$$\lim_{n\to 0}\left(\frac{\cos(2\cdot\pi\cdot n)}{n} - \frac{2\cdot\cos(\pi\cdot n)}{n} + \frac{1}{n}\right) = 0$$

All these results are clearly visible in the plot of the function f(n) = COS(2πn)/n – 2 COS(πn)/n + 1/n. Just as clearly as when we compare the plots of SIGN(SIN(x)) SIN(nx) for even integers with those for odd integers (pay attention to the symmetry). So it´s not an error it´s a feature.





Let’s compare earlier DERIVE results with recent ones:

$$\mathrm{VECTOR}\left(\frac{\cos(2\cdot\pi\cdot n)-2\cdot\cos(\pi\cdot n)+1}{n},\ n,\ 0,\ 10\right)$$

$$\left[?,\ 4,\ 0,\ \frac{4}{3},\ 0,\ \frac{4}{5},\ 0,\ \frac{4}{7},\ 0,\ \frac{4}{9},\ 0\right]$$

$$\mathrm{VECTOR}\left(\frac{\cos(\pi\cdot n)\cdot(\cos(2\cdot\pi\cdot n)-1)}{n\cdot(\cos(\pi\cdot n)+1)},\ n,\ 0,\ 10\right)$$

$$[?,\ ?,\ 0,\ ?,\ 0,\ ?,\ 0,\ ?,\ 0,\ ?,\ 0]$$

$$\mathrm{VECTOR}\left(\lim_{n\to k}\frac{\cos(\pi\cdot n)\cdot(\cos(2\cdot\pi\cdot n)-1)}{n\cdot(\cos(\pi\cdot n)+1)},\ k,\ 0,\ 10\right)$$

$$\left[0,\ 4,\ 0,\ \frac{4}{3},\ 0,\ \frac{4}{5},\ 0,\ \frac{4}{7},\ 0,\ \frac{4}{9},\ 0\right]$$

Harald Lang, Stockholm, Sweden lang@math.kth.se

I have DERIVE 3.14 classic. When solving î SIN(w) – 3 COS(w) = 2 for w DERIVE produces the erroneous answer:

$$w=\frac{\pi}{2}+i\cdot\left(\frac{\pi}{4}-\mathrm{LN}(\sqrt{2})\right)$$

Do later versions of DERIVE solve this equation correctly?

DNL:

See DERIVE version 4´s answer:

```
î·SIN(w) - 3·COS(w) = 2

SOLVE(î·SIN(w) - 3·COS(w) = 2, w)
```

$$\left[w = \frac{3\cdot\pi}{4} - \frac{î\cdot LN(2)}{2},\ w = -\frac{3\cdot\pi}{4} - \frac{î\cdot LN(2)}{2},\ w = \frac{5\cdot\pi}{4} - \frac{î\cdot LN(2)}{2}\right]$$

DERIVE 6 gives the same results.

Steve Smith, Powell, Wyoming smiths@mail.nwc.whecn.edu

I am trying to plot Euler´s Spiral: $B(t) = \int_0^t \cos\left(\frac{\pi x^2}{2}\right) dx + i \int_0^t \sin\left(\frac{\pi x^2}{2}\right) dx$.

When I try to plot the following:

```
[INT(COS(π/2 x^2),x,0,t), INT(SIN(π/2 x^2),x,0,t)]
```

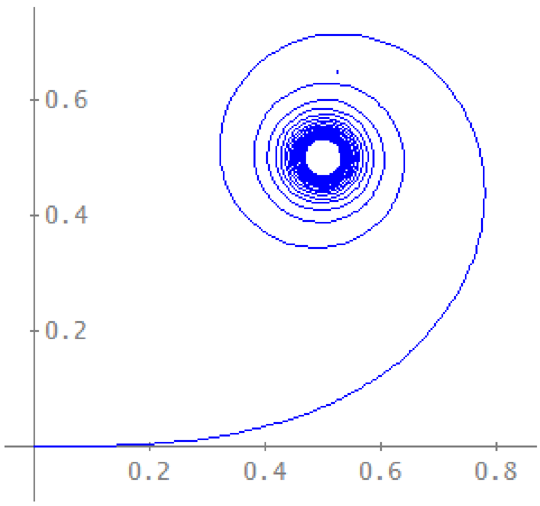
DERIVE says "sorry, the highlighted expression cannot be plotted". I assume it is bailing out because of the integrals, but I don´t see why that should be a problem. I can compute a few of the individual x and y coordinates using vector expressions such as

```
VECTOR(INT(COS(π/2 x^2),x,0,t),t,0,10)
VECTOR(INT(SIN(π/2 x^2),x,0,t),t,0,10)
```

But combining the results by hand for DERIVE to plot is time consuming. Is there a nice way for DERIVE to plot this type of function?
Thanks, Steve

Peer van de Sanden, Netherlands peer.marijke@gironet.nl

Hello, Steve,
I have copied your vector to DfW 4.1 and tried to plot it. No problem at all. Then I tried DERIVE XM 3.05 for DOS and tried to plot the vector. It was slow, but this version has also no problems with the plot.



No problems to plot the parameter representation of the spiral with DERIVE 6.

The Erlang C formula story

Ray Girvan, England ray.girvan@zetnet.co.uk

I've just offered (rashly, it turns out) to confirm, with Derive in exact mode, some results for the Erlang C formula.

```
E(m,u,rho):=u^m/m! / (u^m/m! + (1-rho)*SUM(u^k/k!,k,0,m-1))
```

My timings, using DFW on a 133MHz Pentium, are:

E(320,319,0.9999) ... 8 seconds	(0.406 with DERIVE 6.10)
E(640,639,0.9999) ... 68 seconds	(2.75)
E(1280,1279,0.9999) ... 1098 seconds (ie 18 minutes)	(5.55)
E(2500,2499,0.9999) ... 5359 seconds (ie 89 minutes)	(15.9)

I'm wondering how much can be trimmed off the time. Is there anyone with a faster machine who'd be prepared to try out some/all of the above for timing comparisons? Thanks in advance, Ray

Thomas Lingefjard, Sweden

My timings on a 266 MHz Pentium II Notebook (80 MB RAM; Win98, DfW 4.11)
....
E(2500,2499,0.9999) ... 2464.4 seconds (i.e. 41 minutes 4 seconds)

On a 450 MHz Pentium II Workstation (264 MB RAM; WinNT, DfW 4.11)

E(2500,2499,0.9999) ... 1375.6 seconds (almost 23 minutes)

Dr N.B. Backhouse, England

Timings on a PC fitted with a 300 MHz AMD K/6-2 processor (64 MB RAM; Win98; DfW 4.11)

E(320,...	1.9s
E(640,...	15.5s
E(1280,...	150.5s
E(2500,...	1109.2s

Impressive!

Johann Wiesenbauer J.Wiesenbauer@tuwien.ac.at

Hi all,

Rather than using faster and faster Pentium PCs it may be worthwhile to look into Ray's formula itself which is certainly a far cry from being optimized. For example, the simple approach of rewriting it in the form

```
E(m,u,ρ):=u^m/(u^m+m!(1-ρ)LIM(TAYLOR(EXP(x),x,0,m-1),x,u))
```

will already about half the computation time, but there should be even more time saving implementations. How about a competition on that score?

Cheers, Johann

Stefan Welke, Germany Spwelke@aol.com

Hallo derivers, I can offer an improvement of Ray's Erlang C formula. This is my version:

```
AUX(x, n) := n·ITERATE([1, 1] + v_↓1·[1, v_↓2/x], v_, [1, 1], n - 1)/x

E2(n, x, r) := 1/(1 + (1 - r)·(AUX(x, n))↓2)
```

I found the following execution times on my PentiumII, 300Mhz:

E(640,639,0.9999) : 24.3 seconds (Ray's version)
E2(640,639,0.9999) : 6.9 seconds (My version)

E(2500,2499,0.9999) (0.39 sec with DERIVE 6.10)

This is about twice as fast as Johann's version utilizing the Taylor polynomials (5.34 sec with DERIVE 6.10).

Cheers, Stefan

Johann Wiesenbauer J.Wiesenbauer@tuwien.ac.at

Very nice, indeed! Although I would prefer a more concise representation such as

$$E3(m,\ u,\ \rho) := \frac{u}{u + (1 - \rho)\cdot m\cdot\left(\mathrm{ITERATE}\left([1,\ 1] + v\_{_1}\cdot\left[1,\ \frac{v\_{_2}}{u}\right],\ v\_,\ [1,\ 1],\ m - 1\right)\right)_2}$$

from an algorithmic point of view this could well be the very best that can be achieved. Isn't it consolatory that in the first place it's the gray matter of the programmer that counts and not the number of MHz of his machine?

Cheers, Johann

From Josef Böhm to Ray Girvan:

.... I have a very stupid question: For which purpose do you need Erlang C? I looked into several formula collections, but nowhere Erlang C is even mentioned?

Ray Girvan, England ray.girvan@zetnet.co.uk

Josef,

This has turned into a very interesting problem (and thank you, everyone, for the improvements).

I had never heard of it either. It's something to do with queuing theory in telecommunications, though I'm not sure what, as I don't use it myself. The reason for needing to test this was to check a Mathcad worksheet I wrote. Someone in the Mathcad discussion group was having problems with this function, since Mathcad's symbolic solver was falling over, and its numerical solver overflows at 10^307.

I tried recasting the Erlang C function as an approximation using the logarithm of the Stirling formula, so that all the power and factorial work could be done in the log domain to keep the magnitude safe.

```
LOG_GAMMA(n):=LOG(SQR(2*pi))-n+(n-0.5)*LOG(n)+LOG(1+1/(12*n)+1/(288*n^2)-
              139/(51840*n^3)-571/(2488320*n^4)+163879/(209018880*n^5)+
              5246819/(75246796800*n^6)-534703531/(902961561600*n^7)-
              4483131259/(86684309913600*n^8))

F(k,m,u):=EXP(LOG_GAMMA(m+1)+(k-m)*LOG(u)-LOG_GAMMA(k+1))

ERLC(m,u,rho):=1/(1+(1-rho)*SUM(F(k,m,u),k,0,m-1))
```

It's messy and inefficient in Derive, but runs extremely fast in Mathcad, to an acceptable 13 or 14 digits of accuracy. For instance,

ERLC(5000,4999,0.9999) = 0.99114905600313 ... 14 seconds!

(I think this speed is due to the way Mathcad throws away small terms below the set tolerance level).

But my problem then was to prove its accuracy against exact solutions.

I have to admit, when I first saw the name Erlang C, it sounded like a dialect of C developed at Erlangen! I did a small web search; Erlang turned out to be a person, who developed loading formulae for telephone exchanges early this century. See:

http:// http://plus.maths.org/content/os/issue2/erlang/index

Ray

End of the Erlang C formula story

Comment: the URL given above is valid, Josef

<u>Subject Pythagorean Triples</u> (Requested by T.P.Woods) bjallen@ibm.net

I note that all Pythagorean Triples can be commutatively expressed, as the sum of consecutive, odd, integers, two within the confines of the third e.g. 5^2 = 3^2 + 4^2 = 4^2 + 3^3 = = [1+3+5+7+9] = [1+3+5] + [7+9] = [1+3+5+7]+[9], by virtue of the commutative property of addition, and as correspondingly bracketed.

Similarly 13^2 = 5^2 + 12^2 = 12^2 + 5^2 = [1+3+5+7+9+11+13+15+17+19+21+23+25] = = [1+3+5+7+9] + [11+13+15+17+19+21+23+25] = [1+3+5+7+9+11+13+15+17+19+21+23] + + [25] as correspondingly bracketed, and so on for all Pythagorean Triples, and there is an infinite number of them.

Surely, there must be a formal algebraic proof of this simple commutative property that is unique to squares, and never to cubes, or any other higher power?

I have searched everywhere without success. Perhaps, your mathematics department may contribute. I´d appreciate your help. Thank you. T.P.Woods
(From the cti-maths Newsgroup)

Do you know "Vampire Numbers"?

Project: The Great Canadian World Wide Web Vampire Number Search

Project Overview: My grade 12 computer science class and I are looking for about 1,000 participants who will run a small computer program overnight and email back a data file of results. We would like participants from all over the world.

Details: A vampire number is a whimsical mathematical idea, introduced by computer graphics guru Clifford Pickover, in the June 1995 issue of Discover magazine. It makes note of the fact that when you multiply two numbers, all of the digits in those two numbers occasionally show up in the result. For example 21 × 87 = 1827, or 146 × 938 = 136948. The number 1827 is a four digit vampire number and 136948 is a six digit vampire number. Looking for these numbers makes for an interesting computer programming project. I have several web pages that describe vampire numbers in detail. See http://grenvillecc.ca/faculty/jchilds and follow link(s) to vampire numbers. (Or you can go directly to http://grenvillecc.ca/faculty/jchilds/vamli12.htm)

Comment: the URLs given above are not valid. If you are interested in Vampire Numbers then try the links given below, Josef

http://en.wikipedia.org/wiki/Vampire_number
http://rosettacode.org/wiki/Vampire_number
http://www.dreamincode.net/forums/topic/204318-c-program-to-generate-all-vampire-numbers-upto-a-given-number/

Rüdeger Baumann, Celle, Germany baumann-celle@t-online.de

Dear Josef,
The "Look´n Say"-Sequence {1,11,21,1211,111221,312211,....} seems not to be easily to program in DERIVE because you have to consider the order of the numbers.
The sequence which I presented in DNL#33 {1,3,7,12,18,26,35,...} forms together with the sequence of its differences [2,4,5,6,8,9,....} the set of the natural numbers. It is not difficult to program this sequence in PASCAL.

Another thing: Here is a DERIVE session belonging to Alfonso´s ACDC7 and Leandro´s and Javier´s contribution in DNL#30: Diophantine Equations:

```
#1:   InputMode := Word

#2:   El algoritmo de Euclides

#3:   EUCLIDES(a, b) := ITERATE(IF(y = 0, [x, y, s0, s1, t0, t1], [y, MOD(x, y),
        s1, s0 - FLOOR(x, y)·s1, t1, t0 - FLOOR(x, y)·t1]), [x, y, s0, s1, t0,
        t1], [a, b, 1, 0, 0, 1])

#4:   LET(u, v, w) := ITERATE(u, v, w, 1)

#5:   EUCLID(a, b) := LET([v₁, v₃, v₅], v, EUCLIDES(a, b))
```

```
#6:   Ejemplo de Tortosa & Santacruz
#7:   [a := 29, b := 17, c := 780]
#8:   EUCLID(a, b) = [1, -7, 12]
#9:   [d := 1, s := -7, t := 12]
#10:  Solucion particular:
```

#11: $\left(p0 := \left[\frac{s\cdot c}{d}, \frac{t\cdot c}{d}\right]\right) = p0 := [-5460, 9360]$

#12: Solucion general:

#13: $\left(u := \left[\frac{b}{d}, -\frac{a}{d}\right]\right) = u := [17, -29]$

#14: $p0 + k\cdot u = [17\cdot k - 5460, 9360 - 29\cdot k]$

#15: $(F(k) := (p0 + k\cdot u)^2) = F(k) := 1130\cdot k^2 - 728520\cdot k + 117421200$

#16: $\text{SOLVE}(2260\cdot k - 728529, k) = \left(k = \frac{728529}{2260}\right)$

```
#17:  SOLVE(2260·k - 728529, k) = (k = 322.3579646)
#18:  k0 := 322
#19:  Solucion special:
#20:  p0 + k0·u = [14, 22]
#21:  29·14 + 22·17 = 780
```

Jonathan Devor, USA jdevor@usa.net

Hello all DERIVERS,
I've been trying to solve the following set of non-linear equations:

$$b^2 + b\cdot c + c^2 = x,\ \ a^2 + a\cdot c + c^2 = y,\ \ a^2 + a\cdot b + b^2 = z$$

solving for *a, b, c*.
I use DERIVE 3.12 for DOS, and as is, it gave up on these equations very quickly. Helping it along manually, I got a "solution not verified" or an unending calculation pause (I gave up after 20 hours on my 266MHz Pentium). Does any one out there have a substitution or other trick to get a solution for this, seemingly simple problem.
Thanx.

Johann Wiesenbauer J.Wiesenbauer@tuwien.ac.at

In the enclosed file I have tried to sketch an ad hoc solution (without Groebner bases and all that stuff!) If you feel like trying it out yourself, you are strongly advised to print it out and input or copy all the necessary lines into a new (!) Derive-file. (Note that double assignments must always be simplified subsequently, otherwise they won't work properly!)

I have left out some special cases and details, but would be glad to provide you with further information if needed. One concluding question: Is this "seemingly simple" problem, as Jonathan put it, really simple or not?
I leave it to you to answer this question.

Cheers, Johann

```
x = √6·√(√3·√(– a^4 + 2·a^3·(b + 2·c) – a^2·(b^2 + 2·b·c + 6·c^2) + 2·a·c·(b^2 –
    b·c + 2·c^2) – c^2·(b^2 – 2·b·c + c^2))·ABS(b – c) + 2·a^3 + a^2·(b – 5·c) +
    a·(b^2 – 3·b·c + 4·c^2) + 2·b^3 – 5·b^2·c + 4·b·c^2 – c^3)/(6·√(a^2 – a·(b +
    c) + b^2 – b·c + c^2))
```

(*This one solution for x – there are four. The solutions for y and z are obtained by substitution. Johann changed the unknowns a,b,c to x,y,z and the parameters x,y,z to a,b,c. See* NONLIN.MTH.)

