THE DERIVE - NEWSLETTER #48

THE BULLETIN OF THE

DERIVE®

USER GROUP

+ TI 92

Contents:

December 2002

[1] **Programmieren mit *DERIVE* 5**, Josef Böhm, 64 pages + Diskette, bk-teachware SR-32, ISBN 3-901769-50-1

[2] **Neue Aufgaben für das Unterrichten mit Derive & TI-89/92/92+/Voyage 200 – Band 1**, Josef Böhm, 64 pages + Diskette bk-teachware SR-33, ISBN 3-901769-51-X

[3] **Neue Aufgaben für das Unterrichten mit Derive & TI-89/92/92+/Voyage 200 – Band 2**, Josef Böhm, 63 pages + Diskette bk-teachware SR-34, ISBN 3-901769-52-8

[4] **VISIT-ME 2002**, CD-ROM, bk-teachware SR-31, ISBN 3-901769-49-8

Visit-me

The CD contains more than 1600 pdf-pages, including all plenary lectures (Bruno Buchberger, Miguel de Guzman, Albert Rich, Hans-Georg Weigand), all lectures and workshops, all accompanying files, the complete list of participants, photo galleries and a lot of extras.

You can order your copy of the VISIT-ME-2002 Proceedings very easily. Use the electronic order form at:

`https://secure.bk-teachware.com/visit-me-2002d/` Formular in Deutsch

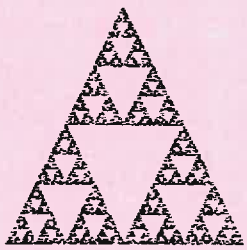
`https://secure.bk-teachware.com/visit-me-2002e/` Order form in English

There are a lot of new publications on `http://www.acdca.ac.at`

Recommended websites: `http://`

`www.mathematik.uni-kassel.de/~koepf`	Wolfram Koepf's homepage
`www.gwdg.de/~cais`	full with CAS-Information
`us.uvm.dk/gymnasie/almen/projekter/caseopgcas.pdf`	A collection of Danish assessment problems including CAS
`shorl.com/shorl.com/fypivovitaho`	An online Linear Algebra Toolkit
`www.stmaurice.ch/pfrache`	Pierre Frachebourge's homepage - (mostly Cabri, in French)
`sunsite.univie.ac.at/Spreadsite/Welcome.html`	Spreadsheet-Applications
`www.landauer/griechische-mathematik/index.htm`	
`www.zahlenjagd.at/`	Hunt for Numbers (Deutsch, English, French)
`btmdx1.mat.uni-bayreuth.de/smart/intro.htm`	Aufgabensammlung
`www.tmg.musin.de/mathematik.htm`	Linksammlung
`b.kutzler.com`	Find information about Math- and CAS-Conferences worldwide

A Spreadsheet – Christmas Tree



Dear DUG Members,

While waiting for the last promised contribution for this *DNL* I write the last letter of the Editor of 2002. I hope that this letter finds you well and in a good mood to enjoy the many contributions of this issue.

What I am waiting for? Elvira Malitte gave a fine presentation at VISIT-ME-2002 in Vienna – in German. I asked her to allow an English translation. Her "Moon Story" is really attractive for our pupils and shows many aspects of using technology in maths education. If you will not find it in this issue, it arrived a bit too late and it will be published in the next one.

Peter Lüke Rosendahl did not only send a great paper on doing geometry with *DERIVE* – to be published in the next *DNL* -, in his opinion I should exchange my picture in the Editor´s Letter and he took a shot during my workshop at VISIT-ME.

The dark spots on my shirt demonstrate that it was very hot in the overfull PC-Lab – that I was so nervous!!

One of the many "highlights" in this issue is doubtless Don Phillips' "Hypotheses Testing". Incited by the Statistics Tools presented earlier, he createded an excellent file, which will be accomplished by his ANOVA-file in one of the next *DNLs*. I tried to accompany his solutions with the *TI*´s way to tackle the problems.

As you can see in this *DNL* – again – a lot of DUGers take the contributions not only as is, but feel inspired to find better solutions, or to use them for their own work like Don Phillips:

"... I'm sorry I didn't get back to you right away, but your descriptive statistics programs were so interesting that I plunged right in to adopting them for my own work. .... And your work inspired me to create some programs for hypothesis testing with DERIVE. ....I was able to do this because I studied your plotting routines to learn how. Thank you very much! "

Dear colleagues, dear friends,

Many thanks to all contributors and supporters of the *DERIVE* User Group. My wife and I wish you and your families a Merry Christmas and a Happy, Successful, Peaceful and Healthy New Year 2003.

Like every year you will find the last issue of the year the Renewal Form Please renew your membership as soon as possible in order to make our administration easier. (My wife will be very grateful for that.)
Please use payment by CreditCard, the rates for charging a cheque are extremely high (sometimes more than 50% of the amount). Many thanks for your cooperation.

2003 Renewal

Wir bitten unsere österreichischen und deutschen Mitglieder, den beigelegten Zahlschein zur Begleichung des Mitgliedsbeitrags für 2003 zu verwenden. Recht herzlichen Dank.

The *DERIVE-NEWSLETTER* is the Bulletin of the *DERIVE & TI-92 User Group*. It is published at least four times a year with a contents of 44 pages minimum. The goals of the *DNL* are to enable the exchange of experiences made with *DERIVE* and the *TI*-89/92/Voyage 200 as well as to create a group to discuss the possibilities of new methodical and didactical manners in teaching mathematics.

As many of the *DERIVE* Users are also using the symbolic *TI*-devices the *DNL* tries to combine the applications of these modern technologies.

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Contributions:
Please send all contributions to the Editor. Non-English speakers are encouraged to write their contributions in English to reinforce the international touch of the *DNL*. It must be said, though, that non-English articles will be warmly welcomed nonetheless. Your contributions will be edited but not assessed. By submitting articles the author gives his consent for reprinting it in the *DNL*. The more contributions you will send, the more lively and richer in contents the *DERIVE & TI-92 Newsletter* will be.

Next issue: March 2003
Deadline 15 February 2003

Preview: Contributions waiting to be published

Tanz der Wallace-Geraden / Dance of Wallace-Lines, R. Baumann, GER
Finite continued fractions, Detection of Periods, St. Welke, GER
Kaprekar´s "Self numbers", R. Schorn, GER
Some simulations of Random Experiments, J. Böhm, AUT
Wonderful World of Pedal Curves, J. Böhm
Another Task for End Examination, J. Lechner, AUT
Tools for 3D-Problems, P. Lüke-Rosendahl, GER
Penrose Inverse of a Matrix, K. Schmidt, GER
Putzer´s Method for the Calculation of e^{At}, F. Fernández, ARG
ANOVA with DERIVE/TI, M. R. Phillips, USA
Hill-Encription, J. Böhm
CAD-Design with DERIVE and the TI, J. Böhm
Sierpinski-Tetrahedrons and Octahedrons, H.-R. Geyer, GER
Ways to Write with TI and DERIVE, M. Lesmes-Acosta
Avoiding Convolution and Transforming Methods, M. Lesmes-Acosta
2D- & 3D-Visualization of Moebius Transformations, T. Comar, USA
and
Setif, FRA; Vermeylen, BEL; Leinbach, USA; Aue, GER; Koller, AUT, Keunecke, GER, ...and others

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I have tried ,using Mathematica 4.0 ,the sequence:
196+691=887 887+788=.... ,problem 2 (Milton's riddle #2) *DNL* 47 p 39.
After 200 000 additions,I got a number of 82927 digits and it was still not a palindrom !

Enric Puig, Barcelona, Spain enricpuig@teleline.es

Hi, everyone!
In his latest Titbits (*DNL* #46), Johann Wiesenbauer uses "the undocumented `SHIFT(n,c)`" (you see, Johann, you do have readers...). He gives a brief explanation of its use. Could we learn more about it? Are there any further hidden treasures in Derive's backyard ?
Regards Enric

(Maybe, that you will discover another undocumented feature on page 32, Josef.)

Valeri Anisiu, Romania anisiu@math.ubbcluj.ro

Hello to all Derivers,

last year in this forum several solutions were given for the sum of the series $\sum_{k=0}^{\infty}\frac{u(k)}{k!}$, $u(k)$ being a polynomial. In the meantime, *DERIVE* has a built-in algorithm for this series.
The purpose of this note is to show that a more general series (which appears frequently in applications) can be summed very easy, and should be built-in too.

The series is $\sum_{k=0}^{\infty}\frac{u(k)\cdot x^k}{k!}$ (it is always convergent).

```
sumfactpow(u, k, x := 1, n := 0, s := 0, a) :=
  Loop
    If u = 0
       RETURN s EXP(x)
    a := SUBST(u, k, n)
    u := QUOTIENT(u, k - n, k)
    s :+ a x^n
    n :+ 1
```

For example, to compute $\sum_{k=0}^{\infty}\frac{\sin(k\,x)\,(k^2+1)}{k!}$, use:

$$\frac{\text{sumfactpow}(k^2+1,\ k,\ \text{EXP}(\hat{\imath}\ x)) - \text{sumfactpow}(k^2+1,\ k,\ \text{EXP}(-\hat{\imath}\ x))}{2\ \hat{\imath}}$$

$$\hat{e}^{\text{COS}(x)}\ (\text{COS}(\text{SIN}(x))\ (\text{SIN}(2\ x) + \text{SIN}(x)) + \text{SIN}(\text{SIN}(x))\ (\text{COS}(2\ x) + \text{COS}(x) + 1))$$

Cheers,
V. Anisiu

Lorenz Kopp, Germany L.Kopp@T-online.de

.... I´d like to contribute to the discussion about "Arrows in 3D-Graphics" (*DNL*#44, page 35). Why not represent the arrowheads as solids. I used my old 2D-solution and "updated" it for 3 dimensions. The arrowheads are represented by pyramids with a square-base. My earlier attempts to use pentagons or hexagons prove to need too much calculation and the results are not significantly better.
As *DERIVE5* now allows editing a vector using the semicolon, I generally write vectors in columns (like it is done in most textbooks). In combination with the "Quote"-Operator, vector expressions can be represented in a very pretty form.
(You can find Lorenz´s contribution on pages 29 – 31, Josef.)

There are very often troubles when working with functions. I met the problems in school and I still face them in DERIVE-workshops and courses. The system doesn´t accept the function, seems to lock up, ... the pupil asserverates: "I did nothing at all!". Have a look and issue Declare > Function or Declare > Variable. In many cases you will be surprised. So I am very happy to have this discussion raised in the Internet printed in the DNL. The discussion was initiated by Aleksey Tetyorko:

Hi!

Try to define some functions, then redefine (not edit!) one of them, and delete the redefinition (by 'del').
Save the file, and quit the program. Open the saved file. You are in the strange situation:
your function is redefined, and you cannot see the redefinition.
Aleskey

Hi Aleskey & Others

A dfw file saves all defined functions whether they have been removed from the Algebra window. When you remove a function definition from the algebra window, the function is still defined in Derive's memory. If my memory serves me the dfw file ability to save 'removed' function definitions was introduced to eliminate parsing errors when a saved Derive file was re-loaded with missing auxiliary functions or definitions that were in the wrong order.

A simple example is

```
#1: FEE(x,y):=x^2+y^2

#2: FI(z):=FEE(z^2,z^3)
```

If you were to swap the definitions round in the Derive window and save the file, then when the file was reload the function FI(z) would be loaded first, but this calls FEE(x,y) which has not yet been defined, so you would receive a parsing error. Indeed if you were to delete the FEE() definition from the algebra window and save, the same thing would happen. So when saving as a dwf file Derive saves ALL defined functions irrespective of whether you have removed (or swapped their order) them from the algebra window.

If you only have Derive expressions and you have no text or embedded objects in your algebra window you can save as an mth file and when you reload all 'removed' definitions will have gone.

To say it is a bug is wrong, it is a feature. A feature that, over the years, has saved me hours of work and frustration I have to say.

Hope this helps, Terence

I've stumbled into that in the past, I've defined a function, accidentally deleted it and later saved the file.
I've found that USUALLY there is a way to see the invisible function:
You know the name of the function: Author and then simplify

f()

and you should see the body of the invisible function with the original argument names, or at least enough details to help you try to remember what you did, even if f had one or more arguments.

Then you can redefine f and resave.

This isn't guaranteed to work, things like if() and some other things can be lost in the simplification. But, now and then, this will be enough to save a lot of work.
Don Taylor

Hello Aleskey and Don,

As you discovered, in Derive you can define a function, delete the definition, but the function remains defined. The following quote from Derive's on-line help for the Declare > Function Definition command explains how to see all the currently defined functions in the current Algebra window:

"Click on the pull-down menu button at the right end of the Function Name and Arguments field to display an alphabetical list of all the currently defined function names. If you click on one of the function names, the current definition of the function is displayed in the Function Definition field."

Similarly, in Derive you can assign a value to a variable, delete the assignment, but the variable remains assigned to the value. In this case, you can use the Declare > Variable Value command to see all the currently assigned variables in the current Algebra window.

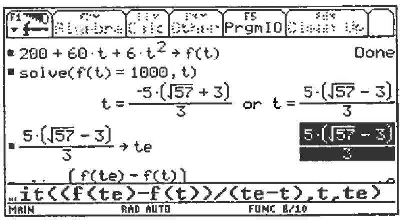
As far as dfw files are concerned, they restore Derive to the exact same state that existed at the time the dfw files are saved. Therefore, if the definition of a function or the assignment to a variable was deleted from the Algebra window when the dfw file was saved, the definition or assignment will not be there when the dfw file is subsequently loaded.

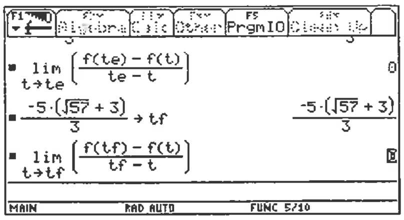
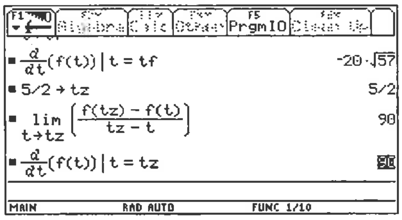
Hope this helps.

Aloha,
Albert D. Rich
Co-author of Derive

Thomas Himmelbauer, Vienna j.himmelbauer@chello.at

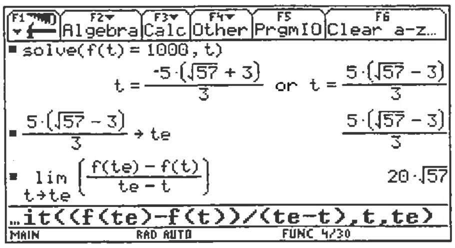
Help, my TI-92PLUS and my Voyage 200 both cannot differentiate!!

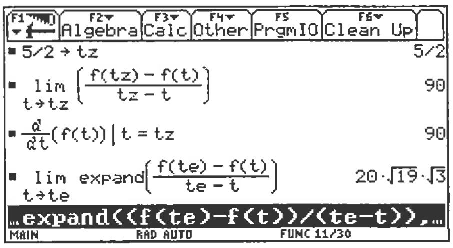


The differential quotient in the zeros of a simple quadratic function, derived as limit of the instantaneous rate of change is wrong (inspect the screen shot at the right side above).

At another location $x_0 = 5/2$ the limit turns out to be correct.
My „old" ordinary *TI*-92 was always right, as shown below.





Solution: This seems to be a bug. However, there is a way to convince the PLUS/Voyage how to find the correct limit, by forcing the device to first expand the fraction and then perform the limit process.

This seems not to be an elegant solution of the problem. When to use Expand, and when not? Do you know??

Peter Schofield, England p_schofield@TASC.AC.UK

Dear All,
I have now sorted out my ideas in relation to 'body-scanning techniques' with DERIVE. Currently they are contained in a .dfw file '2D-&3D-Bodyscans' attached to this email. I have also attached the latest version of 2D-&3D-transformations.dfw because all of the general-purpose transformers and objects of this file are available for use in 2D-&3D-Bodyscans.

If you're interested, I suggest you look at 2D-&3D-Transformations first; followed by the basic bodyscans; followed by the more advanced body-scanning and surface covering tools. I have tried to collect together all of the different methods using a limited number of instruction tools which have wide-ranging and practical use. The examples in 2D-&3D-Bodyscans.dfw are meant to illustrate this.

3D body-scanning techniques bypass the need to select Insert>Plot and then select numbers for panels and ranges (you can still use Insert>Plot to select and customize color schemes). They are data intensive, and so they tend to be slower to calculate than methods based on single parametric equations, say. However, with Approximate Before Plotting ON you can plot straight into both the 2D and 3D plot windows with 2D-&3D-Bodyscans. The data required can be calculated and entered by hand, formulated using VECTOR instructions, or a combination of both.

Initially, both of these files could be placed in the Users Folder of the next issue of DERIVE? However, Theresa and Albert have also suggested that some or all of the tools could also be placed in GRAPHICS.mth – so that they can be made available for more general use with DERIVE. To carry this out I would appreciate your considered opinion of the syntax and terminology used in both 2D-&3D-Trans. and 2D-&3D-Bodyscans, and which of the instructions would be most useful to include in GRAPHICS.mth.

I hope you find a great deal to interest you in these files. Finally, I thought you might like to see a photo of the model I used for babushka (see below). She was a great deal cheaper, and much easier to work with, than some of the other well-known top-class fashion models!

