THE DERIVE - NEWSLETTER #44

THE BULLETIN OF THE

DERIVE®

USER GROUP

+ TI 92

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December 2001

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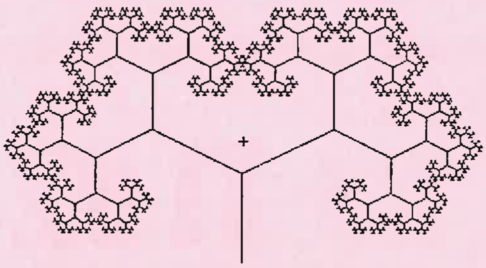
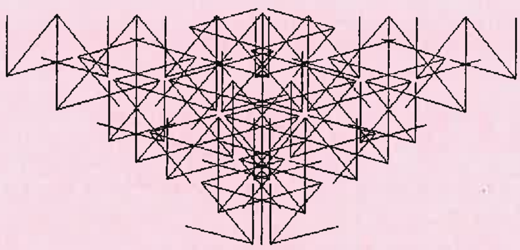

[5] **Elektronische Arbeitsblätter zur Stochastik in der Sek I mit *DERIVE* 5, H-J Kayser,**
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(+ Diskette), bk-teachware SR-27

Aktuelle Informationen auf `http://shop.bk-teachware.com`

[6] **Materialien für einen realitätsbezogenen Mathematikunterricht**
Band 6: Computeranwendungen (DERIVE, TI-92), div Verlag Franzbecker ISBN 3-88120-306-0

[7] **Mathematik, Leistungskurs – Lehrbuch, Weber - Zillmer**
Ein Lehrbuch, in dem durchgängig der TI-92 eingesetzt wird mit den Schwerpunkten auf Analysis, analytische Geometrie, lineare Algebra, Stochastik, Paetec 2000. ISBN 3-89818-100-6

[8] **Reflexionen und Visionen eines technologiegestützten Mathematikunterrichts,**
Tagungsdokumentation der 1. T^3-Winterakademie in Spital/Pyhrn 2001,
B. Barzel & U.Amelung, Hsg, ZKL Text 17, ISBN 3-934064-18-3

[9] **Materialien für den Einsatz von Grafikrechnern und Computeralgebra,**
Teil 1: Differentialrechung, H Knechtel u.a.
Inhalt: Folgenmodelle, Einführung in die Differentialrechnung, Splinefunktionen, Hüllkurven, Krümmung, Westermann 2001, ISBN 3-14-112811-1

[10] **Lineare Funktionen** mit dem TI-83/89/92 in der Sek I, Hsg Herget & Lehmann
(Beiträge von Barzel, Ebenhöh, Herget, Malitte, Richter), Schroedel 2001, ISBN 3-507-73228-9

[11] **Gleichungen** mit dem TI-83/89/92 in der Sek I, Hsg Herget & Lehmann
(Beiträge von Lehmann), Schroedel 2001, ISBN 3-507-73232-7

[12] **Stochastik** mit dem TI-83/89/92 in der Sek I, Hsg Herget & Lehmann
(Beiträge von Grabinger), Schroedel 2001, ISBN 3-507-73231-9

Informationen bei: `http://www.schroedel.de`

[13] **AP Calculus with the TI-92, George Best & Richard Lux**

[14] **Advanced Algebra with the TI-89, Brendan Kelly**

[15] **Exploring Pre-Calculus with the TI-89/92/92PLUS, M Schneider & L Gilligan**

[16] **Exploring Calculus and Differential Equations with the TI-89/92/92PLUS, M Schneider & L Gilligan**

You can get more information about the last four books at `http://www.mathware.com/Books/`

Dear DUG members,

I am sending my best regards by closing the 11th DUG year. In the last Letter of the Editor of 2001 I´d like to put your attention to some contributions of this DNL#44.

Peter Lüke-Rosendahl submitted a paper incited by a Dynamic Geometry lecture at the ICTTM5 in Klagenfurt. Many lectures and even one plenary lecture (presented by our friend and DUG member Eugenio Roanes) dealt with ideas how to connect the power of Computer Algebra Systems with the wonderful features of a Dynamic Geometry Software like Cabri, Cinderella or others. We learned that there are many efforts to proceed in this direction. It is a pity that you cannot admire Peter´s plots in colours. So I recommend to reproduce his contribution using various colours representing the circles and ellipses.

Speaking about colours, I have mentioned DPGraph several times in earlier DNLs. Rainer Wonisch – see the booklist – promised to contribute for one of the next DNLs demonstrating how to use *DERIVE* together with David Parker´s DPGraph.

I announced an article about quaternions for a long time and I always postponed it for the next issue, because I had only a printed version available and it looked to be not to easy to rewrite the paper and to produce the *DERIVE* file. But there was a mail two or three months ago asking especially for the "Quaternions" promised so long. So I had no choice but to type in this – interesting – article and to produce the file – finding a nasty bug in *DERIVE*, which was repaired by Albert Rich for the next update (*DERIVE* 5.05) which should be out in December.

ACDC (- Amazing or Amusing Corner for *DERIVER*´s Curiosity -) was founded by Alfonso Poblacion who was and for a long time the only one who contributed. I found Milton Lesmes´ problems excellent for being presented in this column. He sent some explaining comments on the situation of maths education in his country and on his examples, unfortunately two days too late to be included in this issue. So you will find Milton´s ideas in DNL#45.

I am very glad that the eDUG-discussion list established by our Australian friend David Halprin showed some signs of life recently. You are friendly invited to join eDUG by visiting

`www.egroups.com/group/edug`

and then posting *DERIVE* concerned questions and messages.

I´d like to ask our American members to send articles for the DNL. We have so many DUGers there, but the US are not very often presented by a contribution.

And finally I´d like to remind you all on the next meeting of the *DERIVE* & TI-92 family in Vienna in July 2002. On the back cover you can see two landmarks: The tower of St Stephen´s Cathedral is the landmark of Vienna (the Viennese call it "Steffel") and Albert Rich is one of DERIVE´s landmarks. We had David Stoutemyer giving talks on our Conferences, now we will hear Albert Rich.Together with Miguel de Guzman, an expert on Geometry and CAS, Bruno Buchberger, famous for discovering the Groebner Bases Algorithm and now working on an Automatic Theorem Prover and Hans-Georg Weigand, one of the leading Mathematics Didactics, we can offer a list of splendid plenary speakers. We hope that many of you will submit papers or at least come to make this conference also to a landmark in the rich history of DERIVE & TI-92 events.

At the end of my last letter in 2001 I´d like to thank you all for your cooperation, for so many signs of friendship for my wife and me. We all hope 2002 will bring more peace into our world as it was in 2001.

My wife Noor and I wish you all the best for Christmas and for a healthy and peaceful New Year 2002.

Josef

As in each last issue of a year you find enclosed the Renewal Form. Please renew your membership as soon as possible to make the administration easier. Despite the fact that the mailing costs increased we do not raise the fee, except a small rounding caused by the change of our currency to EURO.

The *DERIVE-NEWSLETTER* is the Bulletin of the *DERIVE & TI-92 User Group*. It is published at least four times a year with a contents of 44 pages minimum. The goals of the *DNL* are to enable the exchange of experiences made with *DERIVE* and the *TI-92/89* as well as to create a group to discuss the possibilities of new methodical and didactical manners in teaching mathematics.

As many of the *DERIVE* Users are also using the *TI-92/89* the *DNL* tries to combine the applications of these modern technologies.

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Contributions:

Please send all contributions to the Editor. Non-English speakers are encouraged to write their contributions in English to reinforce the international touch of the *DNL*. It must be said, though, that non-English articles will be warmly welcomed nonetheless. Your contributions will be edited but not assessed. By submitting articles the author gives his consent for reprinting it in the *DNL*. The more contributions you will send, the more lively and richer in contents the *DERIVE & TI-92 Newsletter* will be.

Next issue: March 2002
Deadline 15 February 2002

Preview: Contributions for the next issues

Inverse Functions, Simultaneous Equations, Speck, NZL
A Utility file for complex dynamic systems, Lechner, AUT
Examples for Statistics, Roeloffs, NL
Various Training Programs
Sand Dunes, Halprin, AUS
Type checking, Finite continued fractions, Welke, GER
Kaprekar´s "Self numbers", Schorn, GER
Some simulations of Random Experiments, Böhm, AUT
Comparing statistics tools: a pie chart with DERIVE,
Stem & leaf diagram with *DERIVE* and on the TI-92
Another Task for End Examination, Lechner., AUT
Cooperation DP-Graph and *DERIVE*, Wonisch, GER
Applications for Random Permutations, Schlöglhofer, AUT
Tools for 3D-Problems, Lüke-Rosendahl, GER
Penrose Inverse of a Matrix, Karsten Schmidt, GER
Differential Equations in Secondary School, Günter Schödl, AUT

and
Setif, FRA; Vermeylen, BEL; Leinbach, USA; Aue, GER; Koller, AUT, Keunecke, GER, ...

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Volker Loose, Germany **vwloose@cityweb.de**

Subject: approximating

Hello all,

using DfW5 I wrote a function

```
schrv(f, g, a, b, d) := VECTOR([i, f(i);i, g(i)], i, a, b, d)
```

Approximating schrv(f(x),g(x),-1,1,0.2) with f(x):=x^2-1 and g(x):=-x^2+1 I got

```
[[-1, f(-1); -1, g(-1)], [-0.8, f(-0.8); -0.8, g(-0.8)], [-0.6, f(-0.6);
 -0.6, g(-0.6)], [-0.4, f(-0.4); -0.4, g(-0.4)], ...   , g(1)]]
```

Approximating this I got

```
[[-1, 0; -1, 0], [-0.8, -0.36; -0.8, 0.36], [-0.6, -0.64; -0.6, 0.64], ...
```

the result I wanted. Why don't I get this result already in the first step?

Volker Loose

David Stenenga, Honolulu, Hawaii

Subject: approximating1

Hi Volker,

I entered your definitions and approximated schrv(g(x),f(x),-1,1,0.2), i.e., I switched the order of your arguments. Sure enough, I got the same result as before which means that your definition is ignoring the f,g parts on the right hand side and just literatlly substituting f(i), g(i) in the definition.

I suggest you modify your definition to:

```
schrv(u, v, x, a, b, d) := VECTOR([i, SUBST(u,x,i);i, SUBST(v,x,i)],
                           i, a, b, d)
```

and then approximate: `schrv(x^2 - 1, -x^2 + 1, x, -1, 1, .2)`

Aloha, Dave

Albert Rich, Honolulu, Hawaii

Subject: approximating2

Hello Volker Loose,

In Derive, only expressions, NOT functions, can be passed as arguments to user-defined functions. Therefore, the variable needs to be included along with the expressions in a function's formal argument list. For example, if your function is defined as

```
schrv(u, v, x, a, b, d) := VECTOR([[x,u],[x,v]], x, a, b, d)
```

then, given the assignments

`f(x) := x^2-1` and `g(x) := -x^2+1`

approximating either of the expressions

```
schrv(f(x), g(x), x, -1, 1, 0.2)
```

or

```
schrv(f(y), g(y), y, -1, 1, 0.2)
```

gives the approximated matrix that you desire. Hope this helps.
Aloha,
Albert D. Rich, Co-author of Derive

Good News for *DERIVE* and TI-Users as well:

Christian Richner **chris123@gmx.ch**

Subject: [eDUG] Derive vs. Ti92

We would like to use Derive at the same time as the pocket calculator TI-92. During evaluation we have discovered that a certain functionality of the calculator seems not to be present in Derive.

This is eg:

Functions for graph discussion like finding zero crossings or extrems.
Is it true that these functions were not implemented in Derive or did we simply not detect them yet?

Who can help?

Christian Richner

Theresa Shelby, Hawaii

Subject: Re Derive vs. Ti92

Hello Christian Richner,

Thank you for your English translation!

The 2D-plot window in Derive 5 does not offer commands for automatically finding zero crossings and extremas like the TI-92. In Derive 5, you may use a combination of plot tracing, zooming and equation solving as described in Chapter 2 Polynomial Zero Finding of the "Introduction to Derive 5" manual.

> We plan to add this capability to a future version of Derive as well as increase the inter-connectivity and symmetry between Derive and the TI-89/92.

Sincerely,Theresa Shelby
Texas Instruments

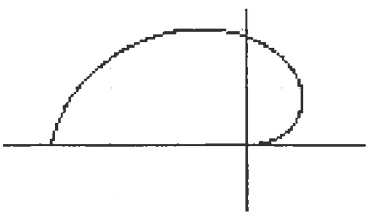
Problems with Polar Plottings

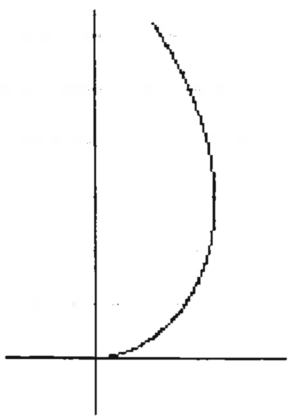
Hugh Williams, Nottingham, UK **hugh.williams@NTU.AC.UK]**

Subject: polar plotting - a bug?

If we plot ln(1 + *theta*) in polar co-ordinates from 0 to π we get the curve we expect but if we plot ln(1 + *x*) also in polar co-ordinates from 0 to π we get a different and incorrect curve. What is going on? Changing the naming of variables does not usually effect the graph that *DERIVE* produces.
We are using *DERIVE* 5.04.

Any ideas.
Hugh Williams





Joe Frisbee, United Space Alliance, USH-483L

I offer the following explanation:

It seems that the difference in the plot is also affected by the form of the plotted function. If "[theta, LN(1+theta)]" is plotted it agrees with the plot of "LN(1+x)". The correct plot of "LN(1+theta)" disagrees with both.

Referring to the Derive 5 Polar Plot help section (excerpts below) I suggest that when plotting "LN(1+*theta*)" theta is correctly interpreted as the angle component and the function of *theta* as the radius of the polar coordinate pair. When *x* is used in place of *theta* the *x* is being interpreted as the radius and the function is interpreted as the angle. This is confirmed by the fact that if we plot "[LN(1+*x*),*x*]" we will be in agreement with the expected form of the plot. This form of the plotted pair forces *x* to be the angle.

In rectangular coordinates, [*x*, *y*] represents the point *x* units from the *y*-axis, y units from the *x*-axis.

In polar coordinates, [*r*, *theta*] represents the point *r* units from the origin, *theta* radians from the positive *x*-axis.

Albert D. Rich, Honolulu, Hawaii

When plotting expressions in polar coordinates, there are two ways to ensure that Derive interprets the expression's variables the way that you intend:

1 The radius (i.e. the distance from the origin) and the angle from the positive x-axis are explicitly determined when making parametric plots. Thus, if u(t) is a univariate expression involving the variable t, plotting the vector [u(t), t] generates a parametric plot where t is interpreted as an angle and u(t) is the corresponding radius. Whereas, plotting the vector [t, u(t)] generates a parametric plot where t is interpreted as a radius and u(t) is the corresponding angle.

2 In polar coordinates, Derive always interprets r as the radius variable and theta as the angle variable. Thus, if u(t) is a univariate expression involving some variable t other than r or theta, plotting the equation r = u(t) generates a parametric plot where t is interpreted as an angle and u(t) is the corresponding radius. Whereas, plotting the equation theta = u(t) generates a parametric plot where t is interpreted as a radius and u(t) is the corresponding angle.

Note that similar comments about the interpretation of variables apply to 2D rectangular plots; and to 3D rectangular, spherical, and cylindrical plots. Also, if the Degree angular mode is selected using the Algebra window's Declare > Simplification Settings command, angle variables are interpreted as specifying degrees rather than radians.

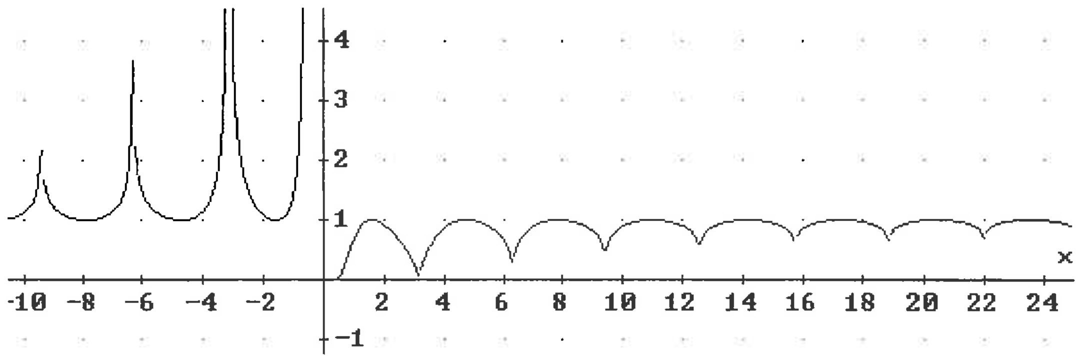
Hope this helps.
Aloha,
Albert D. Rich
Co-author of Derive

Valeriu Anisiu, Cluij, Romania **anisiu@math.ubbcluij.ro**

Subject: Some bugs in *DERIVE*

DERIVE is a wonderful program, but of course it cannot be "bug free". The following ones were recently discovered by me (perhaps some are already known). I hope that the list will be obsolete soon.

$$\lim_{x\to\infty} (\mathrm{SIN}(x)^2)^{1/x} = 1$$



This limit does not exist!

Has been improved in version 5.05!

Incorrect treatment of a limit in connection with an IF-construction:

$$\lim_{x\to\infty} \mathrm{EXP}(-x) = 0$$

$$\mathrm{IF}(\lim_{x\to\infty} \mathrm{EXP}(-x) = 0,\ 1,\ 2,\ 3) = 2$$

$$\mathrm{IF}(\lim_{x\to\infty} \mathrm{EXP}(-x) \le 0,\ 1,\ 2,\ 3) = 1$$

$$\mathrm{IF}(\lim_{x\to\infty} \mathrm{EXP}(-x) \ge 0,\ 1,\ 2,\ 3) = 1$$

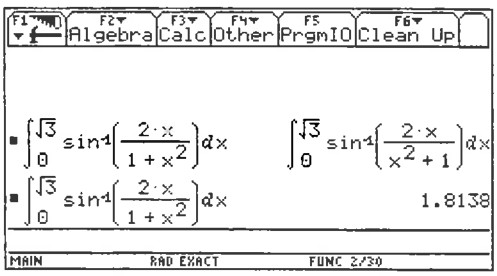
Has been improved in version 5.05!

DERIVE refuses to expand polinomials of form x!/(x-n)! and similar ones for n > 10:

$$\left[\frac{x!}{(x-5)!}, \frac{x!}{(x-11)!}\right] = \left[x\cdot(x-1)\cdot(x-2)\cdot(x-3)\cdot(x-4), \frac{x!}{(x-11)!}\right]$$

The following integral should result in $\frac{\pi}{3}\sqrt{3}$, but:

$$\int_0^{\sqrt{3}} \mathrm{ASIN}\left(\frac{2\cdot x}{1+x^2}\right) dx = 2\cdot \mathrm{LN}(2) + \frac{\sqrt{3}\cdot\pi}{3}$$



The reason is, that *DERIVE* does not find a continuous antiderivative for ASIN(2x/(1+x^2) - as it usually does!

$$\int_0^{\sqrt{3}} \text{ASIN}\left(\frac{2\cdot x}{1+x^2}\right) dx$$

1.813799364

Approximating the integral leads to the correct result!

$$\frac{\sqrt{3}\cdot\pi}{3}$$

1.813799364

POLY_GCD(0, 5·x) = 5   Should return 5 x

Has been improved in version 5.05!

Multiple substitutions for subexpressions are not correctly handled:

$$\text{SUBST}(x^3\cdot y^2 + x\cdot y,\ y,\ x^2,\ x^3,\ y) = x^4\cdot y + x^3$$

instead of x^7 + y.

Has been improved in version 5.05!

In several cases the SORT-function doesn´t work properly if containing a function. See the following example:

Suppose that we want to sort a list of integers such that the odd numbers come first.

```
v := [2, 8, 9, 44, 43, 155, 41, 13, 10, 91, 100]
odd(u) := ODD?(u)
f(x, y) := (odd(x) ∧ ¬ odd(y)) ∨ (odd(x) ∧ odd(y) ∧ x < y) ∨ (¬ odd(x) ∧ ¬ odd(y) ∧
  x < y)

        SORT(v, f(x, y), x, y) = [2, 8, 9, 10, 13, 41, 43, 44, 91, 100, 155]
```

But if one changes the definition of odd(u) then it works:

```
odd(u) := MOD(u, 2) > 0
SORT(v, f(x, y), x, y) = [9, 13, 41, 43, 91, 155, 2, 8, 10, 44, 100]
```

Is improved in version 5.05!