Ray Girvan, England ray.girvan@zetnet.co.uk

This is not as rigorous as Johann´s solution, but a brute force approach seems to work (in DfW)

```
#1: b^2 + b*c + c^2 - x
#2: a^2 + a*c + c^2 - y
#3: a^2 + a*b + b^2 - z
```

Solve #1 for b gives two solutions of form b = f(c,x) and solve #3 for b gives two solutions of form b = f(a,z). Pair all combinations of these solutions to give four equations of the form f(a,c,x,z) = 0. Solve each of these for c, giving eight solutions c = f(a,x,z).
At this point, plug in trial values .... which shows four are duplicates, which you can reject. Then plug these c results into equation #2, simplify, and solve for a. This gives sixteen solutions, which again contain duplicates. They look like, for instance:

```
a = SQR(6)*SQR(-SQR(3)*SQR(-x^2+2*x*(y+z)-y^2+z*(2*y-z))
    *ABS(x^2-x*(y+z)+y*z)-x^3+4*x^2*(y+z)-x*(5*y^2+3*y*z+5*z^2)
    +2*y^3+y^2*z+y*z^2+2*z^3)/(6*SQR(x^2-x*(y+z)+y^2-y*z+z^2))
```

And these can then be substituted back into 1#, #2, #3 to get expressions for b and c.

Ray

Johann Wiesenbauer J.Wiesenbauer@tuwien.ac.at

Hi all,

[Unfortunately there had been a lot of mistakes in my last email. For your convenience, what follows is a (hopefully!) correct version. I'm very sorry!]

Since I have some time left right now, some overdue comments on Jonathan's "simple" set of nonlinear equations might be in order. In the first place it shouldn't be too difficult to come to the agreement in the community of mathematicians that it is more appropriate to use the following notations:

```
x^2 + xy + y^2 = a
x^2 + xz + z^2 = b
y^2 + yz + z^2 = c
```

(Otherwise even DERIVE gets confused in this topsy-turvy world, where a,b,c are variables and x,y,z constants!) Anybody out there who objects to it?
Here we go!

As I have found out as well by forming the difference of any two of these equations one gets

```
(y-z)(x+y+z) = a-b
(x-z)(x+y+z) = a-c
(x-y)(x+y+z) = b-c
```

As a matter of fact it is easy to see that the system consisting of any two of these equations together with one of the original equations is equivalent to the original system of equations, e.g.

```
(x-z)(x+y+z) = a-c
(y-z)(x+y+z) = a-b
  x^2+xy+y^2 = a
```

Now you can use Derive (or your loaf!) to see that the third equation can be rewritten in the form (this is actually the main trick!)

```
(((x - z) + (y - z) + 2(x + y + z))^2 + 3((x-z) - (y-z))^2)/12 = a
```

Thus after setting

```
u:=x-z, v:=y-z, w:=x+y+z
```

we get for u,v,w the equations

```
                          uw = a-c
                          vw = a-b
((u+v+2w)^2+3(u-v)^2)/12 = a
```

Because of

```
x = (2u - v + w)/3
y = - (u - 2v - w)/3
z = - (u + v - w)/3
```

it is obviously sufficent now to solve this new system in u,v,w.

I know that people like Jonathan would have said "from here on things are fairly straight forward" or something like that long ago, but I beg for your pardon if I continue mercilessly all the same.

So let's differ between two fundamental cases:

1. w = 0 (only possible, if also a = b = c !! Therefore this is exactly the exceptional case, Ray has mentioned!)

   In this case we have only one equation that must be fulfilled, viz.

   ```
   u^2 - u·v + v^2 = 3a
   ```

   This means that we can choose an arbitrary value for v and accordingly get two values for u (or only one, if we had chosen $v = 2\sqrt{a}$).

2. $w \neq 0$

   In this case the third equation can be rewritten in the form

   ```
   ((uw+vw+2w^2)^2+3(uw-vw)^2)/12 = aw^2
   ```

   or using the first two equations

   ```
   w^4 - (a + b + c)w^2 + a^2 + b^2 + c^2 -ab - ac - bc = 0
   ```

This is actually a quadratic equation for w^2 (I hear Jonathan moan in the background!), so you can solve it even without DERIVE. One solution is

```
w1 := sqrt((sqrt(3((a+b+c)^2-2(a^2+b^2+c^2)))+a+b+c)/2)
```

and the other three you get by changing the signs of the two square roots independently. The corresponding values of u and v are now given by

```
u = (a-c)/w,   v = (a-b)/w
```

and x,y,z are finally obtained by the formulas above.

One more word to Jonathan's linear equation

```
(a-b)x+ (c-a)y +(b-c)z = 0
```

which I have also found myself. I can understand that he enjoyed this beautiful equation, but it's more interesting from a philosophical point of view (why are all solutions - typically 4 as we have seen - on this plane and what does it mean from a geometric point of view?) and doesn't provide that extraordinary shortcut in my eyes. (Furthermore, it doesn't exist in the case a = b = c, as Ray has noticed!) Look at my previous DERIVE-solution again and you'll see that I arrived very fast (and without any manual computations!) at a fourth equation of the form (#17)

```
x^2(a - 2b + c) - xy(2a - b - c) + y^2(a + b - 2c) = -(b - c)^2
```

This is what you can also expect when using this linear equation. But the actual struggle began thereafter! (Well, with the benefit of hindsight maybe I should have covered this up by saying "the rest is plain sailing"!)

Cheers, Johann

DERIVE 6 has no problems solving this nonlinear system using its built-in Groebner bases. The two lines of code given below produce all solutions within 17 seconds, Josef

$$\left[eq1 := b^2 + b\cdot c + c^2 = x,\ eq2 := a^2 + a\cdot c + c^2 = y,\ eq3 := a^2 + a\cdot b + b^2 = z\right]$$

```
SOLUTIONS(eq1 ∧ eq2 ∧ eq3, [a, b, c])
```

Arnaldo Struzberg, Rio de Janeiro, Brasil aesse@marlin.com.br

How can I obtain with DERIVE the cartesian graph of a binary relation?

Examples: S = {(x,y) ∈ R × R | x^2 y < y^2}
S = {(x,y) ∈ R × R | x^2 + y^2 < 4}

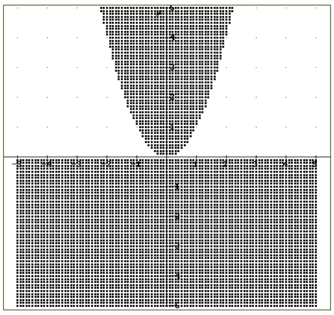
Thanks

DNL: nojo.boehm@pgv.at

What about this? Here is an attempt to graph your relation as a collection of points. Approximate and plot. Josef

There are rumors, that DfW5 will be able to plot the graph of a relation directly! In the meanwhile I could test DfW5 and I can confirm, it does.

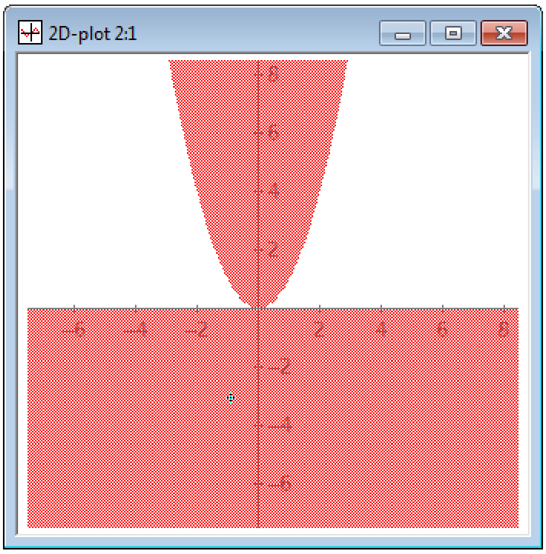
```
REL(rm, xa, xe, dx, ya, ye, dy) := VECTOR(VECTOR(IF(rm, [x, y], ?, ?),
                                   x, xa, xe, dx), y, ya, ye, dy)

rm := x ≥ 2 ∧ |x| = |y|

REL(rm, -10, 10, 1, -10, 10, 1)

rm2 := x ≥ -2 ∧ |x| ≤ |y|

REL(rm2, -5, 5, 0.2, -5, 5, 0.2)

REL(y ≥ |x - 1| ∧ y ≤ |4 - x|, -10, 10, 0.5, -10, 10, 0.5)

REL(MOD(y, x) = 2, -10, 10, 1, -10, 10, 1)

        2      2
REL(x   + y   < 4, -3, 3, 0.1, -3, 3, 0.1)

        2        2
REL(x  ·y  <  y , -5, 5, 0.1, -5, 5, 0.1)
```



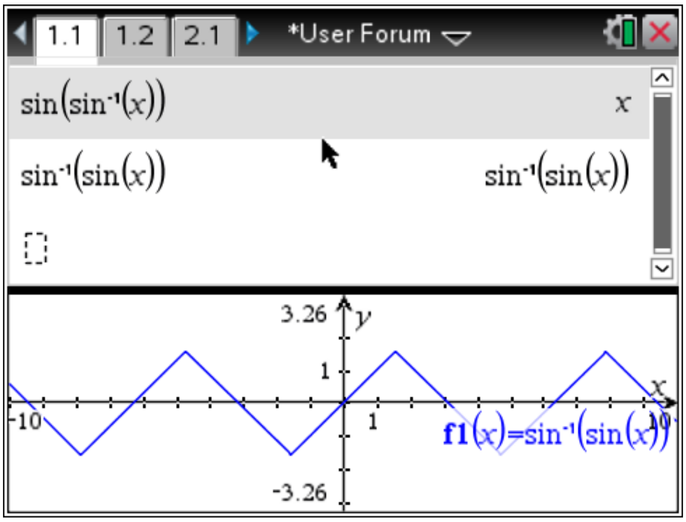
Above you can see the DERIVE 4 procedure but …

… rumours were right. You can now plot the binary relations directly by simply entering the relation and the plotting it, Josef

#1: $x^2 \cdot y < y^2$

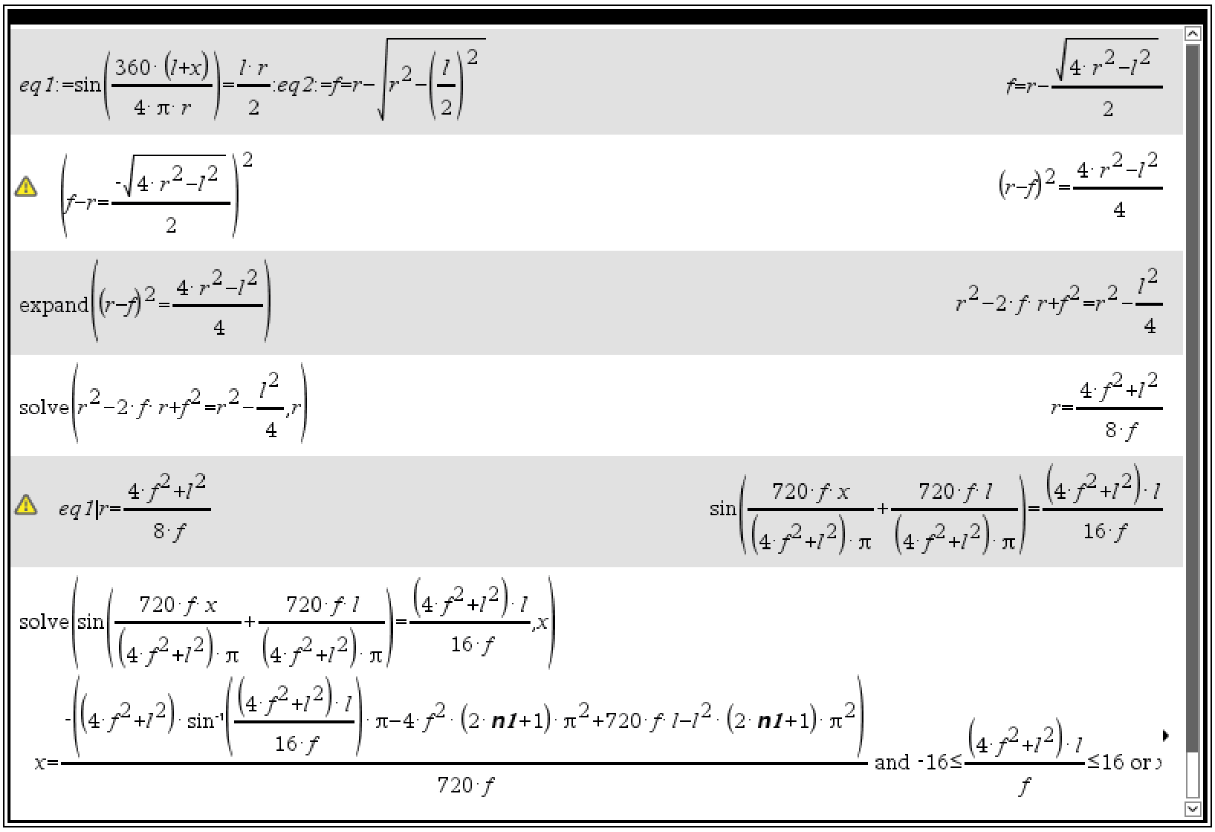


See how TI-Nspire reacts on some User Forum requests from 1999 on the next pages.



arcsin(sin(x)) does not simplify further.

Johan Vegter's system can be solved!



On the next page you will find that

(1) Ray Girvan's integral can not be given in general form,

(2) Harald Lang's equation can not be solved exactly, only numerically,

(3) Peer van de Sanden's spiral can be plotted (needs some time).

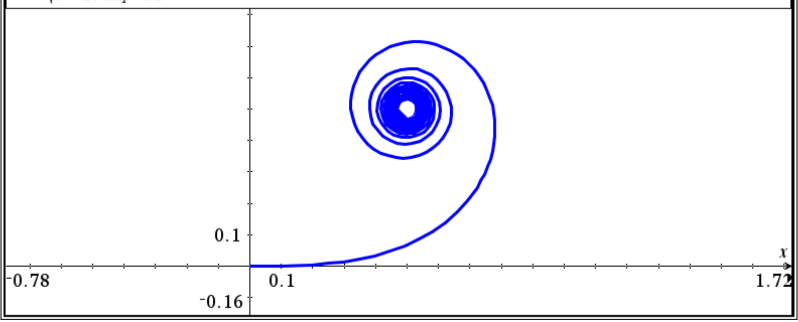
$\int_0^{2\cdot\pi} (\text{sign}(\sin(x))\cdot\sin(x))\,dx$	$\int_0^{2\cdot\pi} \lvert\sin(x)\rvert\,dx$
$\int_0^{2\cdot\pi} (\text{sign}(\sin(x))\cdot\sin(x))\,dx$	4.00000000
$\int_0^{2\cdot\pi} (\text{sign}(\sin(x))\cdot\sin(n\cdot x))\,dx$	$\int_0^{2\cdot\pi} (\sin(n\cdot x)\cdot\text{sign}(\sin(x)))\,dx$
$\int_0^{2\cdot\pi} (\text{sign}(\sin(x))\cdot\sin(n\cdot x))\,dx \mid n=1$	4.00000000
$\int_0^{2\cdot\pi} (\text{sign}(\sin(x))\cdot\sin(n\cdot x))\,dx \mid n=3$	$\int_0^{2\cdot\pi} (\sin(3\cdot x)\cdot\text{sign}(\sin(x)))\,dx$
$\int_0^{2\cdot\pi} (\text{sign}(\sin(x))\cdot\sin(n\cdot x))\,dx \mid n=3$	1.33333333

© There is no generalized result for the integral!

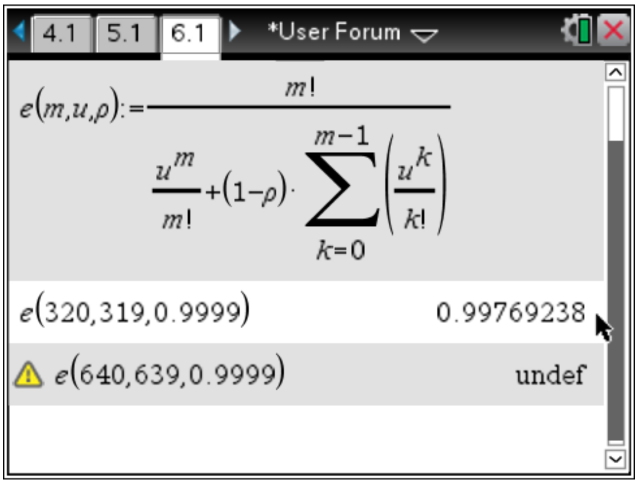
$\text{solve}(i\cdot\sin(w)-3\cdot\cos(w)=2,w)$	false
$\text{cSolve}(i\cdot\sin(w)-3\cdot\cos(w)=2,w)$	$3\cdot\cos(w)+{}^{-}i\cdot\sin(w)+2=0$
$\text{cSolve}(i\cdot\sin(w)-3\cdot\cos(w)=2,w)$	$3\cdot\cos(w)+e^{\frac{-i\cdot\pi}{2}}\cdot\sin(w)+2=0$
⚠ $\text{cSolve}(i\cdot\sin(w)-3\cdot\cos(w)=2,w)$	$w=2.35619449-0.34657359\cdot i$
$\frac{3\cdot\pi}{4}-\frac{i\cdot\ln(2)}{2}$	$2.35619449-0.34657359\cdot i$

cSolve works only by approximation (*Ctlr+Enter*), *no* exact

$$\begin{cases} x1(t)=\int_0^{t}\cos\left(\frac{\pi\cdot x^2}{2}\right)dx \\ y1(t)=\int_0^{t}\sin\left(\frac{\pi\cdot x^2}{2}\right)dx \end{cases} \quad 0\le t\le 10 \;\; tstep=0.1$$



The Erlang C formula can be defined and works in a very restricted amount.



I didn't try solving the nonlinear system from page 17.

But I started developing a package for solving linear difference equations of first and second order. DERIVE provides a utility file RecurrenceEquations.mth for finding exact solutions of first and second order difference equations.