Peter Schofield

The files are on the diskette and you can download Peter´s files from the two well known websites:

http://www.acdca.ac.at/t3/dergroup/index.htm

http://www.bk-teachware.com/main.asp?session=375059

For mouthwatering have a look on page 34 to admire some of Peter´s products. Please contact Peter if you have any proposals or comments on his work, Josef.

Hypothesis Testing

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donphillips@starpower.net

In general, hypothesis testing takes one of three forms, where Ho is the null hypothesis and Ha is the alternative hypothesis. The alternative hypothesis is generally what you are trying to prove by rejecting the null hypothesis. Therefore, it is best to form the alternative hypothesis first and let that determine the null hypothesis. The alternative hypothesis is sometimes called the research hypothesis; it is what your research is trying to show or prove if the null hypothesis can be rejected.

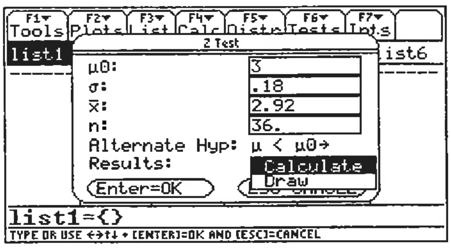
Form 1	Form 2	Form 3
Ho: $\mu \geq \mu 0$	Ho: $\mu \leq \mu 0$	Ho: $\mu = \mu 0$
Ha: $\mu < \mu 0$	Ha: $\mu > \mu 0$	Ha: $\mu \neq \mu 0$

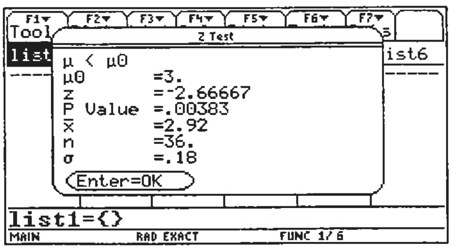
The test routines require entry of the alternative hypothesis, the statistic(s), number of observations, and the level of significance α. The default for α is 0.05. Every hypothesis test has two routines: one to show the numerical results and another to plot the results. The plot will show the appropriate probability distribution curve, the rejection region(s), and the test statistic.

1 One Sample Z Test

A certain manufacturer claims that its can of coffee has a mean weight of 3 pounds or more. A consumer advocate group claims that the manufacturer is cheating consumers, that there is less than 3 pounds of coffee in a can. Suppose that a sample of 36 cans is found to have an average weight of 2.92 pounds with a standard deviation σ of 0.18. Who is correct?
The research hypothesis in this case is Ha: $\mu < 3$.
Use a 5 percent level of significance, i.e., $\alpha = 0.05$.

ZTest(μ < 3, 2.92, 0.18, 36)



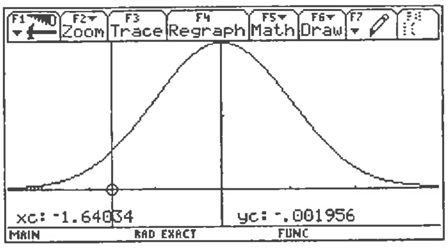


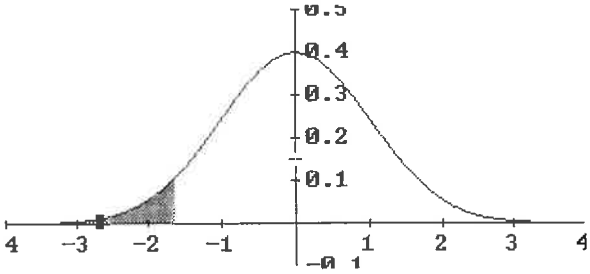
Ho:	μ ≥ 3
Ha:	μ < 3
xbar:	2.92
σxbar:	0.03
N:	36
Zxbar:	−2.666666666
Zα:	−1.644853651
P(Z):	0.003830380557
α:	0.05
P(Type II Error):	0.153434704
Conclusion:	Reject Ho

The results show that Ho should be rejected. Therefore, on average the cans of coffee have less than 3 pounds in them and the consumer advocacy group is correct in its claim. The output shows the Z value of the test statistic: -2.67, the Z value of the beginning of the rejection region: -1.64, the p-value of Z: 0.0038, and the probability of making a type II error, i.e., the probability of accepting Ho when in fact $\mu = 2.92$ is true. This probability is 0.153.

This test can also be shown as a plot.

ZTestPlot(μ < 3, 2.92, 0.18, 36)



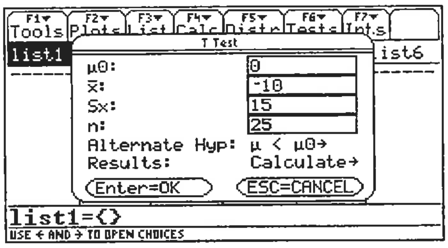


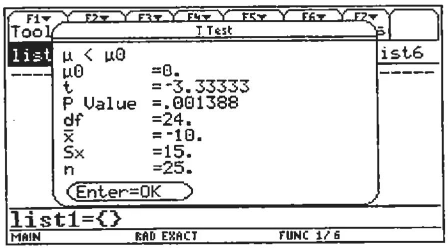
The plot shows the normal curve with the rejection region beginning at -1.64 and the test statistic Z = -2.67 falling within the rejection region.

2 One Sample t Test

A study of a drug designed to reduce blood pressure used a sample of 25 men between the ages of 45 and 55. If the average reduction in blood pressure was -10 with a standard deviation of 15, what can be said about the effectiveness of the drug? Can we conclude that it actually reduces blood pressure? Since we want to conclude that it actually does lower blood pressure, Ha is $\mu < 0$. Since n is less than 30, a t test is called for. Assume α is 0.05.

tTest(μ < 0, -10, 15, 25)



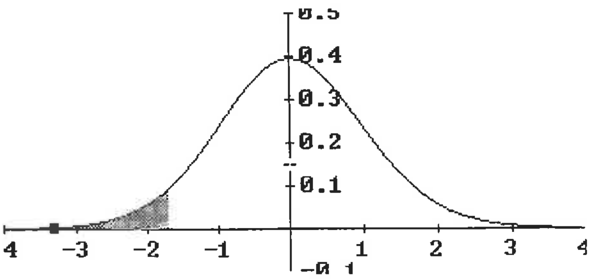


Ho:	μ ≥ 0
Ha:	μ < 0
xbar:	-10
σxbar:	3
N:	25
Degrees of Freedom:	24
txbar:	-3.333333333
tα:	-1.710882084
P(t):	0.001388157091
α:	0.05
P(Type II Error):	0.05888562198
Conclusion:	Reject Ho

At a 5 percent level of significance, the results of the test allow us to reject Ho and conclude that the drug does reduce blood pressure. The t value of the test statistic is -3.33 which is less that the t value of α which is -1.71. The p-value of the t statistic is 0.0014.

A plot of the test shows the same thing.

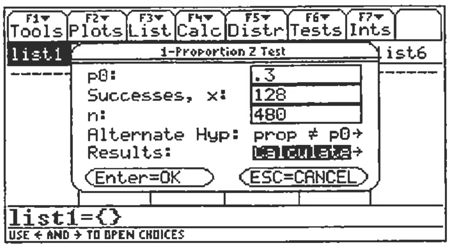
tTestPlot(μ < 0, -10, 15, 25)

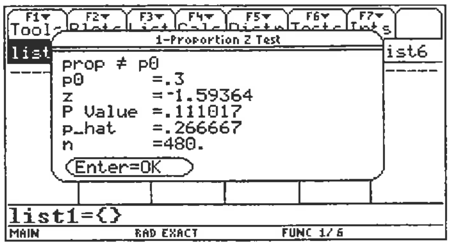


3 One Sample Proportion Test

A city wants to adopt a water-by-request rule at restaurants to conserve the use of water. One restauranteur stated that 30 percent of the partons never touch their water. A sample of 480 patrons at restaurants showed that 128 never touched their water. At a 5 percent level of significance, test the claim that Ho: p = 0.3. The alternative is Ha: p ≠ 0.3.

$$\text{ProportionTest}\left(p \neq 0.3,\ \frac{128}{480},\ 480\right)$$

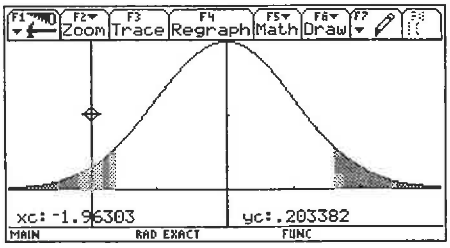


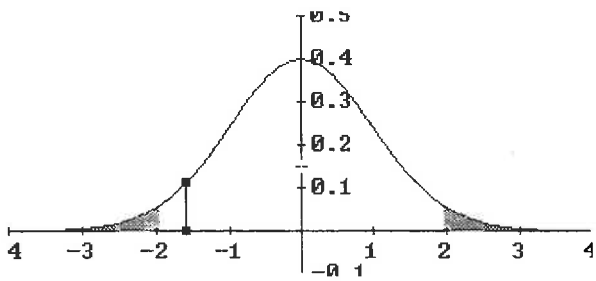


Ho:	p = 0.3
Ha:	p ≠ 0.3
pbar:	0.2666666666
σpbar:	0.02091650066
N:	480
Zpbar:	-1.593638145
Zα:	±1.959963962
P(pbar):	0.1110171055
α:	0.05
P(Type II Error):	0.6429390162
Conclusion:	Do Not Reject Ho

The observed proportion, pbar, is 0.267. Its Z value is -1.59 which is not in either of the rejection regions Zα of ≠1.96. The probability of getting a pbar of 0.267 from a sample of 480 is 0.111. The conclusion is to not reject Ho.
Again, this can be represented in a plot.

$$\text{ProportionTestPlot}\left(p \neq 0.3,\ \frac{128}{480},\ 480\right)$$





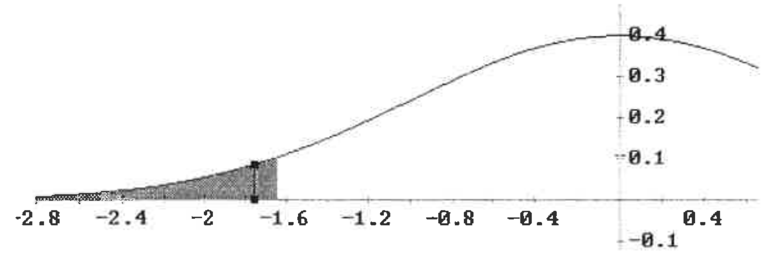
Another example:

A die is rolled 600 times and we count 84 times six points. Is this a fair die?
Test Ho = (≥) 1/6 using first α = 5% and then α = 1%
Ha is p < 1/6.

$$\text{ProportionTest}\left(p < \frac{1}{6},\ \frac{84}{600},\ 600\right)$$

Zpbar:	-1.75271218401653
Zα:	-1.64485362693085
P(pbar):	0.0398257116266295
α:	0.05
P(Type II Error):	0.457053945964595
Conclusion:	Reject Ho

$$\text{ProportionTestPlot}\left(p < \frac{1}{6}, \frac{84}{600}, 600\right)$$

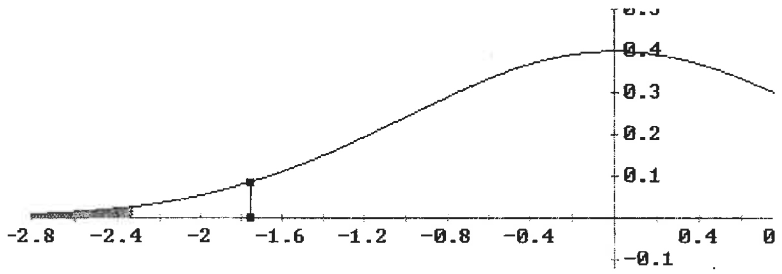


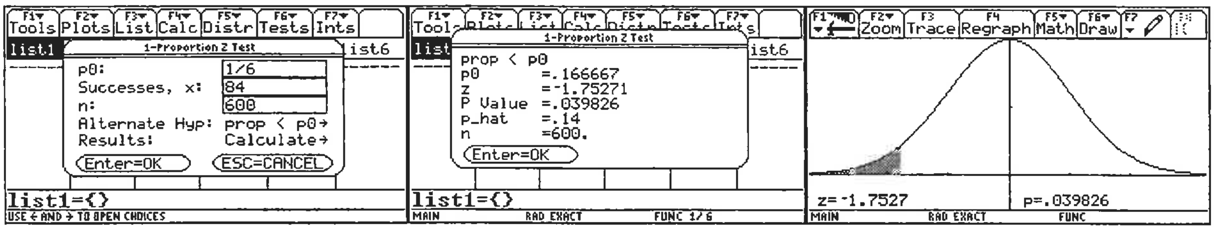
but

$$\text{ProportionTest}\left(p < \frac{1}{6}, \frac{84}{600}, 600, 0.01\right)$$

Zpbar:	-1.75271218401653
Zα:	-2.32634787404094
P(pbar):	0.0398257116266295
α:	0.01
P(Type II Error):	0.716892822457503
Conclusion:	Do Not Reject Ho

$$\text{ProportionTestPlot}\left(p < \frac{1}{6}, \frac{84}{600}, 600, 0.01\right)$$





For α = 0.05 → z = -1.645 → reject Ho; for α = 0.01 → z = -2.326 → don´t reject Ho. You see that it needs some additional considerations to find the correct answer. Don´s tool does this for you.

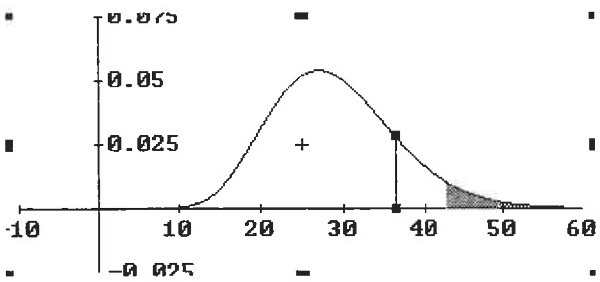
4 One Sample Variance Test

Besides testing the means and proportions of populations, the variances can also be tested. In many cases, not only must the mean or proportion of a population be a certain value, its variance needs to be a certain value also. For instance, a production process may fill a container with x amount of stuff on average. However, if the average is correct, but the variance very high, some containers may be way underfilled and others way overfilled. The variance must be kept within a small range.

A certain part must be machined to very close tolerances, or it is not acceptable to customers. Specifications call for a maximum variance in the lengths of the parts of 0.0004. Suppose that the sample variance for 30 parts turns out to be $\sigma^2 = 0.0005$. Using α = 0.05, test to see if the specification is being violated. Ha is $\sigma^2 > 0.0004$.

The variance test uses the χ^2 distribution with a degrees of freedom of N - 1. χ^2 statistic is 36.25 with a probability of 0.166. Since the χ^2 of α is 42.56, Ho is not rejected.
The variance seems to be within specification.
And, of course, the variance test can be shown in a plot.

```
VarianceTest(σ² > 0.0004, 0.0005, 30)
```



Ho:	$\sigma^2 \le 0.0004$
Ha:	$\sigma^2 > 0.0004$
s^2:	0.0005
N:	30
Degrees of Freedom:	29
Chi2:	36.25
Chi2α:	42.55696775
P(Chi2):	0.1663945047
α:	0.05
Conclusion:	Do Not Reject Ho

5 Two Sample Z Test

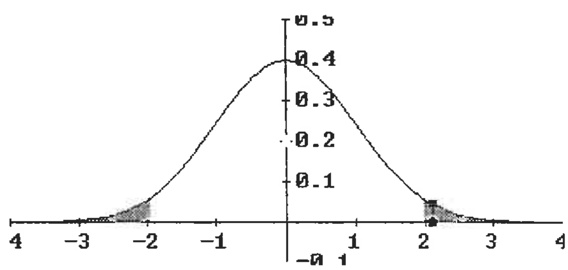
The two sample tests can be used to test the differences between means, proportions, and variances. In most instances, the difference is assumed to be zero, but that does not have to be the case. The routines accept any difference in Ho.

A firm runs two training centers. The examination results in Center A show an average score of 82.5 for 30 students with a standard deviation of 8. Center B shows an average of 78 for 40 students with a standard deviation of 10. Determine if the average scores are different at a 5 percent level of significance. Ho is $\mu1 - \mu2 = 0$ while Ha is $\mu1 - \mu2 \neq 0$.

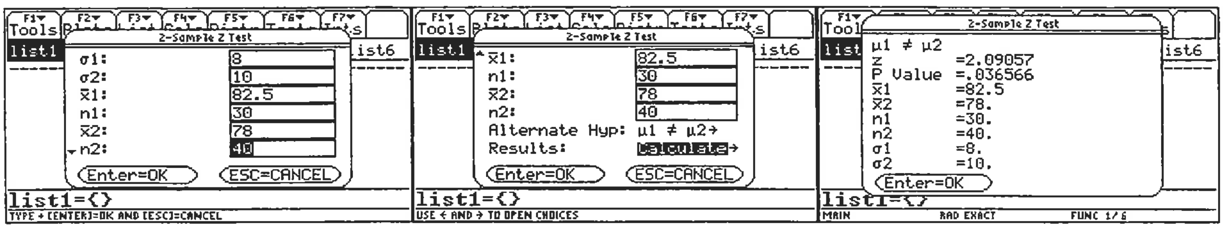
```
TwoSampZTest(µ1 - µ2 ≠ 0, 82.5, 8, 30, 78, 10, 40)
```

n2:	40
s(x1-x2);	2.152517905
Z:	2.090574944
Zα:	±1.959963962
P(Z):	0.03656618373
α:	0.05
Conclusion:	Reject Ho

```
TwoSampZTestPlot(µ1 - µ2 ≠ 0, 82.5, 8, 30, 78, 10, 40)
```



We conclude that there is a difference in test scores. The test statistic is Z = 2.09 with a probability of P(Z) = 0.0366 while Zα is ≠1.96. Z lies within the upper tail rejection region. In addition, the standard error of the difference in the means is calculated, s(x1-x2) = 2.15.