Pedro Tytgat, Belgium

I just asked *DERIVE* to factorize the polynomial $a^6 - a^3 + 1$. *DERIVE* gave me a surprising result: Among others I received the following factors:

$$a - (-1)^{1/9} \qquad 1$$

$$a - (-1)^{-1/9} \qquad 1$$

and reasoning with real numbers I get – even without needing support of *DERIVE*:

Branch := Real

$$a - (-1)^{1/9} = a + 1$$

(cont page 22)

Solution of Linear Equations in the Quaternion Algebra

E.R.Sirota, Tobolsk, Russia

Let **H** be the Quaternion Algebra. This is an associative but non-commutative ring with identity and division (non-commutative field of quaternions).

We will use the following model of **H**:

$\mathbf{H} = \mathbf{R} + \mathbf{i} \cdot \mathbf{R} + \mathbf{j} \cdot \mathbf{R} + \mathbf{k} \cdot \mathbf{R}$ with **i**, **j** and **k** having the following multiplication table:

	1	**i**	**j**	**k**
1	1	i	j	k
i	i	-1	k	-j
j	j	-k	-1	i
k	k	j	-i	-1

If $\mathbf{x} = a_0 + a_1\,\mathbf{i} + a_2\,\mathbf{j} + a_3\,\mathbf{k}$ is a quaternion, then $\mathbf{x} = a_0 + \mathbf{x}'$ where a_0 is the *real part* of **x** and $\mathbf{x}' = a_1\,\mathbf{i} + a_2\,\mathbf{j} + a_3\,\mathbf{k}$ its *imaginary part*. The quaternion $\mathbf{con(x)} = a_0 - a_1\,\mathbf{i} - a_2\,\mathbf{j} - a_3\,\mathbf{k}$ is called the *conjugate quaternion* to **x**.

$\mathbf{N(x)} = a_0^2 + a_1^2 + a_2^2 + a_3^2$ is the *Norm* of **x**
and the *Inverse Quaternion* to **x**, $\mathbf{x}^{-1} = \mathbf{con(x)/N(x)}$.

If $\mathbf{y} = b_0 + b_1\,\mathbf{i} + b_2\,\mathbf{j} + b_3\,\mathbf{k}$, then it is easy to check, that $\mathbf{x}' * \mathbf{y}' = -(\mathbf{x}',\mathbf{y}') + [\mathbf{x}' \times \mathbf{y}']$, where $(\mathbf{x}',\mathbf{y}')$ is the scalar product of **x'** and **y'**, and $[\mathbf{x}' \times \mathbf{y}']$ is their vector product. (Here and further $*$ denotes the operation of the multiplication of the quaternions). Hence

$$\mathbf{x} * \mathbf{y} = a_0 b_0 - (\mathbf{x}',\mathbf{y}') + a_0\,\mathbf{y}' + b_0\,\mathbf{x}' + [\mathbf{x}' \times \mathbf{y}'] \qquad \textbf{(1)}$$

Later on we will consider only the subalgebra $\mathbf{Q}_k$ of the **H**, containing all the quaternions with rational coefficients. Let us consider the subring $\mathbf{Z}_k$ of ring $\mathbf{Q}_k$:

$$\mathbf{Z}_k = \{\, (a_0 + a_1\,\mathbf{i} + a_2\,\mathbf{j} + a_3\,\mathbf{k})/2 \mid a_0, a_1, a_2, a_3 \in \mathbf{Z},\ a_0 \equiv a_1 \equiv a_2 \equiv a_3 \ (\mathrm{mod}\ 2) \,\} \qquad \textbf{(2)}$$

We will call the ring $\mathbf{Z}_k$ the *Integral Ring of the Quaternions*, with its elements called the *Integral Quaternions*. For each $\mathbf{x} \in \mathbf{Z}_k$ we have $\mathbf{N(x)} \in \mathbf{Z}_k$. This ring has the group, consisting of the **24** invertible elements (divisors of unit):

$$\pm\mathbf{1}, \pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{k}, \frac{\pm\mathbf{1} \pm \mathbf{i} \pm \mathbf{j} \pm \mathbf{k}}{2}.$$

Theorem 1: $\varepsilon \in \mathbf{Z}_k$ is invertible in the ring $\mathbf{Z}_k$ if and only if $\mathbf{N}(\varepsilon) = 1$.

Definition 1: $\mathbf{p} \in \mathbf{Z}_k$ is called the *prime quaternion* if $\mathbf{N(p)} \neq 1$ and it follows from the equality $\mathbf{p} = \mathbf{q} * \mathbf{t}$ with $\mathbf{q}, \mathbf{t} \in \mathbf{Z}_k$ that **p** or **t** is an invertible element.

Theorem 2: $\mathbf{p} \in \mathbf{Z}_k$ is the prime quaternion if and only if **N(p)** is the prime number.

Definition 2: Let $\mathbf{q}, \mathbf{r} \in \mathbf{Z}_k$, $\mathbf{r} \neq 0$. We will call **q** *divisible by* **r** *from the right (or from the left)* if $\mathbf{q} * \mathbf{r}^{-1} \in \mathbf{Z}_k$ or $*\mathbf{r}^{-1} * \mathbf{q} \in \mathbf{Z}_k$), The words "from the right" and "from the left" can be omitted if the divisor $\mathbf{r} \in \mathbf{Z}_k$.

Theorem 3: If **a, b** $\in$ **Z**$_k$ with **b** $\neq 0$ then there exist **q, r** and $\mathbf{q_1}$, $\mathbf{r_1}$ belonging to **Z**$_k$ such that

$$\mathbf{a} = \mathbf{b} * \mathbf{q} + \mathbf{r}, \qquad N(\mathbf{r}) < N(\mathbf{b}) \tag{3}$$

$$\mathbf{a} = \mathbf{q_1} * \mathbf{b} + \mathbf{r_1}, \qquad N(\mathbf{r_1}) < N(\mathbf{b}) \tag{4}$$

We will call **q** in **(3)** the *right quotient* ($\mathbf{q_1}$ the *left quotient*) and **r** the *right remainder* (note that if **r** = 0 then **a** is divisible by **b** from the left).

To prove this theorem let us designate the $\mathbf{b}^{-1} * \mathbf{q} = a_0 + a_1\,\mathbf{i} + a_2\,\mathbf{j} + a_3\,\mathbf{k}$. If $\{a_0\} = \{a_1\} = \{a_2\} = \{a_3\} = 1/2$ then $\mathbf{b}^{-1} * \mathbf{q}$ is the integral quaternion and we may take $\mathbf{q} = \mathbf{b}^{-1} * \mathbf{q}$, $\mathbf{r} = 0$ ({x} is the fractional part of x). In other cases by setting $t = \|a_o\|, u = \|a_1\|, v = \|a_2\|, w = \|a_3\|$ ($\|x\|$ is the least distance of x from an integer), $\mathbf{q} = t + u \cdot \mathbf{i} + v \cdot \mathbf{j} + w \cdot \mathbf{k}$, we obtain that $N(\mathbf{b}^{-1} * \mathbf{a} - \mathbf{q}) < 1$. Hence for $\mathbf{r} = \mathbf{a} - \mathbf{b} * \mathbf{q}$: $N(\mathbf{r}) = N(\mathbf{b}) \cdot N(\mathbf{b}^{-1} * \mathbf{r}) = N(\mathbf{b}) \cdot N(\mathbf{b}^{-1} * \mathbf{a} - \mathbf{q}) < N(\mathbf{b})$ and so we proved **(3)**. The proof for **(4)** runs similarly.

Using this theorem we can construct the algorithm which is similar to Euclid´s Algorithm for computation of the greatest common divisors from the left (and from the right). The following theorem can be proved.

Theorem 4: Let $\mathbf{a_1}, \ldots, \mathbf{a_n} \in \mathbf{Z}_k$ and all these quaternions are not simultaneously equal to zero. Then there exists such $\mathbf{d} \in \mathbf{Z}_k$ that:

1) $\mathbf{a_1}, \ldots, \mathbf{a_n}$ are divisible by **d** from the left;
2) for each $\mathbf{c} \in \mathbf{Z}_k$ which divides $\mathbf{a_1}, \ldots, \mathbf{a_n}$ from the left follows that **d** is divisible by **c** from the left;
3) there exist $\mathbf{x_1}, \ldots, \mathbf{x_n} \in \mathbf{Z}_k$ such that

$$\mathbf{a_1} \cdot \mathbf{x_1} + \mathbf{a_2}\ \mathbf{x_2} + \ldots + \mathbf{a_n}\ \mathbf{x_n} = \mathbf{d} \tag{5}$$

The quaternion **d** satisfying the conditions of this theorem will be called the *Greatest Common Divisor* of the elements $\mathbf{a_1}, \ldots, \mathbf{a_n}$ from the left and will be designated as **GCD_L(**$\mathbf{a_1}, \ldots, \mathbf{a_n}$**)**.

Theorem 5: If **d** and $\mathbf{d_1}$ are the greatest common divisors of the quaternions $\mathbf{a_1}, \ldots, \mathbf{a_n}$ from the left in **Z**$_k$ then there exists $\mathbf{t} \in \mathbf{Z}_k$ such that $\mathbf{d_1} = \mathbf{d} * \mathbf{t}$ and $N(\mathbf{t}) = 1$ (**t** is the invertible element in **Z**$_k$). If **d** is the GCD of the quaternions $\mathbf{a_1}, \ldots, \mathbf{a_n}$ from the left then $\mathbf{d_1} = \mathbf{d} * \mathbf{t}$ with $N(\mathbf{t}) = 1$ is the GCD of the quaternions $\mathbf{a_1}, \ldots, \mathbf{a_n}$ from the left, too.

Note that it is not valid if **d** is multiplied by **t** from the left. Let $< a_1, a_2, \ldots, a_n >$ designate the set of all GCDs of the quaternions $\mathbf{a_1}, \ldots, \mathbf{a_n}$ from the left.

Let us consider the problem of finding the general solution in **Z**$_k$ of the linear equation:

$$\mathbf{a_1} * \mathbf{x_1} + \mathbf{a_2} * \mathbf{x_2} + \ldots\ldots + \mathbf{a_n} * \mathbf{x_n} = \mathbf{b} \tag{6}$$

for given $\mathbf{a_1}, \mathbf{a_2}, \ldots, \mathbf{a_n}, \mathbf{b}$. This equation has solutions if and only if

$$< \mathbf{a_1}, \mathbf{a_2} \ldots, \mathbf{a_n} > = < \mathbf{a_1}, \mathbf{a_2} \ldots, \mathbf{a_n}, \mathbf{b} >.$$

If **A** and **B** are two matrices with elements from the non-commutative ring then we will consider their product in just the same way as in case of the commutative ring adding the condition: the elements of the rows of matrix **A** are multiplied by the elements of the columns of **B** from the left. It is not difficult to check that the above defined operation of the product of two matrices is associative. Then equation **(6)** can be written in matrix form:

$$\mathbf{X} \cdot \mathbf{A} = \mathbf{B}. \qquad (7)$$

where $\mathbf{A} = (\mathbf{a}_1\ \mathbf{a}_2\ \ldots\ \mathbf{a}_n)'$, $\mathbf{X} = (\mathbf{x}_1\ \mathbf{x}_2\ \ldots\ \mathbf{x}_n)$, $\mathbf{B} = (\mathbf{b})$.

We can connect the elementary matrices with the elementary transformations as in the case of the commutative ring. Here the multiplication of **A** by the elementary matrices $\mathbf{V}_p(\varepsilon)$ and $\mathbf{V}_{p,t}(\mathbf{q})$ from the left result in the fulfillment of the corresponding elementary transformations of matrix **A** in which the multiplication by ε and **q** is realized from the right. So we can apply the method mentioned above to find the common solutions of equations **(6)** in the integral ring of the quaternions $\mathbf{Z}_k$.

Implementation in *DERIVE*

We will represent the quaternions as four-dimensional vectors in *DERIVE*. The row matrix and the column matrix of the **n** quaternions will form a matrix of order **n** × **4**. The vector consisting of the matrix components will be called *tensor of rank three* (in our case each component of the tensor is the column of the quaternions, so the tensor represents the matrix of the quaternions). To have a utility for finding the common (general) solution of equation **(7)** we have to add some new functions to the main utility file and make some slight changes of its functions. Further we will define new functions and give some comments on them.

(I refer to the files quaternion.mth and quaternion.dfw which can be found on the diskette. Josef)

Given are two quaternions **a** and **b**

```
[a := [2, 3, -2, 4], b := [3, 2, 3, -1]]
```

Then **VEC(x)** gives the vector or imaginary part of **x**.

```
VEC(x):=VECTOR(x SUB i,i,2,DIMENSION(x))
```

and **PROD(x,y)** returns the product of two quaternions **x** and **y** according to formula **(1)**.

```
PROD(x,y):=APPEND([x SUB 1*y SUB 1-VEC(x)*VEC(y)],
           x SUB 1*VEC(y)+y SUB 1*VEC(x)+CROSS(VEC(x),VEC(y)))
PROD(a,b)=[10,3,11,23]
```

(It is a nice exercise to check this multiplying **2 + 3i – 2j + 4k** and **3 + 2i + 3j –k** by hands to receive the result **10 + 3i + 11j + 23k** applying the multiplication table from above and the formula **(1)** as well. Josef)

PROD_VEC(u,v) returns the product of the row (column) **v** (a matrix in *DERIVE*) of the quaternions by quaternion **u** from the left.

```
PROD_VEC(u,v):=VECTOR(PROD(u,v SUB i),i,DIMENSION(v))
PROD_VEC(a,[b])=[[10,3,11,23]]
PROD_VEC(a,[b,[3,4,5,6]])=[[10,3,11,23],[-20,-15,2,47]]
```

SUBTR_NEW(v,i,j,qq) subtracts $\mathbf{v}_j * \mathbf{qq}$ from $\mathbf{v}_i$ in matrix **v** (**qq** is a quaternion)

```
SUBTR_NEW(v,i,j,qq):=VECTOR(IF(m_=i,v SUB i-PROD(v SUB j,qq),
                     ELEMENT(v,m_)),m_,DIMENSION(v))
```

For demonstrating this we introduce a matrix of 4 quaternions `quatm`:

```
quatm:=[a,b,[-4,2,3,1],[-1,3,4,5]]
SUBTR_NEW(quatm,3,2,[1,2,3,4])
[[2,3,-2,4],[3,2,3,-1],[2,-21,1,-10],[-1,3,4,5]]
```

MA_SUB(v,k,i,j,qq) applies the previous transformation on the **k**-th component of tensor **v**

```
MA_SUB(v,k,i,j,qq):=SUBTR_NEW(v SUB k,i,j,qq)
```

and **SUB_TENZ(v,i,j,qq)** does the same for the whole tensor.

NORM(v) gives the Norm of a quaternion **v** while **CON(v)** returns its conjugate quaternion and **INV(v)** returns the inverse of a quaternion according to the definitions given in the introduction.

```
PROD(a,CON(a))=[33,0,0,0]
PROD(CON(a),a)=[33,0,0,0]
PROD(a,INV(a))=[1,0,0,0]
PROD(INV(a),a)=[1,0,0,0]
```

Then we have **R_QUOT(x,y)** and **R_REM(x,y)** resulting in the right quotient of division **x / y** and its right remainder according to Theorem 3 .

Additionally you can find:

e_matr	producing the identity matrix with elements from $\mathbf{Z}_{k}$.
scal(u,v)	the product of row matrix **u** and columnn matrix **v** of the quaternions
prod_ve_ma(v,u)	the product of the row matrix of the quaternions **v** and the tensor **u** (matrix of the quaternions)

How to use this file – One Example

$\mathbf{Z_k} = \{ (a_0 + a_1\,\mathbf{i} + a_2\,\mathbf{j} + a_3\,\mathbf{k})/2 \mid a_0, a_1, a_2, a_3 \in \mathbf{Z},\ a_0 \equiv a_1 \equiv a_2 \equiv a_3 \pmod 2 \}$

We will solve the the following equation:

$(-8 + 10\,\mathbf{i} + 22\,\mathbf{k})\,\mathbf{x}_1 + (-10 - 14\,\mathbf{i} + 4\,\mathbf{j} + 18\,\mathbf{k})\,\mathbf{x}_2 + (-5 + \mathbf{j} + 8\,\mathbf{k})\,\mathbf{x}_3 = -7 - 3\,\mathbf{i} + 5\,\mathbf{j} + \mathbf{k}$

#40: $\left[\text{koef} := \begin{bmatrix} -8 & 10 & 0 & 22 \\ -10 & -14 & 4 & 18 \\ -5 & 0 & 1 & 8 \end{bmatrix},\ \text{r\_part} := [-7, -3, 5, 1]\right]$

#41: GCD_EUC_L(koef) = [1, 0, 1, 2]

#42: IS_SOL(koef, r_part)

#43: $\left[\begin{bmatrix} 2 & -1 & 1 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ -4 & 1 & -1 & -2 \\ -1 & 6 & 6 & -9 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & -2 & -1 & 1 \end{bmatrix}, \begin{bmatrix} -2 & 1 & 1 & 0 \\ 8 & -8 & 0 & 2 \\ -9 & -22 & -4 & 18 \end{bmatrix}\right]$

Then introduce the variable `matr`, assigning the solution from above to find the general solution `com_sol` which contains a bundle of parameters. We will show only a part of its output.

#44: matr :=
$$\left[\begin{bmatrix}2&-1&1&0\\-1&0&-1&0\\0&0&1&0\end{bmatrix}, \begin{bmatrix}1&0&0&0\\-4&1&-1&-2\\-1&6&6&-9\end{bmatrix}, \begin{bmatrix}0&0&0&0\\1&0&0&0\\1&-2&-1&1\end{bmatrix}, \begin{bmatrix}-2&1&1&0\\8&-8&0&2\\-9&-22&-4&18\end{bmatrix}\right]$$

#45: com_sol

#46:
$$\begin{bmatrix} 17\cdot p_{2,4} - 6\cdot p_{2,3} + 2\cdot p_{2,2} + p_{2,1} - 20\cdot p_{1,4} + 14\cdot p_{1,3} - 18\cdot p_{1,2} - p_{1,1} + 18 \\ -3\cdot p_{2,4} + 2\cdot p_{2,3} + p_{2,2} + p_{2,1} + 4\cdot p_{1,4} - 5\cdot p_{1,3} + p_{1,2} - 2 \\ -42\cdot p_{2,4} - 5\cdot p_{2,3} + 12\cdot p_{2,2} - 5\cdot p_{2,1} + 71\cdot p_{1,4} - 9\cdot p_{1,3} + 17\cdot p_{1,2} - 3\cdot p_{1,1} - 89 \end{bmatrix}$$

$$6\cdot p_{2,4} + 17\cdot p_{2,3} + p_{2,2} - 2\cdot p_{2,1} - 14\cdot p_{1,4} - 20\cdot p_{1,3} - p_{1,2} + 2\cdot(9\cdot p_{1,1} + 13)$$

First of all we can check if this is a solution by simplifying `check_sol`. This is not very satisfying. But then we can find partial solutions choosing any set of parameters and check this solution again – maybe as an additional exercise doing it by hands, too.