It started very promising. Then I discovered an expression ATAN(√(-d),-p) which is explained in DERIVE's help file as follows: "ATAN(y, x) is the angle of the point (x, y) in the x-y plane measured counterclockwise from the positive x-axis." This was no problem to realize with Nspire. Next difficulty was the reason to give up – at least for me:

h·∑(r·LIM([- ELEMENT(h, 2), ELEMENT(h, 1)]/DET([h, LIM(h, x, x + 1)]), x, x + 1), x)

This sum(u,x) gives the "antidifference" F(x) of u with respect to x. F(x+1) – F(x) = u

$$\sum_x (x^2 + 1) = \frac{x^3}{3} - \frac{x^2}{2} + \frac{7\cdot x}{6}$$

$$\sum_x 10\cdot 3^{-x-2}\cdot(-1)^{-x} = -\frac{5\cdot 3^{-x-1}\cdot(-1)^{-x}}{2}$$

$$\frac{(x+1)^3}{3} - \frac{(x+1)^2}{2} + \frac{7\cdot(x+1)}{6} - \left(\frac{x^3}{3} - \frac{x^2}{2} + \frac{7\cdot x}{6}\right) = x^2 + 1$$

$$-\frac{5\cdot 3^{-(x+1)-1}\cdot(-1)^{-(x+1)}}{2} - -\frac{5\cdot 3^{-x-1}\cdot(-1)^{-x}}{2} = 10\cdot 3^{-x-2}\cdot(-1)^{-x}$$

I was not able to find the antidifference (which is the indefinite sum) in my first attempt with TI-Nspire. Any advice would be highly appreciated. See more on page 57.

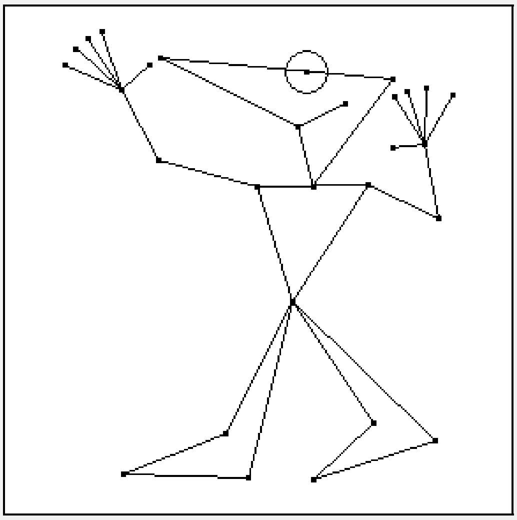
WHAT FILOU HAS BEHIND HIS HEAD

Jean-Jacques Dahan, Toulouse, France

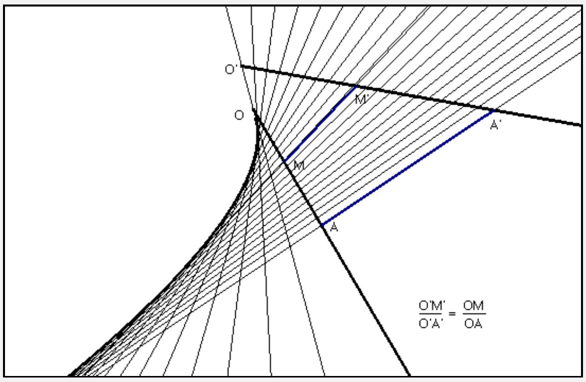
Jean-Jacques DAHAN is a Mathematics Teacher in Toulouse (France) and the author of a book called "An introduction to geometry with the TI-92" in which he has created a small character baptized "Filou" which he uses to show how one can approach the module of geometry using TI-92 in a recreational way.

You are going to attend a conversation between Jean-Jacques and Filou where Jean-Jacques is going to try to show to his little friend that even the most harmless of configurations can hide subtle results. It will perhaps convince the reader of the power of the relationship between Computer Algebra Systems and Geometry using the same instrument.

WHAT FILOU HAS BEHIND HIS HEAD



OR



STORIES ABOUT PARABOLAS

FILOU: Tell me, Jean-Jacques, what's the meaning of this title? How could I have something behind my head without even knowing it myself?

JJD.......You forget that I created you and I am no doubt in a better position than you to know that you are concealing a magnificent parabola.

FILOU: Really! a parabola, this parabola, I don't feel it, , neither in front of my head, nor behind it. You must explain this to me, there is something in what you say but I don't know what and I don't understand.

JJD.......Of course! I'll explain it to you as well to our readers who I feel, are spying in our conversation with certain curiosity. I am going to tell you a mathematics story in three parts and you will see that your head has a very close relationship with a certain parabola !

FIRST EPISODE: ALL POLYNOMIALS OF THE SECOND DEGREE LEAD TO A PARABOLA

1/ DEFINITION: let's call a parabola any curve of the euclidian plane that is the representative curve of a function of the type $x \to ax^2$ in the cartesian coordinate system of this plane with $a \neq 0$

2/ CLASSIC PROPERTIES: Theorem 1 : the representative curve of a function of the type $x \to ax^2+bx+c$ with $a \neq 0$ in a cartesian coordinate system $(O, \vec{i}, \vec{j})$ of the plane is a parabola.

Proof:
It suffices to seek the equation of this curve in the cartesian coordinate system $(I, \vec{i}, \vec{j})$, where I is its extremal point of the abscissa $-\frac{b}{2a}$.. One finds $Y=aX^2$.

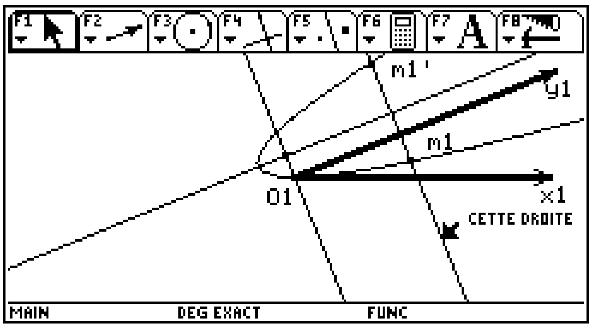
Theorem 2: a parabola is the set of the points M of the plane such that MF=MH where F is a point on the plane called the focus of the parabola and H is the orthogonal projection of M on a line D called the directrix of this parabola. If $y=ax^2$ is the equation of the parabola in a cartesian coordinate system, then F has for its coordinates $\left(\frac{1}{4a}, 0\right)$ and D has for its equation $y = -\frac{1}{4a}$.

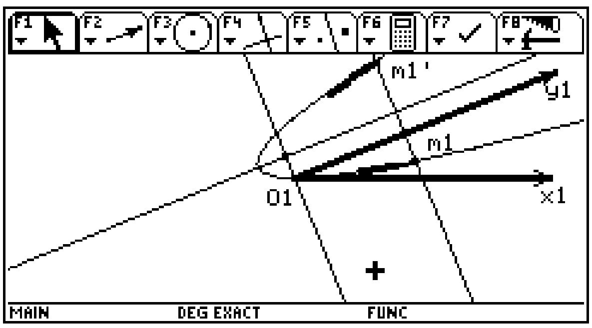
Proof: it is an elementary norm calculation that can be made very easily with the TI-92.

3/ A MORE ORIGINAL PROPERTY:

Constructions, manipulations and conjecture

Let's draw in a cartesian coordinate system $(O, \vec{i_1}, \vec{j_1})$, the curve of the equation $y_1=a_1.(x_1)^2$ with, for instance $a_1=1/2$. It seems that this curve has an axis of orthogonal symmetry parallel to the y axis of our coordinate system. Moreover, one can conjecture that this curve is a parabola as has been defined above.



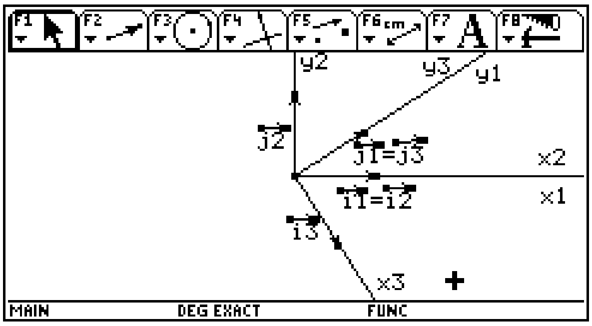


Note that we detect this symmetry by tracing a parallel line to Oy_1 until it matches up. We then draw the symmetrical point m_1' from the generic point m1 of our curve with respect to this line. When we move m_1 on our curve, m_1' also seems to remain on the same curve.

Proof:

Let (P) be the curve of equation $y_1=a_1.(x_1)^2$ in the cartesian coordinate system $(O,\vec{i_1},\vec{j_1})$; let's find its equation in the direct coordinate system $(O,\vec{i_3},\vec{j_3})$. For that, let's use the direct coordinate system $(O,\vec{i_2},\vec{j_2})$.

Note that $(\vec{i_1},\vec{j_1})=\alpha$..



From the vectorial equalities: $\vec{i_1}=\vec{i_2}$ and $\vec{j_1}=\cos\alpha\cdot\vec{i_2}+\sin\alpha\cdot\vec{j_2}$ and $\vec{i_3}=\sin\alpha\cdot\vec{i_2}-\cos\alpha\cdot\vec{j_2}$ and $\vec{j_3}=\cos\alpha\cdot\vec{i_2}+\sin\alpha\cdot\vec{j_2}$ one can easily deduce the formulas :

$x_2=x_1+y_1\cdot\cos\alpha$ $y_2=\quad y_1\cdot\sin\alpha$	Then : $x_2=\sin\alpha\cdot x_3+\cos\alpha\cdot y_3$ $y_2=-\cos\alpha\cdot x_3+\sin\alpha\cdot y_3$

Then :

$x_1=\frac{1}{\sin\alpha}.x_3$ and $y_1=-\frac{\cos\alpha}{\sin\alpha}.x_3+y_3$

All calculations made, (P) has an equation in $(O,\vec{i_3},\vec{j_3})$:

$y_3=\frac{a_1}{\sin^2\alpha}.(x_3)^2+\frac{\cos\alpha}{\sin\alpha}.(x_3)$, so that (P) is indeed a parabola.

NB: if I is the maximum point of this parabola, I ($x_3(I)=-\frac{\sin(2\alpha)}{4a_1}$, $y_3(I)=-\frac{\cos^2\alpha}{4a_1}$), the former has an equation of $y_4=\frac{a_1}{\sin^2\alpha}.(x_4)^2$ in the coordinate system $(I,\vec{i_4},\vec{j_4})$ where $\vec{i_4}=\vec{i_3}$ and $\vec{j_4}=\vec{j_3}$ will have as its focus:

F($x_4(F)=0$, $y_4(F)=\frac{\sin^2\alpha}{4a_1}$) or F($x_1(F)=-\frac{\cos\alpha}{2a_1}$, $y_1(F)=\frac{1}{4a_1}$)

And as directrix, the line (D) has the equations:

$y_4=\frac{\sin^2\alpha}{4a_1}$ or $y_1=(-\cos\alpha).x_1-\frac{1}{4a_1}$.

Case: taking any initial coordinate system $(O,\vec{i},\vec{j})$ with $(\widehat{\vec{i};\vec{j}})=\alpha$

Let us use in this coordinate system, a curve of the equation $y=a.x^2$ with $a\neq0$.

An equation of this curve in the coordinate system. $(O,\vec{i_1},\vec{j_1})$ where: $\vec{i_1}=\frac{\vec{i}}{\|\vec{i_1}\|}=$ and $\vec{j_1}=\frac{\vec{j}}{\|\vec{j}\|}$ is

y1 =a1.(x1)² with $a1=\frac{a\cdot\|\vec{j}\|}{\|\vec{i}\|^2}$:

given by the formulas: $x=\frac{x_1}{\|\vec{i}\|}$ and $y=\frac{y_1}{\|\vec{j}\|}$. It is therefore a parabola and we have proved:

Theorem 3: the curve of a function of the type y=a.x² in any coordinate system $(O,\vec{i},\vec{j})$ of the plane with a≠0 and $\widehat{(\vec{i};\vec{j})} = \alpha$ is a parabola whose symmetry axis is a parallel line to the y axis of the coordinate system , this axis has for its equation : $x = -\frac{\cos\alpha}{2a_1}$, the focus F has for its coordinates ($x(F) = -\frac{\cos\alpha}{2a_1}$, $y(F) = \frac{1}{4a_1}$) and the director (D) has for its equation $y=(-\cos\alpha).x-\frac{1}{4a_1}$, where we have previously proposed $a1 = \frac{a\cdot\|\vec{j}\|}{\|\vec{i}\|^2}$.

End of the first episode

FILOU: I think I've been quite patient up to this point, but where is the parabola that I'm supposed to be hiding ?

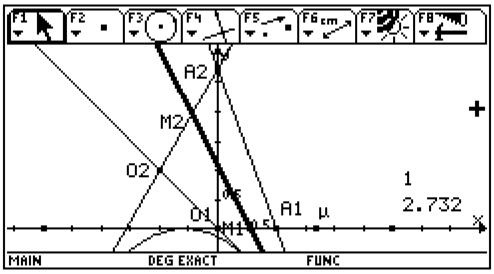
JJD.......I'm getting to it; you should be more patient. This episode was just a necessary introduction.

FILOU: Right ! Well I guess I'll have to summon up some patience. I'm all ears!

SECOND EPISODE:
A PARABOLA, FOUR TANGENTS AND PROPORTIONALITY

Constructions, manipulations and conjecture

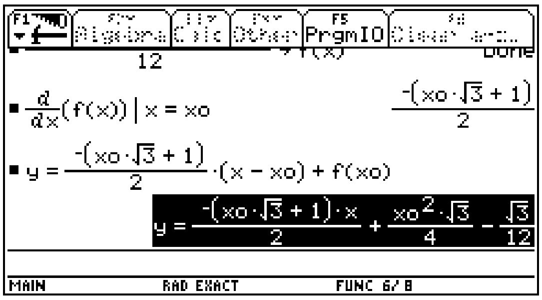
Let the parabola of the equation $y=-\frac{1}{12}.(3.\sqrt{3}x^2+6.x+\sqrt{3})$ be which is defined in the functions' editor under the name y1. This curve is then made to appear on one of the geom-etry files which I have named "courby1" thus allowing it to be worked in on a geometrical way: note that at the same time as the curve y1 appears, we get a tangent at a generic point that can be moved by using the cursor on the point. Note that we draw the following points on our figure : $O_1(0,0)$, $O_2(-1,1)$, $A_1(1,0)$, $A_2(0,1+\sqrt{3})$ and that the four lines (O_1O_2), (A_1A_2), (O_1A_1) and (O_2A_2) are tangents to this parabola.



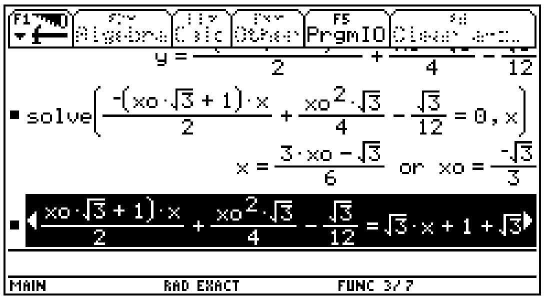
The line drawn in bold is any tangent to the parabola cutting (O_1A_1) at M_1 and (O_2A_2) at M_2. Measurements are made to the machine at the distances O_1M_1, O_1A_1, O_2M_2 and O_2A_2.

The ratios $\frac{O_1M_1}{O_1A_1}$ and $\frac{O_2M_2}{O_2A_2}$ are then displayed. It can be seen on the display that these ratios are equal and remain equal when one modifies the tangent used. The demonstration of this result, in this particular case, is relatively simple.

Proof:

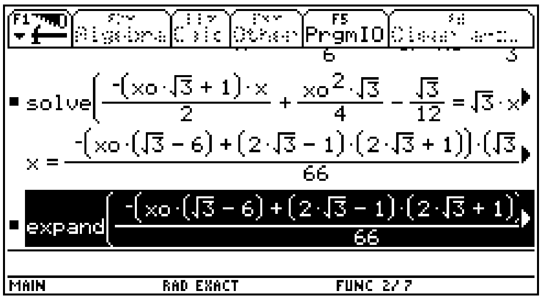


Equation of the tangent to our curve at the generic point $M_0(x_0,y_0)$

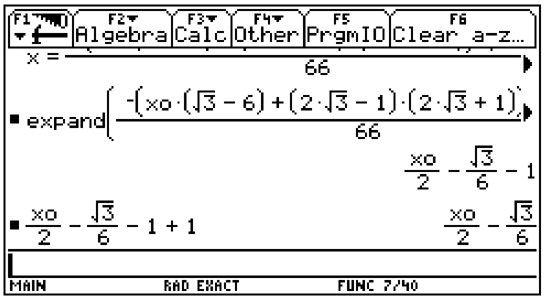


Abscissa of the point M_1:

So that $\overrightarrow{O_1M_1} = \frac{3x_0-\sqrt{3}}{6}.\overrightarrow{O_1A_1}$



Abscissa of the point M_1, which is considered as the intersection of the tangent at the point M_0 and the line O_2A_2 whose equation is $y=\sqrt{3}x+1+\sqrt{3}$



Simple expression of this abscissa,and expression of the abscissa of the vector $\overrightarrow{O_2M_2}$ which also proves that: $\overrightarrow{O_2M_2} = \frac{3x_0-\sqrt{3}}{6}.\overrightarrow{O_2A_2}$

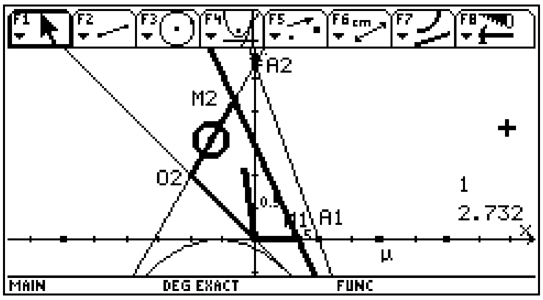
The data of the four chosen tangents initially allows one to deduce, at least in this particular case that these are all lines (M_1M_2) such that $M_1 \in (O_1A_1)$, $M_2 \in (O_2A_2)$ and $\frac{O_2M_2}{O_2A_2} = \frac{O_1M_1}{O_1A_1}$ yet respecting the relationship or non simultaneous relationship of M_1 and M_2 to the half-lines $[O_1A_1)$ and $[O_2A_2)$.