6 Two Sample t Test

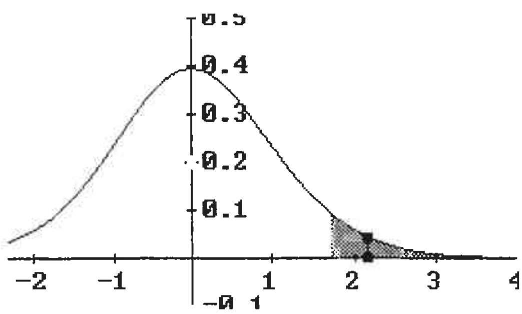
In general, the two sample t test assumes that the populations have normal probability distributions and that the variances of the populations are equal. If the variances are assumed equal, the variances are pooled and the degrees of freedom equals n1 - n2 - 2. If the variances are not assumed to be equal, they are not pooled, and the degrees of freedom is computed by a crazy formula and is not an integer. The default value for pooled is pool = 1, which means that the variances are assumed equal and are pooled. If they are not assumed equal, set pool = 0. The pool variable comes just before α. If the degrees of freedom are not an integer, the inverse t value can take up to 30 seconds or so to calculate.

Using current technology and software, a study of 12 projects indicates that project completion time averages 325 hours with a standard deviation of 40 hours. Using a new software package, a study of 12 projects shows an average project completion time of 288 hours with a standard deviation of 44 hours. Test to see if the diffence in project completion times is greater than 0. Ho is therefore $\mu 1 - \mu 2 \leq 0$ while Ha is $\mu 1 - \mu 2 > 0$. Use a 5 percent level of significance and assume the population variances are equal.

```
TwoSampTTest(μ1 - μ2 > 0, 325, 40, 12, 288, 44, 12)

TwoSampTTestPlot(μ1 - μ2 > 0, 325, 40, 12, 288, 44, 12)
```

s(x1-x2):	17.16585758
t:	2.155441393
Degrees of freedom:	22
tα:	1.71714439
P(t):	0.02116565905
α:	0.05
Conclusion:	Reject Ho



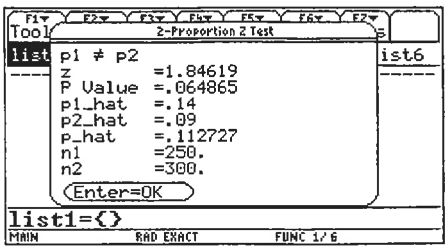
The results of the test allow us to reject Ho and conclude that the new software package provides a lower average project completion time. The t value of the diffence in means is 2.16 with a probability of 0.021, while the t value of α is 1.717. The standard error of the diffence in the means is 17.166.

7 Two Sample Proportion Test

A tax preparation firm is comparing the work of two of its regional offices. In Office 1, they found 35 out of 250 returns with errors. In office 2, 27 out of 300 returns had errors. Test whether there is a significant difference in error rates at a 10 percent level of significance. Ho is $p1 - p2 = 0$ while Ha is $p1 - p2 \neq 0$.

$$\text{TwoSampPropTest}\left(p1 - p2 \neq 0,\ \frac{35}{250},\ 250,\ \frac{27}{300},\ 300,\ 0.1\right)$$

Z:	1.84618928
Zα:	±1.644853651
P(Z):	0.06486472684
α:	0.1
Conclusion:	Reject Ho



p1 is equal to 0.14 and p2 is equal to 0.09. Based on the test, we can reject Ho and conclude that there is a diffence in the error rates between the two offices. The Z statistic is 1.846 with a probability of 0.0649 while Zα is ± 1.645. The Z value falls within the upper tail rejection region. The standard error of the difference in proportions is 0.027.
The plot of the test shows the same results.

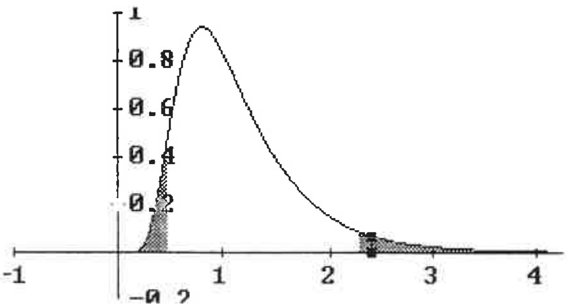
8 Two Sample F Test

The F distribution can be used to test the diffence in two population variances. A school system is deciding between two competing bus companies and it wants to choose the one with the lowest variance in pickup and delivery times. A sample of 25 from Company A shows a variance of 48 while a sample of 16 from Company B shows a variance of 20. Test to see whether the two variances are equal. Ho is $\sigma1^2 = \sigma2^2$ and Ha is $\sigma1^2 \neq \sigma2^2$. Use a 10 percent level of significance.

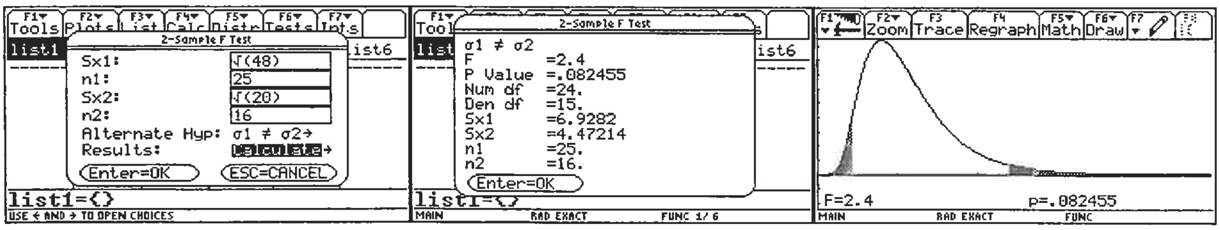
$$\text{TwoSampFTest}(\sigma1^2 \neq \sigma2^2, 48, 25, 20, 16, 0.1)$$

$$\text{TwoSampFTestPlot}(\sigma1^2 \neq \sigma2^2, 48, 25, 20, 16, 0.1)$$

Num df:	24
Den df:	15
F:	2.4
Fα:	[0.4744570435, 2.287826056]
P(F):	0.08245520714
α:	0.1
Conclusion:	Reject Ho



Based on the test, we can reject Ho and conclude that the variances are not equal. The school district can therefore choose the one with the lowest variance, i.e., Company B. The F statistic is 2.4 with a probability of 0.082. Since the F distribution is not symmetric, the lower and upper tail distributions have to be computed separately; they are 0.474 and 2.288. The numerator and denominator degrees of freedom are 24 and 15.



9 Contingency Table Test

Contingency tables are used to test for the independence of two variables using the chi-square distribution. Suppose a simple random sample of 150 beer drinkers reveals the following information by sex and beer preference. Is there a difference in beer preferences based on sex at a 5 percent level of significance? If the variables are independent, beer preferences would be independent of sex.

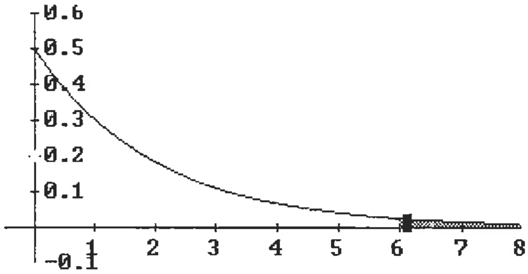
Sex/Beer Pref	Light	Regular	Dark
Male	20	40	20
Female	30	30	10

ContingencyTable $\begin{bmatrix} 20 & 40 & 20 \\ 30 & 30 & 10 \end{bmatrix}$

Ho:	Variables Independent
Ha:	Variables Not Independent
Chi2:	6.122448979
Degrees of Freedom:	2
Chi2α:	5.991464554
P(Chi2):	0.04683031685
α:	0.05
Conclusion:	Reject Ho

Since we can reject Ho at the 5 percent level, we can conclude that beer preference is not independent of sex. The expec-ted frequencies, those that would be expected if beer preference were independent of sex, can be observed by authoring `ef` and entering.

ContingencyTablePlot $\begin{bmatrix} 20 & 40 & 20 \\ 30 & 30 & 10 \end{bmatrix}$



ef

$\begin{bmatrix} 26.66666666 & 37.33333333 & 16 \\ 23.33333333 & 32.66666666 & 14 \end{bmatrix}$

10 Goodness of Fit Test

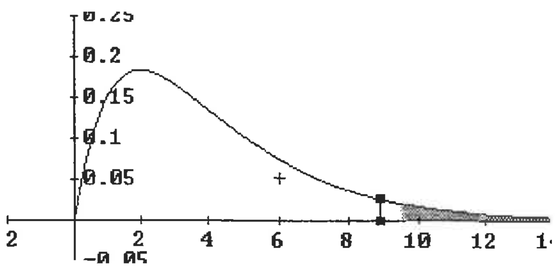
The goodness of fit test uses the χ^2 distribution to determine whether the observed frequencies in a sample are consistent with the expected frequences. Enter the observed list first, then the expected list,then α if different from 0.05.

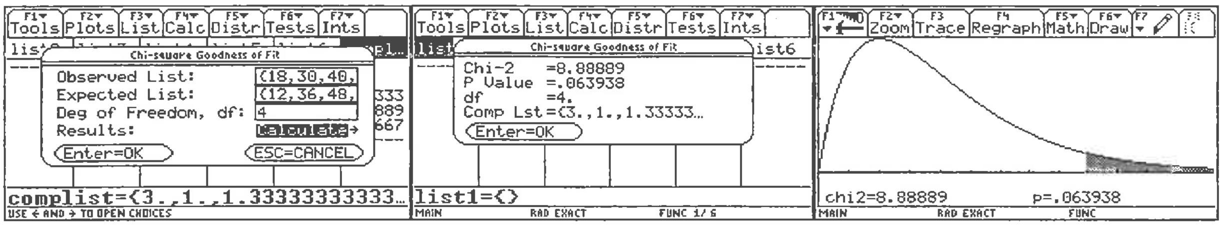
Grade-distribution guidelines at a major university are: 10% A, 30% B, 40% C, 15% D, and 5% F. A sample of 120 statistics grades at the end of a semester showed 18 A's, 30 B's, 40 C's, 22 D's, and 10 F's. Use α = 0.05 and test to see if the actual grades deviate significantly from the grade-distribution guidelines.

```
GoodnessOfFitTest([18, 30, 40, 22, 10], 120 [10%, 30%, 40%, 15%, 5%])
```

Ho:	Observed Consistent with Expected
Ha:	Observed Not Consistent with Expected
Chi2:	8.888888888
Degrees of Freedom:	4
Chi2α:	9.487729072
P(Chi2):	0.06393753267
α:	0.05
Conclusion:	Do Not Reject Ho

Since we cannot reject Ho, we conclude that the observed grade distribution is consistent with the guidelines.





11 Normal Distribution Test

The goodness of fit test can also be used to test whether a sample fits a given statistical distribution, e.g., a normal or Poisson. The difficulty in using the general goodness of fit test is that you have to classify the data into certain ranges based on the type of distriubtion and also compute the expected frequencies based on the type of distribution. The normal distribution test does all that automatically.

The employee aptitude test scores for 50 randomly chosen job applicants are given below. If a normal distribution can be applied to the test scores, the firm can easily determine the scores in the top 10%, top 20%, lower 40%, etc. Test whether the scores are normally distributed with $\alpha = 0.10$.

```
scores := [71, 60, 55, 82, 85, 65, 77, 61, 79, 66, 86, 63, 79, 80, 62, 54, 56, 84,
  61, 70, 56, 76, 56, 90, 64, 63, 65, 70, 62, 68, 61, 69, 74, 80, 54, 73, 76, 53,
  61, 76, 65, 56, 93, 73, 54, 58, 64, 79, 65, 71]
```

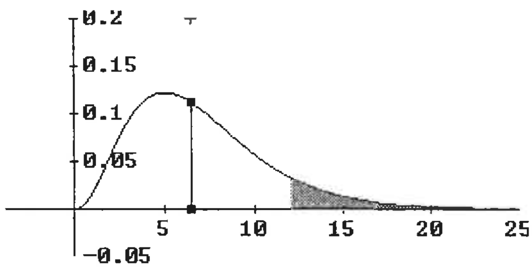
```
NormalDistributionTest(scores, 0.1)
```

Ho:	Normal Distribution
Ha:	Not a Normal Distribution
Chi2:	6.4
Degrees of Freedom:	7
Chi2α:	12.01703662
P(Chi2):	0.4938946499
α:	0.1
Conclusion:	Do Not Reject Ho

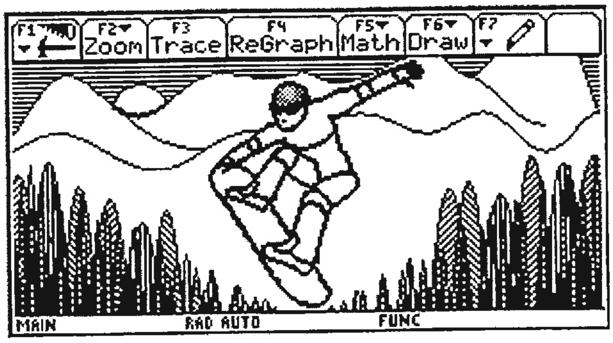
Based on a χ^2 of 6.4 with a probability of 0.494, we cannot reject Ho. The distribution may be assumed to be normal. The observed and exepected frequencies may be observed by authoring `classes`. The class intervals are set so that the same percentage of observations are expected to be in each interval.

```
classes
```

Class	Observed	Expected
$-\infty \le x < 55.0738446$	5	5
$55.0738446 \le x < 59.65530566$	5	5
$59.65530566 \le x < 62.95886144$	7	5
$62.95886144 \le x < 65.78162817$	8	5
$65.78162817 \le x < 68.42$	2	5
$68.42 \le x < 71.05837182$	5	5
$71.05837182 \le x < 73.88113855$	2	5
$73.88113855 \le x < 77.18469433$	5	5
$77.18469433 \le x < 81.76615539$	5	5
$81.76615539 \le x < \infty$	6	5



Artistic Maths or Mathematical Art?



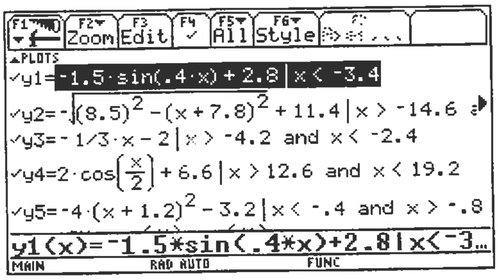
In the last Newsletter I congratulated Karola Hummer for winning an Ars Electronica Award for generating picures – paintings on her *TI*-92. Karola responded on my try to contact her and she sent the GDB-file (Graphic-Data-Base) to reproduce her „Snowboarder in a Tyrolian Landscape".

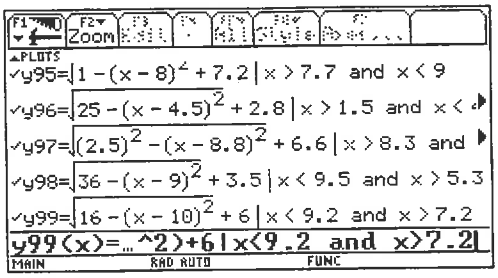
The GDB-file is on the diskette and you can enjoy winter fun on your own device.
Karola gave some additional comments, which might be of interest for you. I try to translate them.

The sequence of functions given in the [Y=]*-Editor seems to be illogical at the first view. But there are good reasons:*

- *Because of the long calculation time I started with the 99th function. So necessary changes in the functions could be performed much easier*
- *There is some strange "logic" in the TI-system. Plot Style "*Square*" obviously counts the type of shading "*Above*" and "*Below*" As I wanted to produce my pictures using 99 functions only, I set the* Square*-functions very intentionally after having performed all my calculations*

See begin and end of Karola's function list, followed by a section of the GDB*-file containing the special style for each function to be plotted.*






```
Below:  ?y17 = -5*(x+11)^2-2
Below:  ?y18 = -2*(x-23)^2+2
Thick:  ?y19 = x/4*(cos(x/4))*sin(x/4)/(tan(x/4))+9|x>-15.2 and x<-10.6
Below:  ?y20 = -4*(x-20.5)^2+1.5
Above:  ?y21 = √((2.2)^2-(x+13)^2)+6
Square: ?y22 = x/x*(-6)|x>0.8 and x<1
Line:   ?y23 = x+9.2|x<-1.2 and x>-2.2
Line:   ?y24 = x+8.8|x<-0.8 and x>-2.2
```

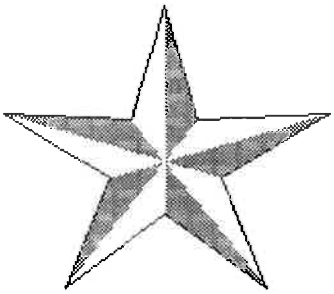
Many thanks Karola for your mail and the tipps. I must confess that I admire your work, and I really appreciate your patience and your knowledge.