#48: p :=
$$\begin{bmatrix}1&0&0&1\\0&1&1&0\\1&0&1&0\end{bmatrix}$$

#49:
$$\text{com\_sol} = \begin{bmatrix}-7&48&-18&-5\\5&-7&7&-1\\-14&-107&37&-54\end{bmatrix}$$

#50: solu1 :=
$$\begin{bmatrix}-7&48&-18&-5\\5&-7&7&-1\\-14&-107&37&-54\end{bmatrix}$$

Step by step one can check the validity of this partial solution:

#51: $\text{PROD}(\text{koef}_1, \text{solu1}_1) = [-314, -58, 1250, -294]$

#52: $\text{PROD}(\text{koef}_2, \text{solu1}_2) = [-158, -130, -190, 30]$

#53: $\text{PROD}(\text{koef}_3, \text{solu1}_3) = [465, 185, -1055, 265]$

#54: [-314, -58, 1250, -294] + [-158, -130, -190, 30] + [465, 185, -1055, 265]

#55: [-7, -3, 5, 1]

Finally let´s find a base solution – setting all parameters equal zero:

#56: p :=
$$\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}$$

#57:
$$\text{com\_sol} = \begin{bmatrix}18&26&30&-16\\-2&-9&-3&2\\-89&-52&-26&37\end{bmatrix}$$

#58: solu2 :=
$$\begin{bmatrix}18&26&30&-16\\-2&-9&-3&2\\-89&-52&-26&37\end{bmatrix}$$

#59:
$$\sum_{i=1}^{3} \text{PROD}(\text{koef}_i, \text{solu2}_i) = [-7, -3, 5, 1]$$

Even if the underlying theory might be too difficult for Secondary School students, learning about the existence of quaternions as a meaningful extension of complex numbers and working with them seems to be considerable. Josef

Shoemaker's Knife from another Point of View

Peter Lüke-Rosendahl, Hannover, Oktober 2001; PeterLR@web.de

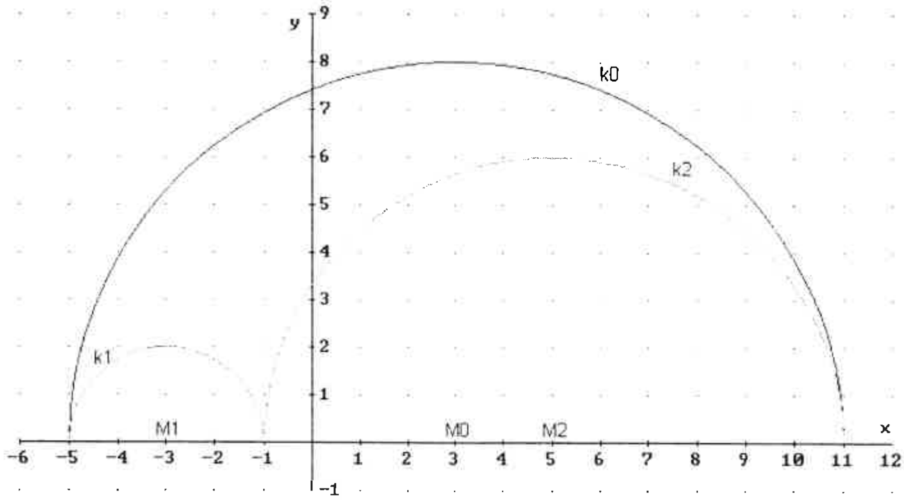
The idea of this reflections was given by a report "Why DGS is such an effective tool in maths education" on ICTMT5 (The Fifth International Conference on Technology in Mathematics Teaching) 2001 in Klagenfurt. The problem was presented, but no solution was given.

Usually by thinking of "Shoemaker's knife" you will have in mind some areas, specially those with same size in connection with the calculation of circles and Pythagoras.

#2: $k0 := y = \sqrt{-x^2 + 6\cdot x + 55}$

#3: $k1 := y = \sqrt{-x^2 - 6\cdot x - 5}$

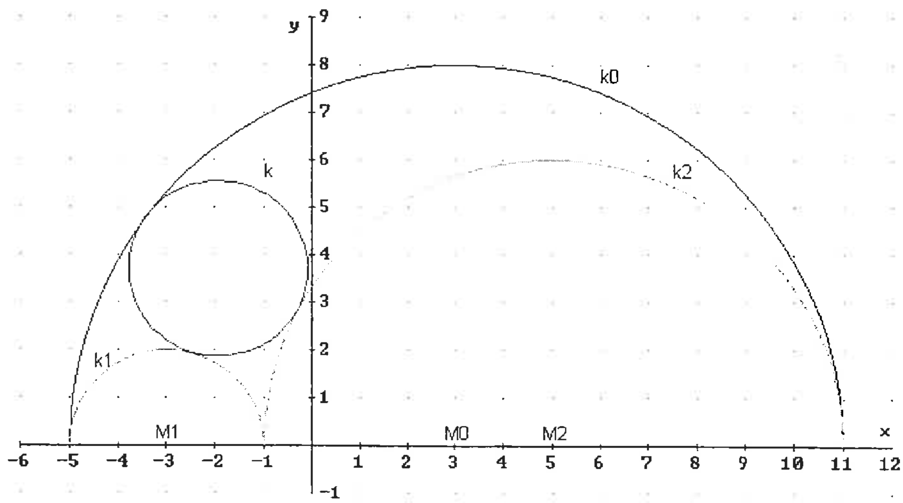
#4: $k2 := y = \sqrt{-x^2 + 10\cdot x + 11}$



It should be possible to construct a circle which will touch the circles k1, k2 and k0 at the same time.

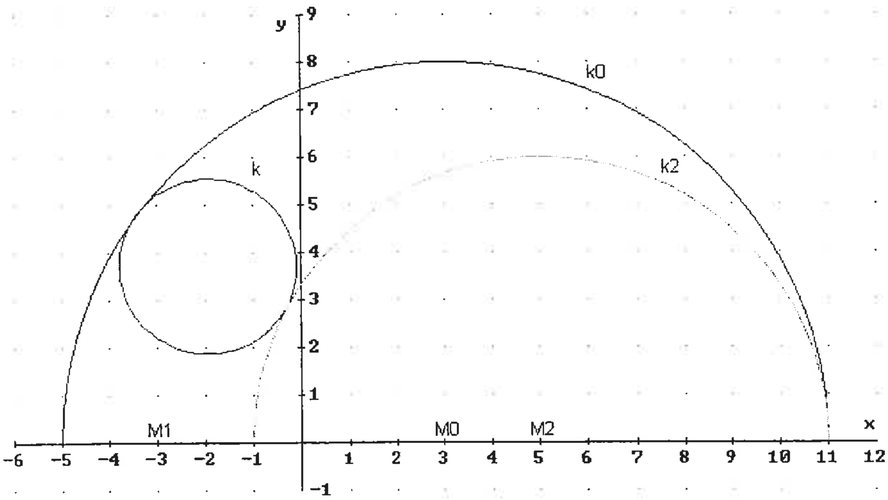
#5: **The equation of this circle will be shown later.**

#6: $k := 169\cdot x^2 + 650\cdot x + 169\cdot y^2 - 1248\cdot y + 2929 = 576$



There is no problem to find such a circle using any DGS by try and error which seems to fulfil the desired conditions.

Let's start our reflections with only two circles, k2 and k0.



First we consider points which are regularly distributed on k2, then we look for those centres of circles which will touch k2 in these points.

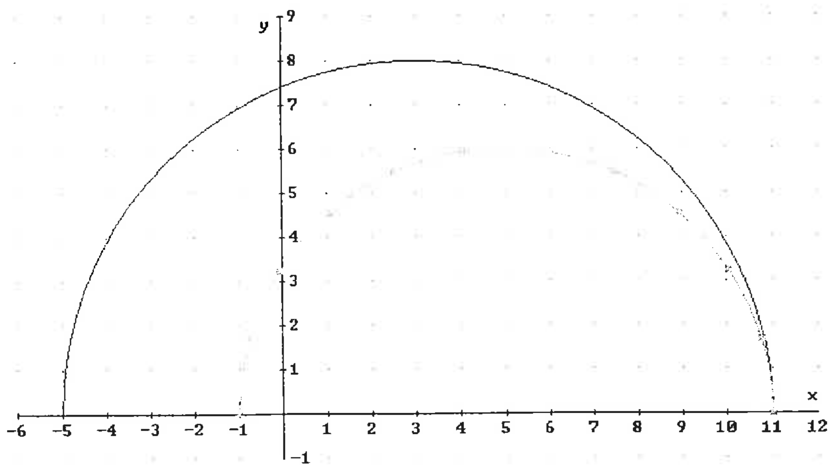
```
#7:   Regular distributed points (·PUNKTE() on a halfcircle mit center on x-axis)
```

#8: $\text{PUNKTE}(xm, r) := \text{VECTOR}\left([r\cdot\text{SIN}(\varphi) + xm, r\cdot\text{COS}(\varphi)], \varphi, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{11}\right)$

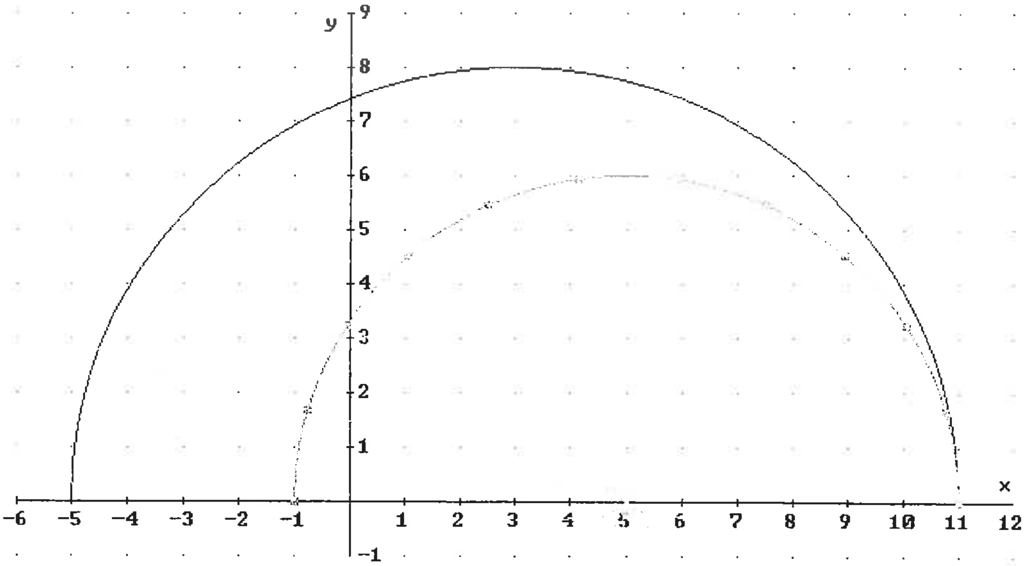
#9: PUNKTE(5, 6)

The steps are chosen in this special way to avoid the problem having a parallel line to y-axis.

-1	0
-0.756957	1.69039
-0.0475206	3.24384
1.07083	4.53449
2.50751	5.45779
4.14611	5.93892
5.85388	5.93892
7.49248	5.45779
8.92916	4.53449
10.0475	3.24384
10.7569	1.69039
11	0



These centres are all on a pencil of lines through M2 and the just constructed points on k2.



We are looking for the locus of all points which are centres of all circles which are touching k2 and k0. When looking at the picture you may guess that the solution is a circle between k0 and k2.

#20: $k3 := y = \sqrt{(- x^2 + 8 \cdot x + 33)}$

#21: Calculation of intersection points from gs with k3

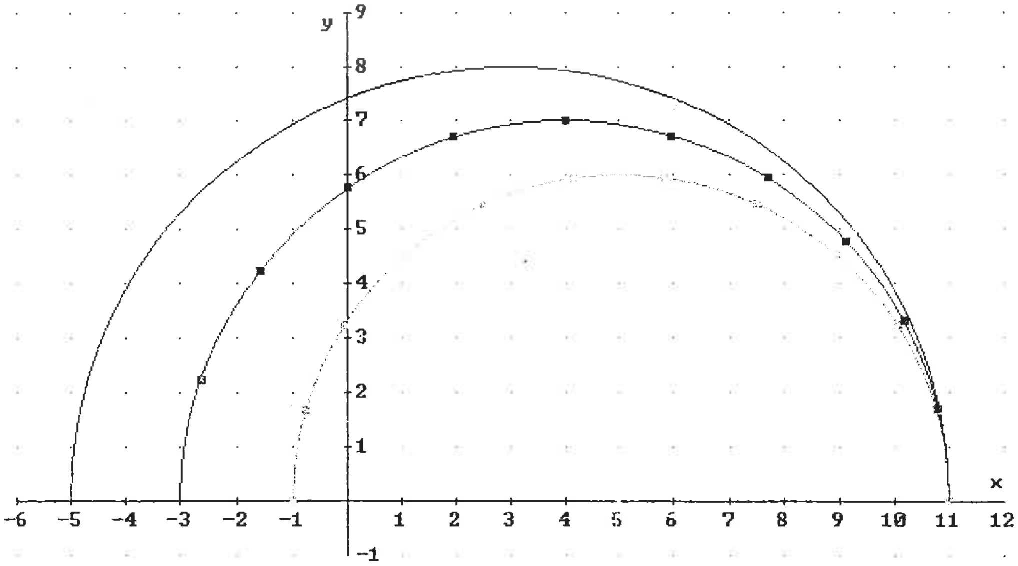
#23: $\text{SOLVE}(\text{RHS}(gs_3) = \text{RHS}(k3), x) = (x = 0.0139265)$

#24: $\text{RHS}(\text{SOLVE}(\text{RHS}(gs_3) = \text{RHS}(k3), x)) = 0.0139265$

```
#25:  VECTOR(RHS(SOLVE(RHS(k) = RHS(k3), x)), k, gs)
#26:  [-2.63163, -1.57889, 0.0139265, 1.94418, 3.99355, 5.96594, 7.71067, 9.12839, 10.1634,
        10.7903]

#27:  xw(Schar, Kurve) := VECTOR(RHS(SOLVE(RHS(k) = RHS(Kurve), x)), k, Schar)
#28:  xw(gs, k3)
#29:  [-2.63163, -1.57889, 0.0139265, 1.94418, 3.99355, 5.96594, 7.71067, 9.12839, 10.1634,
        10.7903]

#30:  Spkte(Schar, Kurve) := VECTOR([k, SUBST(RHS(Kurve), x, k)], k, xw(Schar, Kurve))
#31:  Spkte(gs, k3)
```

#32:

-2.63163	2.24085
-1.57889	4.22799
0.0139265	5.75423
1.94418	6.6913
3.99355	7
5.96594	6.71826
7.71067	5.93555
9.12839	4.76441
10.1634	3.31836
10.7903	1.7002



#33: MPktek3 :=

-2.63163	2.24085
-1.57889	4.22799
0.0139265	5.75423
1.94418	6.6913
3.99355	7
5.96594	6.71826
7.71067	5.93555
9.12839	4.76441
10.1634	3.31836
10.7903	1.7002

#34: BPktek2 :=

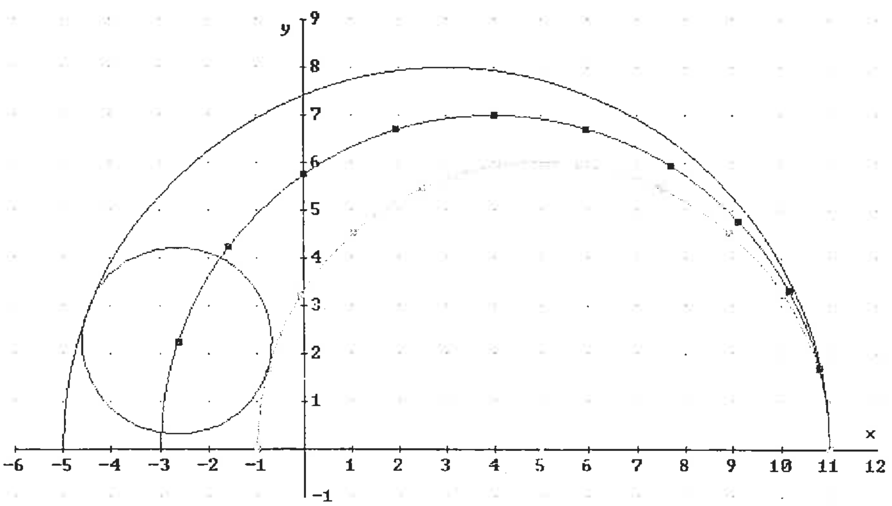
-1	0
-0.756957	1.69039
-0.0475206	3.24384
1.07083	4.53449
2.50751	5.45779
4.14611	5.93892
5.85388	5.93892
7.49248	5.45779
8.92916	4.53449
10.0475	3.24384
10.7569	1.69039
11	0

#40: $\text{Radius1}(i) := \sqrt{\left((\text{BPktek2}_{i+1,1} - \text{MPktek3}_{i,1})^2 + (\text{BPktek2}_{i+1,2} - \text{MPktek3}_{i,2})^2\right)}$

#41: APPROX(Radius1(1)) = 1.953818074

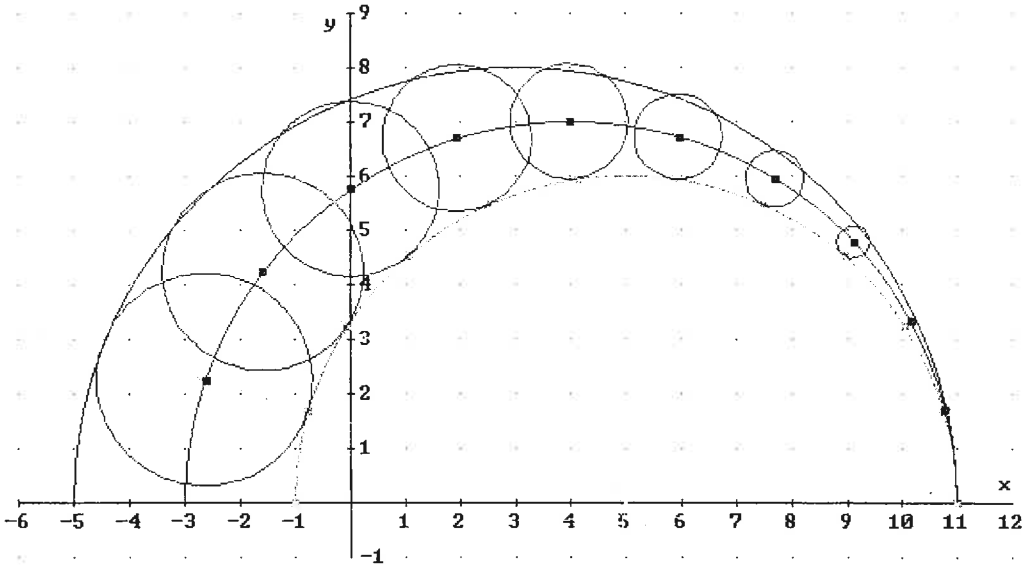
#42: $\text{Kreis1}(i) := (x - \text{MPktek3}_{i,1})^2 + (y - \text{MPktek3}_{i,2})^2 = \text{Radius1}(i)^2$

#43: Kreis1(1)



It seems to fit; let's go on!

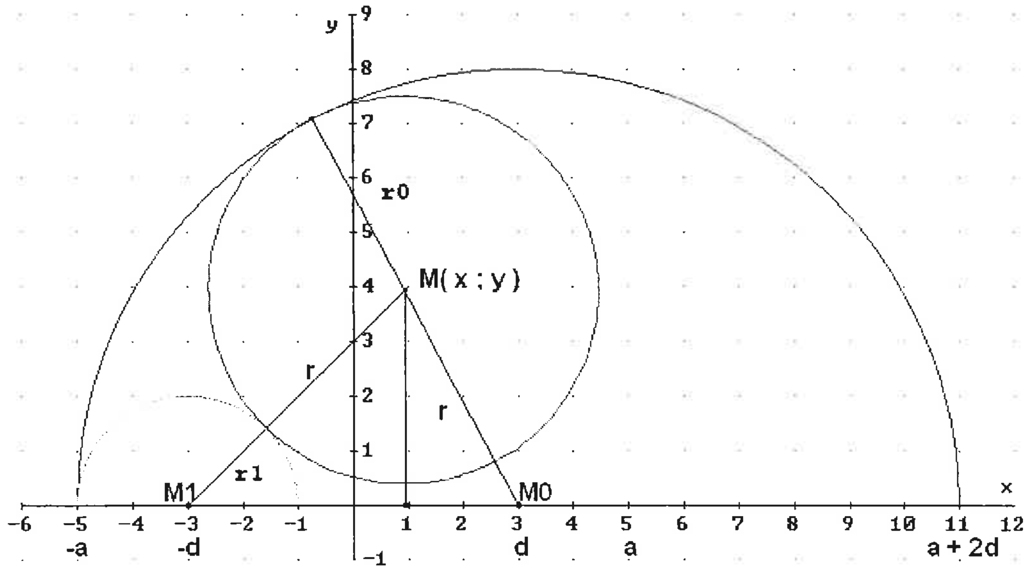
```
#44:   Kreise1 := VECTOR(Kreis1(i), i, 1, 8)
```



Oooups! Even without zooming it's obvious that we were wrong with our idea. That's a pity. You also will be able to show this easily by using DGS and using Cinderella for example you may have a nice animation.