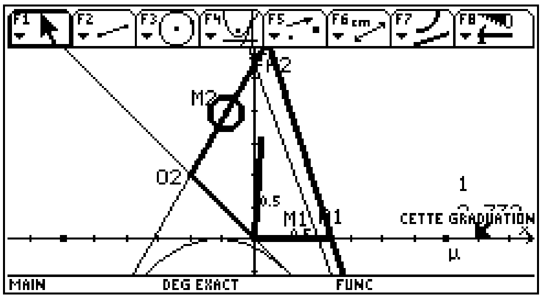
One could re-construct the same experiment with four other tangents and end up with the same results. This results could also be obtained with the CAS module on the calculator.

End of the second episode

FILOU: Jean-Jacques, everything you've showed us is all very well, but I still don't see the relationship between my head and the parabola whose tangents you calculated with such skill.

JJD: My dear Filou, If I may say so, you are suffering from a severe case of myopia. Look carefully at the figure again and tell me what you think of it.





FILOU: Goodness gracious!

JJD........And that's not all!

FILOU: With all due respect, may I point out that although this looks similar to my head, it is not my correct head.

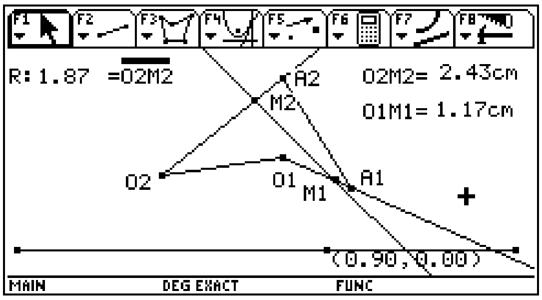
JJD.......Don't worry, be patient and wait until last episode.

THIRD EPISODE:
FILOU ! YOUR HEAD FOR A PARABOLA !

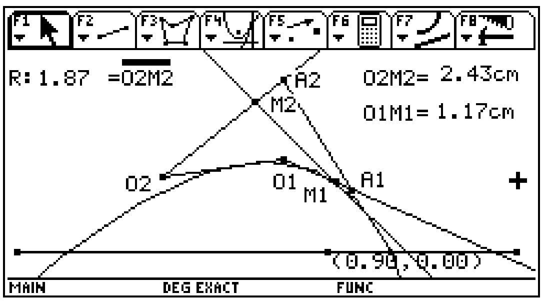
JJD...If youdon't mind, I am going to use only the contours of your head ; your mouth and eye won't be useful.

Constructions, manipulations and conjecture

The point M_1 can move on the line (O_1A_1) when one pulls on the point positioned on the segment situated in the lower part of the screen (note that this segment is included in the axis of the abscissas of the hidden coordinate system). On the other hand M_2 is located on (O_2A_2) so that: $\overline{O_2M_2} = \overline{O_1M_1}.\dfrac{O_2A_2}{O_1A_1}$.

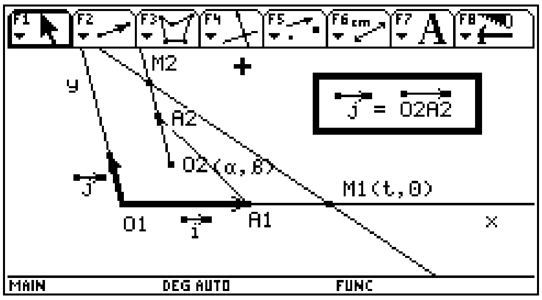


The question that I put forward is the following: does there exist a curve which has the tangents M_1M_2 when M_1 moves along the line (O_1A_1). The calculator is going to answer in an experimental way using the option "Locus" obtained with in menu F4 (one looks for the locus of M_1 when the point situated on the lower segment covers this lower segment). We will therefore only obtain a small portion of the locus sought but this will suffice.



It seems to be that one gets something that looks like a parabola. We must now prove it.

Proof:



Choosing the cartesian coordinate system $(O,\vec{i},\vec{j})$ as the one drawn on the figure on the left. The hypothesis of proportionality is translated into :

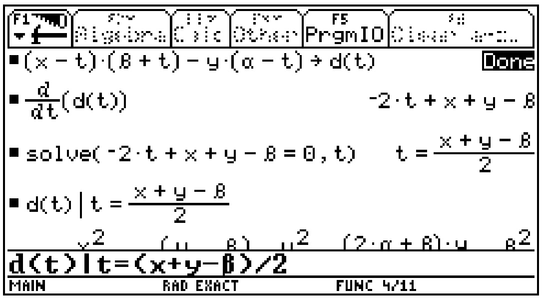
$\overrightarrow{O_1M_1} = t \cdot \overrightarrow{O_1A_1}$ and $\overrightarrow{O_2M_2} = t \cdot \overrightarrow{O_2A_2}$

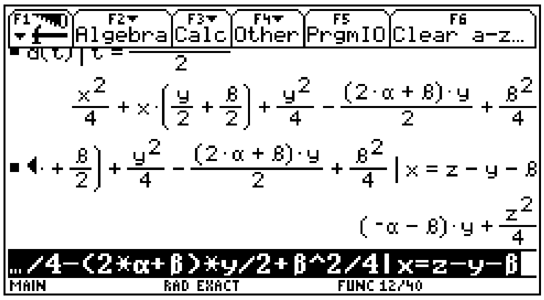
As $\overrightarrow{M_1M_2} \cdot (\alpha\text{-}t,\beta\text{+}t)$, an equation of the line (M_1M_2) can be written as:

d(t) = 0 with d(t) =(x-t). (β+t)-y.(α-t).

One will obtain an equation of the envelope sought by eliminating t in the system:

d(t) = 0 and d'(t) = 0, which we will do with the TI-92 :





These calculations show, after the change of variable x=z-y-β, which corresponds to a change of coordinate system, that our envelope has for its equation in a suitable coordinate system:

(-α-β).y+z²/4=0. It is therefore, in the general case, a parabola according to the first episode; the new coordinate system has for its origin $O_3(-\beta,0)$ and as its basis vector $\vec{i}$ and $-\vec{i}+\vec{j}$; its symmetry axis is directed by $-\vec{i}+\vec{j}$.

The particular case (-α-β)=0 corresponds to the case where $(O_1O_2) \parallel (A_1A_2)$.

Finally, the equation which has allowed us to conclude, can be written: $y = \frac{1}{4.(\alpha+\beta)}.z^2$

Theorem 4: given the head of Filou... no...on a serious note, given a quadrilateral whose sides are not two by two parallel, the envelope of lines (M_1M_2) such as defined above, is a parabola tangent to the four sides of this quadrilateral.

NB: if you want to know how I got this result using TI-92, you can read the brochure of the IREM of Toulouse that deals with this subject (among others things) entitled « De la proportionnalité aux coniques en passant par les lignes de niveaux avec la TI-92 » (IREM 1998).

End of the third episode.

JJD: So, are you satisfied?

FILOU: I must confess that I am surprised to discover that my head like any other head could conceal such complicated calculaions, considering the fact that I'd alwayshad the impression that when looked at, it was after all my head.

JJD.......We must now say to our readers that this article gives them the mathematical explanation of the pretty drawing which is on the cover of the book that I have published "INTRODUCTION A LA GEOMETRIE AVEC LA TI-92" (ELLIPSE Publishings) and which can be seen onthe front page of this article.

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Many thanks to Henri LEMBERG who translated this article into English and above all to Caroline RENNIE who made the French to English translation comprehensible for English speakers.

ON THE SOLUTION OF A LINEAR DIFFERENTIAL EQUATION OF THE ORDER *n* WITH CONSTANT COEFFICIENTS (2)

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Example 3.3 *Let us solve the following differential equation with DERIVE :*

$$y'''(x) - 5\,y''(x) + 17y'(x) - 13y(x) = 0 \quad . \tag{14}$$

In this case vector $v = [1, -5, 17, -13]$, *and we have:*

```
DIFERENCIAL_HOMOGENEA_COEFICIENTES_CONSTANTES([1, -5, 17, -13], x)

 2·x                  2·x                  x
e   ·COS(3·x)·d     + e   ·SIN(3·x)·r    + e ·c
               2,1                  2,1        1,1

          2·x                  2·x                  x
Y(x) := e   ·COS(3·x)·d     + e   ·SIN(3·x)·r    + e ·c
                       2,1                  2,1        1,1

Y'''(x) - 5·Y''(x) + 17·Y'(x) - 13·Y(x) = 0
```

We can immediately check that the solution obtained is a solution of (14).

Example 3.4 *Let us solve the following differential equation with DERIVE – and check the solution:*

$$y^{(4)}(x)-6y'''(x)+12y''(x)-8y'(x)=0 \qquad (15)$$

In this case vector $v=[1,-6,12,-8,0]$, *and we have:*

```
DIFERENCIAL_HOMOGENEA_COEFICIENTES_CONSTANTES([1, -6, 12, -8, 0], x)
```

$$x^2\cdot e^{2\cdot x}\cdot c_{2,3}+x\cdot e^{2\cdot x}\cdot c_{2,2}+e^{2\cdot x}\cdot c_{2,1}+c_{1,1}$$

$$Y(x):=x^2\cdot e^{2\cdot x}\cdot c_{2,3}+x\cdot e^{2\cdot x}\cdot c_{2,2}+e^{2\cdot x}\cdot c_{2,1}+c_{1,1}$$

$$Y''''(x)-6\cdot Y'''(x)+12\cdot Y''(x)-8\cdot Y'(x)=0$$

Example 3.5 *Let us solve the following differential equation with DERIVE:*

$$y^{(4)}(x)+4y'''(x)+8y''(x)+8y'(x)+4y(x)=0 \qquad (16)$$

In this case vector $v=[1,4,8,8,4]$, *and therefore we have:*

```
DIFERENCIAL_HOMOGENEA_COEFICIENTES_CONSTANTES([1, 4, 8, 8, 4], x)
```

$$x\cdot e^{-x}\cdot \mathrm{COS}(x)\cdot d_{1,2}+e^{-x}\cdot \mathrm{COS}(x)\cdot d_{1,1}+x\cdot e^{-x}\cdot \mathrm{SIN}(x)\cdot r_{1,2}+e^{-x}\cdot \mathrm{SIN}(x)\cdot r_{1,1}$$

$$Y(x):=x\cdot e^{-x}\cdot \mathrm{COS}(x)\cdot d_{1,2}+e^{-x}\cdot \mathrm{COS}(x)\cdot d_{1,1}+x\cdot e^{-x}\cdot \mathrm{SIN}(x)\cdot r_{1,2}+e^{-x}\cdot \mathrm{SIN}(x)\cdot r_{1,1}$$

$$Y''''(x)+4\cdot Y'''(x)+8\cdot Y''(x)+8\cdot Y'(x)+4\cdot Y(x)=0$$

Example 3.6 *Let us solve the following differential equation with DERIVE:*

$$y^{(4)}(x)+2y'''(x)+3y''(x)+2y'(x)+y(x)=0 \qquad (17)$$

In this case vector $v=[1,2,3,2,1]$, *and therefore we have:*

```
DIFERENCIAL_HOMOGENEA_COEFICIENTES_CONSTANTES([1, 2, 3, 2, 1], x)
```

$$x\cdot e^{-x/2}\cdot d_{1,2}\cdot \mathrm{COS}\left(\frac{\sqrt{3}\cdot x}{2}\right)+e^{-x/2}\cdot d_{1,1}\cdot \mathrm{COS}\left(\frac{\sqrt{3}\cdot x}{2}\right)+x\cdot e^{-x/2}\cdot r_{1,2}\cdot \mathrm{SIN}\left(\frac{\sqrt{3}\cdot x}{2}\right)+e^{-x/2}\cdot r_{1,1}\cdot \mathrm{SIN}\left(\frac{\sqrt{3}\cdot x}{2}\right)$$

Factor the above expression!

$$e^{-x/2}\cdot\left(\mathrm{COS}\left(\frac{\sqrt{3}\cdot x}{2}\right)\cdot(x\cdot d_{1,2}+d_{1,1})+\mathrm{SIN}\left(\frac{\sqrt{3}\cdot x}{2}\right)\cdot(x\cdot r_{1,2}+r_{1,1})\right)$$

$$Y(x):=e^{-x/2}\cdot\left(\mathrm{COS}\left(\frac{\sqrt{3}\cdot x}{2}\right)\cdot(x\cdot d_{1,2}+d_{1,1})+\mathrm{SIN}\left(\frac{\sqrt{3}\cdot x}{2}\right)\cdot(x\cdot r_{1,2}+r_{1,1})\right)$$

$$Y''''(x)+2\cdot Y'''(x)+3\cdot Y''(x)+2\cdot Y'(x)+Y(x)=0$$

3.1 Particular solutions.

We can use the following function

DIFERENCIAL_HOMOGENEA_COEFICIENTES_CONSTANTES (v,x),

in order to calculate particular solutions of differential equation (6) that verify

$$\left[y(x_0)=e_1,\ y'(x_0)=e_2, \ldots\ldots\ldots, y^{n-1)}(x_0)=e_{n-1}\right]$$

For that purpose we will proceed like in the following example.

Example 3.7 *Let us consider the differential equation already solved in example 3.6*

$$y^{(4)}(x)+2y'''(x)+3y''(x)+2y'(x)+y(x)=0 \ . \qquad (18)$$

In this case vector $v=[1,2,3,2,1]$ *and the solution written in a compact way is given above.*

Let us suppose that we are looking for the particular solution of (18) which verifies

$$[y(0)=1, y'(0)=2, y''(0)=3, y'''(0)=1]. \qquad (19)$$

In this case, we calculate the successive derivatives of the solution function in the point considered, thus obtaining:

```
[[Y(0), Y'(0), Y''(0), Y'''(0)]]'
```

$$\begin{bmatrix} d_{1,1} \\ d_{1,2} - \frac{d_{1,1}}{2} + \frac{\sqrt{3}\cdot r_{1,1}}{2} \\ -\frac{2\cdot d_{1,2} + d_{1,1} - 2\cdot\sqrt{3}\cdot r_{1,2} + \sqrt{3}\cdot r_{1,1}}{2} \\ -\frac{3\cdot d_{1,2} - 2\cdot d_{1,1} + 3\cdot\sqrt{3}\cdot r_{1,2}}{2} \end{bmatrix} \quad \begin{bmatrix} d_{1,1} = 1 \\ d_{1,2} - \frac{d_{1,1}}{2} + \frac{\sqrt{3}\cdot r_{1,1}}{2} = 2 \\ -\frac{2\cdot d_{1,2} + d_{1,1} - 2\cdot\sqrt{3}\cdot r_{1,2} + \sqrt{3}\cdot r_{1,1}}{2} = 3 \\ -\frac{3\cdot d_{1,2} - 2\cdot d_{1,1} + 3\cdot\sqrt{3}\cdot r_{1,2}}{2} = 1 \end{bmatrix}$$

We now proceed to equal each one of the results obtained to the conditions given in (19) in order to obtain an equation system (right matrix) which can be solved (using a substitution!):

```
"d11 = d, d12 = e, r11 = r, r12 = s"
```

$$\text{SOLVE}\left(\left[d = 1,\ e - \frac{d}{2} + \frac{\sqrt{3}\cdot r}{2} = 2,\ -\frac{2\cdot e + d - 2\cdot\sqrt{3}\cdot s + \sqrt{3}\cdot r}{2} = 3,\ -\frac{3\cdot e - 2\cdot d + 3\cdot\sqrt{3}\cdot s}{2} = 1\right], [d, e, r, s]\right)$$

$$\left[d = 1 \wedge e = -6 \wedge r = \frac{17\cdot\sqrt{3}}{3} \wedge s = 2\cdot\sqrt{3}\right]$$

The particular solution of (18) verifying the condition (19) will therefore be:

$$Y\_P(x) := \left(\left(2\cdot\sqrt{3}\cdot x + \frac{17\cdot\sqrt{3}}{3}\right)\cdot SIN\left(\frac{x\cdot\sqrt{3}}{2}\right) + (1 - 6\cdot x)\cdot COS\left(\frac{x\cdot\sqrt{3}}{2}\right)\right)\cdot ê^{-x/2}$$

which actually verifies (19), since as we check:

$$\lim_{x\to 0}\left[\; Y\_P(x) \quad \frac{d}{dx} Y\_P(x) \quad \left(\frac{d}{dx}\right)^2 Y\_P(x) \quad \left(\frac{d}{dx}\right)^3 Y\_P(x)\;\right] = [\;1 \quad 2 \quad 3 \quad 1\;]$$

$$\left(\frac{d}{dx}\right)^4 Y\_P(x) + 2\cdot\left(\frac{d}{dx}\right)^3 Y\_P(x) + 3\cdot\left(\frac{d}{dx}\right)^2 Y\_P(x) + 2\cdot\frac{d}{dx} Y\_P(x) + Y\_P(x) = 0$$

General Physics Problems with DERIVE

Leon Magiera, Wroclaw, Poland

Examples:

II.8 A point is moving according to:

$$r = r_0\,(1 - c\,t),\ \ \Phi = \frac{c\,t}{1 - c\,t}\ \ (c, r_0 - \text{constants})$$

Evaluate radial and transversal components of velocity and acceleration and magnitudes of the vectors.