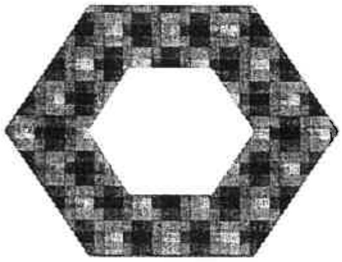
As a former teacher I can report from my experience that we can rise the students' attitude in doing mathematics presenting some challenges as shown on the next page. The "artists" can use various function types according to their level of knowledge. Start with linear functions, quadratics, cubics, trig functions, ... and finally combine them all.

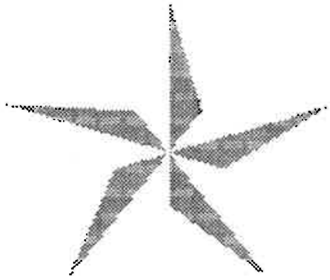
I present some results performed on the *TI* or with *DERIVE*. On the *TI* one has to apply the shade-command. In *DERIVE* are two possibilities to obtain shaded areas:

- Use the AreaOverCurve-, AreaUnderCurve- and AreaBetweenCurves-functions
- Work with Boolean Expressions (which is my favor).

Take for example the „Penta-Star" from below. It needs a lot of mathematics – for pupils, of course - to find the corner points of the figure. Using a CAS it is not so difficult to calculate the coordinates of the points (depending on the mathematical knowledge) and then using a function lin(pt1, pt2) combine all the linear functions forming the boundaries of the regions.

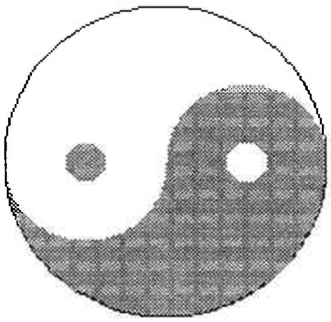


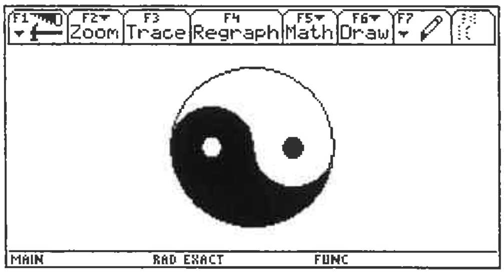


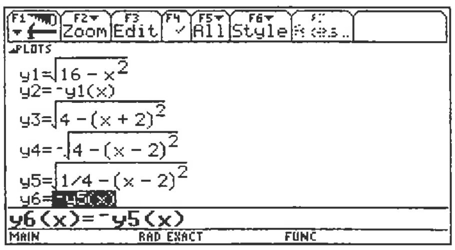


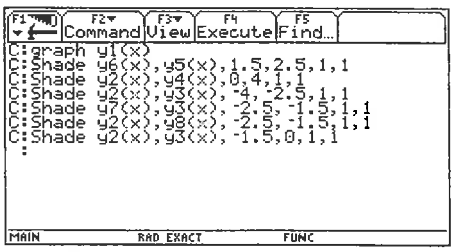
```
[x > 0 ∧ y < lin(A1, e1) ∧ y > lin(o, e1), y < lin(a1, B1) ∧ y < lin(o, a1) ∧ y >
  lin(o, B1), y < lin(b1, o) ∧ y > lin(C1, o) ∧ y < lin(C1, b1), x > 0 ∧ y > lin(c1,
  D1) ∧ y < lin(o, D1), y < lin(o, E1) ∧ y > lin(o, d1) ∧ y > lin(d1, E1)]
```

Work with circles to paint the famous YINYANG-Symbol[1].



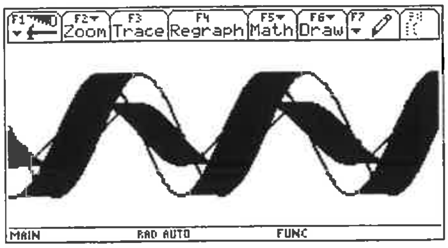






Possible additional tasks: Rotate the figure by 180° or 90, or by 45°!

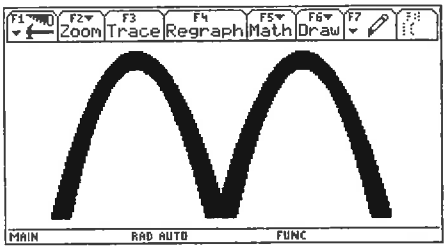
[1] *Programmieren mit Derive*, Josef Böhm, bk-teachware SR-32

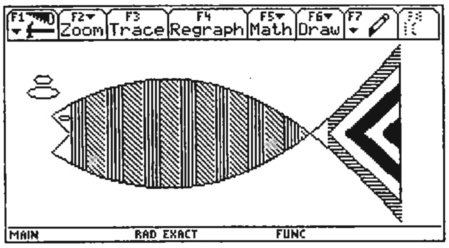


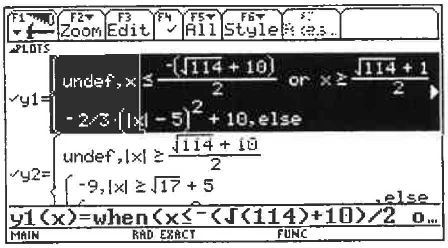
Very spectacular patterns can be created by using trigonometric functions. Talking as a teacher again, I recommend to pose a task and ask the students to reproduce the given pattern as acurate as possible. They shall not work free in the beginning. It is not so easy to find the functions and their parameters which result in a similar looking picture. Give it a try!!

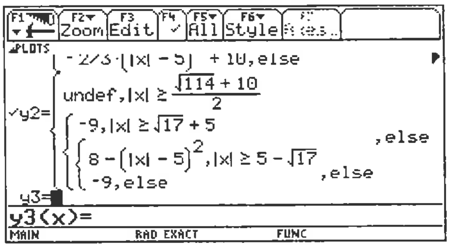
Two more pictures at the end:

Everybody knows the Twin-Parabolas as a worldwide known brand, but I am quite sure that only very few of you have met the funny fish "surfing and bubbling" on the *TI*-Screen. Having wonderful memories on Hawaii, I call it my Humu Humu Nuku Nuku Apuaa – it's the state fish of Hawaii.









shade y2(x) y1(x),-11.5,11.5,1,1 gives the shaded figure. Enjoy your (Burger-) arches.

The intersection points of the curves with $y = -9$ are calculated separately.
Appropriate [WINDOW]-settings are: $xmin = -11.9$, $xmax = 11.9$, $ymin = -10.5$, $ymax = 10.5$.

It might be demanding to define each of the "Twin-Parabolas" as one single function and further to define both curves as piecewise defined functions. So we can wrap some mathematical ideas and concepts into our artistic approach.

And finally let loose phantasy and creativity. You will be surprised about the results.

I'll take the occasion to present a nice website, which shows a lot of pictures produced on the TI-83. From there I got a lot of ideas. Go to

`http://inaz4husains.netfirms.com`

Many thanks to Bernhard Kutzler, who sent a note recommending this website.

Advanced Problem Solving using *DERIVE*

Stefan Welke, Bonn, Germany, Spwelke@aol.com

Introduction

We start with a problem which is familiar to every math teacher:

The volume of a cylinder is 1 litre, its height is 7 cm. Determine radius and surface area!

Pupils are expected to do the following steps:

(1) to remember the formulae for volume and surface area of a cylinder

(2) to substitute the given values for volume and height of the cylinder

(3) to solve the system of equation for the remaining unknowns radius and area

(4) to select the correct, meaningful solution to the problem out of a set of solutions

In practise pupils do not fail the first step, the second step is a moderate challenge, and the third step is often a real problem, especially with systems of nonlinear equations. We will show in this paper how a CAS like *DERIVE* and others can perform steps (2), (3), and probably (4) automatically. What remains is to find an appropriate system of equations and a sufficient number of values as input.

Consider the following solution with *DERIVE*. It resembles the first three steps listed above.

```
#1:   CaseMode := Sensitive
#2:   InputMode := Word
#3:   [V = π·r^2·h, Area = 2·π·r^2 + 2·π·r·h]
#4:   SUBST([V = π·r^2·h, Area = 2·π·r^2 + 2·π·r·h], [V, h], [1000, 7])
#5:   [1000 = 7·π·r^2, Area = 2·π·r^2 + 14·π·r]
#6:   SOLVE([1000 = 7·π·r^2, Area = 2·π·r^2 + 14·π·r], [r, Area])
#7:   [r = 6.743355313 ∧ Area = 582.3025429, r = -6.743355313 ∧ Area =
        -10.87397147]
```

An algebraic problem solver

The previous solution is unsatisfactory for two reasons:

We encounter negative solutions which make no sense in the context of the given problem.

A small change of the problem, say the radius is given instead of the height, needs a new substitution and solve procedure.

Our goal is to develop a function that takes two systems of equations as input and a system of solutions as output. The first input system is a conjunction of equations that assigns the given values to their variables, e.g.:

```
#8:    a = 3 ∧ b = 7
```

We must separate the variables from their values on the right side of the equation sign:

```
#9:    [VARIABLES(a = 3 ∧ b = 7), (SOLUTIONS(a = 3 ∧ b = 7, VARIABLES(a = 3
         ∧ b = 7)))_1]

#10:   [ a  b ]
       [ 3  7 ]
```

Now we are in a position to automate the substitution process. The second input is the system of algebraic equations that describes the context of the problem in question. We call the resulting function `frame_1`, and we still keep our first example:

```
       frame_1(v, w) :=
         Prog
#11:       u_ := VARIABLES(v)
           w_ := SUBST(w, u_, FIRST(SOLUTIONS(v, u_)))
           SOLVE(w_, VARIABLES(w_))

#12:   frame_1(V = 1000 ∧ h = 7, [V = π·r^2·h, Area = 2·π·r^2 + 2·π·r·h])
#13:   [Area = 582.3025428 ∧ r = 6.743355313, Area = -10.87397147 ∧ r =
         -6.743355313]
```

What is left is to suppress the negative solutions. Since the output of the function `frame_1` is a vector of conjunctions of equations, we just have to separate variables and values and then we must skip solutions with negative, or better with non-real or non-positive numbers. The next example shows how to proceed:

```
#14:   a = 3 ∧ b = -3.2 ∧ c = ê + 0.5·î
#15:   FIRST(SOLUTIONS(a = 3 ∧ b = -3.2 ∧ c = ê + 0.5·î, [a, b, c]))
#16:   [3, - 16/5, ê + î/2]
#17:   SELECT(REAL_TYPE?(k) ∧ k > 0, k, [3, - 16/5, ê + î/2])
#18:   [3]
```

If we want all values to be strictly positive we can use the following predicate function:

```
       POS_VEC?(v) :=
         If DIMENSION(SELECT(REAL_TYPE?(k_) ∧ k_ > 0, k_, v)) = DIMENSION(v)
#19:        true
            false
            false

#20:   POS_VEC?([3, -3.2, ê + 0.5·î])
#21:   false
#22:   POS_VEC?([3, 3.2, ê])
#23:   true
```

A combination of the functions `frame_1` and `POS_VEC?` yields a function `frame_2` that filters only positive solutions to a problem.

```
         frame_2(v, w) :=
           Prog
             u_ := VARIABLES(v)
#24:         w_ := SUBST(w, u_, FIRST(SOLUTIONS(v, u_)))
             z_ := VARIABLES(w_)
             FIRST(SELECT(POS_VEC?(FIRST(SOLUTIONS(n_, z_))), n_, SOLVE(w_,z_)
             ))
```

```
                                          ┌          2                   2         ┐
#25:     frame_2(V = 1000 ∧ h = 7,  │V = π·r ·h, Area = 2·π·r  + 2·π·r·h│)
                                          └                                       ┘
#26:                      Area = 582.3025428 ∧ r = 6.743355313
```

We take the following exercise as a second example:

> One side of a rectangle is $7.3cm$, the area is $45cm^2$. Determine the length of the other side and the perimeter of that rectangle.

```
#27:     frame_2(a = 7.3 ∧ A = 45, [A = a·b, C = 2·(a + b)])
#28:                       C = 26.92876712 ∧ b = 6.164383561
```

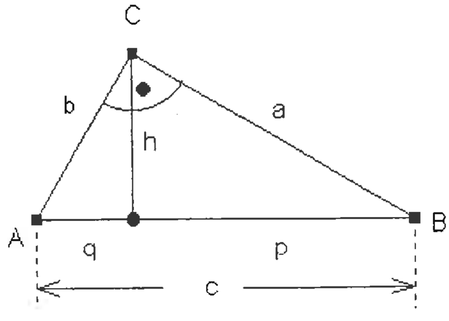
At this point we have developed a framework for a whole bundle of similar problems. Now the way of problem solving goes like this:

(1) Find a complete system of equations which contains all variables of the given and of the requested quantities!
(2) Apply one of the functions `frame_1` or `frame_2`!
(3) Select eventually the appropriate solution(s)!

Needless to say that the scope of this method is limited by the limits of the built-in solve function of the CAS.

The computation of the right-angled triangle

Up to now we encountered rather trivial problems. The next one is of higher complexity and touches the limits of the built-in equation solver of *DERIVE*.



It is generally possible to compute all quantities of a triangle if three appropriate quantities are given, e.g. two edges and the enclosed angle. Now we pose a system of equations which contains all three edges, angles, the height from the right angle to the hypotenuse, and the segments on the hypotenuse of a right-angled triangle according to the figure:

We have six equations and eight quantities: a,b,c,h,p,q, and as usual $\alpha := \measuredangle BAC$, and $\beta := \measuredangle CBA$. Thus a substitution of two variables, the third one is the right angle, yields a non-linear system of six equations with six variables.

```
#29:   [c = a·cb + b·ca, b = h·cb + q·ca, a = h·ca + p·cb, b = c·ca, a =
         c·cb, a·b = h·c]
```

Note that ca and cb are the cosines of α and β. We are working with the cosines for two reasons: (1) #29 is a system of algebraic equations, (2) None of these angles is greater than a right angle, so the cosines are always positive and they correspond in a one-to-one way to the angles. Now the function frame_2 can filter out the desired positive solutions.

In contrast to our previous examples the output will be a complete system of all quantities of the triangle. Since the system of equations remains the same, we need only the equations, which define the given quantities, as input.

```
#30:   rt1(v) := v ∧ frame_2(v, [c = a·cb + b·ca, b = h·cb + q·ca, a = h·ca
         + p·cb, b = c·ca, a = c·cb, a·b = h·c])

#31:   rt1(a = 3 ∧ b = 4)

#32:   a = 3 ∧ b = 4 ∧ c = 5 ∧ ca = 0.8 ∧ cb = 0.6 ∧ h = 2.4 ∧ p = 1.8 ∧ q
         = 3.2
```

We need two conversion functions: ang2co converts an input in degrees to the corresponding cosine, and co2ang converts a cosine-value to its corresponding angle.

```
       ang2co(v, w) :=
         Prog
#33:       u_ := (SOLUTIONS(v, w))↓1
           SUBST(v, [α = u_↓1, β = u_↓2], [ca = COS(u_↓1/180·π),
                     cb = COS(u_↓2/180·π)])

       co2ang(v, w) :=
         Prog
#34:       u_ := (SOLUTIONS(v, w))↓1
           SUBST(v, [ca = u_↓1, cb = u_↓2], [α = ACOS(u_↓1)·180/π,
                      β = ACOS(u_↓2)·180/π])
```

Finally we arrive at the function right_tiangle combining frame_2 with the previous conversion functions:

```
       right_triangle(v) :=
         Prog
#35:       eq_ := [c = a·cb + b·ca, b = h·cb + q·ca, a = h·ca + p·cb,
                   b =  c·ca, a = c·cb, a·b = h·c]
           v_ := ang2co(v, [α, β, a, b, c, h, p, q])
           co2ang(v_ ∧ frame_2(v_, eq_), [ca, cb, a, b, c, h, p, q])

#36:   right_triangle(a = 3 ∧ b = 4)

#37:   a = 3 ∧ α = 36.86989764 ∧ b = 4 ∧ β = 53.13010235 ∧ c = 5 ∧ h = 2.4
         ∧ p = 1.8 ∧ q = 3.2

#38:   right_triangle(a = 3 ∧ α = 36)

#39:   a = 3 ∧ α = 36 ∧ b = 4.129145761 ∧ β = 54 ∧ c = 5.10390485 ∧ h =
         2.427050983 ∧ p = 1.763355756 ∧ q = 3.340549093
```

There is no restriction to the case of a right-angled triangle. It is possible to construct similar functions for general triangles.