You will be able to construct the curve of the „true solution“ (as far as you know the equation) by DGS as conic section or as locus, but (at the moment I think) it's not possible to use this curve as "street" for an animation.

Let's try to find an algebraic solution.



Explanation for the next *DERIVE* screen:

#51: The ellipse is the locus for all points for which the sum of the distances to two fixed points is 2a. Foci are F1 = M1 (-d ; 0) and F2 = M0 (d ; 0)

#53: Standard form with centre in the origin of this ellipse; transformation takes some time.

#46: $r_0 = a + d$

#47: $r_1 = a - d$

#48: $a - d + r = \sqrt{(x + d)^2 + y^2}$

#49: $a + d - r = \sqrt{(x - d)^2 + y^2}$

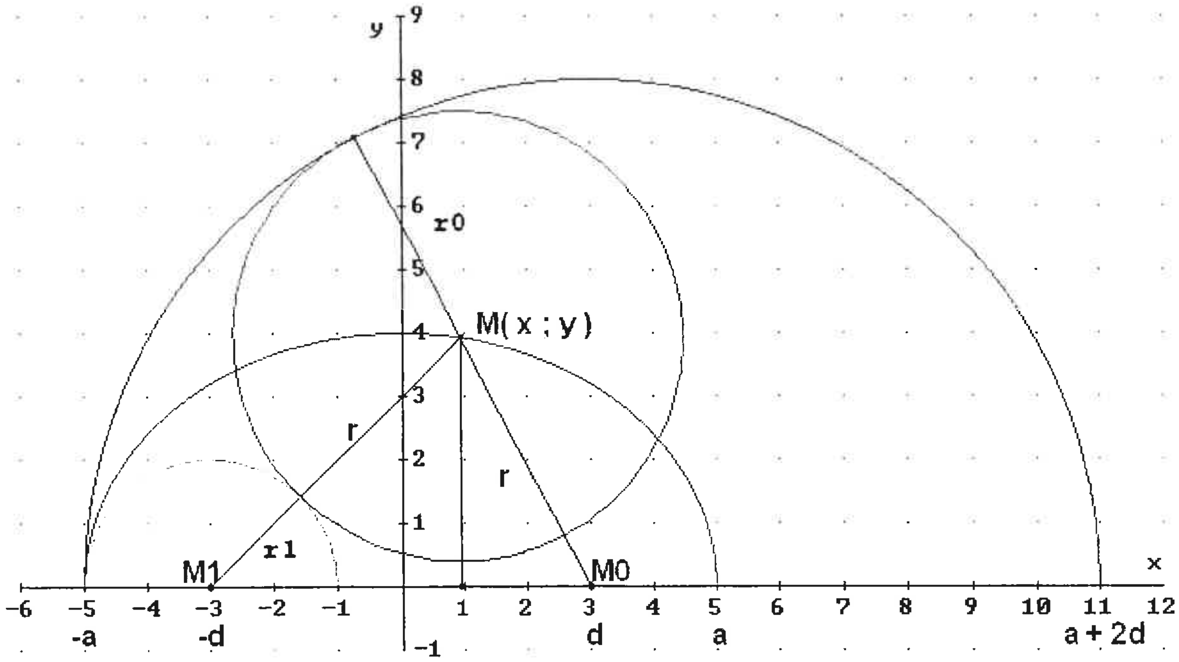
#50: $(a - d + r = \sqrt{(x + d)^2 + y^2}) + (a + d - r = \sqrt{(x - d)^2 + y^2})$

#51: $2 \cdot a = \sqrt{x^2 - 2 \cdot d \cdot x + y^2 + d^2} + \sqrt{x^2 + 2 \cdot d \cdot x + y^2 + d^2}$

#52: Ellipse: a, b; F1 = M1(-d ; 0); F2 = M0(d ; 0)

#53: $\left[b^2 = a^2 - d^2, \frac{x^2}{a^2} + \frac{y^2}{a^2 - d^2} = 1 \right]$

#54: $\text{Ellipse} := y = \frac{4 \cdot \sqrt{25 - x^2}}{5}$



Let's try to imitate a DGS by means of a CAS. We do have the locus of centres of all the desired circles, now only some screenshots of the animation are missing.

#55: PUNKTE(-3, 2)

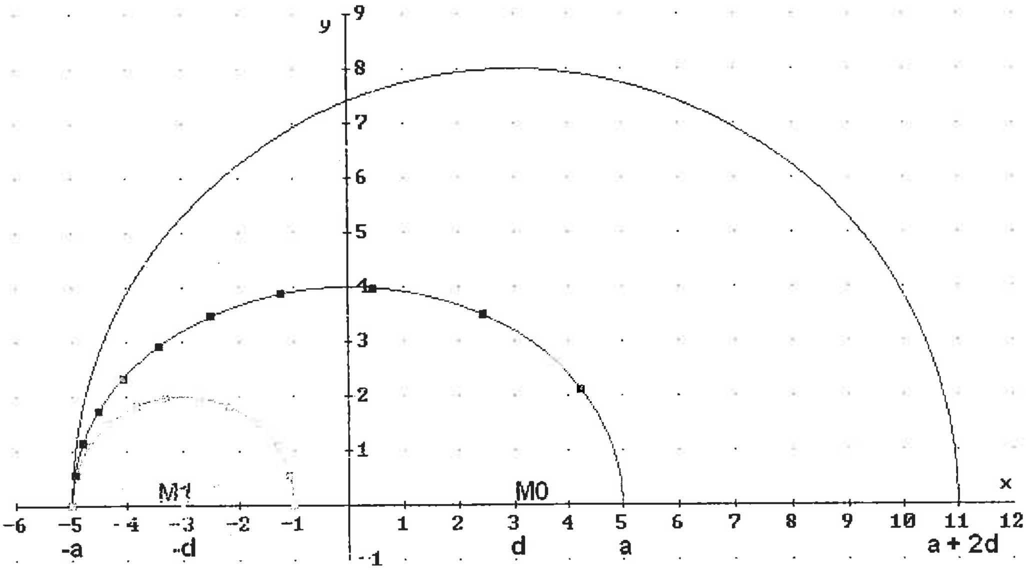
#57: Equations of all straight lines through M1 and points on k1; without x-axis (i = 1)

#58: $gs2 := \text{VECTOR}(\text{ZWPF}((\text{PUNKTE}(-3, 2))_i, [-3, 0]), i, 2, 11)$

#60: Spkte(gs2, Ellipse)

#61:

$$\begin{bmatrix} -4.94858 & 0.572156 \\ -4.789 & 1.14972 \\ -4.50443 & 1.73621 \\ -4.0641 & 2.33005 \\ -3.41957 & 2.91824 \\ -2.50207 & 3.46314 \\ -1.22933 & 3.87721 \\ 0.451838 & 3.98363 \\ 2.43568 & 3.4933 \\ 4.23626 & 2.12475 \end{bmatrix}$$



#62: BPktek1 :=

$$\begin{bmatrix} -5 & 0 \\ -4.91898527 & 0.5634651339 \\ -4.682506887 & 1.081281549 \\ -4.309721175 & 1.51149903 \\ -3.830830039 & 1.819263937 \\ -3.284629629 & 1.979642708 \\ -2.71537037 & 1.979642708 \\ -2.169170047 & 1.819263937 \\ -1.690278546 & 1.51149903 \\ -1.317493143 & 1.081281549 \\ -1.081014245 & 0.5634651339 \\ -1 & 0 \end{bmatrix}$$

#63: MPkteEl :=

$$\begin{bmatrix} -4.948585093 & 0.5721564307 \\ -4.789006342 & 1.14972475 \\ -4.504436411 & 1.736212591 \\ -4.064102564 & 2.330058224 \\ -3.419579533 & 2.918243243 \\ -2.502074688 & 3.46314258 \\ -1.22933552 & 3.877214416 \\ 0.4518384569 & 3.983633612 \\ 2.435682819 & 3.493303111 \\ 4.236264236 & 2.124759152 \end{bmatrix}$$

#64: $\text{Radius2}(i) := \sqrt{\left((\text{BPktek1}_{i+1,1} - \text{MPkteEl}_{i,1})^2 + (\text{BPktek1}_{i+1,2} - \text{MPkteEl}_{i,2})^2\right)}$

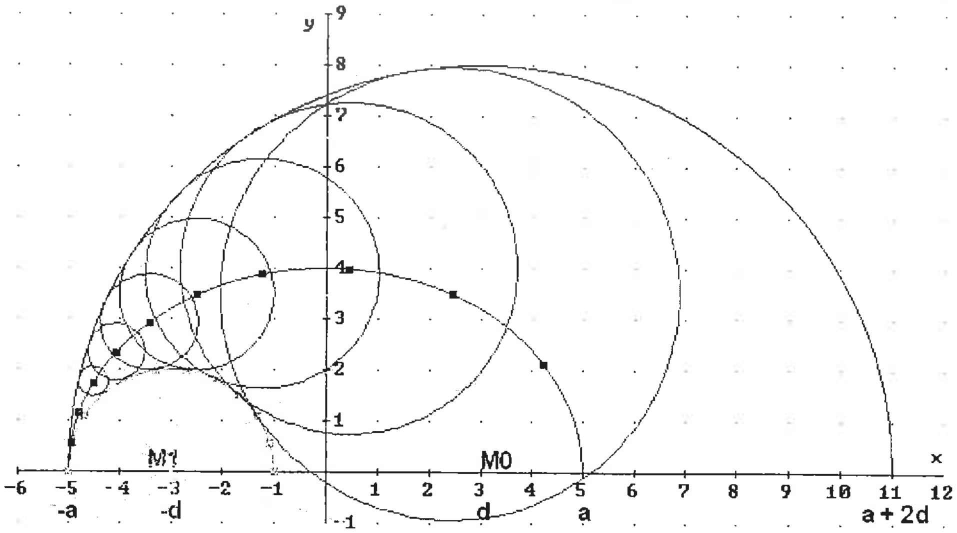
#65: $\text{Kreis2}(i) := (x - \text{MPkteEl}_{i,1})^2 + (y - \text{MPkteEl}_{i,2})^2 = \text{Radius2}(i)^2$

#66: APPROX(Radius2(2)) = 0.1265962308

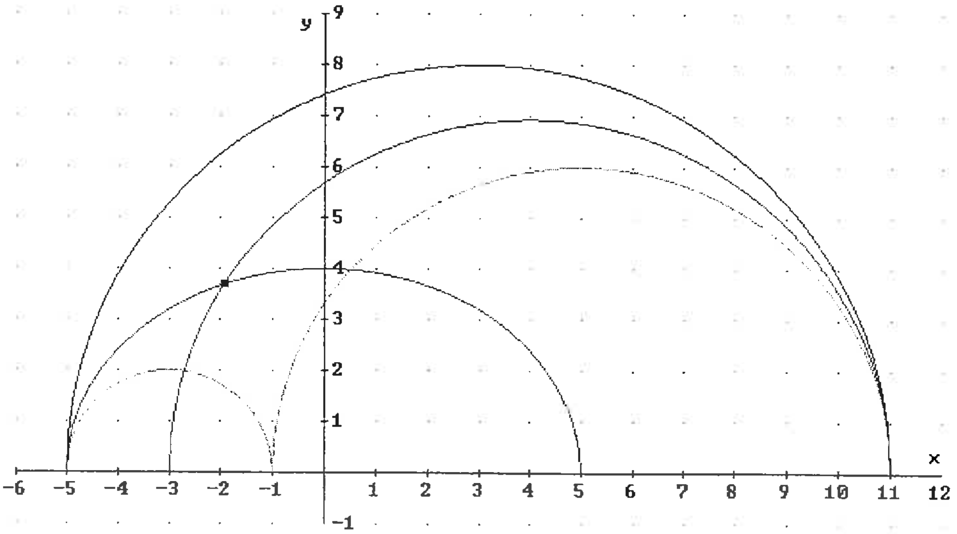
#67: Kreis2(4)

#68: $x^2 + 8.1282 \cdot x + y^2 - 4.66011 \cdot y + 21.946 = 0.315317$

#69: Kreise2 := VECTOR(Kreis2(i), i, 3, 9)



According to the first ellipse you may develop the equation of the second ellipse. The intersection point of both ellipses is obviously the centre of the circle we have been looking for.



#75: Schnittgerade := ZWPF(Schnittpunkt, [-3, 0])

#76: $$y = \frac{24\cdot x}{7} + \frac{72}{7}$$

#75: „Schnittgerade“ = ‚intersection line'; „Schnittpunkt“ = ‚intersection point'

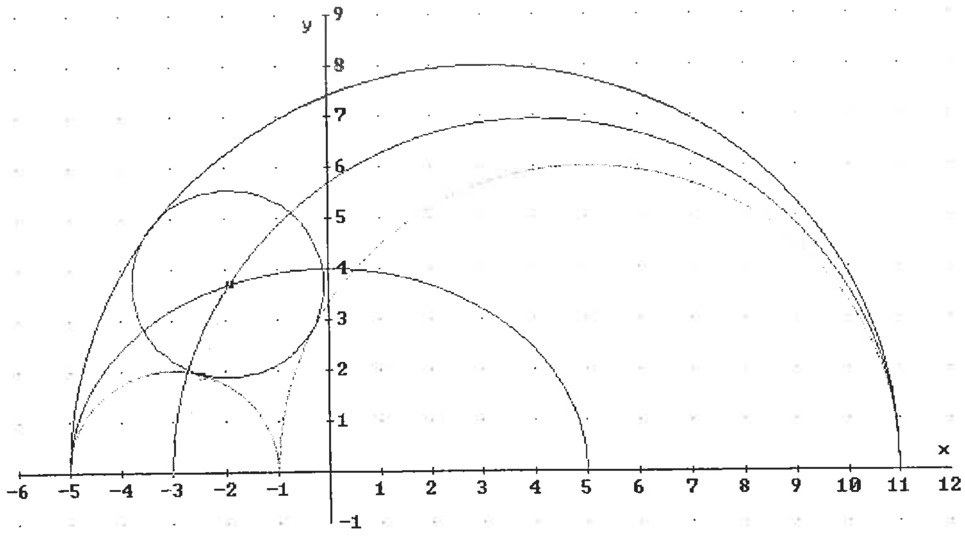
#77: SOLVE(RHS(k1) = RHS(Schnittgerade), x) = $\left[x = -\frac{61}{25}\right]$

#78: $\left[\text{Berührpunkt} := \left[\text{RHS(SOLVE(RHS(k1) = RHS(Schnittgerade), x))}, \text{RHS}\left(\text{SUBST}\left(\text{Schnittgerade}, x, -\frac{61}{25}\right)\right)\right]\right] = \left[-\frac{61}{25}, \frac{48}{25}\right]$

#79: $\left[\text{Radius} := \sqrt{\left((\text{Schnittpunkt}_1 - \text{Berührpunkt}_1)^2 + (\text{Schnittpunkt}_2 - \text{Berührpunkt}_2)^2\right)}\right] = \frac{24}{13}$

#80: $\text{Kreis} := (x - \text{Schnittpunkt}_1)^2 + (y - \text{Schnittpunkt}_2)^2 = \text{Radius}^2$

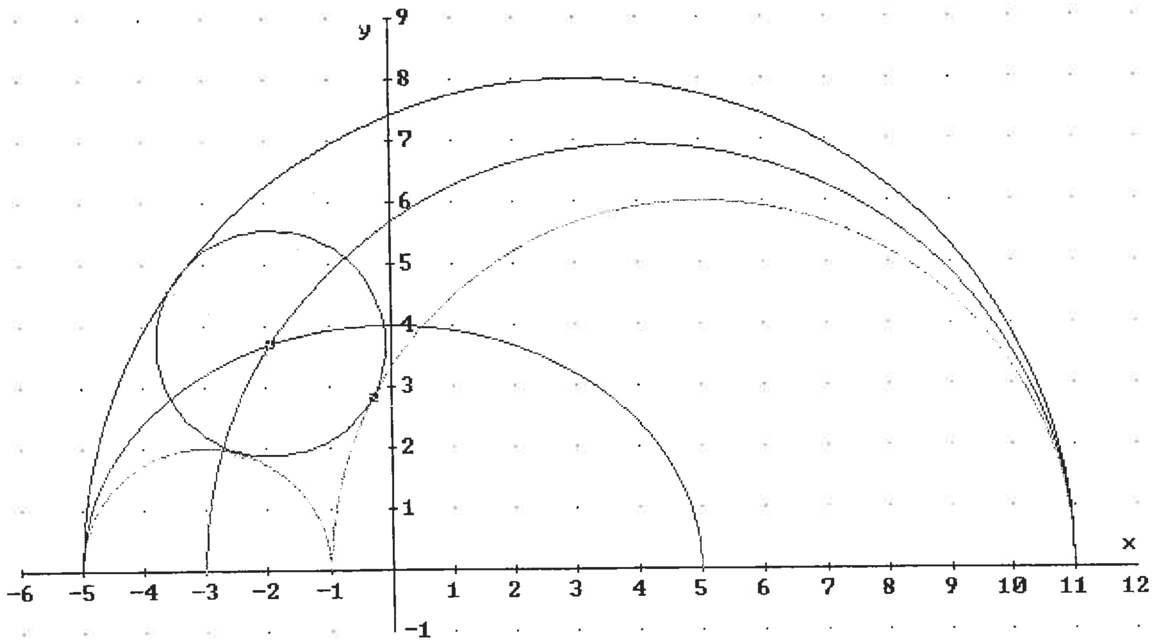
#81: $$\frac{169\cdot x^2 + 650\cdot x + 169\cdot y^2 - 1248\cdot y + 2929}{169} = \frac{576}{169}$$



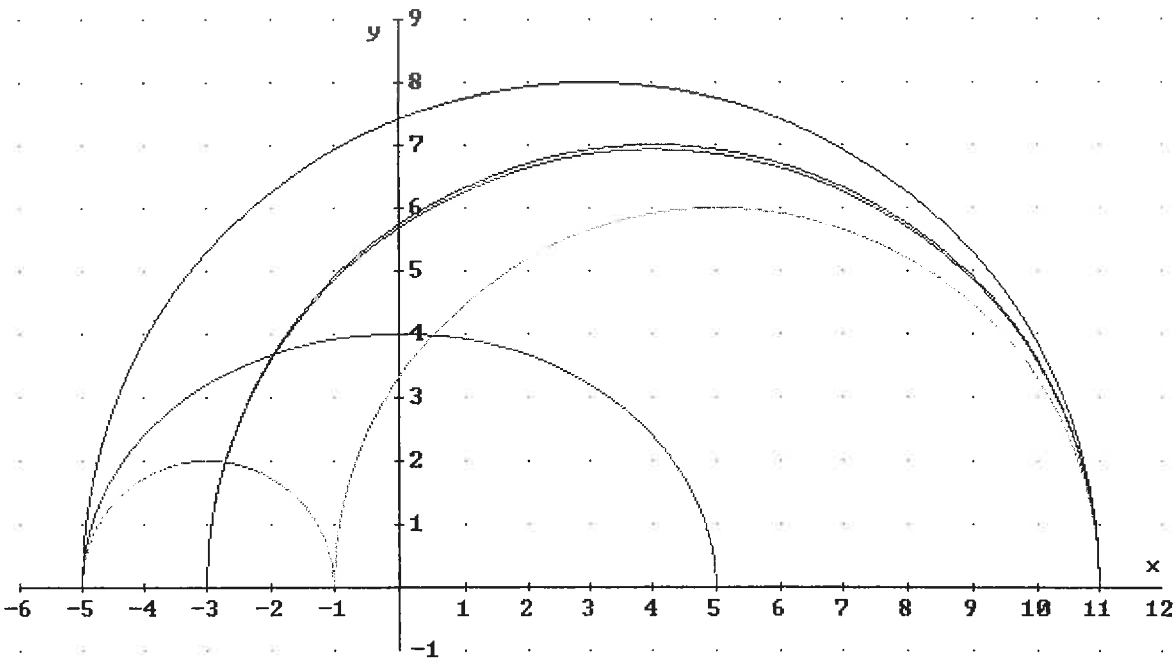
#82: (Schnittgerade2 := ZWPF(Schnittpunkt, [5, 0])) = $\left[y = \frac{8}{3} - \frac{8 \cdot x}{15}\right]$

#83: SOLVE(RHS(k2) = RHS(Schnittgerade2), x) = $\left[x = -\frac{5}{17}\right]$

#84: $\left(\text{Berührpunkt2} := \left[\text{RHS(SOLVE(RHS(k2) = RHS(Schnittgerade2), x))},\ \text{RHS}\left(\text{SUBST}\left(\text{Schnittgerade2}, x, -\frac{5}{17}\right)\right)\right]\right) = \left[-\frac{5}{17}, \frac{48}{17}\right]$



To have a reparation of our first idea let's have a look on the second ellipse and the first circle.