Solution: In the above problem the trajectory is given in polar coordiantes. From the introduction in the chapter KINEMATICS we know that radial and transversal components of velocity and acceleration can be evaluated from the following relations:

$$v_r = \dot{r},\ v_\Phi = r\,\dot{\Phi},\ a_r = \ddot{r} - r\,\Phi^2,\ a_\Phi = r\,\ddot{\Phi} + 2\,v_r\,\dot{\Phi},$$

We enter the trajectory equations:

```
[r:=r0*(1-c*t), φ:=c*t/(1-c*t)]
```

and the relations for velocity and acceleration components:

```
v_:=[vr:=DIF(r,t),vφ:=r*DIF(φ,t)]

a_:=[ar:=DIF(r,t,2)-r*DIF(φ,t)^2,aφ:=r*DIF(φ,t,2)+2*vr*DIF(φ,t)]
```

#1: InputMode := Word

#2: $\left[r := r0\cdot(1 - c\cdot t),\ \phi := \frac{c\cdot t}{1 - c\cdot t}\right]$

#3: $v\_ := \left[vr := \frac{d}{dt}\ r,\ v\phi := r\cdot\frac{d}{dt}\ \phi\right]$

#4: $v\_ := \left[vr := -\ c\cdot r0,\ v\phi := \frac{c\cdot r0}{1 - c\cdot t}\right]$

#5: $a\_ := \left[ar := \left(\frac{d}{dt}\right)^2\ r - r\cdot\left(\frac{d}{dt}\ \phi\right)^2,\ a\phi := r\cdot\left(\frac{d}{dt}\right)^2\ \phi + 2\cdot vr\cdot\frac{d}{dt}\ \phi\right]$

#6: $a\_ := \left[ar := \frac{c^2\cdot r0}{(c\cdot t - 1)^3},\ a\phi := 0\right]$

(Comment: For further work it is necessary to simplify #3 and #5. This returns the velocity components in #4 and the acceleration components in #6.)

To evaluate then the magnitudes of velocity and acceleration by entering and simplifying:

#7: $[|[vr,\ v\phi]|,\ |[ar,\ a\phi]|]$

#8: $\left[\sqrt{(c^2\cdot t^2 - 2\cdot c\cdot t + 2)}\cdot\left|\frac{c\cdot r0}{c\cdot t - 1}\right|,\ \frac{c^2\cdot|r0|}{|c\cdot t - 1|^3}\right]$

Solution #8 will have simpler form if we declare the domains for parameters ***c*** and ***ro***. We consider two cases: $ro \geq 0$ and $c > 0$ and $ro \geq 0$ and $c < 0$.

For the first case:

#9: $[r0 :\in \text{Real}\ (0,\ \infty),\ c :\in \text{Real}\ (0,\ \infty)]$

#10: $\left[\frac{c\cdot r0\cdot\sqrt{(c^2\cdot t^2 - 2\cdot c\cdot t + 2)}}{|c\cdot t - 1|},\ \frac{c^2\cdot r0}{|c\cdot t - 1|^3}\right]$

Simplification of #8 returns #10.

Notes:

We know that r component of the polar coordinates cannot be negative, hence from the content of the problem we have $(1 - c\,t) \geq 0$, so $|c\,t - 1| = -(c\,t - 1)$. Using this result we can further simplify result #9 to the form:

(Highlight `|c·t - 1|` and substitute / simplify in one step – pressing CTRL+ENTER in DERIVE for DOS)

(Comment: In DERIVE 6 you have to work via Simplify > Subexpression Substitution ....)

$$\#11: \quad \left[\frac{c \cdot r0 \cdot \sqrt{c^2 \cdot t^2 - 2 \cdot c \cdot t + 2}}{1 - c \cdot t}, \frac{c^2 \cdot r0}{(1 - c \cdot t)^3}\right]$$

The same result is valid for $c < 0$ (do check it), but in this case the motion is unlimited in time.

II.9 Trajectory of a moving point is given by the equations (in polar coordinates):

$$r = A\,e^{k\Phi}, \quad \Phi = B\,t, \qquad (A, B, k - \text{constants})$$

Show that the angle between velocity and acceleration does not depend on time.

Solution: The time independence of the angle implies the time independence of the trigonometric function of this angle (for example cos-function). So we will investigate the derivative with respect to time of the cos-function of the angle between velocity and acceleration:

$$\frac{d\left(\frac{\vec{v}\,\vec{a}}{|\vec{v}|\,|\vec{a}|}\right)}{d\,t}$$

First we enter formulae describing trajectory of movement, velocity and acceleration (some of them have already been entered in the previous problem - #2 and #3):

```
[r := A·e^(k·t), φ := B·t]
```

Now entering and simplifying the derivative (#14 and #15 are results of simplifying #2 and #3):

$$\#12: \quad \text{CaseMode} := \text{Sensitive}$$

$$\#13: \quad \left[r := A \cdot e^{k \cdot t}, \; \phi := B \cdot t\right]$$

$$\#14: \quad v\_ := \left[vr := A \cdot k \cdot e^{k \cdot t}, \; v\phi := A \cdot B \cdot e^{k \cdot t}\right]$$

$$\#15: \quad a\_ := \left[ar := A \cdot e^{k \cdot t} \cdot (k^2 - B^2), \; a\phi := 2 \cdot A \cdot B \cdot k \cdot e^{k \cdot t}\right]$$

$$\#16: \quad \frac{d}{dt} \frac{[vr, v\phi] \cdot [ar, a\phi]}{|[vr, v\phi]| \cdot |[ar, a\phi]|} = 0$$

The result #16 proves the time independence of the investigated angle.

Exercise: Try to solve the above problem using the *cross product* of two vectors.

II.13 Given are position vectors of three moving points:

$$\vec{r_1} = ct\vec{i} + b\vec{j},\ \vec{r_2} = ct\vec{i} + b\vec{j} + e\vec{k},\ \vec{r_3} = (ct + R\cos(\omega t))\vec{i} + (b + R\sin(\omega t))\vec{j}$$

where *c, b, e, R* – constants, *t* – time. Prove that these points belong to the rigid body.

Solution: The distance between any two points of the rigid body should not depend on time. Hence the equations:

$$\frac{d}{dt}\left|\vec{r_1} - \vec{r_2}\right| = 0,\ \frac{d}{dt}\left|\vec{r_1} - \vec{r_3}\right| = 0,\ \frac{d}{dt}\left|\vec{r_2} - \vec{r_3}\right| = 0$$

Let us enter the above vectors and simplify the appropriate derivatives. Result #18 proves the statement of the problem.

```
#20:  [r1 := [c·t, b, 0], r2 := [c·t, b, e]]

#21:  r3 := [c·t + R·COS(w·t), b + R·SIN(w·t), 0]

       ⎡d               d               d            ⎤
#22:   ⎢── |r1 - r2|,   ── |r1 - r3|,   ── |r2 - r3| ⎥ = [0, 0, 0]
       ⎣dt              dt              dt           ⎦
```

III.3 A body of mass m is moving in the horizontal plane. The initial velocity is v_0. Find the time dependency of velocity and position if the force acting on the body is proportional to:

a) its velocity,
b) squared velocity.

Solution: Let us write down the equations of motion:

$$\text{a)}\quad m\frac{dv}{dt} = -k_1 v, \qquad b)\quad m\frac{dv}{dt} = -k_2 v^2, \qquad k_1, k_2 \text{ – positive constants}$$

The separation of variables leads to the integral form:

$$a)\ -\frac{m}{k_1}\int_{v_0}^{v}\frac{dv}{v} = \int_0^t dt, \qquad b)\ -\frac{m}{k_2}\int_{v_0}^{v}\frac{dv}{v^2} = \int_0^t dt,$$

Solution of the above equations with respect to v gives the velocities.
Let us start the Computer Algebra.

a) We enter the first of the above equations and solve for v. To get position of the body we assign solution #20 to the variable $\boldsymbol{v}$. Then we enter and simplify the equation to obtain $\boldsymbol{x}$.

$$\#24: \quad \text{SOLVE}\left(-\frac{m}{k1}\cdot\int_{v0}^{v}\frac{1}{u}\,du = \int_0^t 1\,du,\ v\right)$$

$$\#25: \quad v = v0\cdot e^{-k1\cdot t/m}$$

$$\#26: \quad v := v0\cdot e^{-k1\cdot t/m}$$

$$\#27: \quad x = \int_0^t v\,dt = \left(x = \frac{m\cdot v0}{k1} - \frac{m\cdot v0\cdot e^{-k1\cdot t/m}}{k1}\right)$$

b) The evaluation steps for the second force are similar. The results are displayed below:

$$\#28: \quad \text{SOLVE}\left(-\frac{m}{k2}\cdot\int_{v0}^{v}\frac{1}{u^2}\,du = \int_0^t 1\,du,\ v\right) = \left(v = \frac{m\cdot v0}{k2\cdot t\cdot v0 + m}\right)$$

$$\#29: \quad v := \frac{m\cdot v0}{k2\cdot t\cdot v0 + m}$$

$$\#30: \quad x = \int_0^t v\,dt = \left(x = \frac{m\cdot \text{LN}(k2\cdot t\cdot v0 + m)}{k2} - \frac{m\cdot \text{LN}(m)}{k2}\right)$$

$$\#31: \quad v :=$$

$$\#32: \quad \text{SOLVE}\left(m\cdot\int_{v0}^{v}\frac{1}{-k1\cdot u}\,du = \int_0^t 1\,du,\ v\right) = \left(v = v0\cdot e^{-k1\cdot t/m}\right)$$

$$\#33: \quad x = \int_0^t v0\cdot e^{-k1\cdot t/m}\,dt = \left(x = \frac{m\cdot v0}{k1} - \frac{m\cdot v0\cdot e^{-k1\cdot t/m}}{k1}\right)$$

Comparing the results for the considered forces we can see that in both cases the velocity tends to zero with time ($\lim_{t\to 0} = 0$), however the distance passed by the body when time tends to infinity only in the case of the force being proportional to velocity, for the second force (proportional to velocity squared) the distance tends to infinity (when evaluating the limit it is necessary to declare domains for the parameters ***m***, ***k1*** and ***k2***).

Exercise: Solve the problem for the force being the sum of both forces and discuss the asymptotics of solutions as dependent on values of the coefficients k_1 and k_2.

Titbits from Algebra and Number Theory(15)

by Johann Wiesenbauer, Vienna

Programming in DERIVE has always been sort of a challenge. Due to the very limited amount of programming constructs, it's more like riding a wild horse rather than driving a car. At least, that's what I used to say. The advent of version 5 of DfW might bring about a dramatic change on that score though and programming in DERIVE is sure to become a lot easier in the very near future. Hence, with one auspicious and one dropping eye, as it were, let's mount that wild horse for what could be the very last ride. (As always the following programs refer to the latest version of DfW which is 4.11 at present.)

As you may remember I wrote in my last column that STIRLING1(n,k) is also the number of permutations on n letters having k cycles, whereas STIRLING2(n,k) is the number of ways to partition a set of n elements into k disjoint subsets. Let's call those numbers S1(n,k) and S2(n,k), respectively. Then e.g. the implementation of S1(n,k) in order to check the equality STIRLING1(n,k) = S1(n,k) for some values of n and k, could be just the challenge we're looking for. We need quite a few routines to this end, but as you will see, they are all interesting on their own.

In the first place we need a routine to generate all elements of a symmetric group S_n. Assuming that its elements are represented as the permutations of the set $\{1,2,\dots,n\}$, a recursive solution of this problem could look like this.

```
SYMGROUP(n) := IF(n = 1, [[1]], SORT(APPEND(VECTOR(VECTOR(
INSERT_ELEMENT(n, v_, n - k_ + 1), v_, SYMGROUP(n - 1)), k_, 1, n))))

SYMGROUP(3) = [[1, 2, 3], [1, 3, 2], [2, 1, 3], [2, 3, 1], [3, 1, 2], [3, 2, 1]]
```

Next we need a routine that determines the number of cycles of a given permutation in S_n. I'll do a little more by providing a routine that converts a given permutation into its representation as a product of disjoint cycles. Actually I have already written such a program (cf. DNL #28) but the following version is much more straightforward and considerably shorter.

```
TOCYCLES(p) := (ITERATE(IF(m_ = s_, [APPEND(a_, [c_]), [r_SUB1], pSUB(r_SUB1),
DELETE_ELEMENT(r_), r_SUB1], [a_, APPEND(c_, [m_]), pSUBm_, SELECT(t_ /= m_, t_, r_),
s_], [a_, c_, m_, r_, s_]), [a_, c_, m_, r_, s_], [[], [1], pSUB1, [2, ..., DIMENSION(p)], 1]))SUB1

TOCYCLES([2, 3, 1, 4, 6, 5]) = [[1, 2, 3], [4], [5, 6]]
```

In order to understand better how it works we replace ITERATE by ITERATES in TOCYCLES() and delete the SUB1 at its end thereby getting

```
TOCYCLES0(p) := ITERATES(IF(m_ = s_, [APPEND(a_, [c_]), [r_SUB1],
pSUB(r_SUB1), DELETE_ELEMENT(r_), r_SUB1], [a_, APPEND(c_, [m_]), pSUBm_,
SELECT(t_ /= m_, t_, r_), s_], [a_, c_, m_, r_, s_]), [a_, c_, m_, r_, s_], [[], [1], pSUB1, [2,
..., DIMENSION(p)], 1])

TOCYCLES0([2, 3, 1, 4, 6, 5])
```

[]	[1]	2	[2,3,4,5,6]	1
[]	[1,2]	3	[3,4,5,6]	1
[]	[1,2,3]	1	[4,5,6]	1
[1 2 3]	[4]	4	[5,6]	4
[[1,2,3],[4]]	[5]	6	[6]	5
[[1,2,3],[4]]	[5,6]	5	[]	5
[[1,2,3],[4],[5,6]]	[?]	$[2,3,1,4,6,5]_?$	[]	?
[[1,2,3],[4],[5,6]]	[?]	$[2,3,1,4,6,5]_?$	[]	?

Now the meaning of the 5 variables a_,c_,m_,r_ ,s_ should be obvious: a_ is the current list of disjoint cycles. It is empty at the beginning and will contain the result we are looking for at the end of the run. c_ is the cycle that is currently generated. As soon as the numbers m_ and s_ coincide, where m_ is the image of the current end of the cycle under the permutation and s_ is its first element, this cycle is added to a_ and initilized with the first element of r_ which contains all elements that haven't occurred so far. Eventually this r_ becomes empty thereby producing an error which is the signal for us to exit. (Even errors can be useful at times!)

We have all the ingredients now to achieve our goal.

S1(n, k) := DIMENSION(SELECT(DIMENSION(TOCYCLES(p_)) = k, p_, SYMGROUP(n)))

STIRLING1(n, k) := IF(k >= n, MAX(1 - k + n, 0), IF(k, 0, IF(k = 1, (n -1)!, IF(k = n - 1, COMB(n, 2), IF(k > n/2, (ITERATE([n_ DELETE_ELEMENT(v_)+ DELETE_ELEMENT(v_, 1 - n_ + n), n_ + 1], [v_, n_], ITERATE([n_APPEND(DELETE_ELEMENT(v_), [0]) + v_, n_ + 1], [v_, n_], ITERATE([n_APPEND(v_,[0]) + INSERT_ELEMENT(0, v_), n_ + 1], [v_, n_], [[1], 0], n -k), 2k - n), n - k))SUB1SUB1, (ITERATE([n_ DELETE_ELEMENT(v_, 1 - n_ + n) + DELETE_ELEMENT(v_), n_ + 1], [v_, n_], ITERATE([n_v_ +APPEND(DELETE_ELEMENT(v_), [0]), n_ + 1], [v_, n_], ITERATE([n_INSERT_ELEMENT(0, v_) + APPEND(v_, [0]), n_ + 1], [v_, n_], [[1], 0], k), n- 2·k), k))SUB1SUB1)))))

VECTOR(S1(6,k_),k_,0,6)=[0,120,274,225,85,15,1] (72.2s)

VECTOR(STIRLING1(6,k_),k_,0,6)=[0,120,274,225,85,15,1] (0.0s)

Although the execution times on my Pentium 166 PC speak volumes - well, it was clear from the start that S1(n,k) wouldn't be a serious competitor for STIRLING1(n,k), wasn't it - these results show that our routines work. In particular, it took less than 0.015s on average to find the representation as a product of disjoint cycles for each permutation of S_6, which is not so bad.

Stirling numbers of the first and second kind appear in many areas of mathematics, in particular they play an important role in a number of interpolation formulae and in the calculus of finite differences. As I already mentioned there are also a number of applications in algebra and number theory. For example, you can use them to answer the question whether two given polynomial functions g(x) and h(x) are equal over some residue class ring mod m. After substituting f(x):=g(x)-h(x) this leads to the simpler question whether f(x) ≡ 0 mod m holds for all x or not. (Note that for large n checking of all possible residue classes mod n might not be feasible!) Using the formulae

$$x^n := \sum_{k=0}^{n} S2(n,k)\, x^{\underline{k}} \quad (n \ge 0) \qquad (*)$$

where $x^{\underline{k}} := x(x-1)(x-2)\ldots(x-k+1)$ $(k \geq 0)$ are the factorial powers and S2(n,k) are the Stirling numbers of the second kind, we can represent f(x) in the form

$$f(x) = a_n x^{\underline{n}} + \ldots a_1 x^{\underline{1}} + a_0$$

Now it can be proved that f(x) is identically zero mod m if and only if

$$k!a_k \equiv 0 \bmod m, \text{ k=0,1,…,n}$$

which is obviously much easier to check.