Some strange examples

When I developed the function `right_triangle` I tested all thirteen essentially different cases of combinations of possible input. In this way I found a strange behaviour of *DERIVE*. The following examples works correctly.

```
#40:  right_triangle(p = 0.8 ∧ q = 2.2)

#41:  a = 1.549193338 ∧ α = 31.09093035 ∧ b = 2.569046515 ∧ β =
        58.90906964 ∧ c = 3 ∧ h = 1.326649916 ∧ p = 0.8 ∧ q = 2.2
```

Now we take p and q ten times larger, the result should be a similar triangle, all lengths ten times larger. But the input yields no result on my machine, except the message "memory exhausted".

```
#42:  right_triangle(p = 8 ∧ q = 22)
```

But a slight modification of the data returns immediately a plausible result compared to the output #41.

```
#43:  right_triangle(p = 8.00001 ∧ q = 22.00003)

#44:  a = 15.49195339 ∧ α = 31.09092891 ∧ b = 25.6904998 ∧ β = 58.90907108
        ∧ c = 30.00004 ∧ h = 13.26651649 ∧ p = 8.00001 ∧ q = 22.00003
```

Who can explain?

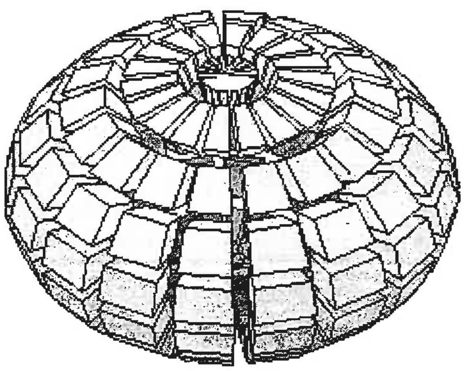
Remarks

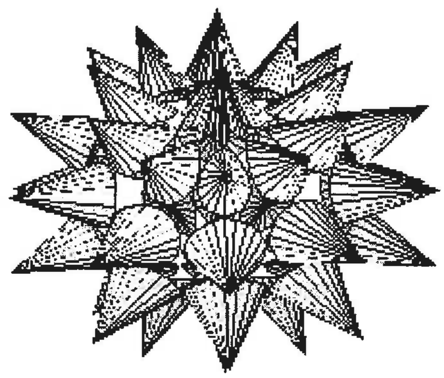
(1) Note that system #29 is the result of a long search. I tried some different but equivalent system that did not work in all possible cases.

(2) The purpose of this note is not to provide lazy or untalented pupils with a means to prevent solving equations. I just wanted to point out that we can use a CAS system to do the equation solving and selection of acceptable solutions to problems of an essentially algebraic nature.

(3) We learn that most of the math and physics exercises in secondary schools are pure routine from the computational point of view. The core of problem solving in this context is to find the correct system of equations.

Two balls to adorn the Christmas Tree!





On Rüdeger Baumann´s Problem of Recognizing the Period in Milton´s Sequence

Dear Josef, dear DERIVE-Community,

It is a pleasure to send you my "homework" on the RB's program for Milton's 3rd riddle. As you requested, the modification provides the position of the starting element of the cycle, and the position of the same element in the next cycle. I hope you like it; at last I find something within my powers !!!

Take this as "a contribution from a Spanish biologist who has discovered maths late in life"

Best regards
Enric Puig

```
QS(n, k) :=
  If n ≤ 0
     0
     MOD(n, 10)^k + QS(FLOOR(n, 10), k)

find_identical(list_) := SELECT(IDENTICAL?(list_           , list_ ), i, 1,
                                               DIM(list_)        i
  DIM(list_))

milton_01(number, exponent, i := 0, list := []) :=
  Loop
    i :+ 1
    list := ADJOIN(number, list)
    If DIM(find_identical(REVERSE(list))) = 2
       RETURN [REVERSE(list), find_identical(REVERSE(list))]
    number := QS(number, exponent)

milton_01(153, 2)

[[153, 35, 34, 25, 29, 85, 89, 145, 42, 20, 4, 16, 37, 58, 89], [7, 15]]

milton_01(136, 4)

[[136, 1378, 6579, 10883, 8274, 6769, 11554, 1508, 4722, 2689, 11969, 14420, 529,
  7202, 2433, 434, 593, 7267, 6114, 1554, 1507, 3027, 2498, 10929, 13139, 6725,
  4338, 4514, 1138, 4179, 9219, 13139], [25, 32]]
```

It is not difficult to change the ouput to present the number of the starting element of the period together with the period´s length.

```
milton_02(136, 4) = [[136, 1378, 6579, 10883, 8274, 6769, 11554, 1508, 4722, 2689,
  11969, 14420, 529, 7202, 2433, 434, 593, 7267, 6114, 1554, 1507, 3027, 2498,
  10929, 13139, 6725, 4338, 4514, 1138, 4179, 9219, 13139], [25, 7]]
```

Stefan Welke contributed to "Detection of Periods", too (- as an Exercise in Functional Programming). His approach will be presented in the next DNL.

Let´s see next Rüdeger Baumann´s try on Johann Wiesenbauer´s Challenge "Currrency Problem". Rüdeger is always fond of recursive solutions.
You can compare with Johann´s proposed solutions in his Titbits 24.

On Johann´s Currency-Problem (DNL#47, page 44)

„How many ways can you make a dollar using pennies, nickels, dimes, quarters, half-dollars and dollars?" Johann Wiesenbauer poses this question in DNL#47, page 44. In €-Land we could ask, how many ways there are to change a 5€ bill into coins.

More general: A state has coins with values $d_1 = 1$, d_2, ..., d_k currency units. In how many ways one can pay n currency units? Let $a(n,k)$ the number of possiblities to pay n currency units using coins from $\{d_1, ..., d_k\}$. Then

$a(n,1) = 1$ for $n \geq 0$
$a(n,k) = 0$ for $n < 0$
$a(n,k) = a(n,k\text{-}1) + a(n - d_k, k)$ else.

1 centime 2 centimes 5 centimes 10 centimes 20 centimes 50 centimes 1 euro 2 euros

Example (EURO-coins): $a(10,3) = a(10,2) + a(5,3)$. Write down all possibilities!
The set of possible payments belonging to a(10,3) is {1111111111,111111112,11111122,1111222, 112222,22222} ∪ {111115,11125,1225,55}. The first part consists of the payments without using the 5-Cent-coin; its cardinality is $a(10,2)$. The second set describes the possible payments using the 5 Cent coins. It is equivalent to {1111,1112,122,5} with cardinality $a(5,3)$. Rewriting this set as {1111,1112,122} ∪ {5} leads to the recursion formula $a(5,3) = a(5,2) + a(0,3)$.
Hence, „Running Back" (lat. *recurrere*) and final summation we finally obtain

$a(10,3) = 6 + 3 + 1 = 10$

In *DERIVE* it reads:

```
a(n, k := 5, d := [1, 5, 10, 25, 50]) :=
  If n < 0
     0
     If k = 1
        1
        a(n, k - 1) + a(n - d↓k, k)

a(100) = 292
```

292 is the number, which Doctor Titibits has asked for.

The recursive program is very transparent and easy to be proved as correct, but it is very inefficient, i.e. for higher amounts of money calculation time increases and one can meet a stack overflow.

Task 1: Accomplish the program, that the output includes the numer of recursive calls
Hint: Calculation can be made more efficient applying repeating commands as ITERATES or LOOP.

Task 2: Write an iterative program to solve the Money-Change-Problem.

Task 3: A letter must be stamped with n = 18 copecks. Available are stamps with value $d_1 = 4$, $d_2 = 6$ and $d_3 = 10$ copecks. Find the recursion formula and the respective *DERIVE*-program for the general case.

Reference: *Engel, A.: Mathematisches Experimentieren mit dem PC, Stuttgart: Klett, 1991*

Hi Albert, Theresa and Josef,

Please find attached a dfw file that solves linear differential equations (of order greater than 2) and sets of simultaneous linear differential equations.

Linear Ordinary Differential Equations

written by

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The definitions of the functions defined in this Derive worksheet are hidden. To view the definitions, issue the Declare > Function Definition command and click on the pull-down menu button at the right end of "Function Name and Arguments" field. But I don't recommend it.

This dfw file finds the analytical solution of linear ordinary differential equations using Laplace transforms. You may ask "why bother, when the packages in ODE1.mth and ODE2.mth can solve first and second order linear equations". Well, the reason is that the Laplace transform method can solve third, fourth, fifth, ... order linear differential equations. This package also finds analytical solutions to systems of linear differential equations.

`LinearODE(v, v0, x, init)` solves linear ordinary differential equations. If a linear differential equation is, say, ay'''+by''+cy'+dy=f : v is the form [a,b,c,d,f]; v0 are the initial conditions; x is the independent variable; init is the independent variables initial value and has a default value of 0. For example to solve the 2nd Order differential equation

x''+6x'+9x=sin(t) where x(0)=0 and x'(0)=0

Simplify

```
LinearODE([1, 6, 9, SIN(t)], [0, 0], t)
```

$$\frac{\hat{e}^{-3t}(5t+3)}{50} - \frac{3\cos(t)}{50} + \frac{2\sin(t)}{25}$$

Notice that the initial value of the independent variable t is 0. If the intial values are say x(1)=0 and x'(1)=0, i.e. t=1 then

Simplifiy

```
LinearODE([1, 6, 9, SIN(t)], [0, 0], t, 1)
```

$$\hat{e}^{-3t}\left(\frac{\hat{e}^{3}(5t-2)\cos(1)}{50} + \frac{\hat{e}^{3}(11-15t)\sin(1)}{50}\right) + \cos(t-1)\left(\frac{2\sin(1)}{25} - \frac{3\cos(1)}{50}\right) + \sin(t-1)\left(\frac{2\cos(1)}{25} + \frac{3\sin(1)}{50}\right)$$

A more demanding example. x'''(t) + 2x(t) = SIN(t), x(0)=0,x'(0)=0,and x''(0)=0

Simplify

```
LinearODE([1, 0, 0, 2, SIN(t)], [0, 0, 0], t)
```

$$-\hat{e}^{2^{1/3}\,t/2}\left(\frac{2^{1/3}\,(2^{2/3}+1)\,\mathrm{COS}\left(\frac{108^{1/6}\,t}{2}\right)}{6\,(2\,2^{1/3}-2^{2/3}+1)}+\frac{108^{1/6}\,(2^{2/3}-1)\,\mathrm{SIN}\left(\frac{108^{1/6}\,t}{2}\right)}{6\,(2\,2^{1/3}-2^{2/3}+1)}\right)+\frac{2^{1/3}\,\hat{e}^{-2^{1/3}\,t}}{6\,(2^{2/3}+1)}+\frac{\mathrm{COS}(t)}{5}+\frac{2\,\mathrm{SIN}(t)}{5}$$

`LinearODESim(v,dep,v0,t)` solves sets of simultaneous Linear Ordinary Differential Equations.

All differential equations should be arranged so that the derivatives and dependent variables are on the left hand side of the equation and the functions of the independent variables on the right hand side. As described below, we enter the differential equations in a matrix form.

A derivative is entered as a 3 element vector [const, variable, order]. For example

3x''(t) = `[3,x,2]`, each derivative is entered as a row of a matrix with the independent variable terms appended at the end. We would enter the differential equation

x''(t) + y'(t) + 3x(t) = 15 e^{-t} as

```
[[1,x,2;1,y,1;3,x,0],15#e^(-t)]
```

So the set of simultaneous equations

x''(t) + y'(t) + 3x(t) = 15 e^{-t}
y''(t) - 4x'(t) + 3y(t) = 15 sin(2t)

would be entered as

```
[[1,x,2;1,y,1;3,x,0],15#e^(-t);[1,y,2;-4,x,1;3,y,0],15*SIN(t)]
```

`dep` is a vector of the dependent variables, `v0` is a matrix of initial conditions in the order presented in the vector `x`.
`e.g. if dep=[x,y]` and the initial conditions are `x(0)=2,x'(0)=1,y(0)=-1,y'(0)=3` then `v0=[2,1;-1,3]`.

`t` is the independent variable.

So to solve the system

x''(t) + y'(t) + 3x(t) = 15 e^{-t}
y''(t) - 4x'(t) + 3y(t) = 15 sin(2t)

with the initial conditions x(0) = 2, x'(0) = 1, y(0) = -1 , y'(0)= 3

Simplify

$$\text{LinearODESim}\left(\left[\begin{bmatrix}\begin{bmatrix}1 & x & 2\\ 1 & y & 1\\ 3 & x & 0\end{bmatrix} & 15\,\hat{e}^{-t}\\ \begin{bmatrix}1 & y & 2\\ -4 & x & 1\\ 3 & y & 0\end{bmatrix} & 15\,\text{SIN}(t)\end{bmatrix}\right], [x, y], \begin{bmatrix}2 & 1\\ -1 & 3\end{bmatrix}, t\right)$$

$$\left[x = 3\,\hat{e}^{-t} - \frac{31\,\text{COS}(3\,t)}{64} + \frac{3\,\text{SIN}(3\,t)}{4} - \frac{33\,\text{COS}(t)}{64} + \frac{(28 - 15\,t)\,\text{SIN}(t)}{16},\ y = - 3\,\hat{e}^{-t} - \frac{3\,\text{COS}(3\,t)}{2} - \frac{31\,\text{SIN}(3\,t)}{32} + \frac{(28 - 15\,t)\,\text{COS}(t)}{8} + \frac{153\,\text{SIN}(t)}{32}\right]$$

Heart-Transplantation for a Matrix

Josef Böhm

Maybe that this is a useful tool for you, if working with matrices. Some time ago I was asked if it is possible to exchange a submatrix of a given matrix by another one. So for example, we would like to exchange the marked submatrix of `a1` by the 2×3-matrix `b1`.

$$\left[a1 := \begin{bmatrix}1 & 2 & 3 & 4\\ 5 & \boxed{6} & \boxed{7} & \boxed{8}\\ 9 & \boxed{10} & \boxed{11} & \boxed{12}\\ 13 & 14 & 15 & 16\end{bmatrix},\ b1 := \begin{bmatrix}x & y & z\\ u & v & w\end{bmatrix}\right]$$

The *DERIVE*-Help file gives some advice how to exchange elements and rows, e.g.

```
a  := [x, y, z, u]
 1

[x, y, z, u]

    [ x   y   z   u ]
    [ 5   6   7   8 ]
a = [ 9  10  11  12 ]
    [13  14  15  16 ]
```

Replaces the 1st row.

```
a↓↓1 = [x, 5, 9, 13]

a↓↓1 := [0, 0, 0, 0]

[0, 0, 0, 0]

    [ x   y   z   u ]
    [ 5   6   7   8 ]
a = [ 9  10  11  12 ]
    [13  14  15  16 ]
```

Does **not** replace the 1st column.

```
r1 := [r, r, r, r]

r2 := [s, s, s, s]

a1        := [r1, r2]
  [1, 3]

[ r  r  r  r ]
[ s  s  s  s ]

     [ r   r   r   r ]
     [ 5   6   7   8 ]
a1 = [ s   s   s   s ]
     [13  14  15  16 ]
```

Replaces rows 1 and 3.

Maybe that with some tricky transposing there and back again, we could *DERIVE* convince to also replace the columns, but:

```
a      ↓↓[2, 3] := [ e  f ]
 [2, 3]             [ g  h ]

[ e  f ]
[ g  h ]

    [ r   r   r   r ]
    [ s   s   s   s ]
a = [ 9  10  11  12 ]
    [13  14  15  16 ]
```

I found no way to replace the center of the matrix!!

So, what do do?

I said to myself: "Write your own program!!"
(It is on the diskette!!)

submat(targetmatrix, upper-left corner of the replacement, replacement-matrix)

$$\text{submat}\left(a,\ [2,\ 2],\ \begin{bmatrix} e & f \\ g & h \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & e & f & 8 \\ 9 & g & h & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}$$

.... and it works properly (without the necessary confirmation of the *DERIVE*-implemented - incomplete – substitution, recommended in the Help-file). How to replace a single column?