Finally special thanks to Steffen Timman for exchanging ideas and helping to find the equation of the ellipse and Josef Böhm to develop a more powerful tool to solve a set of equations at one time.

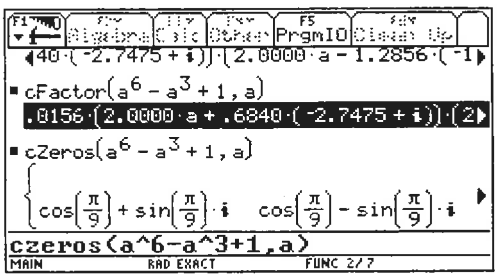
DERIVE & TI-92 User Forum continued from page 7

So apparently, Derive means another complex 9th root of -1, but which one?
Do I have to try using trial and error? Am I missing something? Or just expecting too much?
Regards,

Pedro Tytgat

Answer from Wim de Jong

Pedro,
By applying "Approximate" to (-1)^(1/9) you will see that Derive interprets the expression as cos(pi/9)+isin(pi/9). So it uses the principal value pi for the argument of -1, not 9pi. On other occasions you may have noticed that Derive uses cos(pi.n) for (-1)^n.
Cheers,
Wim de Jong



You can find the TI-92 confirmation of Wim de Jong´s answer looking for the complex roots of the expression from above, Josef.

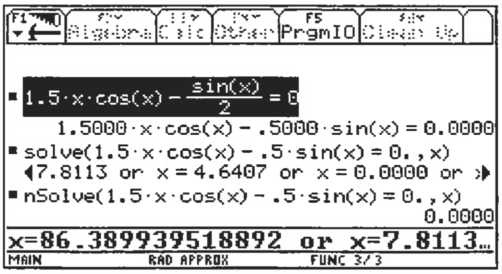
Equations, Equations – Solutions, Solutions

Josef Böhm, Würmla, Austria

Some weeks ago I received a call from a *DERIVE* User who claimed not to find the correct roots of a transcendental equation:

$$\frac{3x}{2}\cos(x)-\frac{\sin(x)}{2}=0$$

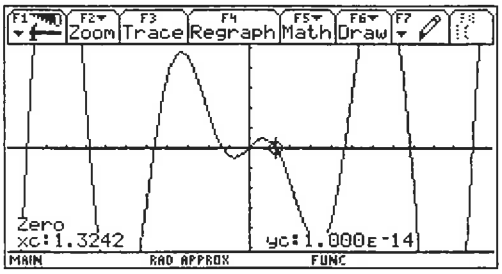
He gave me some of the approximated roots and at first I tried the *TI*-92 and *TI*-92+ to check the solutions. And I was also surprised, because it didn't work in the expected way.



The first screen shows Mode APPROXIMATE. We receive some roots – but between 86.39 and 7.81 seems to be much space for more roots ...

... and indeed the *TI* gives a friendly warning: "More solutions may exist"

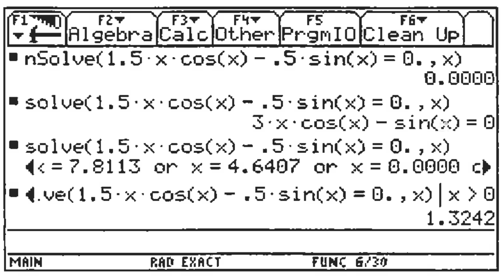
nsolve was completely dissatisfying.

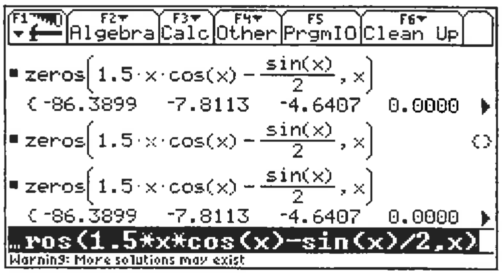


The "ordinary" *TI* showed the same results – but without the warning. So the PLUS justifies the higher price in more politeness, but not in better results (performing this task, of course).

I inspected the graph and found immediately at least two more solutions with ± 1.3242.

Then I switched to the EXACT Mode and only giving one additional restriction using the with-operator I could urge the *TI* to show the expected value – but only one.





Full of hope I turned on the computer and launched *DERIVE*. The output describes exactly how I developed a function to return hopefully all solutions of most of the equations, which do not have exact solutions – I must admit that didn't consider "pathological" equations.

$$\text{mysolutions}\left(1.5\cdot x\cdot \text{COS}(x) - \frac{\text{SIN}(x)}{2}, x, -20, 20, 1\right) = [-17.25944890, -14.11355337,$$

$$-10.96518440, -7.811334475, -4.640683630, -1.324194449, 0, 0, 1.324194449,$$

$$4.640683630, 7.811334475, 10.96518440, 14.11355337, 17.25944890]$$

This utility function searches for roots in intervals of length 1 starting with [-20,-19] and ending with [19,20]. I know that there can be two or even more roots in one intervall. So to go sure one could take smaller intervals.

The *DERIVE* help file says, that NSOLVE and NSOLUTIONS are treating successfully polynomial equations only:

> *As the examples above show, given a polynomial equation with numeric coefficients, NSOLVE and NSOLUTIONS return all the real and complex solutions to the equation. However, for any other type of equation, NSOLVE and NSOLUTIONS return only one solution to the equation.*

#2: $1.5\cdot x\cdot \text{COS}(x) - \frac{\text{SIN}(x)}{2} = 0$

#3: $\text{SOLVE}\left(1.5\cdot x\cdot \text{COS}(x) - \frac{\text{SIN}(x)}{2} = 0, x\right)$

#4: $3\cdot x\cdot \text{COS}(x) - \text{SIN}(x) = 0$

#5: NSOLVE(3·x·COS(x) - SIN(x) = 0, x, Real) = (x = 0)

#6: NSOLVE(3·x·COS(x) - SIN(x) = 0, x, 0, 10) = (x = 0)

#7: NSOLVE(3·x·COS(x) - SIN(x) = 0, x, 1, 10) = (x = 7.81133)

#8: NSOLVE(3·x·COS(x) - SIN(x) = 0, x, 1, 3) = (x = 1.32419)

#9: $\text{VECTOR}\left(\text{NSOLVE}\left(\frac{3\cdot x\cdot \text{COS}(x)}{2} - \frac{\text{SIN}(x)}{2} = 0, x, a, a + 1\right), a, \text{VECTOR}(k, k, 1, 10)\right)$

#10: [x = 1.32419, false, false, x = 4.64068, false, false, x = 7.81133, false, false, x = 10.9651]

Now I used NSOLUTIONS( ) to receive only the numbers:

#11: $\text{VECTOR}\left(\text{NSOLUTIONS}\left(1.5\cdot x\cdot \text{COS}(x) - \frac{\text{SIN}(x)}{2} = 0, x, a, a + 1\right), a, \text{VECTOR}(k, k, 1, 10)\right)$

#12: [[1.32419], [], [], [4.64068], [], [], [7.81133], [], [], [10.9651]]

Next step to avoid the empty brackets:

#13: $\text{APPEND}\left(\text{VECTOR}\left(\text{NSOLUTIONS}\left(1.5\cdot x\cdot \text{COS}(x) - \frac{\text{SIN}(x)}{2} = 0, x, a, a + 1\right), a, [0, ..., 9]\right)\right)$

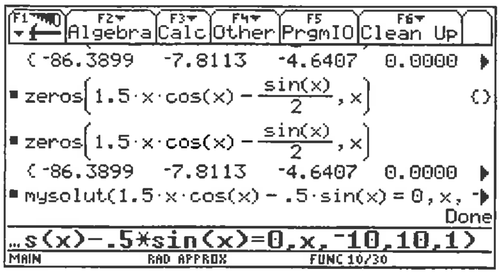
#14: [0, 1.324194449, 4.64068363, 7.811334475]

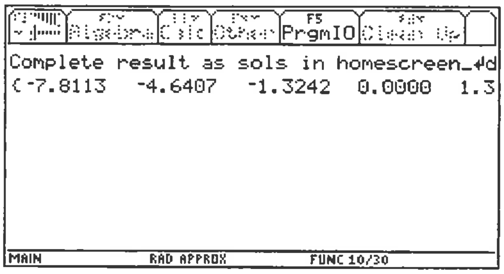
So finally I can define a more general function to find "mysolutions":

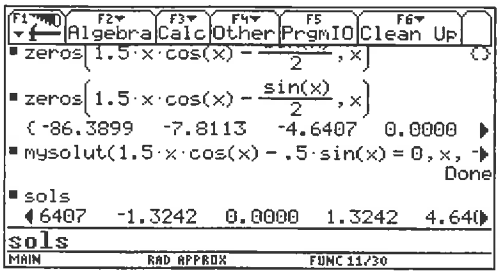
```
#15: mysolutions(equ, equ_v, start, end, step) := APPEND(VECTOR(NSOLUTIONS(equ, equ_v,
     a_, a_ + step), a_, start, end - step, step))
```

#16: $\text{mysolutions}\left(1.5\cdot x\cdot \text{COS}(x) - \frac{\text{SIN}(x)}{2}, x, 0, 10, 0.5\right) = [0, 1.32419, 4.64068, 7.81133]$

What now? What do you think?? Absolutely correct. I turned back to my *TI*-92 and wrote a little program for performing the same task:
I´d like first to present the results of my efforts:







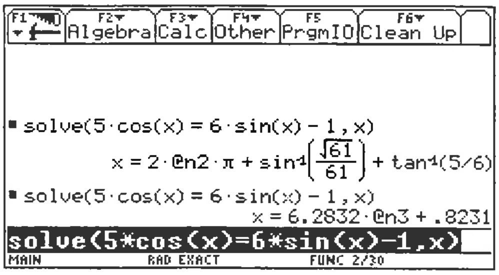
Fortunately I found the same output as before.

The program is not very difficult to follow. Please not the use of the third optionin the when – function.
In case of returning "no solution found", the system cannot decide the value of the variable and in this case the empty set {} is given back.

```
mysolut
(eq_,eqv_,st_,end_,step_)
Prgm
Local  i_,sol_,sols_
{}→sols_
For  v_,st_,end_-step_,step_
{nSolve(eq_,eqv_)|eqv_≥v_ and eqv_<v_+step_}→sol_
when(sol_[1]^2≥0,sol_,sol_,{})→sol_
augment(sols_,sol_)→sols_
EndFor
sols_→sols
Disp  "Complete result as sols in homescreen_
disp"
Disp  sols_
EndPrgm
```

Not many days after we had a seminar in the frame of our nation wide 4[th] ACDCA project. (Austrian Center of Didactics of Computer Algebra). One group is working on a special selection of CAS-bound assessment examples. They came across a simple trigonometric equation, and they also faced unexpected outcomes of their *TI*s. But in this case the equation has an exact solution.

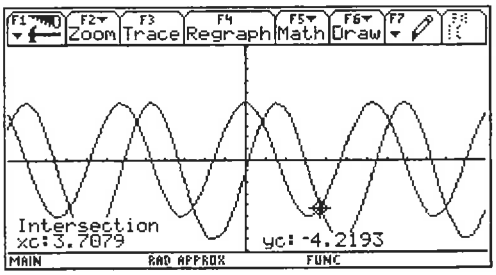
$$5\cos(x) = 6\sin(x) - 1$$



This is what the *TI*-92+ did. So we expect solutions with a period of 2π.

Inspecting the intersection points of the two trig functions we are surprised to find two families of solutions:

3.71 can not be obtained from the general solution given by the calculator!



Rewriting the function gives less and more solutions. Only pressing [ENTER] in EXACT Mode delivers no answer at all. Approximating the solving process gives an uncomplete list of solutions.

What to do?? Any idea??

```
F1 F2 Algebra F3 Calc F4 Other F5 PrgmIO F6 Clean Up
■ solve(5·cos(x) - 6·sin(x) + 1 = 0, x)
                         5·cos(x) - 6·sin(x) + 1 = 0
■ solve(5·cos(x) - 6·sin(x) + 1 = 0, x)
  x = 28.8407 or x = 7.1063 or x = 3.7079 ◄►
■ mysolut(5·cos(x) = 6·sin(x) - 1, x, -10, 10►
                                            Done
■ sols
 ◄34   -5.4601   -2.5752   .8231   3.7079 ►
sols
MAIN        RAD EXACT        FUNC 8/30
```

First of all you could use your knowledge of substituting for $\cos x = \sqrt{1-\sin(x)^2}$ and then solving the quadratic equation ……

Secondly you could use `mysolut()`. But this will not lead to the general solution as desired in this case. Show your competence using the device!

```
F1 F2 Algebra F3 Calc F4 Other F5 PrgmIO F6 Clean Up
 584   -5.4601   -2.5752   .8231   3.7079
■ tCollect(5·cos(x) = 6·sin(x) - 1)
                         5·cos(x) = 6·sin(x) - 1
■ tCollect(5·cos(x) - 6·sin(x) + 1 = 0)
                 1 - √61·sin(x - tan⁻¹(5/6)) = 0
■ solve(1 - √61·sin(x - tan⁻¹(5/6)) = 0, x)
  x = 2·@n5·π - sin⁻¹(√61/61) + tan⁻¹(5/6) + π or►
…(5/6)+π or x=2*@n5*π+sin⁻¹(√(…
MAIN        RAD EXACT        FUNC 11/30
```

You might remember that solving trig equations often needs some tricks, so let the *TI* show its tricks.

Use `tCollect` to rewrite the equation!

You immediately are offered another – simpler ??? – form.

But obviously the *TI* likes this form, and indeed if you would inspect it closer you would see that in this equation only the sin-function is remaining. The rest is mere child's play for the calculator and we finally receive both families of solutions.

I could not convince *DERIVE* to unhide the general solution of this equation:

```
#25:  SOLVE(5·COS(t) = 6·SIN(t) - 1, t, Real)
```

$$\#26:\quad t = \mathrm{ATAN}\left(\frac{4\cdot\sqrt{15} - 11}{119}\right) + \frac{\pi}{4} \vee t = \frac{5\cdot\pi}{4} - \mathrm{ATAN}\left(\frac{4\cdot\sqrt{15} + 11}{119}\right) \vee t = -\mathrm{ATAN}\left(\frac{4\cdot\sqrt{15} + 11}{119}\right) - \frac{3\cdot\pi}{4}$$

```
#27:  t = -5.46005 v t = -2.57524 v t = 0.823127
#28:  mysolutions(5·COS(t) - 6·SIN(t) + 1 = 0, t, -10, 10, 1)
#29:  [-8.85842, -5.46005, -2.57524, 0.823127, 3.70794, 7.10631, 9.99112]
```

Who of you can?

Patrick West [eDUG discussion list] discovered a useful source for a bundle of TI-92 programs:

http://www.math.armstrong.edu/ti92/

The minimum safety distance of cyclists on busy roads

Karl-Heinz Keunecke, Kiel, Germany

Introduction

With the CBL and CBR it is possible to determine and digitally record physical, chemical, and biological measurements. The parameter, number, and time interval of measurements etc. can be transmitted to the devices by graphical calculators. The calculators can also display results and further analyse data. It is also possible to determine measurements outside the physics lab in the „real world" of school children as the devices are small and require no additional power supply. Here is an example relevant to the school environment:

What is the minimum distance a cyclist can ride behind a vehicle in safety?

The following facts will be investigated and determined in a series of physical experiments:

1. Brake acceleration of the cyclist, using the ultrasonic ranger (CBR).
2. Brake acceleration of the vehicle, using the acceleration sensor and the CBL.
3. Reaction time of the cyclist.

Using the results of the experiments the stopping distances of the car and the cyclist as a function of the initial speed can be determined. The relationship between minimum safety distance and the speed of both vehicles will then be displayed on the calculators using both graphs and tables.

1 Brake acceleration of cyclists, using the ultrasonic ranger CBR.

Before students can start with observations they have to be introduced to the use of the TI-xx software *ranger()* and then they have to learn to measure distances by means of an ultrasonic ranger.

Two points are important:

- The radius of action is restricted to 6 – 8m because echos from farer are too weak to be detected.
- The sensor – fixed on the bike like a front lamp - sends out short ultrasonic pulses at a constant rate when the bike is moving towards the wall. In the period between the emission of sound echos are received from objects which are hit by the sound beam. Because of the transition time of the sound signal from the sensor to the object and back again, the repetition rate and the distance of the object are interdependend.

After this introduction the question may be asked to the class:

"Which distance do you need to stop your cycle?"

Then the pupils can start their observations and investigations in the school yard. It takes time before students get results which show the braking phase correctly. (From my experience the most reliable data are received when the CBR is fixed on the cycle and the distance to a wall or a big object is measured.) Back in the classroom the students present and explain their data to each other using the View Screen.

The observation period was 5s – 10s. The brake time is between 1s and 2s. It makes the the discussion easier when only this period is displayed on the screen. The results could look like figure 1 and figure 2.

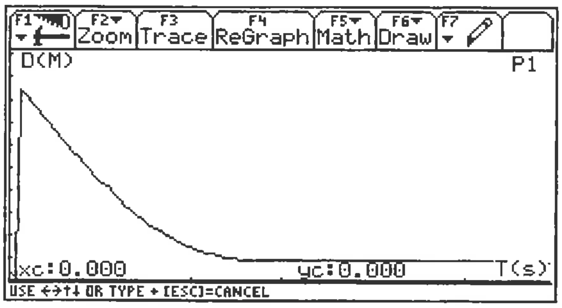


Figure 1
Distance of the cycle to the wall during braking phase

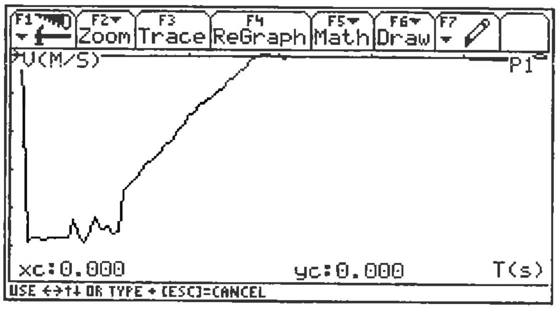


Figure 2
Velocity of the cycle to the wall during braking phase

Students are rather used to displacement-time-diagrams where motion starts at the origin of the co-ordinate system. This can be achieved by an appropriate co-ordinate transformation. The data are stored in the lists l2 (displacement), l3 (velocity) and l4 (acceleration). The list operations

`max(l2)-l2→l2, -l3→l3, -l4→l4`

gives the displays shown in figures 3 and 4.