Although the following example is a very small one (in order to save valuable space!) you will clearly see how this works in general. We are assuming here that the DERIVE-implementation of Stirling numbers of the second kind, namely

```
STIRLING2(n, k) := if(n<k,0,ITERATE(APPEND(v_, [0]) + INSERT_ELEMENT(0, v_),
  v_, [1], k) VECTOR((-1)^(k - j_) j_^n, j_, 0, k)/k!)
```

has already been loaded.

```
m := 2040

               5         4           3           2
F(x) := 17·x  - 170·x  + 1275·x  - 2890·x  + 1768·x + 22440

(a :== lim TERMS(F(x))) = [17, -170, 1275, -2890, 1768, 22440]
       x→1

t :== VECTOR(VECTOR(STIRLING2(n, k), k, 5, 0, -1), n, 5, 0, -1)
```

$$\begin{bmatrix} 1 & 10 & 25 & 15 & 1 & 0 \\ 0 & 1 & 6 & 7 & 1 & 0 \\ 0 & 0 & 1 & 3 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

```
a·t = [17, 0, 680, 0, 0, 22440]

MOD([17·5!, 0·4!, 680·3!, 0·2!, 0·1!, 22440·0!], m) = [0, 0, 0, 0, 0, 0]
```

Note that it may be a bit more difficult to get hold of the vector a of coefficients of f(x) in case there are vanishing coefficients of f(x) after the leading one. Since our construct using TERMS( ) leaves them out you will have to add them manually or use a different idea to compute a.

And here another nice application of Stirling numbers before we leave this topic for good. It deals with computations of powers

$$a^{n^k} \bmod m, \text{ n=0,1,2,…} \qquad (**)$$

where a, k and m are fixed positive integers. Of course, we can't beat the time DERIVE needs for the computation of a single power of this type, but as I will show this changes dramatically if it comes to the computation of a lot of such powers, say for all $n \leq s$, where s is not too small. To this end consider the set S of all unary vectors of a fixed length r and with nonnegative integer components < m, on which a unary operation f: $S \rightarrow S$ is defined in the following way:

$$g((x_1, x_2, \ldots, x_r)) := (x_1 x_2 \bmod m, x_2 x_3 \bmod m, \ldots, x_{r-1} x_r \bmod m, x_r)$$

i.e. $x_i x_{i+1}$ mod m becomes the new x_i for i=1,2,,…,r-1, whereas x_r doesn't change at all. Now let's assume that we start with the vector x whose components are

$$x_i := a^{i!S2(k,i)} \bmod m,\ i=0,1,2,\ldots,k \text{ and } x_{k+1} := 1.$$

Then the first components of the sequence x,g(x), g(g(x)), g(g(g(x))), … are exactly the numbers of the sequence (**) in the same order!

Again let's illustrate this with a small example, where a=2, k=3 and m= 1000000. The obvious way to compute the elements of (**) for n=0,1,2,…,10 would be

$$\left[a := 2,\ k := 3,\ m := 10^6\right]$$

$$\text{VECTOR}\left(\text{MOD}\left(2^{n^k}, m\right), n, 0, 10\right)$$

[1, 2, 256, 217728, 551616, 26432, 977536, 926208, 84096, 734912, 69376]

Now let's check our assertion by means of the following routines:

```
G(x) := APPEND(VECTOR(MOD(xSUBk_·xSUB(k_ + 1), m), k_, 1, k + 1), [1])
ITERATES(G(x), x, APPEND(VECTOR(MOD(a^(i!·STIRLING2(k, i)), m), i, 0, k), [1]), 9)
```

1	2	64	64	1
2	128	4096	64	1
256	524288	262144	64	1
217728	953472	777216	64	1
551616	693952	741824	64	1
26432	248448	476736	64	1
977536	105728	511104	64	1
926208	3712	710656	64	1
84096	955072	481984	64	1
734912	422848	846976	64	1
69376	107648	206464	64	1

In fact, the numbers of the first column coincide with the numbers calculated above! The question remains whether this method is faster than the "brute force" method which computes those powers one by one. Well, if we compute all powers for $n \le s$ and neglect the time for the computation of the start vector x involving Stirling numbers of the second kind (we can certainly do this if s is large) then we get the matrix above by performing roughly ks multiplications. DERIVE uses internally the "square and multiply"-method which takes on average 1.5 k ln n multiplications for one term in (**) and hence 1.5 k ln s! multiplications altogether. Using the approximation

$$\ln s! \approx \int_1^s \ln t\,dt = s(\ln s - 1)$$

we finally get about 1.5 ks (ln s -1) for the average number of multiplications, which is certainly a difference, if s is large!

The fast computation of the sequence (**) by means of Stirling numbers can be used to factor large integers very efficiently if certain preconditions are fulfilled and maybe I will deal with this interesting application another time. For now I would like to keep a promise and turn to a completely different topic which refers to the interesting article by Richard Schorn in DNL #33, p22-23. It goes without saying that I was very delighted to see a nice application of DERIVE to a problem of abstract algebra, which is certainly not very often the case. On the other hand, it soon became clear when skimming over that article that a lot of things had been done manually which should have been handed over to DERIVE. What follows are some ideas on that score.

First of all, we assume that all elements of the group in question are stored in a vector l along with a symbolic name. The vector l typically looks like this

$$l=\begin{bmatrix} e & x \\ a & \dfrac{x}{x-1} \\ b & 3-x \end{bmatrix}$$

which means that we have chosen the names e, a,b for the functions x, x/(x-1) and 3-x, respectively. We now define two operations OP0(u,v) and OP(u,v) which accept only symbolic names as input, but the output will always be a function of x for OP0(u,v), whereas OP(u,v) additionally tries to find the symbolic name for the output, if one is available.

```
OP0(u,v):=ITERATE((SELECT(l_SUB1=u,l_,l))SUB1SUB2,x,(SELECT(l_SUB1=v,l_,l))SUB1
          SUB2,1)
OP(u,v):=ITERATE((APPEND(SELECT(l_SUB2=o_,l_,l),[[o_]]))SUB1SUB1,o_,OP0(u,v),1)
```

Note that only OP0(u,v) refers to the specific group which consists of linear rational functions in x in the case at issue with the composition as binary operation. At any time we can evoke the routine TABLE which yields the current operation table using OP(u,v).

```
table:=VECTOR(VECTOR(OP(u,v),v,l`SUB1),u,l`SUB1)
```

A new variable is defined by means of DEF(newvar,f1,f2) where either newvar is the composition of already defined variables f1 and f2 (f1 and f2 are supposed to be their symbolic names!) or f1 is an expression in x defining a new linear rational function and f2 is left out. Programming buffs might profit a lot when looking into all the tricks I used to achieve this goal! (By the way, for the very last time I used here an auxiliary function like NEWL(u) to make self-assignments for l possible. In DfW 5 they no longer cause problems. Good riddance!)

```
NEWL(u):=(l:=u)
DEF(newvar,f1,f2):=IF(VARIABLES(f1)=[x],l SUB DIMENSION(NEWL(APPEND(l,
        [[newvar,f1]]))),0,l SUB DIMENSION(NEWL(APPEND(l,[[newvar,OP(f1,f2)]]))))
```

Finally I defined a routine ORD(u) that computes the order of u.

```
ORD0(u):=DIMENSION(ITERATES(u,x,x))-1
ORD(u):=ORD0((SELECT(l_SUB1=u,l_,l))SUB1SUB2)
```

To see these routines at work, let’s set up the operation table for the same group that was considered by Richard Schorn in his paper. We start with the following definitions:

```
l:=[]
DEF(e,x)=[e,x]
```

$$\mathrm{DEF}\left(a,\frac{x}{x-1}\right)=\left[a,\frac{x}{x-1}\right]$$

```
DEF(b,3-x)=[b,3-x]
```

Now the vector l consisting of all currently defined variables has the form given above as an example. By evoking TABLE we get

$$\mathrm{table}=\begin{bmatrix} e & a & b \\ a & e & \frac{x-3}{x-2} \\ b & 2-\frac{1}{x-1} & e \end{bmatrix}$$

which clearly shows how we should continue, e.g. by entering

$$\mathrm{DEF}(ab,a,b)=\left[ab,\frac{x-3}{x-2}\right]$$

After entering DEF(ba,b,a) and DEF(aba,a,ba) in the same way (note that all these inputs must be simplified immediately otherwise they won't take effect!) we finally arrive at

table=

e	a	b	ab	ba	aba
a	e	ab	b	aba	ba
b	ba	e	aba	a	ab
ab	aba	a	ba	e	b
ba	b	aba	e	ab	a
aba	ab	ba	a	b	e

$\mathrm{VECTOR}(\mathrm{ORD}(l\_),l\_,l`_1)=[1,2,2,3,3,2]$

Try it out yourself! On this occasion I want to thank Richard Schorn for many suggestions. Among other things he gave me the "homework" to look into the group generated by the two functions 1/x, and (x*(sqrt(3)+2)-1)/(x+sqrt(3)+1) which I would like to pass on to you. Have a lot of fun!

J.Wiesenbauer@tuwien.ac.at

About Difference 1st order Difference Equations:

Solve the difference equations: y(x+1) = 1.5 y(x) with y(0) = 5 and u(n+1) = 1.5 u(n) + 2 with u(0) = 5:

$$\mathrm{LIN1\_DIFFERENCE}\left(\frac{3}{2},\ 0,\ x,\ 0,\ 5\right)=5\cdot\left(\frac{3}{2}\right)^x$$

$$\mathrm{LIN1\_DIFFERENCE}\left(\frac{3}{2},\ 2,\ t,\ 0,\ 5\right)=2^{-t}\cdot 3^{t+2}-4$$

$$lin1\_diff\left(\frac{3}{2},0,n,0,5\right) \quad 5\cdot\left(\frac{3}{2}\right)^n$$

$$lin1\_diff\left(\frac{3}{2},2,n,0,5\right) \quad 2^{-n}\cdot\left(9\cdot 3^n-4\cdot 2^n\right)$$

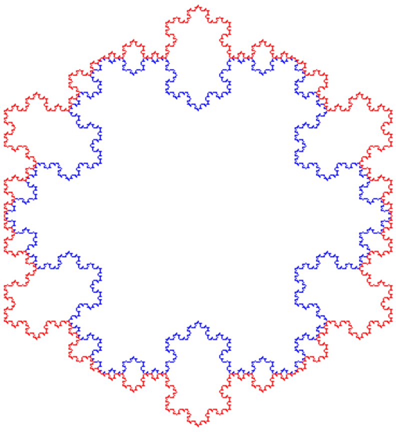
$$\mathrm{expand}\left(2^{-n}\cdot\left(9\cdot 3^n-4\cdot 2^n\right)\right) \quad \frac{9\cdot 3^n}{2^n}-4$$

Computergrafik mit *DERIVE* - Computer Graphics with *DERIVE*

Maria Koth, University of Vienna, Austria

3.4 Snowflake Curve and Koch-polygons

If you compose three Koch Curves to an equilateral triangle then you will get a starlike curve which is called Snowflake Curve.



Substituting v0 from the last example (Koch curves in a Square) by the vector `ECK(n)`, which contains the vertices of a regular polygon, the function `KOCHPOLYGON(k,n)` - similar to `QUADRAT` - produces a figure consisting of n Koch curves of order k over the sides of the given polygon.

(You need `KOCH(1)` and the respective `GENERATOR(x,y)` from DNL#33.)

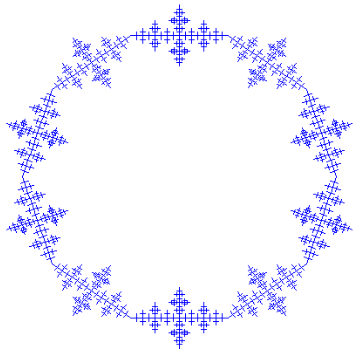
```
ECK(n):=VECTOR([COS(2*pi*i/n),SIN(2*pi*i/n)],i,0,n)

KOCHPOLYGON(k,n):=VECTOR(KOCH1(k,[(ECK(n))↓j,
(ECK(n))↓(j+1)]),j, DIMENSION(ECK(n))-1)
```

You can change `ECK(n)` to run clockwise through the polygon´s vertices. This results in Koch curves outside of the polygon.

```
ECK(n):=VECTOR([COS(2*pi*i/n),SIN(2*pi*i/n)],i,n,0,-1)
```

It is nice to combine both kinds of curves in one figure.



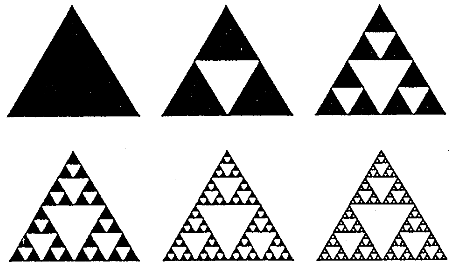
4. The Sierpinski Triangle

The Sierpinski Triangle is also a classic example of a fractal. It was defined by the Polish mathematician Waclaw Sierpinski in 1916.

4.1 The construction

You have to start with an equilateral triangle (side *a*) and remove its center triangle. Now you have the Sierpinski Triangle of order 1, which consists of three congruent equilateral triangles with sides *a*/2. Applying this principle of construction on each of these three triangles leads in the Sierpinski Triangle of order 2, etc.

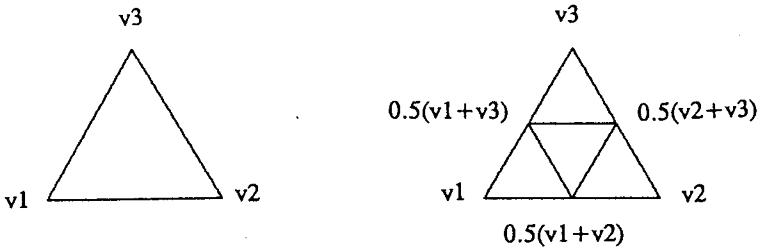
Each step of construction multiplies the number of triangles by three and bisects the length of the sides. So the Sierpinski Triangle of order k consists of 3^k equilateral triangles with sides $a \cdot (0.5)^k$. The **Sierpinski Triangle** is the set of points remaining after having applied the described function infinite times.



To obtain a *DERIVE* function `SIERP(k)` for creating the S.T of order k let us start with the triangle **d0** (side $a = 1$) given by the following vector:

```
v0:=[[0,0],[1,0],[1/2,SQRT(3)/2],[0,0]]
```

The Sierpinski Triangle(1) consists of this initial triangle **v0** together with triangle **v1**, which connects the midpoints of v0´s sides.




```
v1:=[0.5*(v0↓1+v0↓2),0.5*(v0↓2+v0↓3),0.5*(v0↓+v0↓3),0.5*(v0↓1+v0↓2)]
```

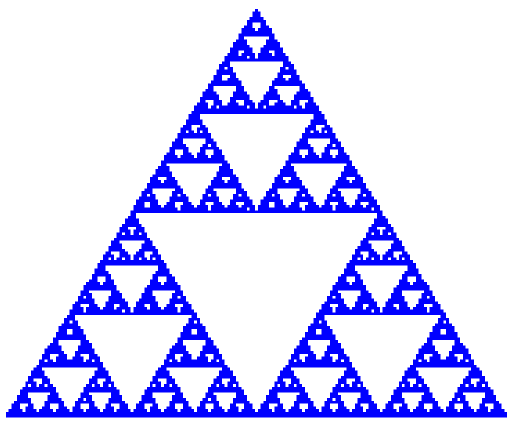
Proceeding to the Sierpinski Triangle(2) we have to add three new triangles, which are obtained applying three centric stretchings onto triangle d1 with v1, v2 and v3 as streching centers and 0.5 as stretching factor.

Applying these stretchings onto each of the newly created triangles we can find 9 triangles which form together with the 1 + 1 + 3 = 5 existing triangles the Sierpinski Triangle of order 3. In the same way we apply the stretchings onto the last created 3^{k-1} triangles when we proceed from Sierpinski Triangle(k) to Sierpinski Triangle(k+1).

```
STRECK(i,v):=VECTOR(VECTOR(0.5*(v0↓i+v↓j↓k),k,1,4),j,1,DIM(v))
```

Being `v` the vector of the triangles created in the last step, the `STRECK(i,v)`-statement causes the stretching of all these triangles by the factor 0.5 with the i-th vertex of `v0` be done. `SIERP(n)` produces a Sierpinski Triangle of order n.

```
SIERP(n):=IF(n=1,v0,APPEND([v0],APPEND(ITERATES(APPEND(STRECK(1,v),
          STRECK(2,v),STRECK(3,v)),v,[v1],n-1))))
```



SIERP(8)

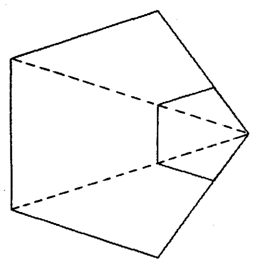
4.2 Sierpinski Polygons

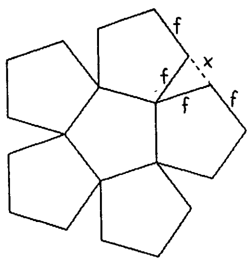
Construction of a Sierpinski Triangle can also be explained as follows:

An equilateral triangle is substituted by three congruent equilateral triangles which

- are parallel to the sides of the initial triangle,
- have one vertex with the initial triangle in common,
- have side lengths such that two neighbouring triangles have one point in common.

This principle of construction can easily be transfered on regular n-gons with $n \geq 5$.





The small n-gons of the next generation are created by an appropriate stretching with one vertex of the parent-gon as center. The stretching factor f depends on the kind of polygon. If it is a triangle, then we need three stretchings with factor f = 0.5 - as we have seen -, if it is a regular hexagon then we need six stretchings with factor f = 1/3 etc.