`submat(a1,[1,3],[0;0;0;0])` results in:

$$\text{submat}\left(a1,\ [1,\ 3],\ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 & 0 & 4 \\ 5 & 6 & 0 & 8 \\ 9 & 10 & 0 & 12 \\ 13 & 14 & 0 & 16 \end{bmatrix}$$

... and this is our example from the begin:

$$\text{submat}(a1,\ [2,\ 2],\ b1) = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & x & y & z \\ 9 & u & v & w \\ 13 & 14 & 15 & 16 \end{bmatrix}$$

Arrows in 3D-Graphics

Lorenz Kopp, 07.2002, e-mail: L.Kopp@T-online.de

Examples to use with the following procedures:

I prefer 3×1-matrices (columns) instead of 1×3-vectors (rows) in 3D-geometry, because it's common practice to show vectors as columns. Two simple procedures (see `Vz` and `Ma`) make it possible to use the different axioms in matrix- and vector-calculation.

$$\left[o := \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},\ a := \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix},\ b := \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix},\ c := \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix},\ d := \begin{bmatrix} -4 \\ 3 \\ 2 \end{bmatrix},\ e := \begin{bmatrix} -2 \\ 6 \\ 2 \end{bmatrix}\right]$$

Procedures:

`Vz`: converts n×1-matrices to 1-n-vectors (for graphs or some calculations)

`Ma`: converts 1×n-vectors to n-1-matrices

$$Vz(a\_) := \text{VECTOR}(a\_{}_{i,1},\ i,\ 1,\ \text{DIM}(a\_))$$

$$Ma(a\_) := \text{VECTOR}([a\_{}_{i}],\ i,\ 1,\ \text{DIM}(a\_))$$

or:

```
Vz(a_):=a_`sub 1
Ma(a_) := [a_]`
```

Two examples for using columns and the quote-operator:

1. Linear combination of 3 vectors, first form for calculations and second form to get a nice term.

$$\text{LinCom3}(a\_,\ b\_,\ c\_) := \lambda\ a\_ + \mu\ b\_ + \nu\ c\_$$

$$\text{LinCom3C}(a\_,\ b\_,\ c\_) := {'}(\lambda\ a\_ + \mu\ b\_ + \nu\ c\_)$$

$$\text{LinCom3}(a,\ b,\ c) = \begin{bmatrix} \lambda + 2\mu + \nu \\ 2\lambda - \mu + \nu \\ 3\lambda + 2\mu - 3\nu \end{bmatrix}$$

$$\text{LinCom3C}(a,\ b,\ c) = \lambda \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \mu \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} + \nu \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}$$

2. Plane, first form also for calculations and plotting, second form to get a nice equation.

```
Plane(a_, s_, u_, t_, v_) := Vz(a_ + s_ u_ + t_ v_)
```

$$\text{PlaneC(a\_, s\_, u\_, t\_, v\_)} := \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \text{'(a\_ + s\_ u\_ + t\_ v\_)}$$

```
Plane(b, s, c, t, d)
PlaneC(b, s, c, t, d)
```

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} + s \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix} + t \begin{bmatrix} -4 \\ 3 \\ 2 \end{bmatrix}$$

Arrowheads plotted as pyramids with quadratic base:

Arrowheads plotted as pyramids with quadratic base:

hp:	height of pyramid,
rp:	radius of circumcircle of base
Per3(a_):	a vector perpendicular to a_ (3×1-mat.),
NV1(a_,s_):	the unity-vector to Lot(AS) (1×3-vec.),
NV2(a_,s_):	unity-vector perpendicular to NV1 and AS (1×3-vec.),
MpPy(a_,s_):	centre of pyramid-base

```
[hp := 0.5, rp := 0.1]

Per3(a_) :=
  If a_↓1↓1 = 0
     [1; 0; 0] + [0; a_↓3↓1; - a_↓2↓1]
     If a_↓2↓1 = 0
        [0; 1; 0] + [a_↓3↓1; 0; - a_↓1↓1]
        [a_↓2↓1; - a_↓1↓1; 0] + [- a_↓3↓1; 0; a_↓1↓1]
```

$$\text{NV1(a\_, s\_)} := \text{Vz}\left(\frac{\text{Per3(s\_ - a\_)}}{|\text{Per3(s\_ - a\_)}|}\right)$$

$$\text{NV2(a\_, s\_)} := \text{CROSS}\left(\text{NV1(a\_, s\_)}, \text{Vz}\left(\frac{\text{s\_ - a\_}}{|\text{s\_ - a\_}|}\right)\right)$$

$$\text{MpPy(a\_, s\_)} := \text{Vz}\left(\text{s\_} - \frac{\text{hp (s\_ - a\_)}}{|\text{s\_ - a\_}|}\right)$$

Available are the following functions (I omit printing the code. You can inspect it opening the file from the diskette, J)

ArHd(a_,s_):	arrowhead of arrow AS (surface area of a pyramid)
Ar3D(a_,s_,v_):	arrow AS, moved with vector v_,
OAr3D(a_):	arrow beginning at O, ending at point a_
ArClass3D(a_,k_,n_):	class of n_ arrows moved with a random-vector

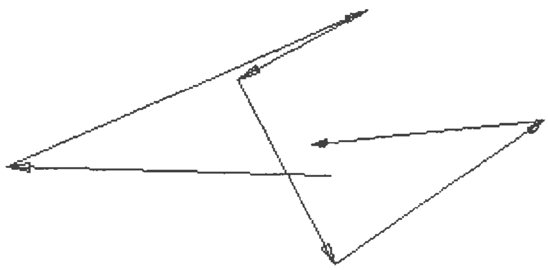
For the Examples:

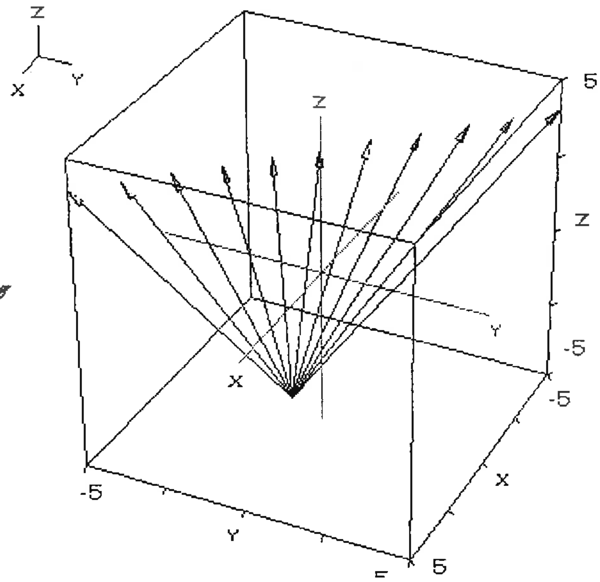
Highlight the expression or parts of a list and plot it in 3D-Graphics.
Settings of 3D-plot window: Window - Tile Vertically, Options - Simplify Before Plotting, Set - Plot Range - Reset (-5 < x, y, z < 5)
.

A Chain of Arrows (left) and a Bundle of Arrows (right):

$$\left[\text{OAr3D}\begin{bmatrix}5\\-5\\0\end{bmatrix},\ \text{Ar3D}\left(\begin{bmatrix}5\\-5\\0\end{bmatrix},\ a,\ o\right),\ \text{Ar3D}(a,\ b,\ o),\ \text{Ar3D}(b,\ c,\ o),\ \text{Ar3D}(c,\ d,\ o),\ \text{Ar3D}\left(d,\ \begin{bmatrix}-5\\-5\\5\end{bmatrix},\ o\right)\right]$$

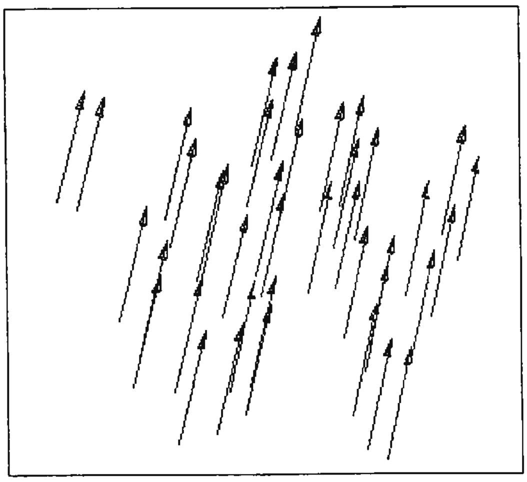
$$\text{VECTOR}\left(\text{Ar3D}\left(\begin{bmatrix}2\\0\\-3\end{bmatrix},\ \begin{bmatrix}-i\\i\\4\end{bmatrix},\ o\right),\ i,\ -5,\ 5\right)$$





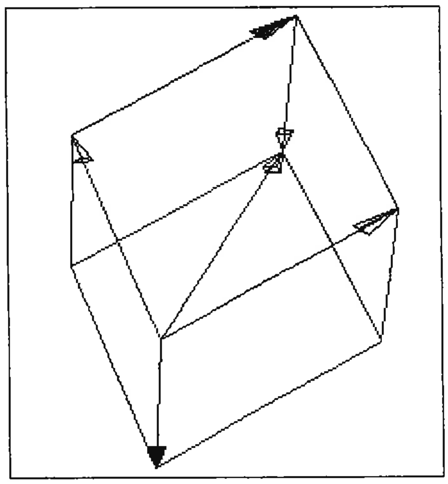
A class of arrows (repeat plotting)

`ArClass3D(-c, 4, 10)`



A spar built of a, b and c:

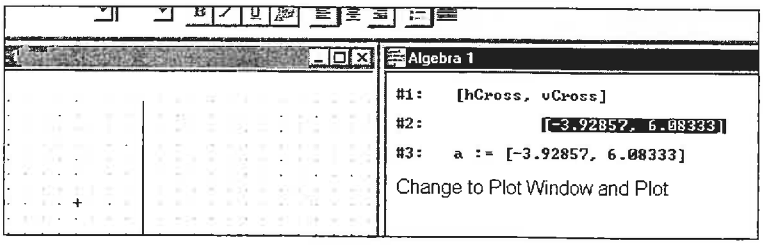
`LinKom3Graph(1, a, 1, b, 1, c)`

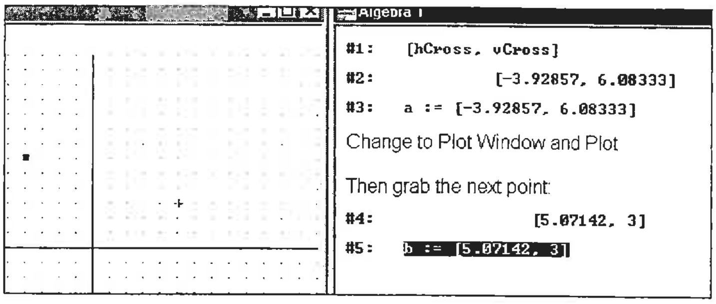


Use your 2D-PLOT- (GRAPH-) Window as a Sketchpad

Josef Böhm, Würmla

Preparing a paper or a worksheet I often need a sketch of a triangle, quadrangle or any other figure including special angles, which I want to be marked by little arcs. It is easy to plot the polygons, but is not so easy to produce the arcs. So use your *DERIVE* 2D-Plot Window as a sketchpad. Let's assume that you want to have any pentagon with some angles specially marked. Tile the screen – or not – and move the cursor to the first point of your figure, edit `[hcross,vcross]`, confirm and approximate in the Algebra Window. Assign the pair of the Cross-coordinates to a variable name, say `a`, you can immediately plot the point (even without approximating).

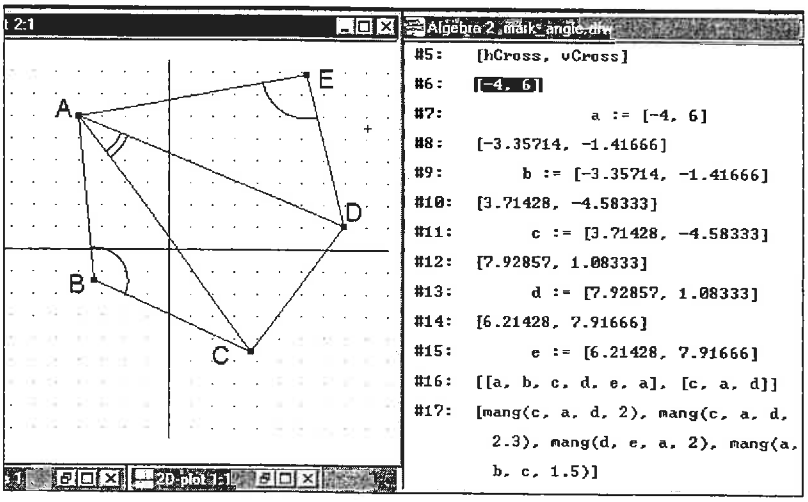




As I want to work with the points, I perform the approximation and the assignation as well.

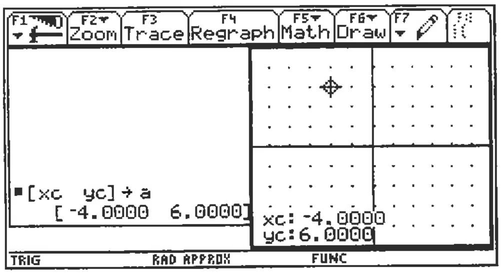
I produce five points. Using variables `a` to `e` in order to be free to work with the coordinates (two point form, working with vectors, ...). But I am still missing the arcs to outline the angles <ABC, <CAD and <AED!

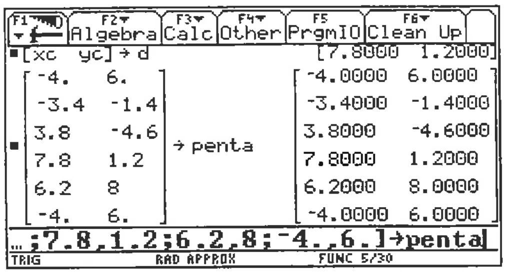
I wrote a utility-function `mang(pt1,pt2,pt3,radius)` (and a German version `wbog( )`) for drawing the arcs. This could be a nice project for pupils, because it might be not so easy to handle the various cases to fix the angle inside of the polygon and to understand the arctans of the direction vectors or the `phases` (*DERIVE*) or `angles` (on the *TI*s) if working with complex numbers, etc.

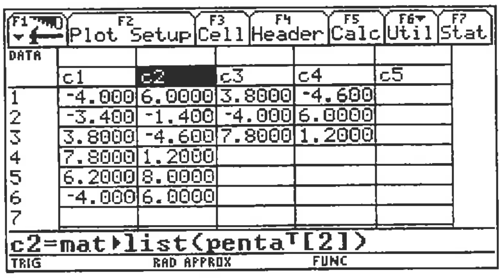


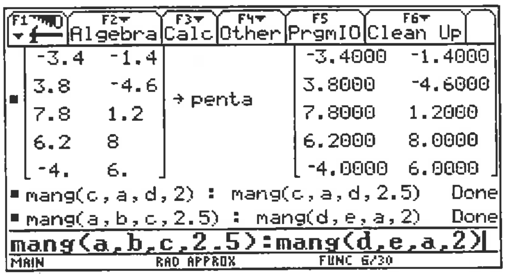
Additionally I don't like to enter for each arc the individual parameter values to plot it. So I reparameterized the function, which allows to enter for the boundaries for the parameter as 0 and 1 for all plots. This is not necessary on the *TI*, because of the `drawparm`-command.

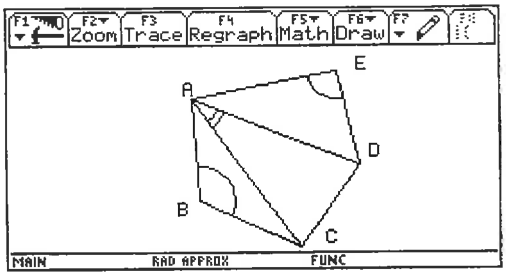
The very same can be done on the *TI*. You can follow the pictures below. `xc` and `yc` are the system variables connected with the cursor position in the [GRAPH]-Window. Navigate across the screen and fix your points (say a to e). For plotting transfer the data in any way to the `Data/Matrix-Editor` and set up an `xyline`-Plot for the pentagon together with the two of its diagonals.
Then call the arcs, like in *DERIVE* with the function `mang()` or `wbog()`.











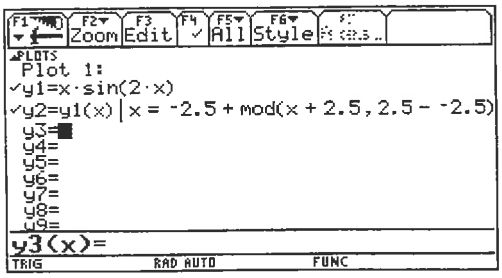
I hope, that I didn't forget any special case. I tried successfully marking the angles of several polygons.

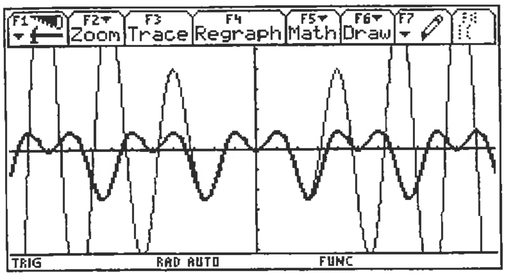
Milton Lesmes Acosta from Bogota, Colombia,

presents a nice tipp for the CAS-users. He wrote:

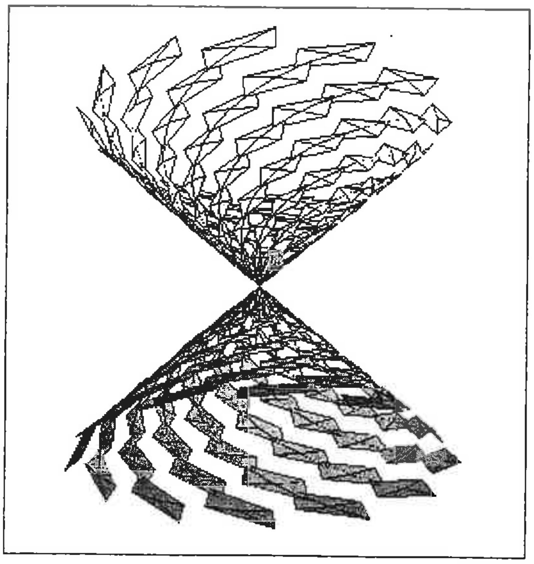
Now I am working with **periodic functions**. I don not know if it is interesting but the following tool which I found in these days is very practical.

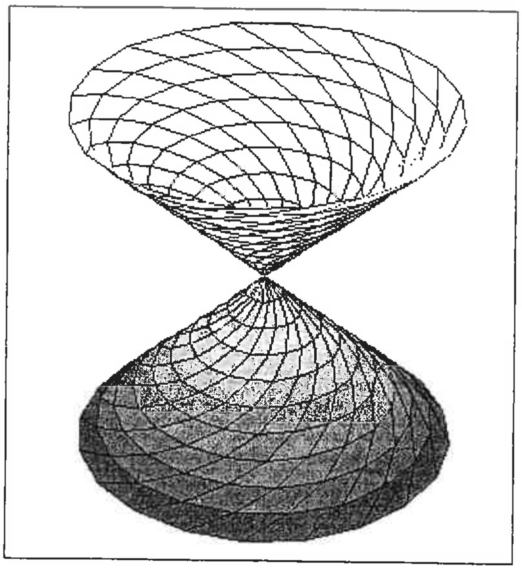
Define a function $f(x)$ in a real interval (a,b), then plot $f(a+\text{mod}(x-a,b-a))$. The result is a periodic function defined in R. This works in the same way in *DERIVE (see function* `mperiod` *on diskette)*.