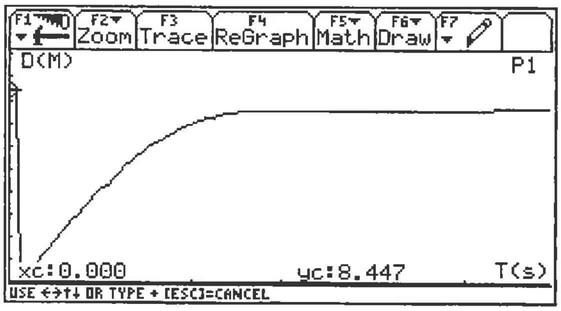


Figure 3
Distance *d(t)* of the cyclist from the wall

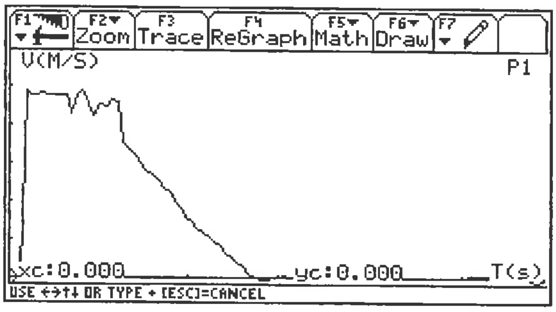


Figure 4
Speed *v(t)* of the cyclist

From these displays one can see that the the cyclist rides first at a constant speed. When he starts to brake the velocity decreases nearly linearly. The slope is then the acceleration. By means of the trace option two points are read from figure 2 and used for calculating the acceleration

$$a = \frac{2{,}743\frac{m}{s} - 1{,}276\frac{m}{s}}{0{,}029s - 3{,}499s} \cong -3{,}37\tfrac{m}{s^2}. \qquad (1)$$

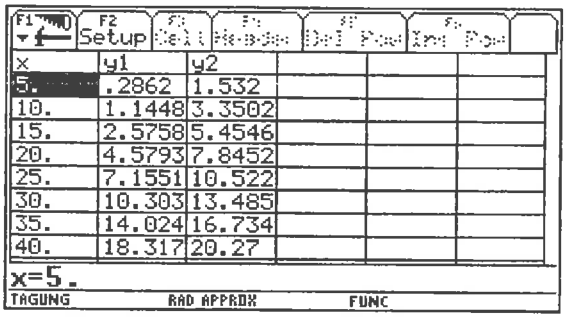
In order to calculate the distance which a cyclist needs to stop from a given speed v_0, a relation between the displacement and the acceleration is required. The displacement and velocity expressed as functions of time are

$$s(t) = v_0 t - \frac{a}{2}t^2 \qquad (2)$$

$$v(t) = v_0 - at \qquad (3)$$

Braking begins at t=0 and ends when v=0. Hence it takes the time $t_{br} = \frac{v_0}{a}$ to stop. Substituted in equation (2) the braking distance is yielded:

$$s(t_{br}) = s_{br} = v_0 \frac{v_0}{a} - \frac{a}{2}\frac{v_0^2}{a^2} = \frac{v_0^2}{2a}. \qquad (4)$$



x	y1	y2
5.	.2862	1.532
10.	1.1448	3.3502
15.	2.5758	5.4546
20.	4.5793	7.8452
25.	7.1551	10.522
30.	10.303	13.485
35.	14.024	16.734
40.	18.317	20.27

Figure 5
$x \approx$ initial speeds, $y_1 \approx$ braking distances,
$y_2 \approx$ safety distances

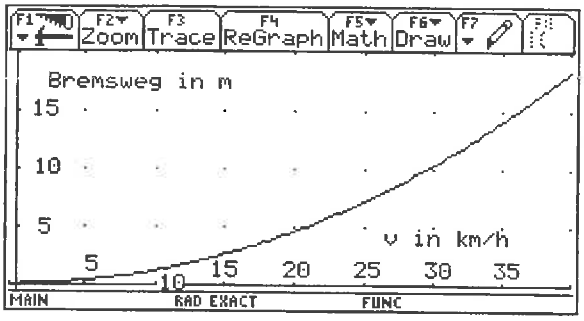


Figure 6
Braking distance versus intitial speed

Students should calculate braking distances for realistic speeds of cyclists from $10\frac{km}{h}$ to $30\frac{km}{h}$. This can be done in a table (figure. 5) or a graphical display (figure 6). It is very important to point out the quadratic relation between distance and initial speed. Students should learn from these experiments that doubling speed results in multiplying the braking distance by 4 – and hopefully will make conclusions for their own behaviour in traffic now and in the future not only using a bike.

2 Brake aceleration of a car with ABS.

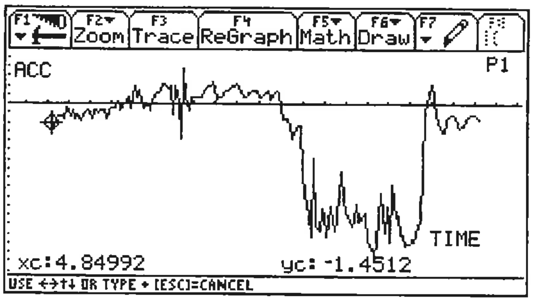


Figure 7
Acceleration of a car which is first speeded up to 50kmh^{-1} and then stopped

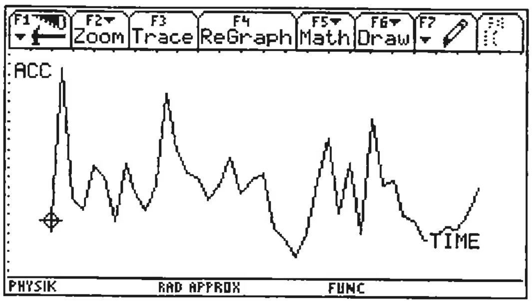


Figure 8
Brake acceleration

The brake acceleration of a motor car can be measured directly by an acceleration sensor with a range of ±5g. In an empty schoolyard the car was speeded up to $50\frac{km}{h}$ and then stopped as quick as possible. The result is shown in figure 7. A positive acceleration is seen as long as the car speeded up. During the braking phase the acceleration is negative. It varies around a constant value. This can be seen from figure 8 where only the braking phase is displayed. In figure 9 the mean acceleration and the standard deviation are calculated:

$$a = (9 \pm 1{,}6)\frac{m}{s^2}. \qquad (5)$$

Using equation (4) the braking distance of a car can be determined.

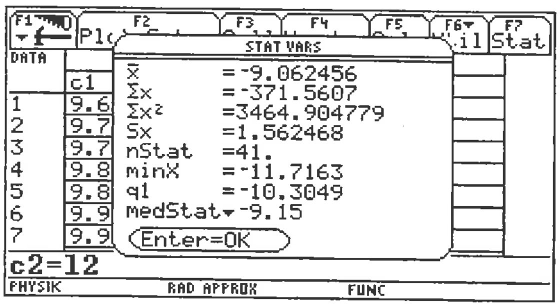


Figure 9
Calculation of the mean acceleration during the brake phase

3 Safety distance of cyclists.

Now the the teacher may ask the question:

What is the minimum distance a cyclist can ride behind a vehicle in safety?

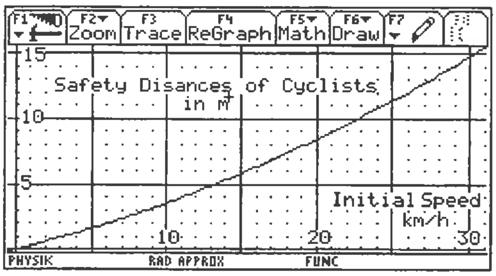


Figure 10
Safety distance versus initial speed

Because the car can stop faster then a cyclist he has to keep a distance between both. The minimum safety distance results from the braking distances of the car and the cyclist and the reaction time of the cyclist:

$$d = v_0 t_{react} + \frac{v_0^2}{2}\left(\frac{1}{a_{cyc}} - \frac{1}{a_{car}}\right) \qquad (6)$$

The safety distances are drawn in figure 10, where the reaction time is assumed to be 1second. It shows that students have to keep a safety distance of 4m at a speed of 10 $\frac{km}{h}$ to 15m at 30 $\frac{km}{h}$.

Milton´s Problems

As I have mentioned in my Letter of the Editor I had the opportunity to visit Colombia in July. Milton Lesmes from the Distrital University of Bogotá took care of me and I enjoyed his hospitality for one week. We had a lot of mathematical discussions and Milton posed some problems and asked how to perform them in DERIVE. I had my notebook with me – as ever – and in the late evenings I sat in my wonderful suite, 11th floor with a marvellous sight to the campus of Javeriana Universidad and tried to find the DERIVE5 answers. Josef

Problem 1 (as Milton posed it to me)

Localizar en el cuadrado [0,1]x[0,1] del plano, en coordenadas rectangulares, todas las parejas de números racionales (p/k,q/k) con denominadores $1 \le k \le r$.

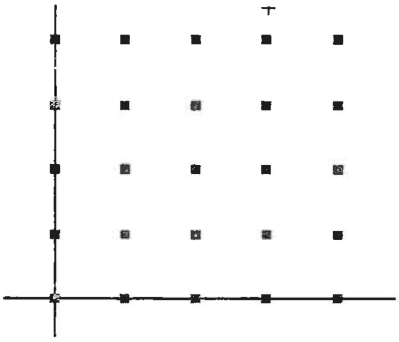
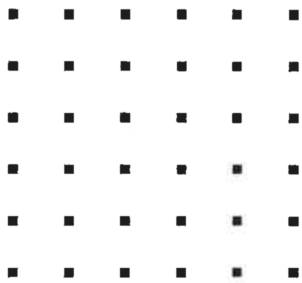
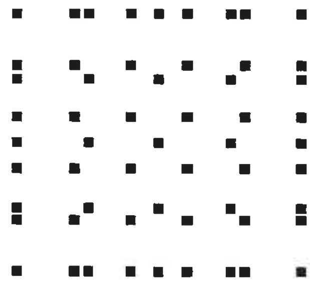
We want to put the pairs of all rational numbers up to a chosen denominator on the set [0,1]x[0,1] of the plane,. We want to investigate the possible patterns appearing.

$$\text{fr(r) := VECTOR}\left(\text{VECTOR}\left(\left[\frac{p}{r}, \frac{q}{r}\right], p, 0, r\right), q, 0, r\right)$$

$$\text{fr(3)} = \left[\begin{bmatrix} 0 & 0 \\ \frac{1}{3} & 0 \\ \frac{2}{3} & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \\ 1 & \frac{1}{3} \end{bmatrix}, \begin{bmatrix} 0 & \frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} \\ 1 & \frac{2}{3} \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ \frac{1}{3} & 1 \\ \frac{2}{3} & 1 \\ 1 & 1 \end{bmatrix}\right]$$

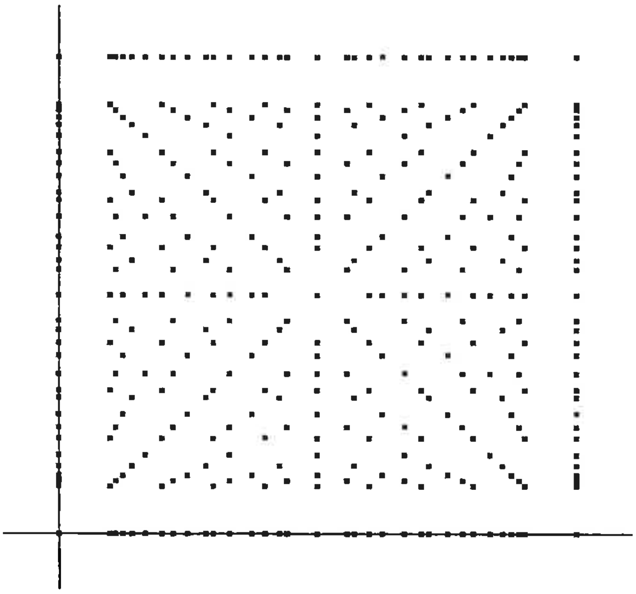
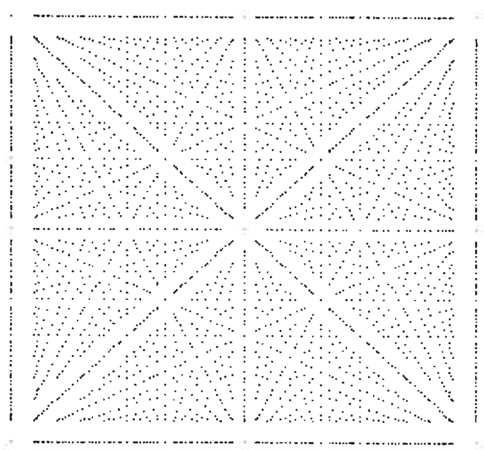
On the next page you can find the visualization of fr(4) and fr(5) and their superimposing.

(*) ACDC Amazing Corner of DERIVERS Curiosity was founded by Alfonso Poblacion, Spain who has been main contributor for this column. It´s is nice that Milton – also Spanish speaking but from another continent – continues with his problems.

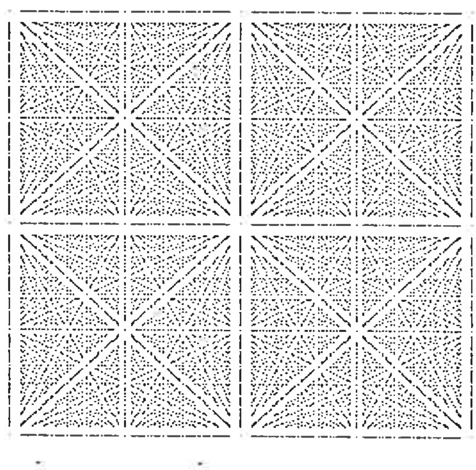
It is a pity that you cannot see the pattern in colours. Then use the powerful VECTOR command to automate the superimposing procedure. You can experiment with increasing values for *r*.

```
VECTOR(fr(r), r, 1, 10)                    VECTOR(fr(r), r, 1, 20)
```

$$\text{fr2(r) := VECTOR}\left(\text{VECTOR}\left(\left[\frac{p}{r}, \frac{q}{r}\right], p, 0, 2\cdot r\right), q, 0, 2\cdot r\right)$$

```
VECTOR(fr2(r), r, 1, 20)
```

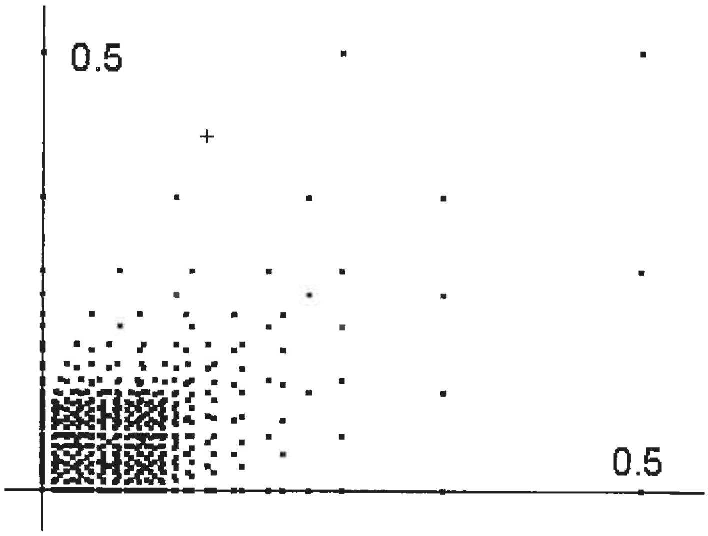


Or you can produce copies of your pattern and produce a nice tiling.

Next question: What happens changing the mapping?
Do you have any idea how the pattern will change squaring the denominator?

$$\text{frr(r) := VECTOR}\left(\text{VECTOR}\left(\left[\frac{p}{r^2}, \frac{q}{r^2}\right], p, 0\right.\right.$$

```
VECTOR(frr(r), r, 1, 10)
```



I missed three points in the screen shot. Which ones?

Problem 2

Find a function, which puts the digits of an integer into reverse order, eg

f(123) = 321
f(420) = 024 = 24
f(1000) = 0001 = 1

Ok, that was it. I got back to my room with specatucalr view and began:

```
rev(x) := REVERSE(STRING(x))
rev(123) = 321
rev(1000) = 0001
```

Looks nice, but doesn't work, because the results are strings, which you can immediately see copying the expressions in the edit line and observing the quotes, or trying to perform a calculation:

```
rev(2000)·rev(40) = 0002·04
```

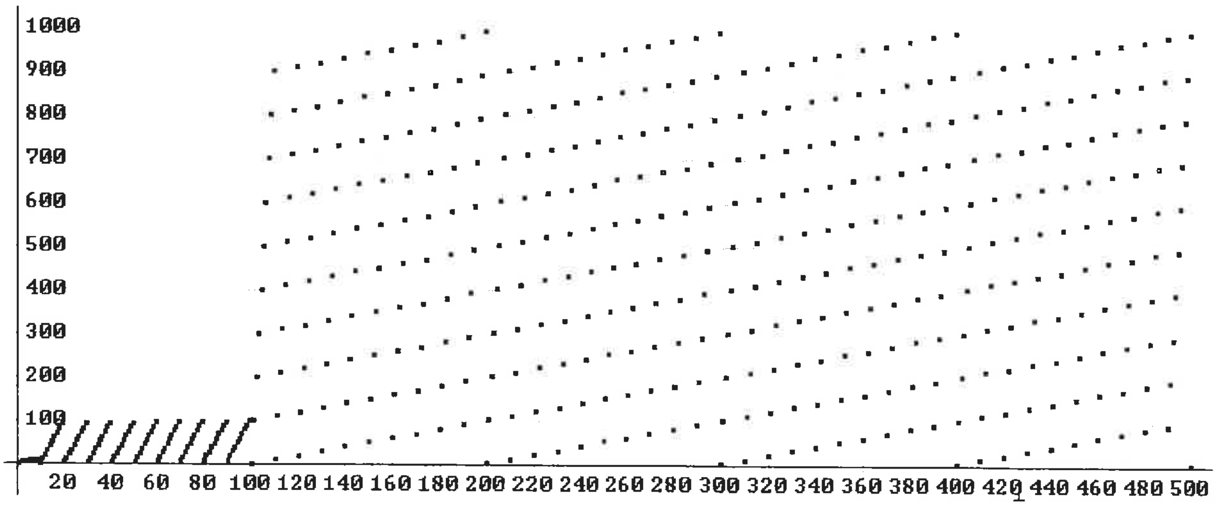
I didn't find an easier way as doing the following:

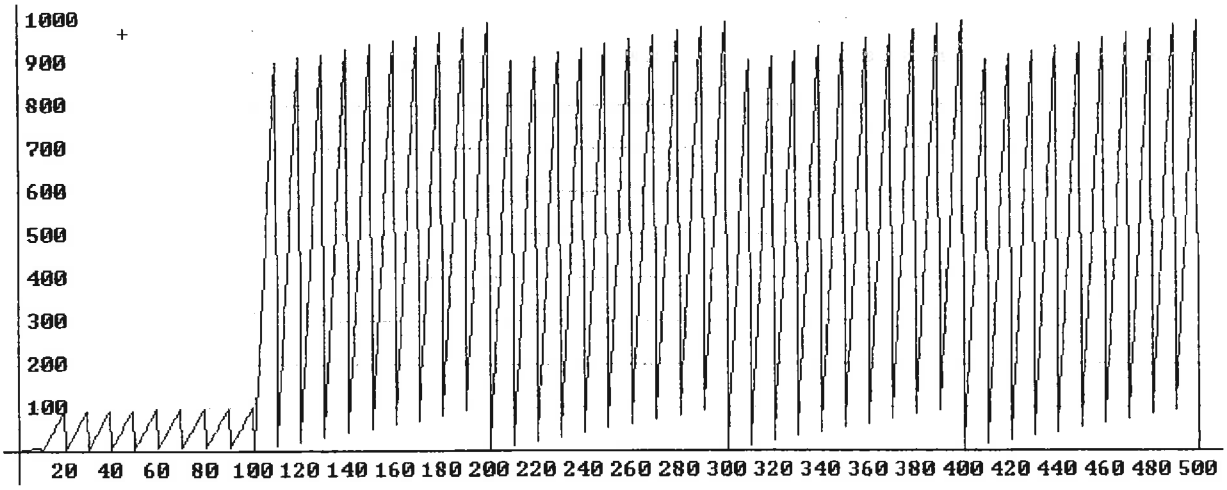
$$\text{CODES\_TO\_NAME}(\text{NAME\_TO\_CODES}(321_1)) = 3$$

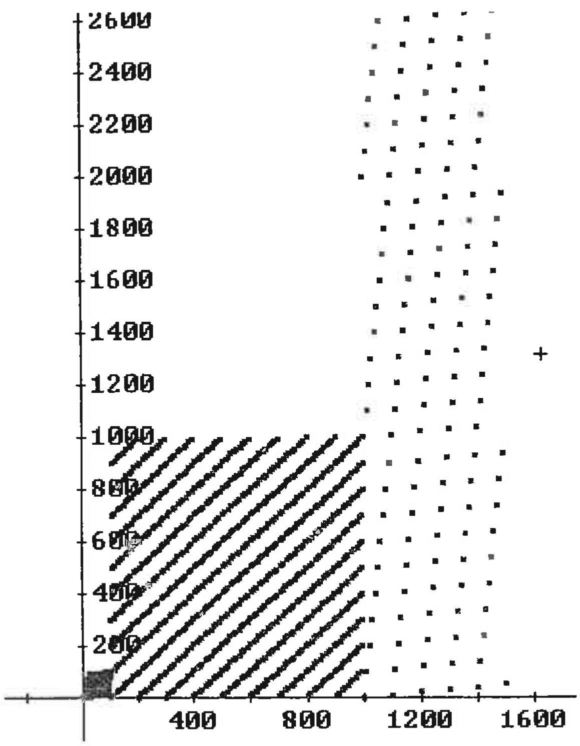
$$\text{revf}(x) := \sum_{i=0}^{\text{DIM}(\text{rev}(x)) - 1} \text{CODES\_TO\_NAME}(\text{NAME\_TO\_CODES}((\text{rev}(x))_{\text{DIM}(\text{rev}(x)) - i}))\cdot 10^{i}$$

```
revf(321) = 123
revf(2000)·revf(40) = 8
```

and finally I wanted to the graph of my `revf`-function for the first 500 and for the first 1500 integers: `TABLE(revf(x), x, 1, 500)`. (Points first Not Connected and then Connected).