If we start with a regular n-gon with side length $1 = 2f + x$, then we will call the side length of one "child" *f*. The sketch above shows how to find *f* in an easy way:

$$f = \frac{1}{2\left(1 + \cos\frac{2\pi}{n}\right)}.$$

For the *DERIVE* procedure we first define *f* and the coordinates *v*0 of the initial n-gon:

```
[f:=1/(2+2*COS(2*pi/n)),v0:=VECTOR([COS(i),SIN(i)],i,0,2*pi,2*pi/n)]
```

Like working with the Sierpinski Triangle the function `TEIL(v,i)` generates the i-th partial polygon of the parent-polygon v.

```
TEIL(v,i):=VECTOR((1-f)*v SUB i+f*v SUB j,j,1,n+1)
```

TEILALL(v) creates the complete next generation of figure v

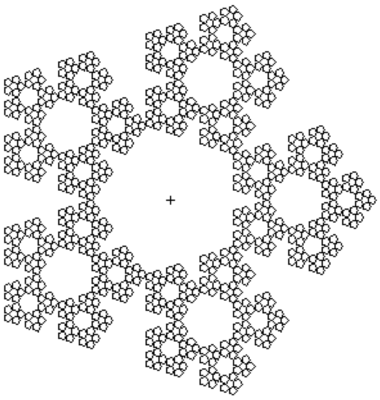
```
TEILALL(v):=VECTOR(TEIL(v,i),i,n)
```

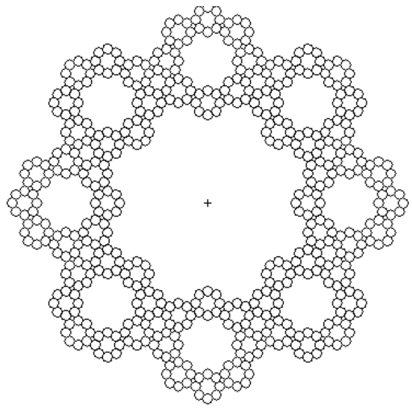
As the Sierpinski polygons of order k - v = SIERPP(k) - consists of n^k polygons v_i, we have to apply TEILALL(v_i) onto each of these polygons v_i. The recursive statement SIERPP(k) creates the polygon of order k - after having defined n!!

```
NECHST(v):=IF(DIMENSION(v)=1,TEILALL(v SUB
1),APPEND(NECHST(DELETE_ELEMENT
           (v,DIMENSION(v))),TEILALL(v SUB DIMENSION(v))))

SIERPP(k):=IF(k=0,[v0],NECHST(SIERPP(k-1)))
```

```
n:=5
SIERPP(4)
n:=8
SIERPP(3)
```

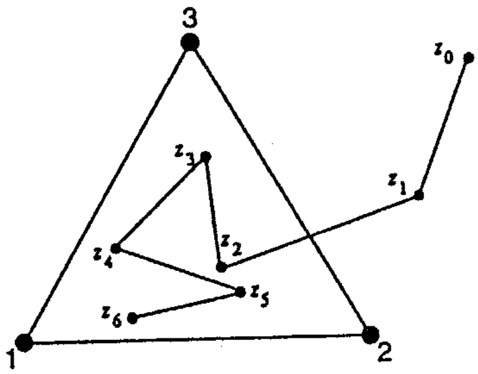




5. The Chaos Game

1985 the English mathematician Michael Barnsley presented a new interesting method to create fractal pictures. He called this method “Chaos Game”.

5.1 The Chaos Game for the Sierpinski Triangle



Start with an equilateral triangle with vertices v1, v2 and v3. Then pick any arbitrary point z0 anywhere in the plane. Now, randomly select one of the three vertices, say v2. The midpoint of the segment v2,z0 gives the next point z1. Select - again randomly - one of the vertices, say v1 → midpoint of segment (v1,z1) gives z3, etc.

In *DERIVE* we can describe this algorithm using the RANDOM(n)-function, which returns for each n ∈ N and n > 0 a randomly chosen element of {0,1,2,......,n-1}.

So 1+RANDOM(3) gives a randomly chosen element of {1,2,3}. The rule to obtain the sequence of points is given by:

$$z_{n+1} = 0.5(z_n + v_{1 + RANDOM(3)}).$$

The *DERIVE* code consists of only two lines. We define the initial triangle v together with the starting point z0, e.g. [1,1]. CHAOS(n) produces n+1 points of the Chaos Game:

```
[v:=[[0,0],[1,0],[0.5,0.5*SQRT(3)]],z0:=[1,1]]

random(0)= 471223100

CHAOS(n):=ITERATES(0.5*(z+v↓(1+RANDOM(3))),z,z0,n)
```

It is important to plot the points in the "Discrete" Mode. What might be the result for large numbers n? We would expect a random distribution of points within the initial triangle (- trivial for a starting point within the circumference, and easy to explain that if starting from outside we will come closer to the triangle and at a certain moment a point will be captured inside and all the following points will stay inside). But the distribution shows - although unpredictible for each point - a predictible pattern:

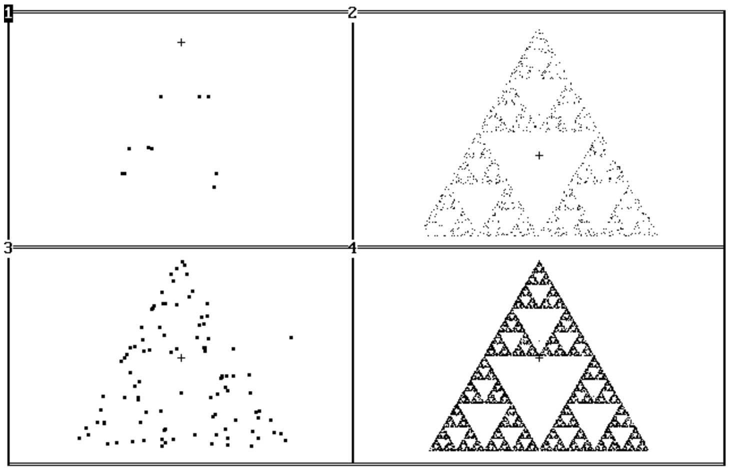
```
CHAOS(10)

CHAOS(100)

CHAOS(1000)

CHAOS(10000)
```

Very clear appears the Sierpinski Triangle out from the shadow of points!!



(You can find a TI-92 program for the Chaos Game in "Mastering the TI-92" from Rich, Rose and Gilligan, GILMAR 1996 - DNL#22,Josef.)

The principle of construction contains the same stretching as the construction of the Sierpinski Triangle in 4.1. If you start the Chaos Game with a point z0 which is an element of the Sierpinski Triangle the sequence of the following points will all be points of the Sierpinski Triangle If you choose another point then the points of the sequence approximate the Sierpinski Triangle Let x be a point of the Sierpinski Triangle and $\delta > 0$*, then there exists an element of the sequence which is lying in the* δ*-surrounding of x.* [Peitgen].

5.2 More Chaos Games

In 4.2 we transfered the Sierpinski Triangle Construction onto polygons. We can do the same with the Chaos Game.

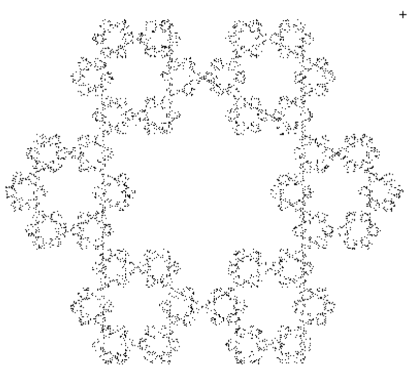
```
[v6:=VECTOR([COS(i),SIN(i)],i,0,5*pi/3+0.1,pi/3),f:=1/3,z0:=[1,1]]

CHAOS6(n):=ITERATES(f*z+(1-f)*v6↓(1+RANDOM(6)),z,z0,n)

CHAOS6(6000)
```

Needs only 3 seconds for being plotted (DERIVE 6).
In 1999 46.5 seconds were necessary!

(I added the 0.1 within the VECTOR command that "Approximate before plotting" can be used in the 2D-Plot Window, Josef)



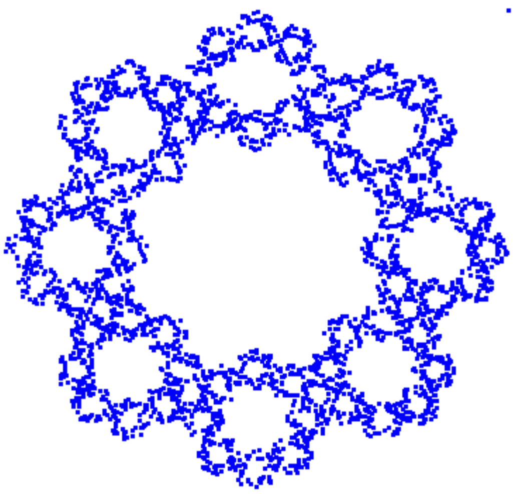
Working with Maria Koth's file from 1999 I had the idea to try a more general function including the kind of polygon and the number of iterations as function parameters: chaosp(k,n) was the result:

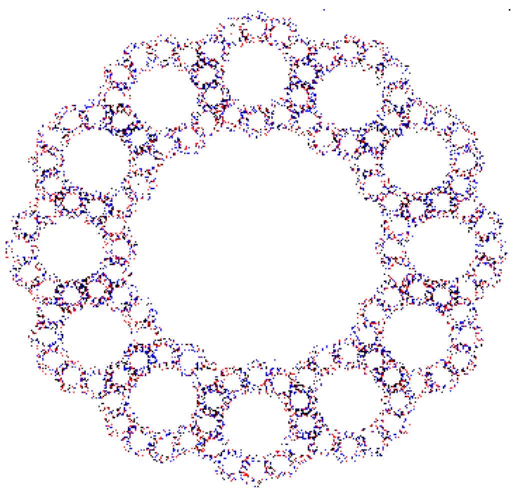
$$f\_(k) := \frac{1}{2\cdot\left(1 + \cos\left(\frac{2\cdot\pi}{k}\right)\right)}$$

$$v\_(k) := \text{VECTOR}\left([\cos(i), \sin(i)],\ i,\ 0,\ \frac{2\cdot\pi\cdot(k-1)}{k} + 0.1,\ \frac{2\cdot\pi}{k}\right)$$

$$\text{chaosp}(k, n) := \text{ITERATES}(f\_(k)\cdot z + (1 - f\_(k))\cdot(v\_(k))_{1 + \text{RANDOM}(k)},\ z,\ z0,\ n)$$

chaosp(8, 4000)





chaosp(12, 4000)

chaosp(12, 4000)

chaosp(12, 4000)

Three plots superimposed (12 000 points)

References:

Endl, K. *Kreative Computergrafik*. VDI Verlag, Düsseldorf 1986.

Hermann, D. *Algorithmen für Chaos und Fraktale*. Addison-Wesley, Bonn 1994.

Koth, M. *Einführung in das Arbeiten mit DERIVE*. PI-Wien, Wien 1994

Peitgen, H.O. u.a. *Fractals for the Classroom, Part 1*. Springer, New York 1992.

It is not difficult to present the Chaos Game with TI-NspireCAS.

I thought that I would need a LUA program in order to produce a plot displaying 2000 points or more. Then I was surprised that – at least on the PC – lists of so many elements can be used for producing a scatter plot.

The parameters for the Game are entered in a Notes page.

```
chaos
Define chaos()=
Prgm
Local v,nst
xl:={st[1,1]}
yl:={st[1,2]}
nst:=st
For i,1,n
v:=randInt(1,3)
nst:=(nst+tr[v])/2
xl:=approx(augment(xl,{nst[1,1]}))
yl:=approx(augment(yl,{nst[1,2]}))
EndFor
EndPrgm
```

CHAOS GAME

The vertices of the given equilateral triangle:

tr:= $\begin{bmatrix}\frac{-9}{\sqrt{3}},-3;\frac{9}{\sqrt{3}},-3;0,6\end{bmatrix}$ ▸ $\begin{bmatrix}-3\cdot\sqrt{3} & -3\\ 3\cdot\sqrt{3} & -3\\ 0 & 6\end{bmatrix}$

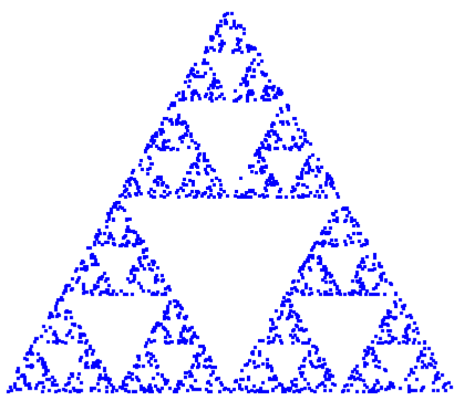
Enter a starting point st with $-10 \le x \le 10$ and $-6 \le y \le 6$: st:=[2,2] ▸ [2 2]

Enter a random seed number: RandSeed 2111 ▸ *Done*

Enter the number of iterations: n:=2000 ▸ 2000

Run chaos(): chaos() ▸ *Done*

Lists xl and yl must be displayed in a Scatter Plot.



CHAOS POLYGONS

Enter k for a k–polygon: k:=8 ▸ 8

Enter a starting point st with −10 ≤ x ≤ 10 and −6 ≤ y ≤ 6: st:=[1,1]

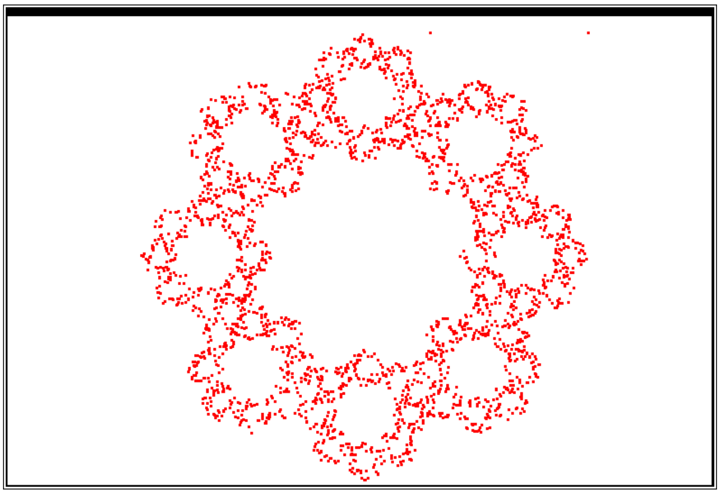
Enter a random seed number: RandSeed 1111

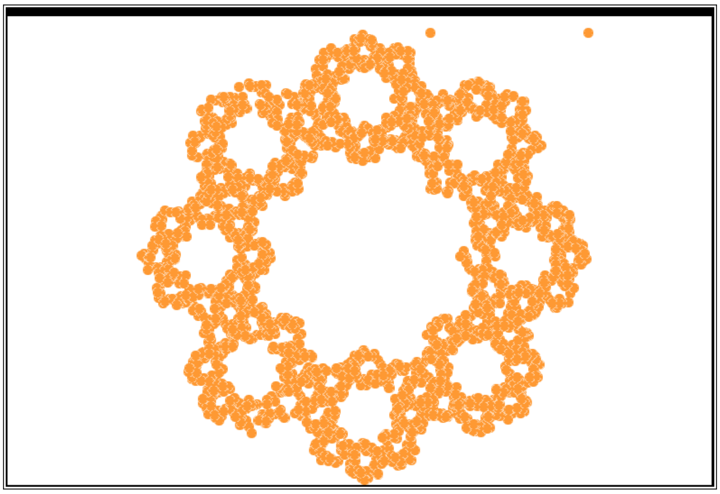
Enter the number of iterations: n:=3000

Run chaosp(): chaosp()

Lists xl and yl must be displayed in a Scatter Plot.

The Chaospolygons need one function and one program. The chaotic octagon is plotted with two different attributes.





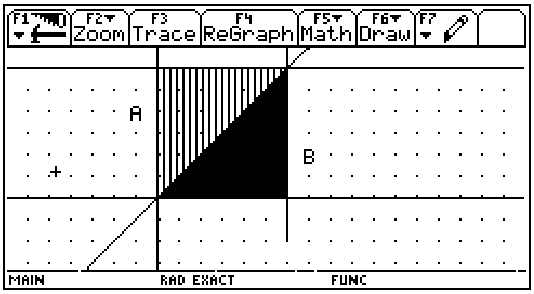
```
chaosp
Define chaosp()=
Prgm
Local v,nst
xl:={st[1,1]}:yl:={st[1,2]}
nst:=st
f:=1/(2·(1+cos(2·π/k)))
For i,1,n
v:=randInt(1,k)
nst:=approx(f·nst+(1−f)·p(k)[v])
xl:=augment(xl,{nst[1,1]})
yl:=augment(yl,{nst[1,2]})
EndFor
EndPrgm
```

```
p
Define p(k)=
Func
Local m,i
m:=newMat(k,2)
For i,1,k
m[i]:=[cos(2·π/k·(i−1))  sin(2·π/k·(i−1))]
EndFor
m
EndFunc
```

An Investigation in Calculus using the TI-92

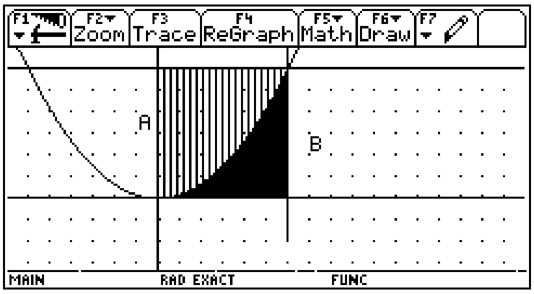
by Michael Meagher, Vienna International School

We start with the problem as follows. Given the graphs of y = x, y =1 and x=1:



We then ask the simple question: What is the ratio of the Area A to the Area B? By straight-forward calculation of the areas of the two triangles we get the answer 1 : 1.

The problem starts to become more interesting if we consider the graphs y=x^2, y=1 and x=1.



The area B is calculated, by hand, as $\int_0^1 x^2 dx$ which is 1/3 and, since the area of A is calculated by subtracting the area of B from a unit square, we get that the area of A is 2/3. The ratio of the areas is therefore 2 : 1.