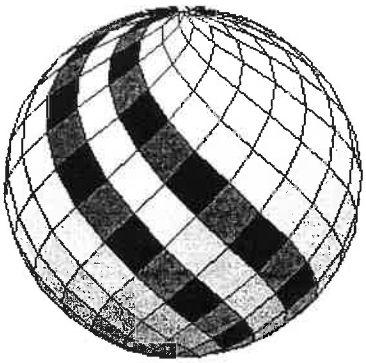


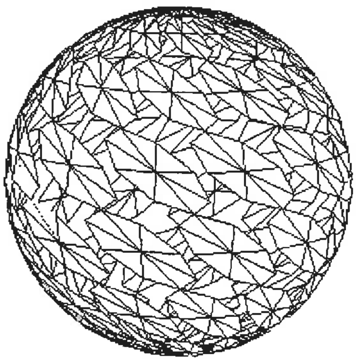
Some figures resulting from Peter Schofield´s collection of 2D- and 3D-tools (page 8).

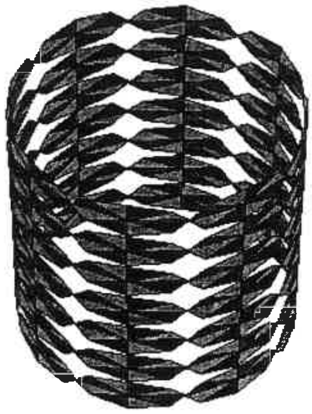




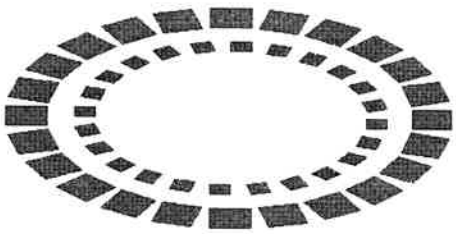
```
conepat1([0, 0, 5], 20, 20)
conepat1(STR(RPOLY(4, 1, [0, 0, 5]), [0.2, 0.4, 1]), 12, 16)
```

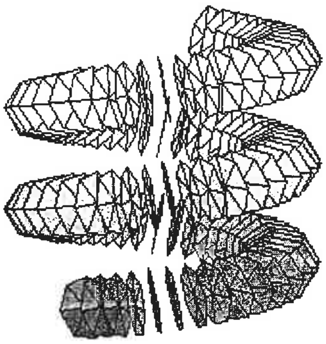


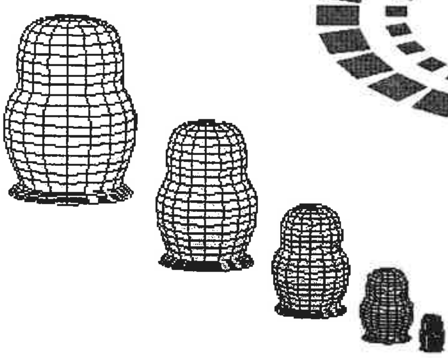




Examples with patterns on spheres and cylinders, patterns in the plane, snakes and spirals consisting of polygons, babushkas, ...
Download Peter´s files and experiment on your own.







When I collected all papers for editing the VISIT-ME-Proceedings and converted them into pdf-format I was attracted by Elvira´s lecture, which I unfortunately could not attend at the Conference because of my several duties as one of the organizers. The lecture was held in German, and the paper is also in German. Immediately I found it a pity that it would not be accessible for non German speaking readers.

I wrote to Elvira and asked for permission to publish a translation in the DNL. Fortunately she agreed, for which I am very grateful, and proposed to send a translation, but unfortunately the translation has not arrived until now and I want to be in time sending the DNL. So I decided to start with the contribution preparing the first part by myself.

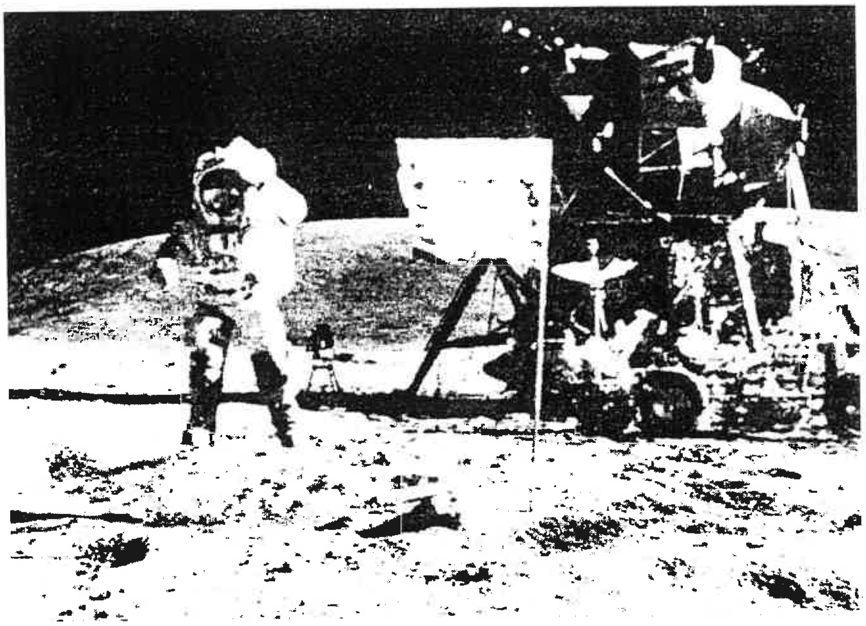
Elvira used in her presentation the TI-83+ and the TI-92 to work with CellSheet and – this is also very interesting – demonstrated how to link Excel and the handheld technology by the CellSheet Converter.

In addition to the TI-83+ I use the TI-92/Voyage screens through the paper and will add some DERIVE related comments. It should be easy for you to decide which parts can be done using graphing calculators, and which not. For our German members being teachers in Secondary school I recommend reading Elvira´s paper (VISIT-ME-2002 CD), containing ready made worksheets for an immediately use in class. A possible "DERIVE-performance" will be given in the next issue of the DNL.

By begin of November my wife and I were in Orlando, FL, at the ICTCM and we took the occasion to visit Kennedy Space Center. This visit was a second motivation to include Elvira´s article. And now, let´s finish the Count Down and have a Lift Off.

With the SINE-function on the Trail of the Moon (1)

Elvira Malitte, Halle, Germany
malitte@mathematik.uni-halle.de



Do you remember Neil Armstrong´s words?

"*How does duration of moonlight change in course of time?*" This question is raised automatically, if pupils accomplish the task collecting data for moonrise and moonset from newspapers. Based on real data we present a sequence of problems , which leads step by step to an appropriate mathematical description.

The data preparation and the mathematical treatment of the functional relation "*Day → Duration of Moonlight*" form basic aspects of this sequence. Graphic visualization leads to the idea of a curve fit by a sine-function. This idea will be considered and realized step by step.

One can use this sequence to motivate and to introduce trigonometric functions, but also in order to practise, consolidate or repeat knowledge acquired earlier. Disclaiming the sine-curve-fit, i.e. setting the functional relation in the center of the problem, you can work in secondary level 1.

Realising the sequence of tasks many tools can be used in a meaningful way. Here we start demonstrating the opportunities using a graphing calculator[1]. The graphing calculator proves to be an appropriate and efficient tool. By using it, mastering the task turns out to become an interesting and multifarious problem. With a CAS available additional considerations are possible. Take e.g. the special `CellSheet` application, which is implemented in the new generation of the symbolic *TI*-devices – and which can be implemented using flash-technology into the *TI*-89/92+. Working on a computer one can use Excel.

In some cases it might be desireable to exchange the data between the various tools/programs. We have now the *TI-CellSeet-Converter* available, which makes possible exchanges of data between the *TI* and Excel. We will demonstrate this process.

The use of various tools opens the opportunity of comparing observations and demonstrates the advantages and boundaries of their use.

Evaluation of worksheets 1 and 2 (with the TI-83+ and the TI-92)

(The worksheets 1 and 2 are on the next two pages.)

The given data are entered as lists or into the `Data/Matrix-Editor`. Supported by the graphic representation – which has to be discussed – one can set up conjectures about the underlying connection between the numbers of the days and the duration.

It is necessary to incite a discussion about the *discrete domain* for the day numbers and the *continous view of the function* "Moonlight Duration" and to reflect on the interpretation of the results.

Prposing the function type *sine* forms the starting point for the fit of the parameters which will be described later.

Essential for an *active* understanding of the discovered relationship is to use the derived function for purposes of extrapolation and interpolation.

The data are entered as lists or colums. The formulae are entered in the heads of the lists/columns.

The requested formula of Worksheet 2, task 1 b) is

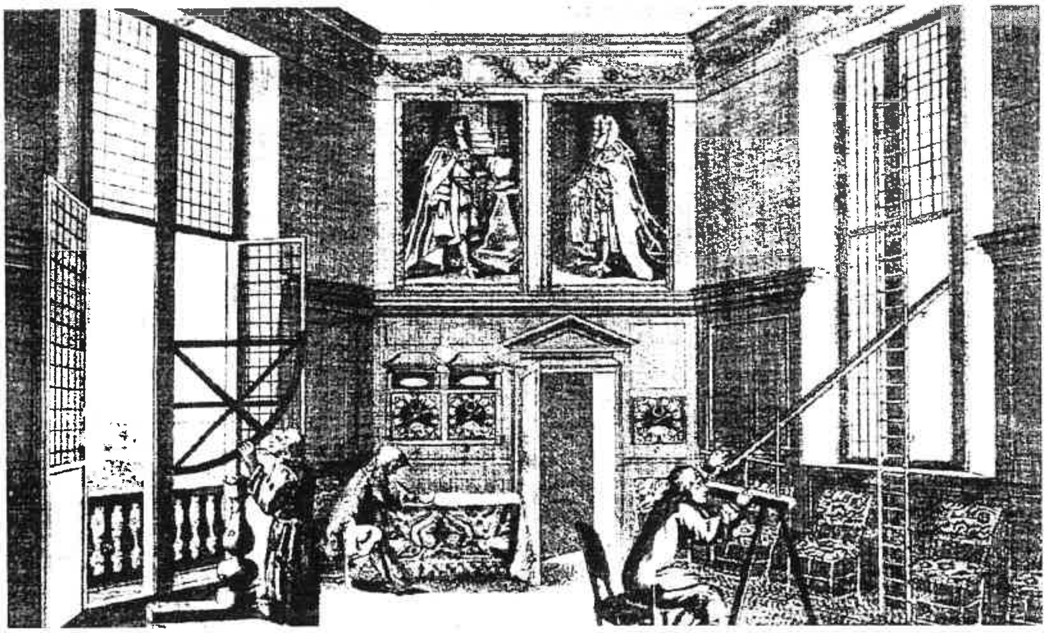
$L_6=L_4-L_2+(L_5-L_3)/60$ (on the CAS-*TI*s take columns `c1` – `c6`.)

If there is no change of days between MR and MS, then the formula returns the correct duration, but if there is a change, then it returns a negative number. (Take 1 July: -12.17) In this case one has to add 24 ("midnight").

Both cases can be combined on the *TI*-83+ using a little trick. The boolean expression $L_6<0$ returns in case of being true the value 1, in case of being false the value 0. (This does not work on the other *TI*s, here we receive `true` or `false`.)

So the final duration is calculated by $L_6+24(L_6<0)$.

[1] Reference: W. Herget, E. Malitte, K. Richter 2002



Observatory in Greenwich: the octogonal room is equipped with extra high windows in order to enable astronomic observations using special devices. Engraving by Francis Place, ~ 1700

Moon has risen

Worksheet 1

From the news one can find the data for moonrise (MR) and moonset (MS).
One student collected the data from 1 July through 28 August and presented the following table. On Sundays the news does not appear and sometimes the pupil forgot to note the times. This is to explain the gaps in the table.

Nr.	Date	MR	MS
1	01.07.	13:02	00:52
2	02.07.	14:06	01:14
3	03.07.	15:09	01:35
4	04.07.	16:13	01:59
5	05.07.		
6	06.07.	17:45	02:24
7	07.07.	19:19	03:29
8	08.07.	20:16	04:12
9	09.07.	21:07	05:03
10	10.07.	21:52	06:02
11	11.07.	22:30	07:09
12	12.07.		
13	13.07.	23:32	09:38
14	14.07.	23:59	10:52
15	15.07.		12:10
16	16.07.	00:25	13:27
17	17.07.	00:52	14:44
18	18.07.	01:22	16:00
19	19.07.		
20	20.07.	02:35	18:23

Nr.	Date	MR	MS
21	21.07.	03:22	19:24
22	22.07.	04:16	20:17
23	23.07.	05:17	21:00
24	24.07.	06:22	21:37
25	25.07.	07:29	22:07
26	26.07.		
27	27.07.	09:42	22:56
28	28.07.	10:48	23:18
29	29.07.	11:52	23:40
30	30.07.	12:56	
31	31.07.		
32	01.08.		
33	02.08.		
34	03.08.	17:05	01:26
35	04.08.	18:03	02:05
36	05.08.	18:57	02:52
37	06.08.	19:45	03:47
38	07.08.	20:26	04:51
39	08.08.	21:02	06:02
40	09.08.		

Nr.	Date	MR	MS
41	10.08.	22:02	08:36
42	11.08.	22:30	09:56
43	12.08.	22:57	11:15
44	13.08.	23:26	12:33
45	14.08.	23:58	13:50
46	15.08.		15:04
47	16.08.		
48	17.08.	01:19	17:17
49	18.08.	02:09	18:11
50	19.08.	03:07	18:57
51	20.08.	04:09	19:36
52	21.08.	05:15	20:08
53	22.08.	06:22	20:35
54	23.08.		
55	24.08.	07:38	21:08
56	25.08.	08:34	21:22
57	26.08.	09:38	21:44
58	27.08.	10:42	22:08
59	28.08.	11:45	22:29
60	29.08.		

► **Can we deduce interesting facts from this table?** **Let us try!**

How does Moonlight Duration change in Course of Time?

Worksheet 2

Requested is a representation of moonlight duration versus number of the day in the system of coordinates.
For this purpose it would be useful to have the duration in decimal numbers.

1 How can we calculate the duration from the data using the calculator?

a) In order to work with the data, transfer them as lists to your device. But use only data records which are complete.

Number of the day — L_1 on the *TI*-83+ (`c1` in the `Data/Matrix-Editor`)
MR-hour
MR-minute
MS-hour
MS-minute

b) Attach an additional list/column (L_6 / `c6`) containing the difference between MS and MR – but in decimal numbers!

Formula for the difference between MR and MA, in decimal notation:	L_6 / `c6` = ...

Attention:
There are two different cases occuring the data records: Between MR and MS we can find a change of days – or we don not. This must be considered in the final formula.

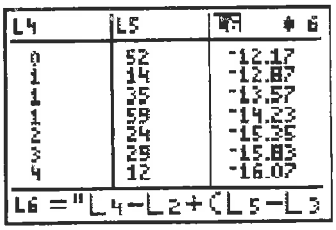
Add another list/column for the correct moonlight duration.

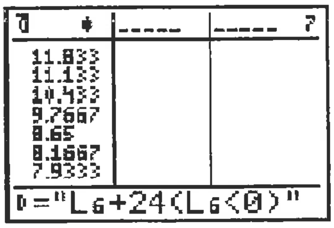
Formula for the duration of moonlight, in decimal notation:	D = ...

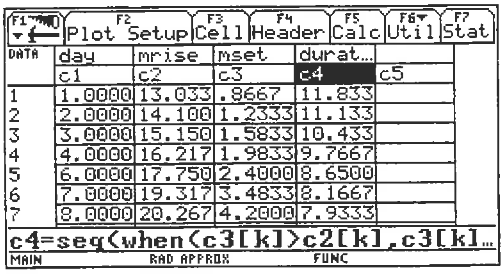
2 Represent graphically the relationship between the number of the day and the respective duration of moonlight.

3 How could we accomplish the missing data of the given table in a meaningful way?

4 Which duration of moonlight can be estimated for 30 September?







DATA	day	mrise	mset	durat...	
	c1	c2	c3	c4	c5
1	1.0000	13.033	.8667	11.833	
2	2.0000	14.100	1.2333	11.133	
3	3.0000	15.150	1.5833	10.433	
4	4.0000	16.217	1.9833	9.7667	
5	6.0000	17.750	2.4000	8.6500	
6	7.0000	19.317	3.4833	8.1667	
7	8.0000	20.267	4.2000	7.9333	

On the TI-92/Voyage I directly entered the data as decimal numbers (13+2/60, ...) *and calculated the duration time using the* `when( )`*-function. But attention:*

You are not able to only define `c4=when(c3>c2,c3-c2,c3-c2+24)`*. This does not work correctly, because of the TI's strange processing of the* `when`*-function in a* `Data`*-table. (See an earlier DNL.)*

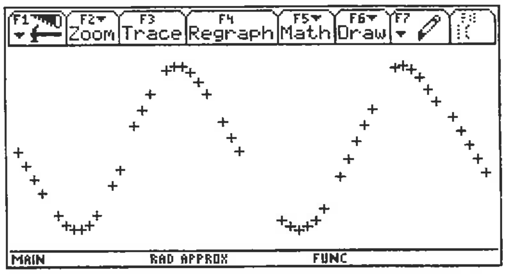
You have to define a sequence, adressing each single element by its index, say k*.*

The complete line reads as follows:

```
c4 = seq(when(c3[k]>c2[k],c3[k]-c2[k],c3[k]-c2[k]+24),k,1,dim(c1))
```

Setting the appropriate WINDOW-values and graphic representation

$0 \leq x \leq 60$ and $7 \leq y \leq 17$



How could we fill the gaps in the table in a meaningful way?