Problem 3

Start with any number, say 128 and form the following sequence:

$$1^2 + 2^2 + 8^2 = 69$$
$$6^2 + 9^2 = 117$$
$$1^2 + 1^2 + 7^2 = 51$$
$$\ldots$$
$$\ldots$$

Then you will end either with 1, 1, 1, … or with a loop starting with 16: 16, 37, 58, 89, 145. 42. 20, 4, 16, …

How can you do this with *DERIVE*?

This was a very welcome opportunity to practise my programming skills in *DERIVE* again using the new features of manipulating with strings.

```
                                           2
next_nr(x_) := Σ(VECTOR(CODES_TO_NAME(k) , k, NAME_TO_CODES(x_)))

                                    next_nr(128) = 69

                                    next_nr(69) = 117

sixteen(n, seq, nn_, k_, kk_) :=
  Prog
    seq := [n]
    nn_ := n
    Loop
      If nn_ = 16 ∨ nn_ = 1 exit
      kk_ := Σ(VECTOR(CODES_TO_NAME(k_)^2, k_, NAME_TO_CODES(nn_)))
      seq := APPEND(seq, [kk_])
      nn_ := kk_
    RETURN seq
```

`next_nr(x_)` was only a test to find the "formula" for the core activity.

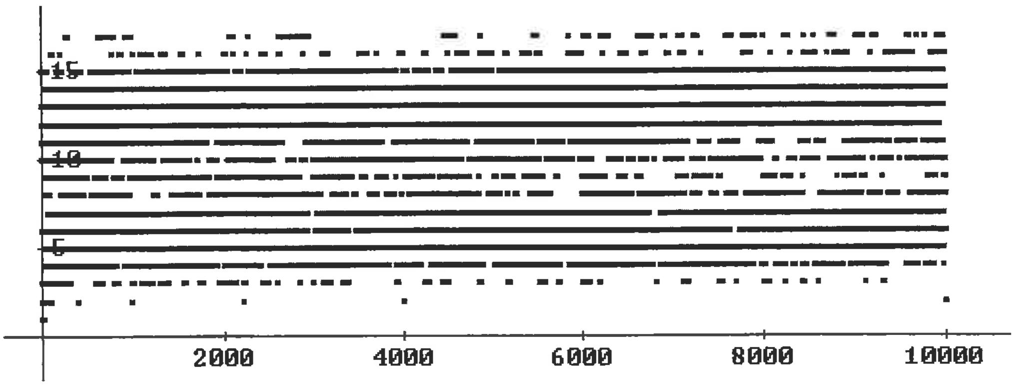
```
sixteen(128) = [128, 69, 117, 51, 26, 40, 16]
sixteen(345) = [345, 50, 25, 29, 85, 89, 145, 42, 20, 4, 16]
sixteen(1001) = [1001, 2, 4, 16]
sixteen(100000) = [100000, 1]
```

You will imagine what is the next question to be answered: Is the conjecture true for a larger set of numbers and how long are the sequences until they run into the loop?

```
VECTOR(DIM(sixteen(k)), k, 101, 200)
[4, 11, 3, 12, 4, 9, 11, 11, 5, 4, 14, 16, 5, 12, 13, 10, 5, 14, 10, 11, 16, 13, 13,
 12, 14, 13, 14, 7, 4, 3, 5, 13, 6, 4, 12, 11, 11, 12, 6, 12, 12, 12, 4, 13, 5, 12,
 14, 12, 7, 4, 13, 14, 12, 5, 5, 4, 13, 13, 12, 9, 10, 13, 11, 12, 4, 9, 4, 5, 15,
 11, 5, 14, 11, 14, 13, 4, 15, 13, 6, 11, 14, 7, 12, 12, 13, 5, 13, 5, 13, 5, 10,
 4, 6, 7, 12, 15, 6, 13, 12, 3]
```

And finally I produced a plot of number versus length of the sequence for the first 10000 numbers receiving a chaotic pattern. It seems that we are not exceeding 17 elements until reaching 16 or 1.

```
TABLE(DIM(sixteen(k)), k, 1, 10000)
```



<u>JYRI-MATTI.AHOKAS@TURVA.FI</u>
Subject: Re: Pi-guestion

I am interested to know how does Derive 5 calculate Pi`s value. I am not talking about digits.
Thanks Jyri-Matti Ahokas

Hello Jyri-Matti Ahokas from Finland
Derive uses the following formula for pi that Ramanajun disçovered:

```
pi_(n):=4/SUM((-1)^m*(1123+21460*m)*PRODUCT(2*r-1,r,1,m)*PRODUCT(2*r-1,r,
        1,2*m)/(882^(2*m+1)*32^m*m!^3),m,0,n)
```

Each increase in n by 1 gives about 4 more digits of accuracy.
Aloha, Albert D. Rich

$$pi\_(n) := \frac{4}{\displaystyle\sum_{m=0}^{n} \frac{(-1)^m\cdot(1123 + 21460\cdot m)\cdot\left(\prod_{r=1}^{m}(2\cdot r - 1)\right)\cdot\prod_{r=1}^{2\cdot m}(2\cdot r - 1)}{882^{2\cdot m + 1}\cdot 32^m\cdot m!^3}}$$

pi_(10) = 3.14159265358979323846264338327950288419716939937510582097494

Concerning 3D-Vectorfields and Arrows

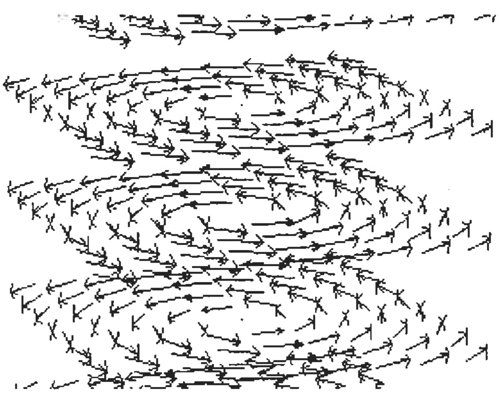
Ludvig Strigéus & Josef

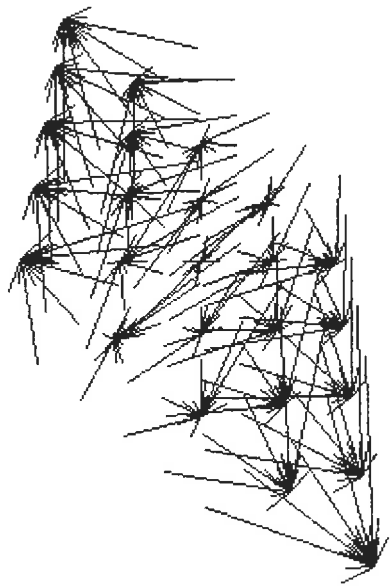
In DNL#43 page 5 I gave a presentation of a 3D-vectorfield. But a vector without an arrowhead isn't a vector! Ludvig Strigéus, a gifted Student of David Sjöstrand (Elof Lindälvs Gymnasium in Kungsbacka, Sweden) produced a function to show also the arrowheads for three dimensioned vectorfields. He sent a mail together with the DERIVE-file `vectorfield3.mth`.
But I also promised to work further to make my vectors arrowheaded, and almost at the same time I had my arrows ready. I accomplished my `v_field.mth` from DNL#43.
You can compare our both results.

First the student:

$$\text{VECTORFIELD3}\left(-4,\ 4,\ 2,\ -4,\ 4,\ 2,\ -4,\ 4,\ 2,\ x,\ -y,\ \frac{z}{2},\ x,\ y,\ z\right)$$

$$\text{VECTORFIELD3}\left(-4,\ 4,\ 1,\ -4,\ 4,\ 1,\ -4,\ 4,\ 1,\ -\frac{y}{\sqrt{x^2+y^2+z^2}},\ \frac{x}{\sqrt{x^2+y^2+z^2}},\ 0,\ x,\ y,\ z\right)$$



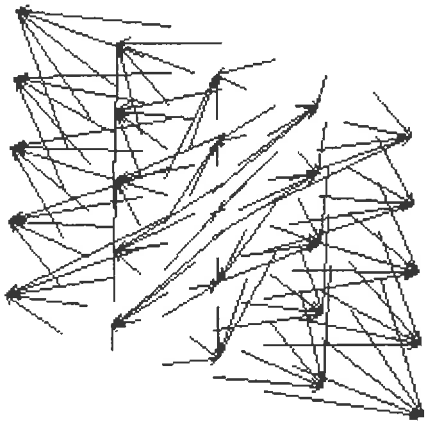


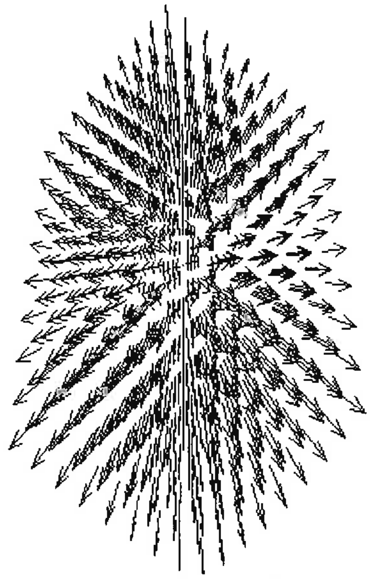
And then my result (the last argument is the length of the barbs):

$$\text{vec3\_arr2}\left(\left[x,\ -y,\ \frac{z}{2}\right],\ -4,\ 4,\ 2,\ -4,\ 4,\ 2,\ -4,\ 4,\ 2,\ 0.5\right)$$

$$\text{vec3\_arr2}\left(\frac{[x,\ y,\ z]}{\sqrt{x^2+y^2+z^2}},\ -4,\ 4,\ 1,\ -4,\ 4,\ 1,\ -4,\ 4,\ 1,\ 0.5\right)$$

NotationDigits := 3





And then I found a mail from Nurit Zehavi in my mailbox concerning Sergey Birjukow´s tool to produce 2D-arrows and Albert Rich´s reaction.

Dear Josef,

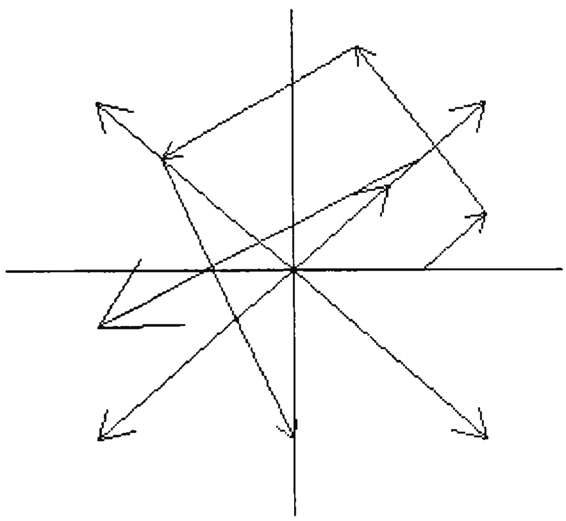
Many thanks for editing so nicely our paper for DNL.
I am glad you used Birjukow's tool for plotting arrows in the last DNL issue. I am currently producing Derive-based activities for teaching Vectors in high school. I found the file from 1994 and copied three lines, and the credit line of course, and it is qiuite nice to see the arrows. I'll ask Al and Theresa to consider including it in the utility file graphic.mth.

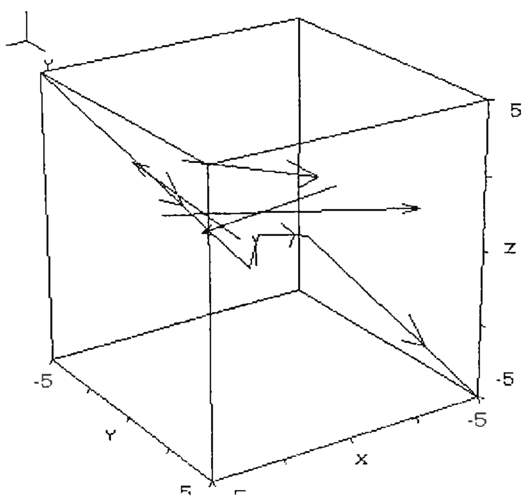
Dear Nurit,

In response to your "wish" for an Arrows package for Derive 5, I sent Biryukov's arrows.dfw file to our Derive plotting expert, Peter Schofield.
He liked the idea so much that he extended its capabilities, and turned it into a full-fledged package for the Derive 5 Users directory. It is attached. Please let me know if you have any comments or suggestions.

Aloha, Albert

I add two examples (one in the plane and one in space) using Peter Schofield´s utility ARROWS.MTH which will be included in the next update of the User Package. You can find it on the diskette of the year.





In this DNL you can find a contribution applying a tool for finding a family of solutions of an intersection problem. As this is a very common problem, I´d like to give an extra note how to use DERIVE´s functionality to tackle this task. The question was:

I´d like to find the intersection points of a family of lines with the upper half of an ellipse. And I´d prefer to have all intersection points as the result of only one calculation so that I can refer to this set of points as a whole in the next calculation steps. The problem is:

$$y = \frac{4\cdot\sqrt{(25 - x^2)}}{5}$$

$$\left[y = \sqrt{3}\cdot x - 2\cdot x + 3\cdot\sqrt{3} - 6,\ y = -\frac{\sqrt{3}\cdot x}{3} - \sqrt{3},\ y = -x - 3,\ y = -\sqrt{3}\cdot x - 3\cdot\sqrt{3},\ y = -\sqrt{3}\cdot x - 2\cdot x - 3\cdot\sqrt{3} - 6,\ y = \infty,\ y = -\infty,\ y = \sqrt{3}\cdot x + 2\cdot x + 3\cdot\sqrt{3} + 6,\ y = \sqrt{3}\cdot x + 3\cdot\sqrt{3},\ y = x + 3,\ y = \frac{\sqrt{3}\cdot x}{3} + \sqrt{3},\ y = -\sqrt{3}\cdot x + 2\cdot x - 3\cdot\sqrt{3} + 6\right]$$

I tried a not very satisfying way, squaring the equations and solving for x, receiving two solutions, rejecting one, ...

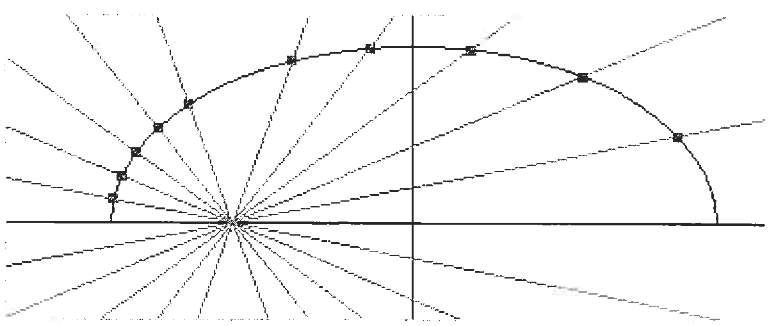
Try this:

$$\text{ellipse} := y = \frac{4\cdot\sqrt{(25 - x^2)}}{5}$$

$$\text{gs} := \left[y = \sqrt{3}\cdot x - 2\cdot x + 3\cdot\sqrt{3} - 6,\ y = -\frac{\sqrt{3}\cdot x}{3} - \sqrt{3},\ y = -x - 3,\ y = -\sqrt{3}\cdot x - 3\cdot\sqrt{3},\ y = -\sqrt{3}\cdot x - 2\cdot x - 3\cdot\sqrt{3} - 6,\ y = \infty,\ y = -\infty,\ y = \sqrt{3}\cdot x + 2\cdot x + 3\cdot\sqrt{3} + 6,\ y = \sqrt{3}\cdot x + 3\cdot\sqrt{3},\ y = x + 3,\ y = \frac{\sqrt{3}\cdot x}{3} + \sqrt{3},\ y = -\sqrt{3}\cdot x + 2\cdot x - 3\cdot\sqrt{3} + 6\right]$$

```
gs_new := SELECT(RHS(k) < ∞ ∧ RHS(k) > -∞, k, gs)
xvals(curve, fam) := VECTOR(RHS(SOLVE(RHS(k_) = RHS(curve), x)), k_, fam)
intspts(curve, fam) := VECTOR([k_, SUBST(RHS(curve), x, k_)], k_, xvals(curve, fam))
intspts(ellipse, gs_new)
```

-4.95685	0.524337
-4.82367	1.05289
-4.58870	1.58870
-4.23076	2.13175
-3.71689	2.67548
-2.01951	3.65920
-0.714285	3.95897
0.930172	3.93017
2.76887	3.33066
4.35165	1.96986

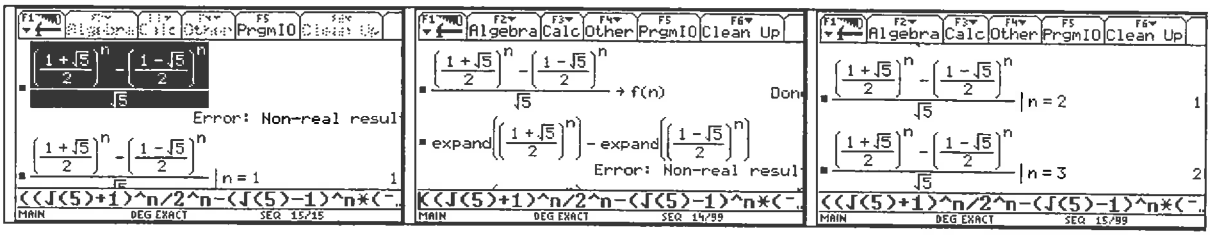


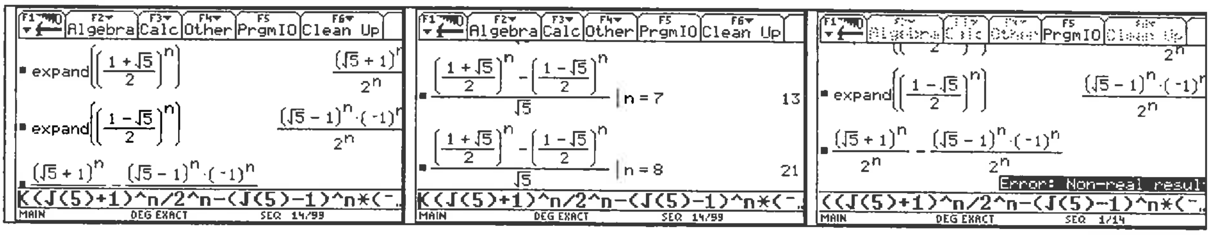
Some Experiences of TI-89/92ers

The Fibonacci-Sequence explicitly on the TI

reported by Klaus Sibum, Gifhorn, Germany

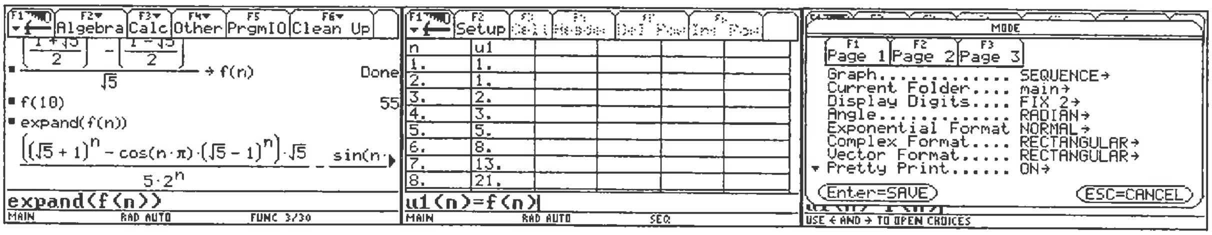
We wanted to check the explicit formula for the Fibonacci Sequence and one colleague came across this strange behaviour of our TI-92+ (OS 2.05). All efforts to use the formula result in the error message `Non-real result`. But as you can see it is no problem to define the function and then ask for a special function value. We are also unable to work in the sequence editor.
By the way, there are no problems with DERIVE!