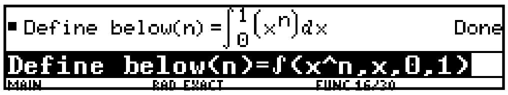
Following the same steps for y = x^3 we find that the area is 3 : 1. We conjecture at this point that there is a relationship between the index of x in the curve and the ratio of Area A : Area B. This ratio can be calculated as follows:

$$\frac{1-\int_0^1 x^n\,dx}{\int_0^1 x^n\,dx} = \frac{1-\left[\frac{x^{n+1}}{n+1}\right]_0^1}{\left[\frac{x^{n+1}}{n+1}\right]_0^1} =$$

$$= \frac{1-\frac{1}{n+1}}{\frac{1}{n+1}} = \frac{\frac{n}{n+1}}{\frac{1}{n+1}} = n$$

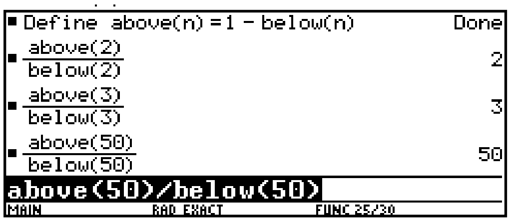
We can therefore see that in general the ratio of Area A : Area B is n : 1.

We would like to develop the problem in a number of ways, in particular by changing the limits of integration. We would like to use the TI-92 for this problem and so, as a first step, we will check our work thus far on the TI-92 by defining some functions to calculate the areas A and B.



Whenever a function is defined on the TI-92, it is always a good idea to test the function on a few known results:

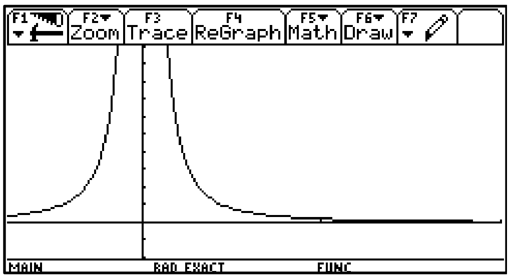
All seems well. We define above(n) and test some results:



And now for the interesting result:

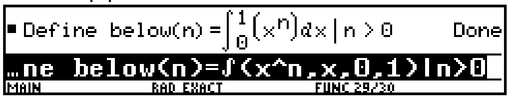
What????

Look at all the examples we have seen so far: n= 2, 3, 4, 50. All of these are positive integers. Of course the original question was posed with n implicitly meant to be a positive integer. A moment's thought about the graph of y = x^n with x a negative integer reveals that there is no area enclosed under the graph between 0 and 1 as the following graph of x^(-2) suggests:



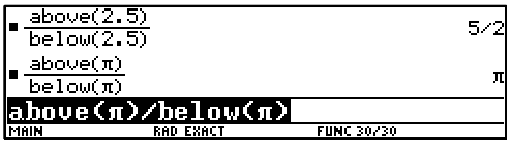
This is a good example of how the TI-92 promotes exact mathematical thinking. Our casual definition of the function below(n) was not quite what we meant but the TI-92 forces us to think more clearly.

We need to modify our definition of below(n) using the "with" (|) operator:



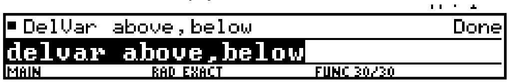
Now we go back to our question:

In fact, the problem works for any real number greater than 0 not just integers:

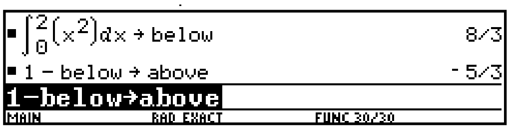


Extension 1

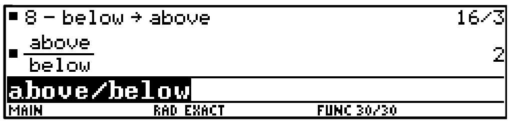
A reasonable first extension to this problem is to ask what happens if we integrate from 0 to 2. Before we start again we want to delete the functions below and above:



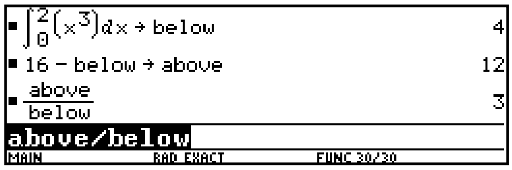
We start again from the basics and can generate the first results by hand or with the TI-92 as follows:



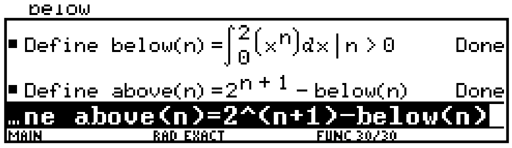
Not very promising, but of course the total of area A and area B is no longer a square. The value of the function f(x) = x^2 at x = 2 is 4 and so we are looking at a two by four rectangle with area 8.



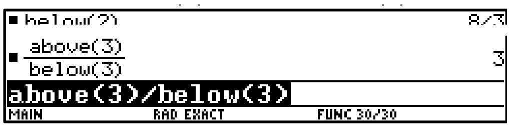
If the function is f(x) = x^3 then the shape is a two by eight rectangle with area 16:



In general the value of f(x) = x^n at x = 2 is 2^n, so the rectangle we need to consider for the total area is 2 * 2^n or 2^(n+1). So proceeding as before:



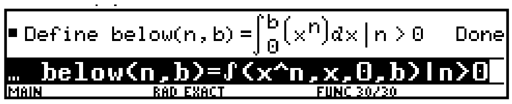
We check a few known values as before:



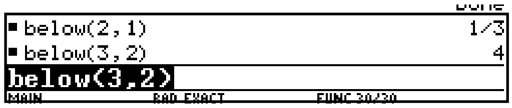
And finally:

Extension 2

So if this property holds while integrating from 0 to 1 and integrating from 0 to 2, will it work from 0 to b, an arbitrary parameter?



We check this definition for known values:



The value of the function f(x) = x^n at x = b is b^n. The rectangle under consideration is then b*b^n or b^(n+1) :

```
■ Define above(n,b)=b^(n+1) - below(n,b)          Done
…bove(n,b)=b^(n+1)-below(n,b)
MAIN        RAD EXACT        FUNC 30/30
```

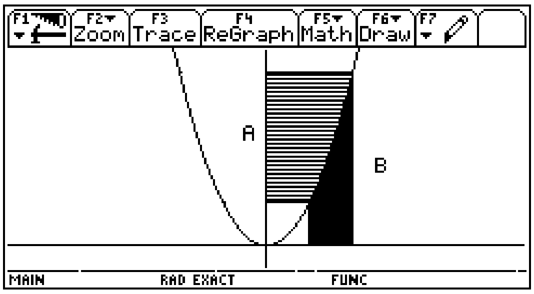
And we look for the general result:

```
■ above(n,b)/below(n,b)                           n
above(n,b)/below(n,b)
MAIN        RAD EXACT        FUNC 30/30
```

And so the property holds on the integral from 0 to an arbitrary parameter b.

Extension 3

The next question that arises is: if it works for 0 to 1 and it works for 0 to 2, does it work from 1 to 2?



As before we delete below and above:

```
■ DelVar above,below                              Done
delvar above,below
MAIN        RAD EXACT        FUNC 30/30
```

To find the new area B we take:

```
■ ∫(x^2, x, 1, 2) → below                         7/3
∫(x^2,x,1,2)→below
MAIN        RAD EXACT        FUNC 30/30
```

As before the total area is a 2 by 4 rectangle with area 8 and there is a unit square left unshaded. So the area A is 8 - 1 - 7/3

```
■ 8 - 1 - below → above                           14/3
■ above/below                                     2
above/below
MAIN        RAD EXACT        FUNC 30/30
```

And again the property seems to hold in this case.

We now turn our mind to the most general result which is for the curve x^n, considered between arbitrary parameters a and b with a < b:

```
■ Define below(n,a,b)=∫(x^n, x, a, b) | n > 0
                                                  Done
…elow(n,a,b)=∫(x^n,x,a,b)|n>0
MAIN        RAD EXACT        FUNC 30/30
```

The values of the function f(x) = x^n at a and at b are a^n and b^n respectively. The total area of the shape is b*b^n or b^(n+1) and the unshaded part is a*a^n or a^(n+1). Therefore:

```
■ Define above(n,a,b)=b^(n+1) - a^(n+1) - be▸
                                                  Done
…b^(n+1)-a^(n+1)-below(n,a,b)
MAIN        RAD EXACT        FUNC 30/30
```

Once more we check the definitions for some known values:

```
■ below(2,1,2)                                    7/3
■ above(2,1,2)                                    14/3
■ above(2,1,2)/below(2,1,2)                       2
above(2,1,2)/below(2,1,2)
MAIN        RAD EXACT        FUNC 30/30
```

Finally, the general result:

```
■ above(n,a,b)/below(n,a,b)                       n
above(n,a,b)/below(n,a,b)
MAIN        RAD EXACT        FUNC 30/30
```

Extension 4

This entire investigation can be expanded to the question of volumes of revolution. It works but doesn't have the same answer as with the area and is dependent on the axis of revolution.

Extension 5

To return finally to the question of a negative value of n. The property does, in fact hold for values of n so long as the limits of integration are either both positive or both negative. In other words the problem is if one of the endpoints of the integral is 0 or the integration is across 0.

To my surprise however, when we define the functions below(n,a,b) and above(n,a,b) with no restriction on n we get

```
F1  F2▾ Algebra  F3▾ Calc  F4▾ Other  F5 PrgmIO  F6 Clear a-z…
■ Define below(n,a,b)=∫(x^n, x, a, b)             Done
■ Define above(n,a,b)=b^(n+1) - a^(n+1) - be▸
                                                  Done
■ above(n,a,b)/below(n,a,b)                       n
■ above(⁻2,a,b)/below(⁻2,a,b)                     ⁻2
above(⁻2,a,b)/(below(⁻2,a,b))
MAIN        RAD EXACT        FUNC 4/30
```

As I mentioned on page 24 I wanted to solve linear difference equations of 1st and 2nd order with the TI-NspireCAS. For this purpose I inspected the respective DERIVE functions in the utility file provided by DERIVE.

```
#1:   LIN2_CCF_HOM_AUX(b, x) :=
        If b < 0
           - COS(π·x)·(-b)^x
           b^x

#2:   LIN2_CCF_HOM(p, q, d, x) :=
        If d = 0
           [(- p/2)^x, x·(- p/2)^x]
           If d > 0
              [LIN2_CCF_HOM_AUX((-p + √d)/2, x), LIN2_CCF_HOM_AUX((-p - √d)/2, x)]
              ((p^2 - d)/4)^(x/2)·[SIN(x·ATAN(√(-d), -p)), COS(x·ATAN(√(-d), -p))]
           ((p^2 - d)/4)^(x/2)·[SIN(x·ATAN(√(-d), -p)), COS(x·ATAN(√(-d), -p))]
```

$$\#3:\quad \text{LIN2\_CCF\_PARTIC}(h, r, x) := h\cdot\sum_x r\cdot\lim_{x\to x+1}\frac{[-\text{ELEMENT}(h, 2), \text{ELEMENT}(h, 1)]}{\text{DET}\left(\left[h, \lim_{x\to x+1} h\right]\right)}$$

```
#4:   LIN2_CCF_AUX(h, r, x, c) := c·h + LIN2_CCF_PARTIC(h, r, x)
```

$$\#5:\quad \text{LIN2\_CCF}(p, q, r, x, c1, c2) := \text{LIN2\_CCF\_AUX}(\text{LIN2\_CCF\_HOM}(p, q, p^2 - 4\cdot q, x), r, x, [c1, c2])$$

Solving one example (stepwise): y(x+2) + 6·y(x+1) + 9·y(x) = 10

$$\text{LIN2\_CCF\_HOM}(6, 9, 0, x) = \left[(-3)^x, x\cdot(-3)^x\right]$$

$$h := \left[(-3)^x, x\cdot(-3)^x\right]$$

$$\text{DET}\left(\left[h, \lim_{x\to x+1} h\right]\right) = 3^{2\cdot x+1}\cdot(1 - 2\cdot\cos(\pi\cdot x)^2) - 2\cdot 3^{2\cdot x+1}\cdot i\cdot\sin(\pi\cdot x)\cdot\cos(\pi\cdot x)$$

$$\frac{[-\text{ELEMENT}(h, 2), \text{ELEMENT}(h, 1)]}{\text{DET}\left(\left[h, \lim_{x\to x+1} h\right]\right)} = \left[3^{-x-1}\cdot x\cdot(-1)^{-x}, -3^{-x-1}\cdot(-1)^{-x}\right]$$

$$10\cdot\lim_{x\to x+1}\frac{[-\text{ELEMENT}(h, 2), \text{ELEMENT}(h, 1)]}{\text{DET}\left(\left[h, \lim_{x\to x+1} h\right]\right)} = \left[-10\cdot 3^{-x-2}\cdot(x+1)\cdot(-1)^{-x}, 10\cdot 3^{-x-2}\cdot(-1)^{-x}\right]$$

$$\sum_x 10\cdot\lim_{x\to x+1}\frac{[-\text{ELEMENT}(h, 2), \text{ELEMENT}(h, 1)]}{\text{DET}\left(\left[h, \lim_{x\to x+1} h\right]\right)} = \left[\frac{5\cdot 3^{-x-1}\cdot(4\cdot x+3)\cdot(-1)^{-x}}{8}, -\frac{5\cdot 3^{-x-1}\cdot(-1)^{-x}}{2}\right]$$

$$h\cdot\sum_x 10\cdot\lim_{x\to x+1}\frac{[-\text{ELEMENT}(h, 2), \text{ELEMENT}(h, 1)]}{\text{DET}\left(\left[h, \lim_{x\to x+1} h\right]\right)} = \frac{5}{8}$$

```
LIN2_CCF(6, 9, 10, x, c1, c2)
```

$$\frac{5}{8} + (c2\cdot x + c1)\cdot(-3)^x$$

I check the result:

$$y(x) := \frac{5}{8} + (c2\cdot x + c1)\cdot(-3)^x$$

$$y(x + 2) + 6\cdot y(x + 1) + 9\cdot y(x) = 10$$

Function l2chom(p,q,x) solves the homogeneous difference equation, which is no problem. It is the equivalent to DERIVE's LIN2_CCF_HOM. I had to define a special ATAN-function with two arguments: "*ATAN(y, x) is the angle of the point (x, y) in the x-y plane measured counterclockwise from the positive x-axis*", (DERIVE Online Help).

$l2chom(6,9,0,x)$ $\quad [(-1)^x\cdot 3^x \quad (-1)^x\cdot x\cdot 3^x]$

$h:=[(-1)^x\cdot 3^x \quad (-1)^x\cdot x\cdot 3^x]$

$[\cos(\pi\cdot x)\cdot 3^x+\sin(\pi\cdot x)\cdot 3^x\cdot i \quad x\cdot\cos(\pi\cdot x)\cdot 3^x+x\cdot\sin(\pi\cdot x)\cdot 3^x\cdot i]$

© the determinant dt:

$l2cp(h,10,x)$ $\quad (3-6\cdot(\cos(\pi\cdot x))^2)\cdot 9^x-6\cdot\sin(\pi\cdot x)\cdot\cos(\pi\cdot x)\cdot 9^x\cdot i$

© the lim dt1 multiplied by r (= 10):

$l2cp(h,10,x)$

$\left[\frac{-10\cdot(x+1)\cdot\cos(\pi\cdot x)\cdot 3^{-x}}{9}+\frac{10\cdot(x+1)\cdot\sin(\pi\cdot x)\cdot 3^{-x}}{9}\cdot i \quad \frac{10\cdot\cos(\pi\cdot x)\cdot 3^{-x}}{9}\right.$ ▸

©(the indefinite sum = the antidifference of the above expression):

$l2cp(h,10,x)$

$$\left[\frac{-10\cdot\sum_{k=1}^{x-1}\left((k+1)\cdot\cos(k\cdot\pi)\cdot 3^{-k}\right)}{9} \quad \frac{-5\cdot\cos(\pi\cdot x)\cdot 3^{-x}}{6}-\frac{5}{18}\right]$$

l2cp 1/5

```
Define l2cp(h,r,x)=
Func
Local dt,dt1
dt:=det((augment(h',(lim x→x+1 (h))')'))
dt1:= lim x→x+1 ((r·[-h[1,2]  h[1,1]])/dt)
adiff(dt1,x)
©dotP(h,adiff(dt1,x))
EndFunc
```

The hard problem was – and still is – to implement the indefinite sum (the antidifference) for not only polynomial functions.

Albert Rich admits in the DERIVE Online Help that it is not easy to find the antidifference in many cases:

"*Even when such an antidifference exists, there is no known method that is guaranteed to find it.*"

Below is a 1st order difference equation:

y(m+1) = m·y(m) + (m+1)!; y(1) = 2 solved with DERIVE and TI-NspireCAS:

$adiff(x^2+1,x)$ $\quad \frac{x^3}{3}-\frac{x^2}{2}+\frac{7\cdot x}{6}-1$

$f(x):=\frac{x^3}{3}-\frac{x^2}{2}+\frac{7\cdot x}{6}-1$ $\quad$ Fertig

$f(x+1)-f(x)$ $\quad x^2+1$

$adiff(10\cdot 3^{-x-2}\cdot(-1)^x,x)$ $\quad \frac{-5\cdot\cos(\pi\cdot x)\cdot 3^{-x}}{6}-\frac{5}{18}$

$f(x):=\frac{-5\cdot\cos(\pi\cdot x)\cdot 3^{-x}}{6}-\frac{5}{18}$ $\quad$ Fertig

$f(x+1)-f(x)$ $\quad \frac{10\cdot\cos(\pi\cdot x)\cdot 3^{-x}}{9}$

```
LIN1_DIFFERENCE(m, (m + 1)!, m, 1, 2)
```

$$\left(\frac{m^2}{2}+\frac{m}{2}+1\right)\cdot(m-1)!$$

$lin1\_diff(m,(m+1)!,m,1,2)$ $\quad \frac{(m^2+m+2)\cdot m!}{2\cdot m}$

See two more 1st order difference equations on page 45.