Let's take day #15:

Way 1

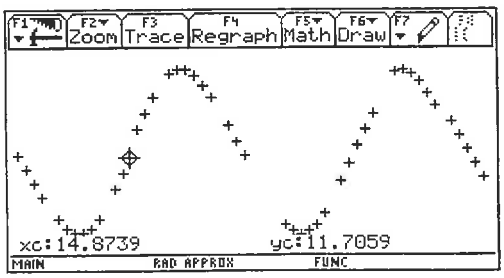
Navigating the Cursor in the Graph-Window enables a rough estimation of intermediate values. Mostly it is not possible to obtain integer values caused by resolution of the screen.

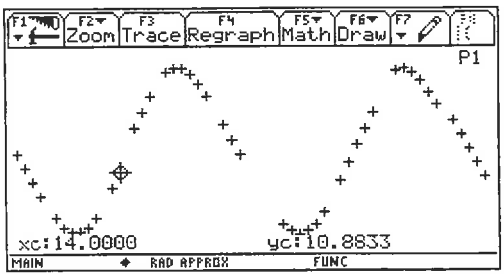
A pupil might obtain an estimation of 11.7 hours, which makes approximately 11 hours and 40 minutes.

Way 2

Using `Trace` we can move directly on the graph, i.e. jump from one point to the next one. Doing so we cannot reach any intermediate values. Only the x-values of list L_1 (c1) are available.

So take the mean of duration(14) and duration(16) = (10.88+13.03)/2 = 11.96 hours which is approximately 11h 57 m.





(Will be continued in DNL#49. We will proceed in demonstrating CellSheet and fitting the curve.)

Titbits from Algebra and Number Theory (24)

by Johann Wiesenbauer, Vienna

Well, did you follow my advise and have a go at one of the "programming challenges" at the end of my last column? Yes? Then make yourself comfortable and enjoy the following pages where I'm going to describe some more or less sophisticated ways to tackle these problems. It's up to you then to compare the solutions given here with your own solution in order to see if you missed something. (Don't worry, I'll give no bad marks though!)

This was the first problem:

How many ways can you make a dollar using pennies, nickels, dimes, quarters, half-dollars and dollar coins? (For European readers here the corresponding amounts in cents: 1,5,10,25,50,100.)

Maybe the most obvious approach is to set up six variables, say p, n, d, q, h, b for the amounts of pennies, nickels, dimes, quarters, half-dollars and bucks, er.. dollars, and to check within nested loops all the possibilities to get 100 for their total value p+5n+10d+25q+50h+100b (in cents). We could even save the loop for p by testing the condition 5n+10d+25q+50h+100b ≤ 100 instead, since the value of p is uniquely determined then.

In the following Derive-programs I considered the more general case, where the total amount of cents to be achieved is n instead of 100 as in the original problem. For one thing, any Derive-program needs at least one variable, even if it is a dummy variable, otherwise it won't be recognized by the parser as a program. For another thing, we will later test the quality of the underlying algorithm by choosing bigger values of n than n=100, say n= 10000 or n= 10^{100} (no joke!). Furthermore, I used a list of "indexed" loop variables, because this method can be more easily generalized to other programs, where even more nested loops are involved.

```
cents0(n, a_, c_ := 0, n_ := 5, p_, w_ := [100, 50, 25, 10, 5]) :=
  Prog
    a_ := VECTOR(0, k_, n_)
    Loop
      If w_·a_ ≤ n
         Prog
           c_ :+ 1
           p_ := n_
           a_↓p_ :+ 1
         Prog
           p_ :- 1
           If p_ = 0
              RETURN c_
           a_↓p_ :+ 1
           j_ := p_ + 1
           Loop
             a_↓j_ := 0
             j_ :+ 1
             If j_ > n_ exit
```

cents0(100) = 293 (0.02s)

cents0(1000) = 2103596 (142s)

I got the solution 293 quite fast for the original problem n=100, namely in 0.02s on my 2GHz-machine. On the other hand the huge increase of computation time for n=1000 clearly shows that this method works only for relatively small n.

When dealing with big n's the following solution is far better. Let's denote by a(n) the solution of our problem for n cents. Setting

a(0)=1, a(k)=0 for k<0

we have the following linear recursion for a(n) a(n)=a(n-1)+a(n-5)+a(n-10)+a(n-25)+a(n-50)+a(n-100)–

–a(n-(1+5)) –a(n-(1+10)) –...–a(n-(50+100) +

+a(n-(1+5+10))+...+a(n-(25+50+100))–

–....

–a(n-(1+5+10+25+50+100))

Actually you may view this as an example of the inclusion-exclusion principle, because in the first line all combinations are summed up, where a fixed coin out of the six different coins is used at least once, in the second line all combinations are subtracted where a fixed pair out of all 15 possible pairs of coins is used at least once, because they were counted twice, etc.

A more concise way to write this formula is

a(n) = v · (a(n-1),a(n-2),...,(n-191))

where

v is the precomputed vector of coefficients of the polynomial

(1-(1-x)(1-x^5)(1-x^10)(1-x^25)(1-x^50)(1-x^100))/x

starting with lowest power of x. In particular, v has got 191=1+5+10+25+50+100 components, but only 64 are nonzero.

$$v := \text{VECTOR}\left(\text{POLY\_COEFF}\left(\frac{1-(1-x)\cdot(1-x^{5})\cdot(1-x^{10})\cdot(1-x^{25})\cdot(1-x^{50})\cdot(1-x^{100})}{x}, x, k\_\right), k\_, 0, 190\right)$$

```
[1, 0, 0, 0, 1, -1, 0, 0, 0, 1, -1, 0, 0, 0, -1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 1, -1, 0, 0, 0, -1, 1, 0, 0, 0, -1, 1,
 0, 0, 0, 1, -1, 0, 0, 0, 0, 0, 0, 0, 0, 1, -1, 0, 0, 0, -1, 1, 0, 0, 0, -1, 1, 0, 0, 0, 1, -1, 0, 0, 0, 0, 0, 0,
 0, 0, -1, 1, 0, 0, 0, 1, -1, 0, 0, 0, 1, -1, 0, 0, 0, -1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 1, -1, 0, 0, 0, -1, 1, 0, 0,
 0, -1, 1, 0, 0, 0, 1, -1, 0, 0, 0, 0, 0, 0, 0, 0, -1, 1, 0, 0, 0, 1, -1, 0, 0, 0, 1, -1, 0, 0, 0, -1, 1, 0, 0, 0,
 0, 0, 0, 0, 0, -1, 1, 0, 0, 0, 1, -1, 0, 0, 0, 1, -1, 0, 0, 0, -1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 1, -1, 0, 0, 0, -1,
 1, 0, 0, 0, -1, 1, 0, 0, 0, 1, -1]
```

```
cents1(n, w_) :=
  Prog
    w_ := ADJOIN(1, VECTOR(0, k, 1, 190))
    Loop
      If n = 0
         RETURN FIRST(w_)
      w_ := DELETE(ADJOIN(w_·v, w_), -1)
      n :- 1
```

cents1(100) = 293 (0.05s)

cents1(1000) = 2103596 (0.45s)

cents1(10000) = 139946140451 (4.89s)

As you can see, the complexity of this algorithm is already much better, but even though we still wouldn't have the ghost of a chance to handle n's with say 100 digits.

But here comes our "Wunderwaffe" (miracle weapon) for this sort of combinatorial problems and its name is "generating power series". To be more precise, we are going to compute the formal power series

$$f(x) = \sum_{n=0}^{\infty} a(n)x^n$$

whose coefficients are exactly the numbers we are looking for. To give an explicit formula for f(x) is surprisingly easy, namely

$$f(x)=\sum_{p=0}^{\infty}x^{p}\cdot\sum_{n=0}^{\infty}x^{5n}\cdot\sum_{d=0}^{\infty}x^{10d}\cdot\sum_{q=0}^{\infty}x^{25q}\cdot\sum_{h=0}^{\infty}x^{50h}\cdot\sum_{b=0}^{\infty}x^{100b}=$$

$$=\frac{1}{(1-x)(1-x^{5})(1-x^{10})(1-x^{25})(1-x^{50})(1-x^{100})}$$

After all, a(n) is by definition the number of ways to write n as a sum p+5n+10d+25q+50h+100b and this is obviously the coefficient of x^n after expanding the product above.

Now let g(x) be the integer polynomial (!) of degree 409 that is defined by the quotient

$$g(x):=\frac{(1-x^{100})^{5}}{(1-x)(1-x^{5})(1-x^{10})(1-x^{25})(1-x^{50})}$$

Then f(x) can be rewritten in the form

$$f(x)=g(x)(1-x^{100})^{-6}=g(x)\sum_{k=0}^{\infty}(-1)^{k}\binom{-6}{k}x^{100k}=g(x)\sum_{k=0}^{\infty}\binom{k+5}{5}x^{100k}$$

It is now easy (though not for a human, but for DERIVE!) to get the coefficient of any power x^n, i.e. the number a(n), from the representation of f(x) above. The reader is referred to the following program for the details (note that simplification of g(x) has been omitted in the following for obvious reasons!)

```
                        100 5
                  (1 - x    )
g(x) := EXPAND -------------------------------------------------------
                       5         10          25          50
                (1 - x)·(1 - x )·(1 - x  )·(1 - x  )·(1 - x  )

list := VECTOR(POLY_COEFF(g(x), x, k), k, 0, 200, 5)
[1, 2, 4, 6, 9, 13, 18, 24, 31, 39, 50, 62, 77, 93, 112, 134, 159, 187, 218, 252, 287, 325, 364, 406, 449, 493, 538,
 584, 631, 679, 722, 766, 805, 845, 880, 910, 935, 955, 970, 980, 985]

cents2(n, r_, s_, t_ := 0) :=
  Prog
    s_ := FLOOR(n, 100)
    r_ := n - 100·s_
    c_ := COMB(s_ + 5, 5)
    Loop
      t_ :+ c_·list↓(FLOOR(MIN(r_, 409 - r_), 5) + 1)
      s_ :- 1
      r_ :+ 100
      If s_ < 0 ∨ r_ > 409
         RETURN t_
      c_ :* (s_ + 1)/(s_ + 6)
      k_ :+ 1

cents2(10000) = 139946140451

                  100
cents2(RANDOM(10     )) =

 603800094568827486286343647588935300139297181260474544576498661753585384335817157550098904933375271584612111328515~
 953158490936501210145208446179648091683563151736525142118205353988626884234190751896597936860273455379829040766542~
 873067284568487511410215455338633944189013665338526480478994942992817718310586694386735758412293230653977548752055~
 091399441742145375598320213019001178882128984986971814075372357719928019430543768003507253241746143769806849907879~
 674268529112544292567272309654552
```

As for the timings, they were 0.000s for all tested numbers. Even the huge 100-digit random number turned out to be a pushover for DERIVE, which it refused to take any notice of!

Let's turn to the second problem now:

Is there a way to relabel the faces of two dice such that the probability of rolling any sum will not change? (Note that the numbers of spots are assumed to be positive integers though not necessarily distinct!)

If $a_1,...,a_6$ and $b_1,...,b_6$ are the labels for the first and second die, respectively, then we obviously have to choose them in such a way that

$$(x^{a_1}+...+x^{a_6})(x^{b_1}+...+x^{b_6})=(x^1+x^2+x^3+x^4+x^5+x^6)^2$$

because this will guarantee the same probability distribution as for an ordinary pair of dice. In principle, we could use Derive for a brute-force approach, where we vary the vector $(a_1,...,a_6)$ just as we did the amounts p,n,..,b of different coins in our first example. I leave this out for lack of space here and give only the much shorter algebraic solution.

For this I used Derive to decompose the right hand side of the last equation into its irreducible factors:

$$\text{FACTOR}((x + x^2 + x^3 + x^4 + x^5 + x^6)^2) = x^2\cdot(x + 1)^2\cdot(x^2 + x + 1)^2\cdot(x^2 - x + 1)^2$$

After setting

$$f(x):= x^{a_1}+...+x^{a_6} \text{ and } g(x):= x^{b_1}+...+x^{b_6}$$

we know that f(1)=g(1)=6, since both dice have six faces, and f(0)=g(0)=0, as all labels are positive integers. From this and the factorization above we may conclude that

$$f(x)=x(x+1)(x^2+x+1)(x^2-x+1)^i \text{ and } g(x)= x(x+1)(x^2+x+1)(x^2-x+1)^j$$

with $i,j\in\{0,1,2\}$ and i+j=2. Since i=j=1 leads to the ordinary pair of dice, we have the essentially unique solution i=0 and j=2. The labels for these dice, the famous Sicherman dice, can be seen from

$$\text{EXPAND}(x\cdot(x + 1)\cdot(x^2 + x + 1)) = x^4 + 2\cdot x^3 + 2\cdot x^2 + x$$

$$\text{EXPAND}(x\cdot(x + 1)\cdot(x^2 + x + 1)\cdot(x^2 - x + 1)^2) = x^8 + x^6 + x^5 + x^4 + x^3 + x$$

namely 1,2,2,3,3,4 and 1,3,4,5,6,8, respectively.

Speaking of programming challenges, this reminds me of Rüdeger Baumann's challenge posed in DNL #46 and treated in some detail by Stephan Welke in DNL #47. As it deals with number theory I was actually supposed to comment on it in this column here, but somehow I forgot about it. (Sorry!)

Let me correct this mistake now. First of all, Rüdeger's sequence fieb(n) is very closely related to a special Lucas-sequence $L_0,L_1,L_2,...$ recursively defined by

$$L_0=2, L_1=1 \text{ and } L_n=L_{n-1}+L_{n-2} \text{ for } n>1$$

Here are the first 11 numbers of this sequence:

```
VECTOR(LUCAS(k_), k_, 0, 10) = [2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123]
```

To be more precise, we have

```
fieb(n) := LUCAS(n) - 1
```

for all $n\geq 0$, if we extend Rüdeger's sequence to the left by defining fieb(0)=1 and fieb(1)=0.

If we reformulate Rüdeger's conjecture, namely that p is always a divisor of fieb(p) if p is a prime it becomes the very well-known theorem that $L_p \equiv 1 \bmod p$, whenever p is a prime. Believe it or not, it was even mentioned in this column and used as a primality test more than once (cf. DNL #26, p 27 and DNL #28, p 40) ! In fact, it was even used there in the more general form

$$n \text{ prime} \Rightarrow V(n,P,Q,n) \equiv P \bmod n$$

where the case considered above is the special case P=1, Q= –1. Hence, the shortest form of Rüdeger's test is

```
test(n) := SOLVE(V_MOD(n, 1, -1, n) = 1)

         100
test(10     + 267) = true
```
(0.04s)

As you can see yourself this test is really very fast, which is the good news. The bad news: It is far from being new and moreover not very good at excluding composite numbers, at least when compared to other primality tests of this sort! Here is a far more sophisticated primality test that tests whether n is a so-called strong Lucas probable prime:

```
SLPP(n, P, Q, s_, t_) :=
  Prog
    s_ := n - JACOBI(P^2 - 4·Q, n)
    Loop
      s_ :/ 2
      If ODD?(s_) exit
    If U_MOD(s_, P, Q, n) exit
    t_ := V_MOD(s_, P, Q, n)
    Loop
      If t_ = 0
         RETURN true
      If 2·s_ = n ± 1
         RETURN false
      t_ := MOD(t_^2 - 2·MOD(Q^s_, n), n)
      s_ :* 2

         100
SLPP(10     + 267, 3, 2) = true
```
(0.09s)

What else are Lucas numbers good for? Well, you could use them not only for primality testing, where they are often used in combination with Rabin-Miller tests, but also for factoring. Have a look at the following example:

```
                                         331              331
GCD(V_MOD(LCM([1, ..., 40000]), 1, -1, 2     - 1) - 2, 2     - 1) = 16937389168607
```

This 14-digit factor p= 16937389168607 of the Mersenne number $2^{331}-1$ was found in only 17.3s due to the fact that p+1 has only relatively small prime factors:

```
                                  5
FACTOR(16937389168607 + 1) = 2 ·3·233·22273·33997
```

On the other hand, it's only a fifty-fifty chance, whether you are successful or not, even if the boundary, which is 40000 in our case, is large enough to cover all prime factors of p+1. In the example at issue, 5 must be a nonsquare mod p and we were lucky enough that this is fulfilled here:

```
JACOBI(5, 16937389168607) = -1
```

Computing square roots mod p, where p is an odd prime, is another nice application of Lucas numbers. This feature could be used in so-called zero-knowledge proofs and also in a nice cryptographic application known as "coin-tossing by telephone". More on this another time though! (j.wiesenbauer@tuwien.ac.at)