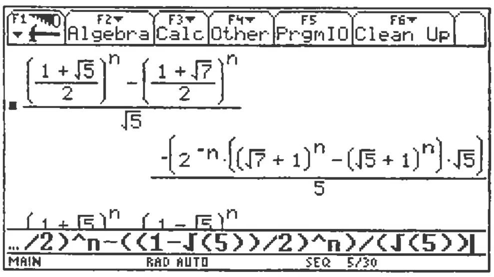




How to overcome this problem:

Problems like this happen very often in class and in the very first moment one might not have any idea, what´s wrong? The secret lies – how very often – in an inappropriate setting of the `Complex Format` in the Mode-Screen. In this formula one faces a power of a negative base which can be a complex number – if n is not an integer. Additionally I´d recommend to work in `RADIAN`-Mode.





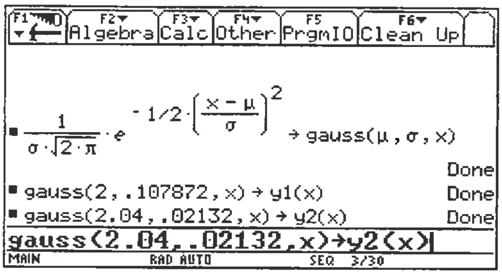
I switch back to the `Complex Format REAL` and now I have no problems with the formula containing two positive bases of the powers, but the Fibonacci-formula again is not accepted.

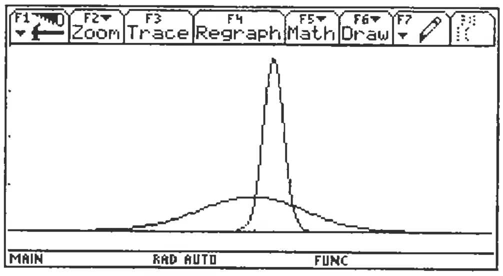
Josef

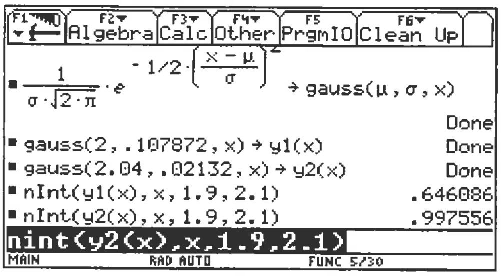
Surprising Results with the Normal Distribution

reported by Tania Koller, St.Poelten, Austria

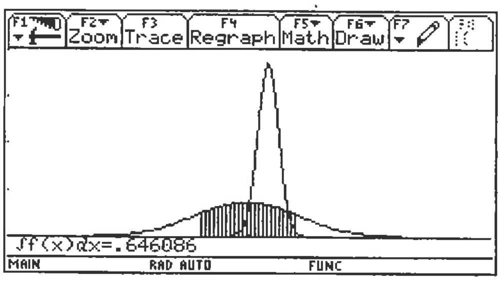
Working in class we came across a strange behaviour of the TI-89/92/92+ family dealing with the normal distribution applying the features of the Graph-Window.

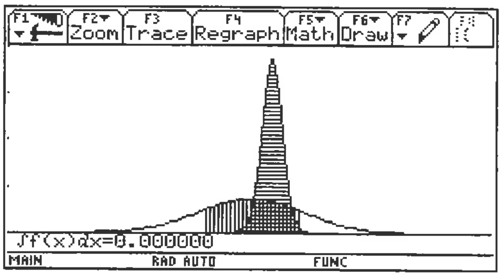


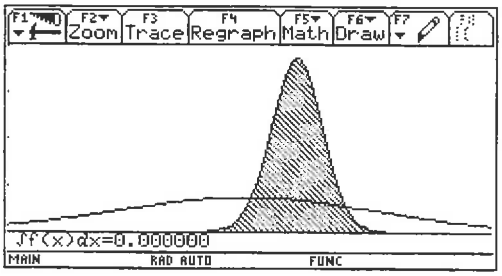




As you can easily follow we tackle a very common probability theory task in a numerical way first and then we want to visualize the numerical results by plotting the density functions and inspecting the respective areas under their graphs.



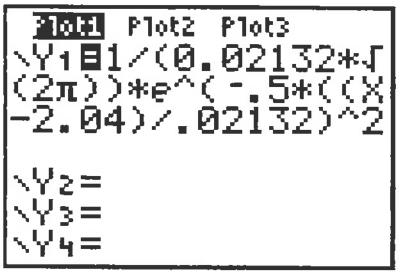


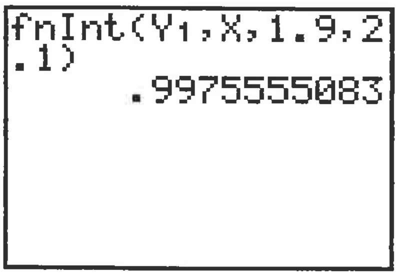


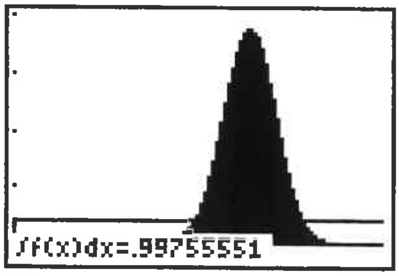
We all were very surprised to receive the answer 0 for one of the areas. We changed the Window-values to enlarge the area, but the TI insisted in making the area = 0???

Any explanation for this strange behaviour? How can we trust in the future in the device´s answers at similar occasions?

Doing quite the same on the TI-83, we have no problems to find the correct answer numerically and graphically as well, as the screenshots given below are showing:





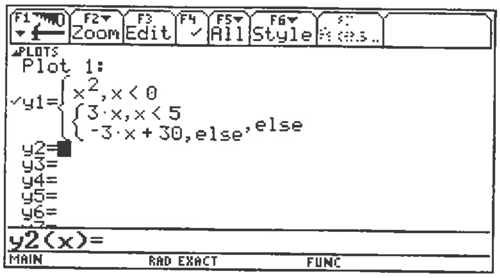


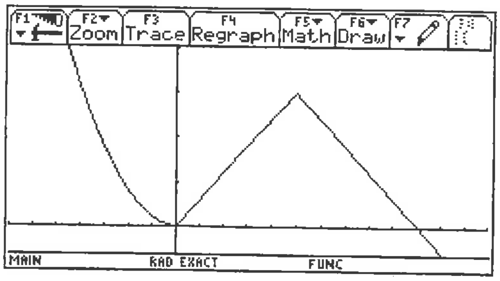
Take Care working with Piecewise Defined Functions

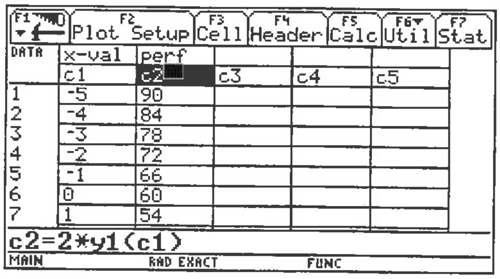
reported by Thomas Himmelbauer, Vienna, Austria

At the occasion of the last ACDCA-Seminar we worked out a collection of sample assessment examples for using technology in maths education. (There is a special project group which focuses on this objective, which you met on page 25).

And there we came across another strange behaviour working with piecewise defined functions using them in the DATA-Editor. Follow the screen shots, please:



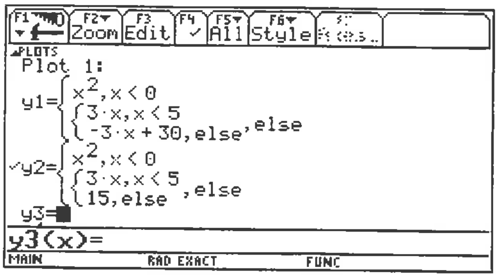


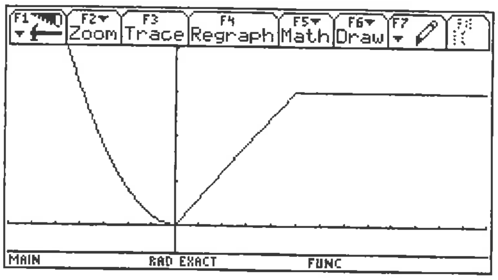


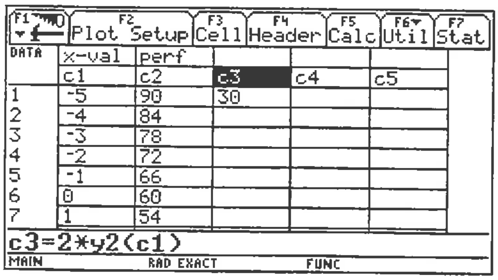
The graph looks quite good, and it is ok.

But the values in column 2 should show the double function values. One glance will say us, that 2*y1(0) cannot be 60!! So this is wrong!!

Take another example which is much more interesting:

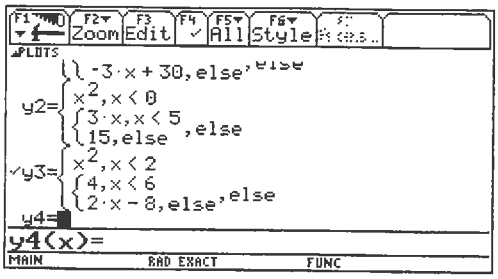


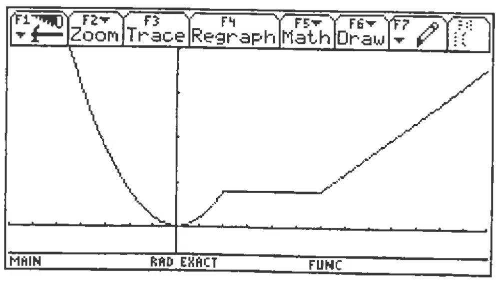


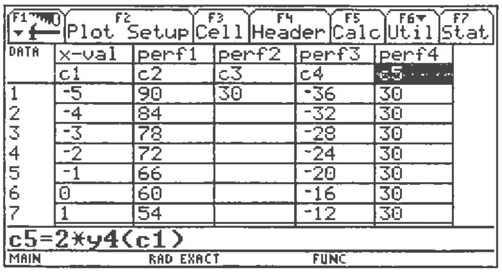


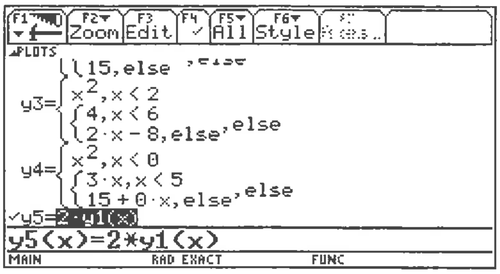
In this case we are not even tempted to believe the TI's output, because we are only delivered one result – which is also wrong!!).

We made some other attempts and could not find either satisfying results nor the underlying "rules"!









As I made a very similar experience preparing my presentation for the Klagenfurt Conference – extending the "From Pole to Pole" – contribution from DNL#41 – I sent a mail to David Stoutemyer asking for an advice how to overcome this obstacle. This was his answer:

Dear Josef,
I have verified the behavior that you observed and eliminated several possible causes, but it will take me awhile to identify the cause.

Thanks for sending the example. We always want to know about such things.

-- aloha, David

...and some hours later there was another message in my mailbox:

Hi Josef,
The cause of the behavior is:

a) DERIVE simplifies expressions such as [-1, 0] <= 0 to [-1 <= 0, 0 <= 0], which simplifies to [true, true] when it is the predicate of a DERIVE IF statement; whereas

b) the TI-89 and TI-92+ simplify expressions such as {-1, 0} <= 0 to false, so the WHEN statement takes the x^2 branch rather than the x^2/8 branch.

In this case the DERIVE approach is more useful. However, there are cases where it is important to decide that a list is never comparable to a scalar -- particularly in programming applications.

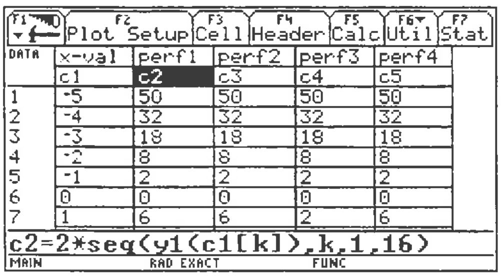
To do this application for lists, you could write a multi-line function that iterates the existing xcol(...) over the elements of a list.

I hope this helps.

I was not very happy with this advice, because it was my intention to specially use the "spreadsheet like" approach in the whole paper and not to change the method for the very last step. As the problem lies in treating the whole list incorrectly, so let´s leave the list and address the single elements of the lists. And this is the solution of the problem – not very elegant, but it works in any case:

Set up a new list, using the sequence-command, and address the elements by their numbers written between brackets. The entry line could read:

```
c2 = seq(2*y1(c1[k]),k,1,16)
```



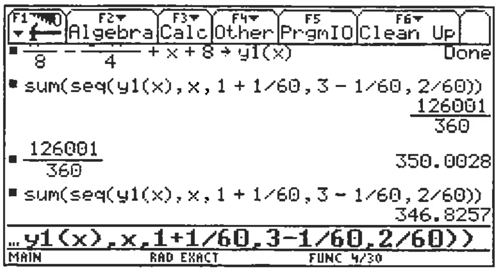
What I am still missing is a system variable giving the number of covered cells in column 1 to be more flexible.

Maybe that this system variable exists, but I don´t know?

Josef

Thomas found another strange thing evaluating a simple sum:

He defined a cubic and wanted to perform the sum. That seems to be no problem, but comparing the results in EXACT-Mode approximated with the result given after direct approximation he found a difference which seems to be too big for being accepted??



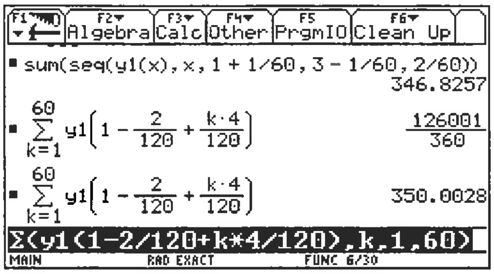
$$y1(x) := \frac{x^3}{8} - \frac{5\cdot x^2}{4} + x + 8$$

$$\text{VECTOR}\left(y1(x),\ x,\ 1 + \frac{2}{120},\ 3 - \frac{2}{120},\ \frac{2}{60}\right)$$

$$\Sigma\left(\text{VECTOR}\left(y1(x),\ x,\ 1 + \frac{2}{120},\ 3 - \frac{2}{120},\ \frac{2}{60}\right)\right) = \frac{126001}{360}$$

$$\Sigma\left(\text{VECTOR}\left(y1(x),\ x,\ 1 + \frac{2}{120},\ 3 - \frac{2}{120},\ \frac{2}{60}\right)\right)$$

350.00277

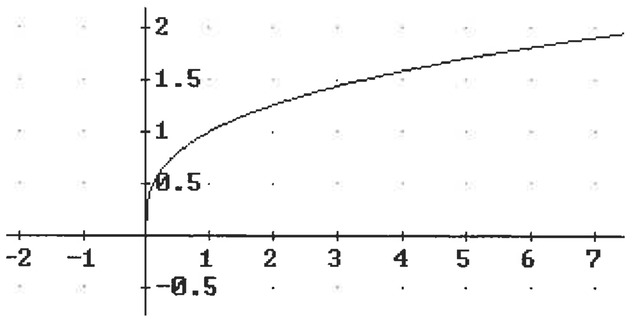


Please compare with the respective *DERIVE* output. Finally I tried to change the calculation order in rewriting the sum, as given below – now I got matching results???

Josef

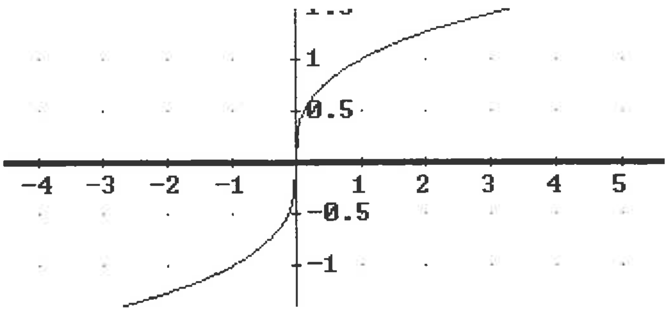
Question: How can I force *DERIVE* to plot the third root of x also for negative x-values?

#1: $y = x^{1/3}$



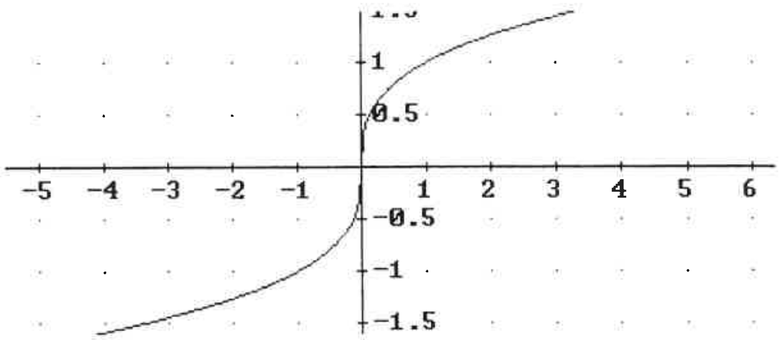
As mentioned in some cases before power functions can lead into troubles, because of possible complex roots. So make sure to set the appropriate Simplification Settings:

```
Declare Simplification Settings Branch

#2:    Branch := Real
```



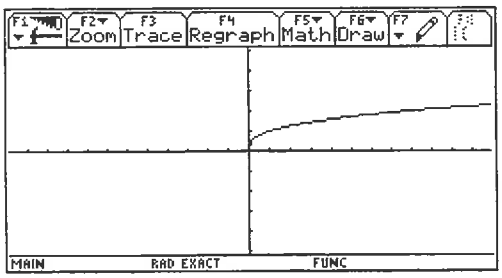
This is quite better now, but not good enough. We can see the branch of the graph even for negative *x*-values, but we are paying for that with an extraordinary thick *real*-axis. Add another setting:

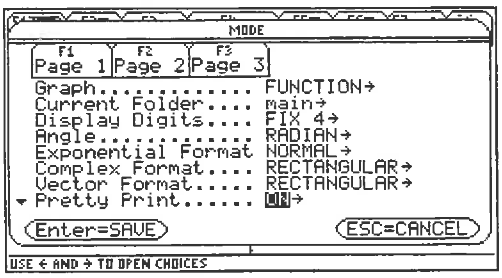
```
Declare Variable Domain

#3:    y :∈ Real
```

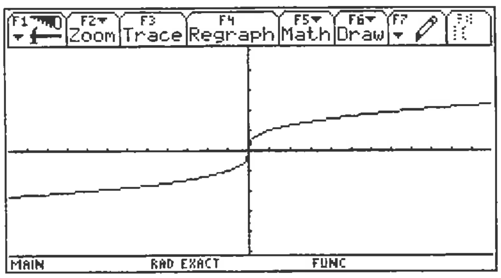


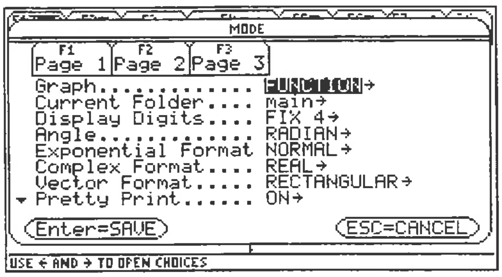
And now everything is ok.

Turning to the TI, we can face the same problem and - if necessary - changing the `COMPLEX FORMAT`, we find a nicely plotted root – function.









VISIT-me

Vienna International Symposium in Integrating Technology in Math Education

Dear friends,

don´t forget to send your submission for our Conference in time. I´d like to remind you on the deadline which is 10 February 2002.

For further information go to

www.acdca.ac.at/visit-me-2002

It is a pleasure for us to announce our four excellent keynote speakers

Bruno Buchberger, Austria

Miguel de Guzman, Spain

Albert D Rich, USA

Hans-Georg Weigand, Germany

To accomplish the collection of pictures of the *DERIVE* makers in the DNL, I am happy to present Albert D Rich, Co-Author of *DERIVE*, who will give a plenary talk.

Find all the DERIVE and TI-files on the following web sites

http://www.acdca.ac.at/t3/dergroup/index.htm
http://www.bk-teachware.com/main.asp?session=375059